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WORKING PAPER N. 14/5

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Abstract

In the present paper uncertainty over the market price of a risk-neutral competitive firm's output and limited liability imply the possibility of bankruptcy, give rise to moral hazard and entail that the firm's output decision depends on its equity holding. Subjecting the firm to a Value-at-Risk constraint induces it to behave in an as-if risk-averse manner, but in a static context moral hazard persists for a certain interval of values of equity. In a dynamic setting the size of equity holding becomes a choice variable and the VaR constraint guides the firm to select equity values outside the moral-hazard interval. Thus it achieves to reconcile two apparently conflicting goals: encourage entrepreneurial activity by means of limited liability and avoid irresponsible gambling due to the incentives provided by it.

JEL classification: C61, D21, D81, G32, G33

Keywords: limited liability, moral hazard,
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1 Introduction

It is well known that the legal provision of limited liability has the drawback that a firm may act less carefully than it would if it were fully liable to the outcome. To counteract this moral-hazard problem we explore in the current paper a novel approach, namely, the use of a Value-at-Risk constraint. In a dynamic context and under certain, as we believe, natural conditions, this will in fact discipline a firm's behavior in the sense that, even if it is risk neutral, it does not quite behave in that way but rather similar to being risk averse. In spite of this, limited liability can be maintained, and so the incentives for investment are preserved.

Value-at-Risk (VaR) is a well-known concept intended to be a measure of the risk of financial investments.[1] Although it is commonly employed by investment banks to evaluate the market risk of the assets they hold in their portfolio, it may also be used by these banks to evaluate the creditworthiness of projects submitted to them by firms and to determine the corresponding terms of credit. Vice versa, a bank may impose a VaR constraint on a firm that asks for funding to limit its exposure to risk. The firm may then combine this with an objective - for example profit maximization - to determine its most preferred action.

More precisely, and to illustrate this, in this paper we study a firm à la Greenwald-Stiglitz (1993). It produces output y financing it partly with retained capital or equity a and partly with debt capital b. Since the output is sold in a competitive market at a price p which is determined only after the firm has contracted b, there is a possibility of bankruptcy. The VaR constraint requires the firm to not take an output decision which give rise to a probability of bankruptcy larger than a predetermined threshold probability α, the *confidence level*. As shown in Tulli and Weinrich (2009) for a static context, this implies that the output decision depends on the size of equity a and, due to the moral-hazard effect of limited liability, this dependence is non-monotone.

[1] See e.g. Duffie and Pan (1997).

In a dynamic framework a becomes endogenous: by deciding how much of its profit realized in period t to distribute to shareholders and how much to keep within the firm for its operations in period $t+1$, a becomes a dynamic variable a_t for which an optimal path can be studied. It turns out that in the case that the subjective discount rate of the firm's shareholders is smaller than the inverse of the interest factor the firm will always choose a path along which the VaR constraint is binding and, moreover, that the dynamically desired sequence (a_t^*) is constant, i.e. $a_t^* = \overline{a}^* \geq 0$. That can be strenghtened to $\overline{a}^* > 0$ if the elasticity of scale of the firm's technology is not too small, i.e. sufficiently close to constant returns. Finally, and most importantly, the presence of the VaR constraint will imply that the moral-hazard effect of limited liability does not bite. Vice versa, without a VaR constraint, in our model not only would there be moral hazard, but $\overline{a}^*$ would be zero.

Value-at-Risk has achieved high status because of being written into industry regulations (see e.g. Jorion (1997)) and is in fact a popular instrument used in practice (e.g. Bauer (2000), Pritsker (1997)). It also has been adopted as a standard tool to assess risk and to calculate capital requirements in the financial industry. It is currently the risk measure contemplated in the European solvency regulation for the insurance sector (Solvency II), and this is also the case of solvency regulation for the banking sector (Basel accords). This is true in spite of the fact that VaR is known to have two weaknesses: capital requirements for catastrophic losses based on the measure may be underestimated and VaR may fail the subadditivity property.[2] Thus there are proposals to overcome these problems, like e.g. by means of Conditional Value-at-Risk (e.g. Lüthi and Doege (2005) and Rockafellar and Uryasev (2000)), Tail Value-at-Risk and GlueVar (Belles-Sampera et al.

[2] A risk measure is subadditive when the aggregated risk is less than or equal to the sum of individual risks. Subadditvity is an appealing property when aggregating risks in order to preserve the benefits of diversification. VaR is subadditive for elliptically distributed losses. However, in general the subadditivity of VaR is not granted.

