Non-Myopic Portfolio Choice with Unpredictable Returns: The Jump-to-Default Case

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Non-Myopic Portfolio Choice with Unpredictable Returns: The Jump-to-Default Case

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Abstract

If the risky asset is subject to a jump-to-default event, the investment horizon enters the optimal portfolio rule even if asset returns are unpredictable. The optimal rule solves a non-linear differential equation that, by not depending on the investor’s pre-default value function, allows for its direct computation. Importantly for financial planners offering portfolio advice for the long term, tiny amounts of costant jump-to-default risk induce a marked time variation in the optimal portfolios of long-run conservative investors.

KEYWORDS. Strategic asset allocation, intertemporal portfolio choice, time-varying hedging demand, investment horizon, investment opportunity set, jump-to-default risk, arbitrage-free markets, risk premia, jump-diffusive processes, return predictability, irreversible regime change, portfolio advice.

JEL: G01, G11, G12, C61.

1 Introduction

By the end of 2014, global private financial wealth reached a total of $164 trillion and the global value of professionally managed assets grew to $74 trillion, according to estimates released by Boston Consulting Group. Financial planners providing asset allocation advice to long-term investors greatly benefit

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from guidance based on tractable models of optimal intertemporal portfolio choice. Understanding the
sources of time variation in the optimal portfolio rule is momentous, as optimal portfolio rebalancing is
a key concern for non-myopic investors. The seminal work of Merton (1969, 1971) has determined that
there is no time variation in the optimal portfolio rule resulting from a continuous-time problem with
purely diffusive risk if the investment opportunity set is constant. Merton (1971) shows that this is the
case also in the joint presence of a risky lognormal asset and of a ‘riskless’ asset that is subject to an
unpredictable default event driven by a Poisson process. While excluding the possibility of a jump-to-
default event for the risky asset, Liu, Longstaff, and Pan (2003) broaden the optimal investment analysis
by considering value disruptions for the risky asset as well as jumps in the stochastic volatility of the
risky asset returns. When focusing on the case of a constant investment opportunity set associated with
unpredictable returns, they find that the optimal portfolio rule remains time-invariant while including
a constant hedging demand against asset value jumps. Das and Uppal (2004) prove the same in a
multi-asset jump-diffusion framework.

Our contribution is to show that, if a jump-to-default event for the risky asset is introduced, unpre-
dictable returns are consistent with a pre-default time-varying optimal portfolio rule. This is because
the sudden disappearance of the risky defaultable asset modifies the dependence of the investor’s value
function on the investment horizon. Hence, unlike in Liu, Longstaff, and Pan (2003) and in Das and
Uppal (2004), the ratio between the post-event marginal indirect utility of wealth and the pre-event one
becomes dependent on the investment horizon. Such a ratio is a key component of the hedging demand
against the jump-to-default risk, injecting time variation in the optimal fractional wealth allocation to
the risky defaultable asset. Our analysis of the time-varying pre-default hedging demand is conducted in
a highly tractable model characterized by different but constant pre-default and post-default investment
opportunity sets and by no arbitrage.

We show that the optimal portfolio rule follows a first-order non-linear ordinary differential equation
in the investment horizon that does not involve the investor’s pre-default value function. This empowers
a direct numerical analysis of the investor’s optimal choice, which we conduct for investment horizons of
up to 15 years. We find that the investment-horizon dynamics of the optimal portfolio rule is particularly
conspicuous when the jump-to-default intensity is small and the degree of relative risk aversion is high.
Hence, in the sheer absence of asset return predictability, minute amounts of constant jump-to-default
risk cause strong time variation in the optimal portfolios of long-run conservative investors (they are
more risk averse than the log utility agent), who will markedly increase the fractional allocation to the
risky defaultable asset as their investment horizon shortens. This is an interesting complement to the
well-known investment-horizon effect that predictability in asset returns generates for related classes of
investors (e.g. Merton (1969, 1971), Kim and Omberg (1996), Balvers and Mitchell (1997), Brennan,
Schwartz, and Lagnado (1997), Campbell (1999), Wachter (2002), Campbell, Chan, and Viceira (2003),
Wu (2003), Liu (2007), Koijen, Rodríguez, and Sbuelz (2009), Detemple and Rindisbacher (2010), and
McCarthy and Miles (2013)).

