THE SCHWARTZ AND SMITH (2000) MODEL WITH STATE-DEPENDENT RISK PREMIA

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Abstract. In this paper, I prove the closed-form extension of the Schwartz and Smith (2000) model of commodity futures pricing to state-dependent risk premia. The extended model exhibits important additional flexibility in representing different term-structure patterns.

Keywords: Commodity futures pricing; Term structure of futures prices; State-dependent risk premia; Normal backwardation; Backwardation.

2010 AMS Subject Classification: 60G10, 60G15.

1. Introduction

The Schwartz and Smith (2000) model of commodity futures pricing has been widely used in the theoretical and empirical literature on commodity spot and derivatives markets, as it provides a way to disentangle the permanent ‘equilibrium’ component of the commodity spot price from its transitory component via futures price data. Primed by the studies of Fama

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Received May 5, 2015
and French (1987) and of Casassus and Collin-Dufresne (2005) on the importance of time-varying risk premia in commodity markets, Mirantes, Población, and Serna (2015) have recently proposed an important extension of the Schwartz and Smith (2000) model that considers state-dependent risk premia. Mirantes, Población, and Serna (2015) work out the general risk-neutral valuation scheme for a range of commodity contingent claims without, however, providing a fully explicit solution for the futures prices. Their main concern is investigating the impact of time-varying risk premia on commodity American options. I contribute (1) by deriving the fully closed form of futures prices from no-arbitrage restrictions written under the physical measure, which highlight the presence of the state-dependent risk premia, and (2) by detailing the incremental impact of such risk premia on the term structure of the futures prices.

2. No-arbitrage futures pricing

Schwartz and Smith (2000) assume that the spot log price of a given commodity is the sum of two components: \( \ln(S_t) = \chi_t + \xi_t \). The non-stationary ‘equilibrium’ component \( \xi_t \) is an arithmetic Brownian motion with \( \mathbb{P} \)-dynamics \( d\xi_t = \mu_\xi dt + \sigma_\xi d\xi_t \), where \( \mathbb{P} \) is the physical probability measure. The stationary component \( \chi_t \) is assumed to revert toward zero following an Ornstein-Uhlenbeck process with \( \mathbb{P} \)-dynamics \( d\chi_t = -\kappa\chi_t dt + \sigma_\chi d\chi_t \) (\( \kappa > 0 \)). Under no arbitrage in the commodity derivatives markets, the state price density \( \zeta_t \) has \( \mathbb{P} \)-dynamics

\[
d\zeta_t = \zeta_t \left( -r dt - \Lambda_{\xi,t} d\xi_t^\mathbb{F} - \Lambda_{\chi,t} d\chi_t^\mathbb{F} \right),
\]

where \( r \) is the riskfree rate. I depart from the Schwartz and Smith (2000) model by assuming state-dependent risk premia.

**Assumption** The market prices of risk are state-dependent,

\[
\Lambda_{\xi,t} = \lambda_\xi + \phi_\xi \chi_t \quad \text{(price of } \xi\text{-type risk)},
\]

\[
\Lambda_{\chi,t} = \lambda_\chi + \phi_\chi \chi_t \quad \text{(price of } \chi\text{-type risk)},
\]

and the speed of mean reversion remains positive after risk adjustment, \( \kappa + \sigma_\chi \phi_\chi > 0 \).

The original Schwartz and Smith (2000) model ensues by assuming away the dependence of \( \Lambda_{\xi,t} \) and \( \Lambda_{\chi,t} \) from the state \( \chi_t \) (\( \phi_\xi = 0 \) and \( \phi_\chi = 0 \)). Let \( F(\xi_t, \chi_t, \tau) \) be the current futures price
of the commodity for delivery in \( \tau \) years. The no-arbitrage restriction under \( \mathbb{P} \) for \( F(\xi_t, \chi_t, \tau) \) emphasizes the presence of the state-dependent risk premia:

\[
\begin{align*}
E_t^{\mathbb{P}}[dF] &= (F_{\xi} \sigma_{\xi} \Lambda_{\xi,t,t} + F_{\chi} \sigma_{\chi} \Lambda_{\chi,t,t}) dt, \\
F(\xi_t, \chi_t, 0) &= \exp(\chi_t + \xi_t).
\end{align*}
\]

The resulting no-arbitrage futures price is characterized in the following proposition.