2014). Whether these extended - and necessarily more complex - concepts will be adopted by practitioneers remains to be seen. The model presented in this paper, by using VaR as a constraint in profit maximization à la Greenwald-Stiglitz (1993), is elementary enough to not give rise to these conceptual complications.[3]

Regarding the use of VaR in an optimization context, various aspects have been discussed in, for example, Kast et al. (1998), Rockafellar and Uryasev (2002) and Yiu (2004). None, however, have, to the best of our knowledge, tackled the issue of limited liability and moral hazard dealt with in the present paper.

A further branch of the literature to which there can be seen a connection of the present paper is principal-agent theory. The bank is the principal financing, at least in part, the action of the agent, i.e. the firm, which enjoys limited liability. To not have the firm commit moral hazard, the bank subjects it to the VaR constraint. This is similar in spirit to what can be found for example in Bias, Mariotti, Rochet and Villeneuve (2010). They analyze a dynamic scenario in which a firm's manager has limited liability in preventing large but infrequent losses. It gives rise to a contract between the firm's financier and the manager in which downsizing of the firm and investment decisions are made contingent on accumulated performance. This provides a rationale for prudential regulations that request that the scale at which firms operate be proportionate to their capital. In our paper we do not design contracts, but we do obtain the result that capital holding is elicited as a consequence of the introduction of the VaR constraint. Moreover, unless the firm's shareholders are very shortsighted, the size of equity holding increases with the strenghthening of the VaR constraint, i.e. a reduction in the confidence level α, which is comparable to a sharpening of prudential regulation.

In section 2 we present the model and analyze the static set-

[3]For example, the problem usually related to VaR that it provides no handle on the extent of the losses that might be suffered beyond the threshold amount will be overcome in our model by assuming - realistically, due to limited liability - a constant bankruptcy cost, i.e. independent of the size of the firm's chosen action.

up. Section 3 extends it to a dynamic framework and derives the main results. Section 4 concludes and an appendix collects some auxiliary facts regarding the static model.

2 The Static Model

2.1 Assumptions

Let X be a real-valued random variable with continuous and strictly increasing distribution function $F(x)$ on the (by assumption) non-empty set $F^{-1}(0,1)$. Then, for given confidence level $\alpha \in (0,1)$ we define $VaR_\alpha[X]$ as that real number for which

$$F(-VaR_\alpha[X]) = \alpha. \tag{1}$$

This means that the probability that a realization of X is smaller than or equal to $-VaR_\alpha[X]$ is α. Equivalently, the probability that a realization of $-X$ (the loss) is larger than or equal to $VaR_\alpha[X]$ is α. If $F(x)$ is influenced by variables (y, a), where y is a choice variable and a a parameter, then X has distribution function $F(x, y, a)$ and becomes a random function $X(y, a)$. Therefore $VaR_\alpha[X(y, a)]$ is determined by $F(-VaR_\alpha[X(y, a)], y, a) = \alpha$. If in addition an agent desires to maximize $EX(y, a)$ under the constraint that $VaR_\alpha[X(y, a)]$ be smaller or equal to some a priorily contemplated $v \in \mathbb{R}$, then her decision problem becomes

$$\begin{aligned} &\max_y EX(y, a) \\ &s.t.\ VaR_\alpha[X(y, a)] \leq v. \end{aligned}$$

Thus we have obtained a decision problem involving VaR as an element to control risk.

Following Greenwald and Stiglitz (1993) we now consider a firm that produces a single output y using labour n as the only input, with labour requirement function $n = \Phi(y)$.[4] The corresponding labour cost, wn, with w the nominal wage, is covered by the firm

[4] Properties of Φ will be specified in assumption (A2).

partly with own funds or retained capital a and partly with debt capital, b, obtained in the form of a loan from banks. Thus $b = wn - a$.[5] The firm sells its output on a competitive market and thus the output price, p, is considered as not controlled by the firm but determined by the market. Moreover, the firm does not know with certainty the value of p when it contracts labour because there is a time lag in production and the firm has to hire workers before the uncertainty is resolved. Denoting with $R = 1 + r$ the interest factor and r the interest rate, the resulting profit, according to Greenwald and Stiglitz (1993), is

$$\pi = py - R[w\Phi(y) - a] =: \pi(y, a, p). \tag{2}$$

Note that π is positive if the firm does not produce at all: in that case the firm can act as a lender of its capital a and earn Ra.