Unpredictable default causes an irreversible change in the investment opportunity set. There is a
vast literature on optimal asset allocation with regime switching dynamics for the asset returns (e.g., Ang and Bekaert (2002), Gra‡und and Nilsson (2003), Honda (2003), Guidolin and Timmermann (2007, 2008), Jang, Koo, Liu, and Loewenstein (2007), Tu (2010), Konermann, Meinerding, and Sedova (2013) and Liu and Loewenstein (2013)). In contrast to our model, these authors examine reversible regimes and do not consider defaultable assets.

The rest of the paper is structured as follows. Section 2 describes the risky asset returns and the related risk premia. Section 3 introduces the optimal investment problem. Section 4 characterizes the non-linear dynamics of the optimal risky portfolio weight and numerically discusses the time-varying nature of the resulting optimal asset allocation. Section 5 concludes.

2 Risk premia

The markets consist in a traded riskless security with constant rate of return $r$ and in a traded risky security, whose value dynamics loads a diffusive shock as well as a Poisson-type shock (its intensity under the objective probability measure $\mathbb{P}$ is $\lambda$):

$$\frac{dS}{S_-} = \mu dt + \sigma dz - \eta (dN - \lambda dt).$$

The risky security defaults in the wake of the first Poisson event and may have recovery value ($0 < \eta \leq 1$). As in the dynamic portfolio problems with jump-diffusive risks studied by e.g. Liu and Pan (2003) and Branger, Schlag, and Schneider (2008), we assume that markets are arbitrage-free with the state-price density process $\{\zeta\}$ having the dynamics

$$\frac{d\zeta}{\zeta_-} = -r dt - \xi dz + \eta \zeta (dN - \lambda dt), \quad dzd\zeta = \rho dt,$$

where $\xi$ is the market price of systematic diffusive risk and $\rho$ is the systematic fraction of the risky-security diffusive risk. We assume that the unpredictable default risk can be systematic ($\eta \geq 0$), with a corresponding impact on the investment opportunity set. Indeed, by no arbitrage, the per-annum expected return $\mu$ on the risky defaultable security loads the default risk parameters $\lambda$ and $\eta$ via the systematic-risk parameter $\eta$. 

Proposition 1 The no-arbitrage assumption implies that

$$\mu = r + \xi \sigma \rho + \eta \lambda.$$

Proof. By no-arbitrage, the deflated value process $\{\zeta S\}$ must be a martingale over any finite time horizon. Hence, it must be driftless:

$$E_t \left[ \frac{d(\zeta S)}{\zeta_- S_-} \right] = (\mu + \eta \lambda) + (-r - \xi \lambda) + (-\xi \sigma \rho) + ((1 + \eta \zeta) (1 - \eta) - 1) \lambda = 0. \quad \square$$
The optimal investment problem

The fraction of wealth allocated to the risky defaultable security is $\omega$. The investor’s time horizon and value function are $\tau$ and

$$J(W, \tau) = \max_\omega \mathbb{E}_0 \left[ \frac{(W_x)^{1-\gamma}}{1-\gamma} \right] \quad \text{s.t.} \quad dW = W \left[ rdt + \omega \left( \frac{dS}{S} - rdt \right) \right],$$

respectively. The boundary condition if the default event has just occurred is

$$J(W, \tau) = \frac{(We^{\tau})^{1-\gamma}}{1-\gamma}.$$

In contrast to our model, Puopolo (2015) focuses on the impact of transaction costs on a dynamic asset allocation problem with unpredictable default risk and states a horizon-independent boundary condition at default.

Proposition 2 The terminal condition (at $\tau = 0$) for the optimal investment rule is

$$\omega^*(0) = \frac{\xi \sigma \rho}{\gamma \sigma^2} + \frac{(\eta + 1) \eta \lambda}{\gamma \sigma^2} - \frac{(1 - \omega^*(0) \eta)^{-\gamma} \eta \lambda}{\gamma \sigma^2}.$$

Proof. The optimal myopic investment problem with unpredictable default risk is equivalent to the one considered by Liu, Longstaff, and Pan (2003).