**Proposition** The function \( F(\xi_t, \chi_t, \tau) \) that solves the problem (1) is

\[
F(\xi_t, \chi_t, \tau) = \exp(\xi_t + \chi_t A(\tau) + B(\tau)),
\]

with

(2) \[ A(\tau) = \left( 1 + \frac{\sigma_{\xi} \phi_{\xi}}{\kappa + \sigma_{\chi} \phi_{\chi}} \right) e^{-(\kappa + \sigma_{\chi} \phi_{\chi}) \tau} - \frac{\sigma_{\xi} \phi_{\xi}}{\kappa + \sigma_{\chi} \phi_{\chi}}, \]

(3) \[ B(\tau) = D \tau + G \left( 1 - e^{-(\kappa + \sigma_{\chi} \phi_{\chi}) \tau} \right) + H \left( 1 - e^{-(\kappa + \sigma_{\chi} \phi_{\chi}) \tau} \right), \]

\[
D = \left( \mu_{\xi} - \sigma_{\xi} \lambda_{\xi} + \frac{\sigma_{\xi}^2}{2} \right) - \left( \sigma_{\xi} \sigma_{\chi} \rho_{\xi \chi} - \sigma_{\chi} \lambda_{\chi} \right) \frac{\sigma_{\xi} \phi_{\xi}}{\kappa + \sigma_{\chi} \phi_{\chi}} + \frac{\sigma_{\chi}^2}{2} \frac{(\sigma_{\xi} \phi_{\xi})^2}{(\kappa + \sigma_{\chi} \phi_{\chi})^2},
\]

\[
G = \frac{\sigma_{\chi}^2 (\kappa + \sigma_{\chi} \phi_{\chi} + \sigma_{\xi} \phi_{\xi})^2}{4(\kappa + \sigma_{\chi} \phi_{\chi})^3},
\]

\[
H = \frac{(\kappa + \sigma_{\chi} \phi_{\chi} + \sigma_{\xi} \phi_{\xi}) \left( (\kappa + \sigma_{\chi} \phi_{\chi}) (\sigma_{\xi} \sigma_{\chi} \rho_{\xi \chi} - \sigma_{\chi} \lambda_{\chi}) - (\sigma_{\xi} \phi_{\xi}) \sigma_{\chi}^2 \right)}{(\kappa + \sigma_{\chi} \phi_{\chi})^3}.
\]

**Proof.** Under \( \mathbb{P} \), the ex-ante marking-to-market instantaneous gain on being long the futures contract is

\[
E_t^{\mathbb{P}}[dF] = \left( -F_{\tau} + F_{\xi} \mu_{\xi} - F_{\chi} \kappa_{\chi} + \frac{1}{2} F_{\xi}^2 \sigma_{\xi}^2 + F_{\xi} \sigma_{\xi} \sigma_{\chi} \rho_{\xi \chi} + \frac{1}{2} F_{\chi} \sigma_{\chi}^2 \right) dt,
\]
where \( d \langle z^\xi_t, z^\chi_t \rangle = \rho_{\xi \chi} dt \). Given the Ansatz \( \exp (\xi_t + \chi_t A(\tau) + B(\tau)) \), the no-arbitrage pricing problem (1) turns out to be a system of first-order ordinary differential equations in the time-to-maturity variable \( \tau \):

\[
\begin{align*}
-A' - A \kappa &= \sigma_\xi \phi_\xi + A \sigma_\chi \phi_\chi, \\
-B' + \mu_\xi + \frac{1}{2} \sigma^2_\xi + A \sigma_\xi \sigma_\chi \rho_{\xi \chi} + \frac{1}{2} A^2 \sigma^2_\chi &= \sigma_\xi \lambda_\xi + A \sigma_\chi \lambda_\chi, \\
A(0) &= 0, \\
B(0) &= 0.
\end{align*}
\]

Its solution is given by (2) and (3). This completes the proof.

Importantly, the exposure of the market price of \( \xi \)-type risk to the transitory component \( \chi_t \) \((\phi_\xi \neq 0)\) implies that, even if deprived of full unit-root persistence \((\kappa > 0)\) and \(\kappa + \sigma_\chi \phi_\chi > 0\), \(\chi_t\) has a futures-price impact that does not vanish as the delivery date diverges \((\tau \to +\infty)\):

\[ A(\infty) = -\frac{\sigma_\xi \phi_\xi}{\kappa + \sigma_\chi \phi_\chi} . \]

The next section visualizes and discusses the additional impact of state-dependent risk premia on the term structure of the futures prices.