Since p is uncertain, from the firm's point of view it can be considered a random variable which we denote by P and the realizations of which are governed, as believed by the firm, by some probability density $g(p)$. Then this induces for the random variable $\Pi(y, a) := \pi(y, a, P)$ the density

$$f(y, a, \pi) = g(\Psi(y, a, \pi))\frac{\partial \Psi(y, a, \pi)}{\partial \pi}, \tag{3}$$

where

$$\Psi(y, a, \pi) := \frac{\pi + R[w\Phi(y) - a]}{y} \tag{4}$$

is the inverse of the function $\pi(y, a, \cdot)$. Depending on the realization of P, profit may be negative. In that case the firm is not able to fully repay its debt b and we consider it to be bankrupt. Being an organization with limited liability, the firm then has to give up its assets and bear other possible costs which add up to a constant bankruptcy cost $c \geq 0$. The expected gain for a firm producing

[5] Greenwald and Stiglitz (1993) assume that, due to asymmetric information, firms cannot finance their production cost by issueing new equity. See e.g. Myers e Majluf (1984) for a formal justification of this argument.

the output $y \geq 0$ endowed with capital $a \geq 0$ is therefore

$$\Gamma(y,a) := \int_{-\infty}^{0} -cf(y,a,\pi)d\pi + \int_{0}^{+\infty} \pi f(y,a,\pi)d\pi \qquad (5)$$

If the firm were risk neutral and had no further constraint, it would maximize $\Gamma(y,a)$. If, however, the firm has to limit the probability of going bankrupt to $\alpha \geq 0$, then, setting

$$F(y,a) := \int_{-\infty}^{0} f(y,a,\pi)d\pi \ , \qquad (6)$$

its problem becomes

$$\begin{array}{c} \max_y \Gamma(y,a) \\ s.t.\ F(y,a) \leq \alpha\ . \end{array} \qquad (7)$$

Recalling the definition of Value-at-Risk in (1), this is obviously equivalent to

$$\begin{array}{c} \max_y \Gamma(y,a) \\ s.t.\ VaR_\alpha\left[\Pi(y,a)\right] \leq 0\ . \end{array} \qquad (8)$$

Note that the firm's degree of "risk aversion" (not risk aversion in the conventional sense) is expressed by the constant $\alpha \geq 0$. Any risk can be excluded by setting $\alpha = 0$. However, in that case the firm may be severely limited in its opportunities to realize a satisfactory profit. Therefore $\alpha > 0$ may be a better choice.

Denote a solution to (8) by $\widehat{y}(a)$. To characterize it, we make the following assumptions which regard the price distribution, the technology and the cost of bankruptcy:

(A1) The price of the product is a random variable P with density

$$g(p) = \begin{cases} 1/(2\sigma), & \text{if } 1-\sigma \leq p \leq 1+\sigma \\ 0, & \text{otherwise} \end{cases}$$

where $0 < \sigma \leq 1$.

(A2) $\Phi(y)$ is differentiable, strictly increasing and strictly convex. Moreover, $\Phi(0) = \Phi'(0) = 0$ and $\lim_{y\to\infty} \Phi'(y) = +\infty$.

(A3) The bankruptcy cost c is zero.

Although the above specification of the price distribution is the simplest to work with, it will not be easy to solve explicitly for $\widehat{y}(a)$; still, it will enable us to show the main point.[6] The parameter σ allows us to monitor the robustness of the results with respect to varying price dispersion.[7] The normalization to $EP = 1$ will simplify the calculations but not be essential in any way to the results. Specifically, (A1) implies, recalling (2)-(4),

$$f(y,a,\pi) = \begin{cases} \frac{1}{2\sigma y} & \text{if } (1-\sigma)\,y - R\,[w\Phi(y) - a] < \pi \\ & \quad < (1+\sigma)\,y - [Rw\Phi(y) - a] \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

From this we obtain by (6)

$$F(y,a) = \begin{cases} 0 & \text{if } 0 \le (1-\sigma)\,y - R\,[w\Phi(y) - a] \\ \frac{R[w\Phi(y)-a]-(1-\sigma)y}{2\sigma y} & \text{if } (1-\sigma)\,y - R\,[w\Phi(y) - a] < 0 \\ & \quad \le (1+\sigma)y - R\,[w\Phi(y) - a] \\ 1 & \text{if } (1+\sigma)y - R\,[w\Phi(y) - a] < 0 \end{cases} \tag{10}$$

Regarding (A2), note that, since labour is the only input of production, strict convexity of $\Phi(y)$ as assumed in (A2) is equivalent

[6] In measure-theoretic terms, both P and $\Pi\,(y,a,P)$ can be thought of as random variables defined on a probability space (Ω, F, Q) with values in $(\mathbb{R}, \mathcal{B})$, i.e. $p = P(\omega)$ and $\pi = \Pi\,(y,a,P(\omega))$, $\omega \in \Omega$, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$. According to (A1) and the definitions of f and Ψ, the distributions of P and $\Pi\,(y,a,P)$ are then given by $g\lambda = Q \circ P^{-1}$ and $f(y,a,\cdot)\lambda = Q \circ P^{-1} \circ \Psi(y,a,\cdot)$, respectively, where λ is the Lebesgue-Borel measure on $(\mathbb{R}, \mathcal{B})$.