The significant addition to the zero-horizon portfolio analysis of Liu, Longstaff, and Pan (2003) is the no-arbitrage assumption. It renders the speculative component of the optimal myopic portfolio $\omega^*(0)$, whose key term is the weighted sum $\frac{\xi \sigma \rho}{\gamma \sigma^2} + \frac{\eta \gamma \lambda}{\gamma \sigma^2}$ of the risk premia, dependent on the default-risk parameters $\lambda$ and $\eta$. The speculative demand includes the term $\frac{\eta \lambda}{\gamma \sigma^2}$ because of the extra security-value drift due to the Poisson-process compensation.

The optimal myopic portfolio $\omega^*(0)$ embeds a negative hedging component,

$$-\frac{(1 - \omega^*(0) \eta)^{-\gamma} \eta \lambda}{\gamma \sigma^2} < 0,$$

which is meant to create a windfall should the value of the risky security be taken down by the unpredictable default event over the next instant. The hedging component makes sure that $\omega^*(0)$ stays below $1/\eta$ and is conducive to an endogenous borrowing constraint as highlighted by Liu, Longstaff, and Pan (2003).

The boundary condition at default causes the dependence of the pre-default optimal portfolio $\omega^*(\tau)$ on the investment horizon $\tau$ for any non-log-utility investor, that is with $\gamma \neq 1$. This is because the pre-default sensitivity of the investor’s value function to wealth shocks structurally differs from the post-default sensitivity $W^{-\gamma}e^{(1-\gamma)\tau}$, which is log-linear in the investment horizon.

Proposition 3 The optimal investment rule $\omega^*(\tau)$ is such that

$$\omega^*(\tau) = \frac{\xi \sigma \rho}{\gamma \sigma^2} + \frac{(\eta + 1) \eta \lambda}{\gamma \sigma^2} - \frac{(1 - \omega^*(\tau) \eta)^{-\gamma} e^{-A(\tau)+(1-\gamma)\tau} \eta \lambda}{\gamma \sigma^2}.$$
where $A(\tau)$ is the $\tau$-dependent component of the investor’s pre-default value function:

$$J(W, \tau) = \frac{W^{1-\gamma}}{1-\gamma} e^{A(\tau)} \quad \text{with} \quad A(0) = 0.$$ 

**Proof.** The Hamilton-Jacobi-Bellman equation for the investment problem reads

$$0 = \max_\omega \left( -A' + (1-\gamma) \left( r + \omega \left( \xi \sigma \rho + (\eta_\zeta + 1) \eta \lambda \right) \right) - \frac{1}{2} \gamma (1-\gamma) \omega^2 \sigma^2 + \left( (1-\omega \eta)^{1-\gamma} e^{-A+(1-\gamma)r\tau} - 1 \right) \lambda \right).$$

The result follows from the first order condition (F.O.C.) with respect to $\omega$. □

Importantly, the pre-default hedging demand is related to the $\tau$-dependent ratio

$$\frac{(1 - \omega^* (\tau) \eta)^{-\gamma} e^{(1-\gamma)r\tau}}{e^{A(\tau)}}$$

between the post-default marginal indirect utility of wealth and the pre-default one. By ensuring that $\omega^* (\tau)$ is less than $1/\eta$, the pre-default hedging demand grants that the investor’s wealth remains positive when the default event strikes.

### 4 The dynamics of the optimal risky portfolio weight

The following proposition states the non-linear dynamics of the optimal investment rule. It makes no reference to $A(\tau)$, whose prior knowledge becomes then unnecessary in working out $\omega^* (\tau)$.