3. Term-structure patterns

The analysis requires the expected spot price in \( \tau \) years from now, which Schwartz and Smith (2000) work out to be (in log levels)

\[
\ln E_t^p [S_{t+\tau}] = \xi_t + \chi_t e^{-\kappa \tau} + \left( \mu_\xi + \frac{\sigma^2_\xi}{2} - \frac{1}{4} \kappa \right) \tau + \frac{\sigma^2_\chi}{4 \kappa} (1 - e^{-2 \kappa \tau}) + \frac{\sigma_\xi \sigma_\chi \rho_{\xi \chi}}{\kappa} (1 - e^{-\kappa \tau}) .
\]

It will be plotted in black in the following figures. Another important pricing benchmark is the futures price prevailing at distant delivery dates, which is (in log levels)

\[ \xi_t + \chi_t A(\infty) + D \tau + G + H . \]

It will be plotted in grey. The futures log price \( \ln F(\xi_t, \chi_t, \tau) \) will be plotted in red. I fix \( \mu_\xi = 7\%, \sigma_\xi = 20\%, \kappa = 0.4 \) (that is a “half-life” of the transitory component \( \chi_t \) of about 21 months under \( P \)), \( \sigma_\chi = 15\% \), and \( \rho_{\xi \chi} = 0.5 \). The permanent component \( \xi_t \) of the spot log price is normalized to 1.
I begin with focusing on the pricing impact of $\Lambda_{\xi,t}$. Figure 1 shows the term-structure implications of the original Schwartz and Smith (2000) model with $\phi_\xi = \phi_\chi = \lambda_\chi = 0$ and $\lambda_\xi = 1$. The positive risk premium implies normal backwardation (i.e. $\ln E^P_t[S_{t+\tau}] > \ln F(\xi_t, \chi_t, \tau)$ for $\tau > 0$) and its size ($D < 0$) generally causes backwardation (i.e. $\ln F(\xi_t, \chi_t, \tau)$ decreases with $\tau$) but for large negative transitory deviations from $\xi_t$, which prompt contango over the short-to-medium maturity dates (i.e. $\ln F(\xi_t, \chi_t, \tau)$ increases there with $\tau$). Figure 2 visualizes the effect of switching on the state-dependent nature of $\Lambda_{\xi,t}$. Given $\phi_\xi = 1$, the changes in the slope $D$ of the long-term futures log price and in its constant-intercept terms $G$ and $H$ are not substantial. What makes the difference is $\chi_t$’s long-run futures-price impact $A(\infty)$, which loads the state $\chi_t$ in the intercept of the long-term futures log price. Large positive transitory deviations foster a stronger backwardation ($\chi_t A(\infty) < 0$), whereas large negative deviations strengthen the contango over the short-to-medium maturity dates ($\chi_t A(\infty) > 0$).
I now turn to the pricing impact of $\Lambda_{\chi,t}$. Figure 3 depicts the term-structure implications of the original Schwartz and Smith (2000) model with $\phi_\xi = \phi_\chi = \lambda_\xi = 0$ and $\lambda_\chi = 1$. Again, the positive risk premium brings about normal backwardation. However, the slope $D$ of the long-term futures log price is only slightly affected by $\lambda_\chi$ and remains positive, generating long-term contango. Backwardation over the short-to-medium maturity dates stems only from a large positive $\chi_t$. Figure 4 shows that activating the state-dependent nature of $\Lambda_{\xi,t} (\phi_\chi = 1)$ has a subdued impact on the term structure of futures prices, the main change being their faster convergence toward the long-term benchmark ($\kappa + \sigma_\chi \phi_\chi > \kappa$).

4. Conclusions

For a generic commodity, I work out the proof of the closed-form extension of the celebrated Schwartz and Smith (2000) model of spot/futures pricing to state-dependent risk premia and point out that state dependence in the market price of the permanent spot-price risk plays an important role in shaping the term structure of futures prices.

Conflict of Interests

The author declares that there is no conflict of interests.
REFERENCES