[7] The actual standard deviation under (A1) is $\sigma/\sqrt{3}$.

to increasing marginal cost, which, moreover, tends to infinity as $\lim_{y\to\infty} \Phi'(y) = +\infty$.

(A3) is technically convenient without altering the substance of the results. The case $c \geq 0$ has been dealt with in Tulli and Weinrich (2009) under the assumption that $\sigma = 1$.

2.2 Solution of the static problem

Returning to problem (8), denote a solution to $VaR_\alpha\left[\Pi(y,a)\right] = 0$ by $\widehat{y}_I(a)$. Under (A1) it is implicitly determined, according to (7) and (10), by

$$R\left[w\Phi(\widehat{y}_I(a)) - a\right] = (2\sigma\alpha + 1 - \sigma)\,\widehat{y}_I(a)\ . \tag{11}$$

As we show in the Appendix (Lemma 2), $\widehat{y}_I(a)$ is a strictly increasing function.

Next set $\widehat{y}_{II}(a) := \arg\max_y \Gamma(y,a)$. To understand its behavior, we first determine $\Gamma(y,a)$. To this end we define $\underline{y}(a)$ by means of

$$(1-\sigma)\,\underline{y}(a) - R[w\Phi(\underline{y}(a)) - a] = 0\ . \tag{12}$$

It implies

$$\underline{y}'(a) = -\frac{R}{1-\sigma - Rw\Phi'\left(\underline{y}(a)\right)} > 0$$

since

$$1-\sigma = \frac{R[w\Phi(\underline{y}(a)) - a]}{\underline{y}(a)} \leq \frac{Rw\Phi(\underline{y}(a))}{\underline{y}(a)} < Rw\Phi'(\underline{y}(a))$$

where the last inequality holds for $a > 0$ by strict convexity of Φ and $\Phi(0) = 0$. $\underline{y}(a)$ is thus the maximum quantity the firm can produce without any risk of bankruptcy, given a. For $y \leq \underline{y}(a)$, by (2) and (A1) $\Gamma(y,a)$ coincides with

$$\mu\,(y,a) := \int_{-\infty}^{+\infty} \pi f(y,a,\pi)d\pi = E\Pi(y,a) = y - R[w\Phi(y) - a]\ . \tag{13}$$

Instead for (y,a) such that $y \geq \underline{y}(a)$ consider, using (9),

$$\mu_1(y,a) := -\int_{-\infty}^{0} \pi f(y,a,\pi) d\pi = -\frac{1}{4\sigma y}\left[\pi^2\right]_{(1-\sigma)y-R[w\Phi(y)-a]}^{\overline{\pi}} \tag{14}$$

where

$$\overline{\pi} := \min\left\{(1+\sigma)y - R\left[w\Phi(y)-a\right], 0\right\} .$$

Then we obtain from (5) and (A3)

$$\Gamma(y,a) = \begin{cases} \mu(y,a), & \text{if } y \leq \underline{y}(a) \\ \mu(y,a) + \mu_1(y,a), & \text{if } y \geq \underline{y}(a) \end{cases} \tag{15}$$

By (13)

$$y^* := \arg\max_y \mu(y,a) = \left(\Phi'\right)^{-1}\left(\frac{1}{Rw}\right) \tag{16}$$

is constant and coincides with $\widehat{y}_{II}(a)$ for $a \geq \overline{a}$, where

$$\overline{a} := w\Phi(y^*) - (1-\sigma)\, y^*/R \tag{17}$$

is the minimum value of a such that, when the firm maximizes expected profit $\mu(y,a)$, there is no risk of bankruptcy. Moreover, if σ is small enough, there is no risk of bankruptcy even if $a = 0$. More precisely, set

$$\underline{\sigma} := \frac{y^* - Rw\Phi(y^*)}{y^*} \tag{18}$$

which is equivalent to

$$(1-\underline{\sigma})\, y^* - Rw\Phi(y^*) = 0. \tag{19}$$

Then $\underline{\sigma}$ is the maximum value of σ such that the expected-profit maximizing firm faces no risk of bankruptcy whatever is the value of a. It is obvious that $\underline{\sigma} < 1$. To see that $\underline{\sigma} > 0$ note that, from (A2), $\Phi(y) < \Phi'(y)\, y$ for all $y > 0$. This implies, using (16),

$$y^* - Rw\Phi(y^*) > y^* - Rw\Phi'(y^*)\, y^*$$

$$= \left(\Phi'\right)^{-1}\left(\frac{1}{Rw}\right) - Rw\Phi'\left(\left(\Phi'\right)^{-1}\left(\frac{1}{Rw}\right)\right)\left(\Phi'\right)^{-1}\left(\frac{1}{Rw}\right) = 0.$$

Since for $\sigma \leq \underline{\sigma}$ there is no risk of bankruptcy when maximizing expected profit - and hence no problem with limited liability and moral hazard - this case is not interesting, and therefore we henceforth assume $\sigma \geq \underline{\sigma}$.