**Proposition 4** The optimal investment rule $\omega^* (\tau)$ is the solution to the following first-order non-linear ordinary differential equation

$$\frac{\omega'}{\xi \sigma \rho \gamma + \eta \gamma \lambda + \omega} + \gamma \frac{\omega' \eta}{1 - \omega \eta} + (1-\gamma) r = H(\omega),$$

where the boundary condition for $\omega(0)$ is specified in Proposition 2. $H(\omega)$ is quadratic in $\omega$:

$$H(\omega) = (1 - \gamma) \left( r + \omega \left( \xi \sigma \rho + (\eta_\zeta + 1) \eta \lambda \right) \right) - \frac{1}{2} \gamma (1-\gamma) \omega^2 \sigma^2 + \frac{1}{\eta} \left( (1-\omega \eta) \left( \xi \sigma \rho + (\eta_\zeta + 1) \eta \lambda \right) - \gamma (1-\omega \eta) \omega \sigma^2 \right) - \lambda.$$

**Proof.** Since

$$A(\tau) = -\ln \left( \frac{(1 - \omega^* (\tau) \eta)^{\gamma}}{\lambda \eta} \left( \xi \sigma \rho + (\eta_\zeta + 1) \eta \lambda - \gamma \sigma^2 \omega^* (\tau) \right) \right) + (1-\gamma) r \tau,$$
deriving both sides with respect to $\tau$ implies

$$A' = \frac{\omega^{\ast}}{\xi \sigma^2} \left( \frac{(\eta + 1) \eta \lambda}{\gamma \sigma^2} - \omega^* \right) + \gamma \frac{\omega^* \eta}{1 - \omega^* \eta} + (1 - \gamma) r.$$ 

On the other hand, multiplying both sides of the F.O.C. by $1 - \omega^* \eta$ yields

$$\frac{1}{\eta} \left( (1 - \omega^* \eta) \left( \xi \sigma^2 + (\eta + 1) \eta \lambda \right) - \gamma (1 - \omega^* \eta) \omega^* \sigma^2 \right) = (1 - \omega^* \eta)^{1 - \gamma} e^{-A + (1 - \gamma) r \lambda}.$$

The result follows from substitution in the Hamilton-Jacobi-Bellman equation. \( \square \)

Tables 1, 2, and 3 exhibit the optimal risky portfolio weight $\omega^* (\tau)$ corresponding to different degrees of relative risk aversion, for two recovery rate sizes ($1 - \eta = 30\%, 0\%$) and three finite jump-frequency levels expressed in years ($1/\lambda = 1, 5, 25$). The no-default lognormal case with time-invariant optimal weight $\frac{\xi \sigma^2}{\gamma^2}$ is represented by taking $1/\lambda = \infty$. The other parameters are fixed at $r = 2\%, \sigma = 20\%, \rho = 1, \xi = 1/2$ and $\eta = 1/2$. For any degree of risk aversion, an increase either in the loss-given-default level $\eta$ or in the jump-to-default intensity $\lambda$ is associated to a reduction of the optimal exposure to the risky defaultable security. An aggressive investor ($\gamma = 1/2$, Table 1) has a hedging demand that is negative but increasing with the horizon $\tau$. Optimal portfolio rebalancing as $\tau$ changes is more pronounced when the jump-to-default intensity is high. By contrast, the hedging demand of conservative investors ($\gamma = 2$ and $\gamma = 5$, Table 2 and Table 3) becomes more negative as $\tau$ increases and rebalancing is conspicuous when the jump-to-default intensity is low. If the frequency of the jump-to-default event is 25 years, a very conservative investor ($\gamma = 5$) with a 15-year horizon chooses an exposure to the risky defaultable security which is 30\% lower than the one chosen by a myopic investor with the same degree of relative risk aversion. Unpredictable default causes an irreversible change in the investment opportunity set. When the chance of such a change is small, long-term conservative investors implement a markedly time-varying hedging demand against it.

5 Conclusions

Asset return predictability is a classic source of time variation in the optimal portfolio rule of non-myopic investors. We uncover a novel channel of investment-horizon dynamics for optimal non-myopic portfolios in the presence of unpredictable returns. The introduction of an unpredictable jump-to-default event for the risky asset renders the optimal pre-default exposure to the risky asset horizon-dependent even in the presence of distinct but constant pre-default and post-default investment opportunity sets. While the hedging demand against the value-jump risk of a non-defaultable asset is known to be constant, we highlight that the hedging demand against the jump-to-default risk depends on the investment horizon.