Coming back to $\widehat{y}_{II}(a)$, for $a \leq \overline{a}$ it is given by

$$y^{**}(a) := \arg\max_{y} \left[\mu(y,a) + \mu_1(y,a)\right]. \tag{20}$$

To see more precisely what $\mu_1(y,a)$ is, notice that, if (y,a) is such that $(1+\sigma)y - R[w\Phi(y) - a] < 0$, then there is bankruptcy for sure. It is clear that in that case, since then $\Gamma(y,a) = 0$, $\mu_1(y,a) = -\mu(y,a)$. If $(1+\sigma)y - R[w\Phi(y) - a] \geq 0$, then $\overline{\pi} = 0$ and from (14)

$$\mu_1(y,a) = \frac{1}{4\sigma y}\left\{(1-\sigma)y - R[w\Phi(y) - a]\right\}^2. \tag{21}$$

In the Appendix we show that $y^{**}(a)$ is a strictly decreasing function.

To complete the analysis of the decision function $\widehat{y}(a)$ we have to combine $\widehat{y}_{II}(a)$ with the function $\widehat{y}_I(a)$. From $\partial F(y,a)/\partial y > 0$ (see the proof of Lemma 2) it is clear that $\widehat{y}(a)$ is the minimum of $\widehat{y}_I(a)$ and $\widehat{y}_{II}(a)$. Recalling (11) let us now temporarily write α explicitly as an argument of $\widehat{y}_I$, i.e. $\widehat{y}_I(a,\alpha)$. Then observe that by (11) and (12) $\widehat{y}_I(a,0) = \underline{y}(a)$ for all a and, by (12), (15) to (17) and (20), $\widehat{y}_I(\overline{a},0) = \underline{y}(\overline{a}) = y^* = y^{**}(\overline{a})$. When α is positive, $\widehat{y}_I(a,\alpha) > \underline{y}(a)$ for all a, as (11) yields

$$\begin{aligned}
\frac{\partial \widehat{y}_I(a,\alpha)}{\partial \alpha} &= \frac{2\sigma\widehat{y}_I(a,\alpha)}{Rw\Phi'(\widehat{y}_I(a,\alpha)) - 2\sigma\alpha + 1 - \sigma} \\
&= \frac{2\sigma\widehat{y}_I(a,\alpha)}{Rw\Phi'(\widehat{y}_I(a,\alpha)) - \frac{R[w\Phi(\widehat{y}_I(a,\alpha) - a]}{\widehat{y}_I(a,\alpha)}} \\
&= \frac{2\sigma\widehat{y}_I(a,\alpha)}{Rw\left[\Phi'(\widehat{y}_I(a,\alpha)) - \frac{\Phi(\widehat{y}_I(a,\alpha)}{\widehat{y}_I(a,\alpha)}\right] + \frac{Ra}{\widehat{y}_I(a,\alpha)}} > 0
\end{aligned}$$

by strict convexity of Φ and $\Phi(0) = 0$. In particular, $\widehat{y}_I(\overline{a}, \alpha) > y^{**}(\overline{a}) = y^*$ for $\alpha > 0$.

Since by (11) $\widehat{y}_I(0,0)$ satisfies $(1-\sigma)\,\widehat{y}_I(0,0) - Rw\Phi(\widehat{y}_I(0,0)) = 0$, from (19) for $\sigma = \underline{\sigma}$ we have $\widehat{y}_I(0,0) = y^*$ and, again from (11), $\widehat{y}_I(0,\alpha) \leq y^*$ for any pair (σ, α) such that $2\sigma\alpha + 1 - \sigma \leq 1 - \underline{\sigma}$. For $\alpha < 1/2$ this is equivalent to $\sigma(1 - 2\alpha) \geq \underline{\sigma}$. By continuity of $\widehat{y}_I(a, \alpha)$ we obtain the following result:

Lemma 1. *For $0 < \alpha < 1/2$ there holds $\widehat{y}_I(0, \alpha) \leq y^*$ if and only if (σ, α) is such that $\sigma(1 - 2\alpha) \geq \underline{\sigma}$. In that case there exists $\underline{a} \geq 0$ such that $\underline{a} < \overline{a}$ and $\widehat{y}_I(\underline{a}, \alpha) = y^*$.*

Finally, since $\alpha > 0$ implies $\widehat{y}_I(\overline{a}, \alpha) > y^{**}(\overline{a})$, and since $\widehat{y}_I(\underline{a}, \alpha) = y^* = y^{**}(\overline{a})$ and $\widehat{y}_I$ is strictly increasing and y^{**} strictly decreasing in a, by continuity there is $\widehat{a}$ between $\underline{a}$ and $\overline{a}$ such that $\widehat{y}_I(\widehat{a}, \alpha) = y^{**}(\widehat{a})$. It is clear that $\widehat{y}(a) > y^*$ for all $a \in (\underline{a}, \overline{a})$. We summarize these findings in the following proposition.[8]

Proposition 1. *Under assumptions* (A1), (A2) *and* (A3), *for given confidence level $\alpha \in [0, 1]$ the firm's problem*

$$\begin{gathered} \max_y \Gamma(y, a) \\ s.t.\ VaR_\alpha[\Pi(y, a)] \leq 0 \end{gathered}$$

admits a solution $\widehat{y}(a)$ for any $a \geq 0$. Moreover, for (σ, α) such that $0 < \alpha < 1/2$ and $\sigma(1 - 2\alpha) > \underline{\sigma}$ there exist $\underline{a}$, $\widehat{a}$ and $\overline{a}$ with $0 \leq \underline{a} < \widehat{a} < \overline{a}$ such that $\widehat{y}(a)$ is given by

$$\widehat{y}(a) = \begin{cases} \widehat{y}_I(a) & if \quad 0 \leq a \leq \widehat{a} \\ y^{**}(a) & if \quad \widehat{a} \leq a \leq \overline{a} \\ y^* & if \quad a \geq \overline{a} \end{cases} .$$

The solution is continuous but non-monotone in the equity base a: strictly increasing between zero and $\widehat{a}$, strictly decreasing between $\widehat{a}$ and $\overline{a}$, and constant beyond $\overline{a}$. This furthermore entails that the firm's behavior gives rise to moral hazard whenever $a \in (\underline{a}, \overline{a})$.[9]

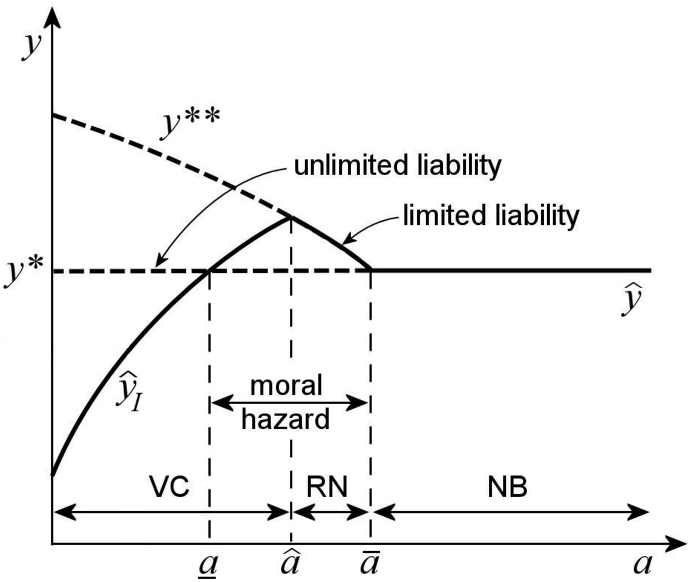

Figure 1: The function $\widehat{y}(a)$ (thick line)

For an economic explanation of these results let us consider Figure 1 and start with looking at what the firm would do in the case of unlimited liability. Then its objective function would be $\mu(y,a)$ as defined in (13) and consequently its optimal output y^* given in (16) would be constant, *for all* $a \geq 0$. To see why and how things change when limited liability is taken into account, i.e. the objective function is $\Gamma(y,a)$ as given by (5), let us parametrically diminish a from large values towards zero. For a sufficiently large ($a \geq \overline{a}$) nothing changes relative to the case with unlimited liability because for $a \geq \overline{a}$ there is no risk of bankruptcy (region NB).