We show that the non-linear investment-horizon dynamics of the optimal portfolio rule is an ordinary differential equation that does not entail the investor’s pre-default value function. This empowers a direct calculation of the optimal dynamic asset allocation. Optimal portfolio rebalancing across different horizons is most significant for conservative investors facing a risky asset with a small probability of
sudden immediate default. By highlighting a new element that leads long-term investors to choose different portfolio strategies from short-term investors, our paper contributes to the important and broad literature that delivers grounded guidance to financial planners offering portfolio advice to long-term investors.

REFERENCES


Table 1

Optimal risky portfolio weights with relative risk aversion $\gamma = 1/2$

This table reports the weights $\omega^*(\tau)$ for $\gamma = 1/2$, $r = 2\%$, $\sigma = 20\%$, $\rho = 1$, $\xi = 1/2$ and $\eta_\xi = 1/2$ $\quad (\mu = r + \xi \sigma + \eta_\xi \eta \lambda)$

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<thead>
<tr>
<th>Default frequency</th>
<th>recovery rate $1 - \eta = 30%$</th>
<th>expected return $\mu$</th>
<th>horizon $\tau$ (years)</th>
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<th>expected return $\mu$</th>
<th>horizon $\tau$ (years)</th>
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Table 2

Optimal risky portfolio weights with relative risk aversion $\gamma = 2$

This table reports the weights $\omega^*(\tau)$ for $\gamma = 2$, $r = 2\%$, $\sigma = 20\%$, $\rho = 1$, $\xi = 1/2$ and $\eta_\xi = 1/2$ 

$$\mu = r + \xi \sigma \rho + \eta_\xi \eta \lambda$$

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<th>Default frequency</th>
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<th>horizon $\tau$ (years)</th>
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recovery rate $1 - \eta = 30\%$

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recovery rate $1 - \eta = 0\%$
Table 3

Optimal risky portfolio weights with relative risk aversion $\gamma = 5$

This table reports the weights $\omega^*(\tau)$ for $\gamma = 5$, $r = 2\%$, $\sigma = 20\%$, $\rho = 1$, $\xi = 1/2$ and $\eta_z = 1/2$ \hspace{1cm} ($\mu = r + \xi \sigma \rho + \eta_z \eta \lambda$)

| Recovery rate $1 - \eta$ = 30\% |
|-------------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| Default frequency              | expected return $\mu$ | horizon $\tau$ (years) |
| $1/\lambda$ (years)            |                  | 0    | 1    | 2    | 3    | 4    | 7    | 10   | 15   |
| 1                             | 47.0\%           | 0.1291 | 0.1126 | 0.1088 | 0.1079 | 0.1077 | 0.1077 | 0.1077 | 0.1077 |
| 5                             | 19.0\%           | 0.1796 | 0.1675 | 0.1589 | 0.1528 | 0.1485 | 0.1419 | 0.1397 | 0.1388 |
| 25                            | 13.4\%           | 0.2846 | 0.2744 | 0.2650 | 0.2564 | 0.2486 | 0.2294 | 0.2155 | 0.2009 |
| $\infty$                      | 12.0\%           | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 |

| Recovery rate $1 - \eta$ = 0\% |
|-------------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $1/\lambda$ (years)            | expected return $\mu$ | horizon $\tau$ (years) |
| 1                             | 62.0\%           | 0.0877 | 0.0769 | 0.07433 | 0.0737 | 0.0736 | 0.0736 | 0.0736 | 0.0736 |
| 5                             | 22.0\%           | 0.1187 | 0.1115 | 0.1063 | 0.1025 | 0.0998 | 0.0956 | 0.0941 | 0.0934 |
| 25                            | 14.0\%           | 0.1978 | 0.1913 | 0.1852 | 0.1797 | 0.1747 | 0.1624 | 0.1533 | 0.1434 |
| $\infty$                      | 12.0\%           | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 |