When a is below $\overline{a}$, more precisely in region RN ($\widehat{a} \leq a < \overline{a}$, risk neutrality), the probability of bankruptcy while producing $y = y^*$ is positive and, if bankruptcy occured, the firm would suffer a loss. With limited liability, however, the loss is contained (with zero bankruptcy cost it is zero) and the firm has an incentive to run a higher risk of bankruptcy, which is tantamount to producing a quantity larger than y^*. Therefore $y^{**}(a) > y^*$ for $a < \overline{a}$. With increasing output - i.e. decreasing a - the probability of bankruptcy increases until at $\widehat{a}$ it reaches the confidence level α. At this point, the VaR constraint becomes binding (region VC) and, with equity a further diminished, output $\widehat{y}_I(a)$ decreases, too. But as long as $a > \underline{a}$, $\widehat{y}_I(a) > y^*$, and the probability of bankruptcy is larger than while producing y^*. Thus for $\underline{a} < a < \overline{a}$ there is moral hazard, whereas for $a \leq \underline{a}$ this is no longer true.

Of course, moral hazard could be avoided by setting the confidence level α to zero in which case $\underline{a} = \widehat{a} = \overline{a}$, but that might severely hamper the firm's business prospects. In particular, if the

[8]This is a modified version of an analogous result in Tulli and Weinrich (2009) where we have assumed $c \geq 0$ and $\sigma = 1$.

[9]We refer to the common definition of moral hazard according to which it is the prospect that a party insulated from risk may behave differently from the way it would behave if it were fully exposed to the risk. Moral hazard arises because an individual or institution does not bear the full consequences of its actions, and therefore has a tendency to act less carefully than it otherwise would, leaving another party to bear responsibility for the consequences of those actions.

firm had zero equity, since $\widehat{y}_I(0) = 0$ in case $\alpha = 0$, the firm, from a dynamic point of view, could not take off while with $\alpha > 0$ it can. A more satisfactory solution to the moral hazard problem will be achieved in the dynamic framework of the next section where it will be shown that, also when α is positive (but $\sigma(1-2\alpha) \geq \underline{\sigma}$), the firm chooses, induced by the VaR constraint, the equity base a so that no moral hazard occurs.

Example 1. *Assume $\Phi(y) = ky^2$, $k > 0$. Then, setting $\alpha_\sigma := \sigma\alpha + \frac{1}{2}(1-\sigma)$ it can easily be checked that*

$$\widehat{y}_I(a) = \frac{\alpha_\sigma + \sqrt{\alpha_\sigma^2 + R^2wka}}{Rwk},\ y^* = \frac{1}{2Rwk}, \tag{22}$$

$$\underline{\sigma} = 1/2,\ \underline{a} = \frac{2\sigma - 1 - 4\sigma\alpha}{4R^2wk},\ \overline{a} = \frac{2\sigma - 1}{4R^2wk}. \tag{23}$$

Moral hazard is a potential problem for $\sigma > 1/2$, since then $\overline{a} > 0$ which allows for $a < \overline{a}$. If $a > \underline{a} \geq 0$, too, it definitely occurs. For example when $\alpha = 0.01$ we get $\underline{a} \geq 0$ if $\sigma \geq 0.51$ while for $\alpha = 0.05$ we obtain $\underline{a} \geq 0$ if $\sigma \geq 0.56$.[10]

Furthermore it can be shown that $y^{**}(a)$ is that real solution to the equation

$$3(Rwk)^2 y^4 - 8R\sigma wky^3 + \left(4\sigma + 1 - \sigma^2 - 2R^2wka\right) y^2 - (Ra)^2 = 0$$

which gives the highest value to the function $\Gamma(\cdot, a)$; for $\sigma = 1$ it is [11]

$$y^{**}(a) = \frac{1 + \sqrt{1 - 3R^2wka}}{3Rwk} \tag{24}$$

and then

$$\widehat{a} = \frac{1 - 2\alpha - 3\alpha^2}{4R^2wk}\ .$$

[10] In all numerical examples values are reasonably rounded off.

[11] To the interested reader the derivation can be provided upon request.

3 Dynamic Analysis

3.1 The problem

We now extend the previous analysis by embedding it in a dynamic framework. The realized profit can be seen as the outcome of a particular period t, $\pi = \pi_t$, giving rise to the next period's equity base $a_{t+1} = (1 - \eta_t)\,\pi_t$, where $\eta_t \in [0, 1]$ is the share of profits that is paid out to shareholders as dividends, and $d_t = \eta_t \pi_t = \pi_t - a_{t+1}$ is their total value. This raises the question of the optimal choice of retained capital and the accumulated value of dividends paid over time.

Let $v(a) := \Gamma(\widehat{y}\,(a)\,, a)$ denote the value function of the static problem. Then by Proposition 1

$$v(a) = \begin{cases} \Gamma(\widehat{y}_I(a), a) & \text{if } a \leq \widehat{a} \\ \Gamma(y^{**}\,(a)\,, a) & \text{if } \widehat{a} \leq a \leq \overline{a} \\ \Gamma(y^*, a) & \text{if } a \geq \overline{a} \end{cases} \tag{25}$$

We assume that the operation of the firm in any period t is guided by the managers' objective to maximize $\Gamma(\cdot, a_t)$. Then the expected dividend payment to the firm's shareholders in that period is $v\,(a_t) - a_{t+1}$, provided the firm has survived until then. The risk-neutral shareholders maximize the discounted stream of all these expected payments which means that, given a_0, they have to optimally choose a sequence of equity amounts $(a_t)_{t\geq 1}$.[12] This gives rise to the problem

$$\begin{gathered} \max_{(a_t)_{t\geq 0}} \ \textstyle\sum_{t=0}^{\infty} \beta_t' \left[v\,(a_t) - a_{t+1}\right] \\ s.t.\ a_{t+1} \in A_{t+1}\ \forall t \geq 0,\ \ a_0 \text{ given,} \end{gathered} \tag{26}$$

[12]Starting with Jensen and Meckling (1976) a huge literature has developed to show that managers' objectives typically are different from those of shareholders and, in particular, that the former may be more short-sighted; see also footnotes 14 and 15. That shareholders are risk neutral is common in this literature, see e.g. Biais et al. (2010).

where $\beta'_t = \Pi_{\tau=0}^t \beta_\tau$, and β_τ, $\tau = 0, 1, \ldots$, are (one-period) discount factors to be determined later. In particular, they will have to capture the possibility of bankruptcy. Of course, if $\beta_\tau = \overline{\beta}$ for all τ, then $\beta'_t = \overline{\beta}^t$. Regarding the constraint sets A_{t+1} in (26), they will take care of the fact that, depending on the realizations π_t, not necessarily any solution to the *unconstrained problem, i.e. when* $A_{t+1} = \mathbb{R}$, will be feasible. We shall solve (26) in two steps: first when the problem is unconstrained and then when it is constrained. In the latter case we shall specify A_{t+1} accordingly.

The necessary Euler-Lagrange condition for a solution $a^* = (a_t^*)_{t\geq 1}$ is, setting $u(a_t, a_{t+1}) := v(a_t) - a_{t+1}$,

$$u\left(a_{t-1}^*, a_t^*\right) + \beta_t u\left(a_t^*, a_{t+1}^*\right) \geq u\left(a_{t-1}^*, a_t\right) + \beta_t u\left(a_t, a_{t+1}^*\right) \quad (27)$$

for any $a_t \in A_t$ and $t \geq 1$. For a solution with non binding constraints this means

$$u_2\left(a_{t-1}^*, a_t^*\right) + \beta_t u_1\left(a_t^*, a_{t+1}^*\right) = 0 \ \forall t \geq 1 \ ^{13} \quad (28)$$

which in the present set-up yields

$$-1 + \beta_t v'\left(a_t^*\right) = 0 \ \forall t \geq 1. \quad (29)$$

Note that a_t^* can be determined knowing β_t only, i.e. without knowing a_τ^*, $\tau \neq t$. On the other hand, β_t may, as we shall see soon, be influenced by a_τ^*, $\tau < t$. However, as we shall also see, at the solution to the unconstrained problem β_t will be constant for (almost) all t, $\beta_t = \overline{\beta}$, so that $a_t^* = \overline{a}^*$. In particular a_t^* will then be independent of all the other a_τ^*.

3.2 The discount factors

Let us now determine the discount factors β'_t and β_τ. Since in any period τ there is a possibility of going bankrupt, say $\delta_\tau \in [0, 1]$, the probability of survival until period $t \geq 2$ is $\Pi_{\tau=1}^{t-1}(1 - \delta_\tau)$. Thus the dividend payment in period t, $v(a_t) - a_{t+1}$, has to be

[13] u_i denotes the partial derivative w.r.t. the i-th argument.

multiplied by this probability to obtain its expected value. Moreover, the shareholders of the firm may have a constant subjective one-period discount factor $\beta \leq 1$. Then the coefficients to be effectively applied as discount factors to the outcome of any period t, $v(a_t) - a_{t+1}$, as seen from the initial period, are

$$\beta_0' = 1,\ \beta_1' = \beta,\ \beta_t' = \beta^t \Pi_{\tau=1}^{t-1} (1 - \delta_\tau)\ \forall t \geq 2.$$

More precisely still, the probability of bankruptcy in any given period $t \geq 1$ depends on a_{t-1} and is

$$\delta_t = \delta(a_{t-1}) := \begin{cases} \alpha & \text{if } a_{t-1} \leq \widehat{a} \\ F(y^{**}(a_{t-1}), a_{t-1}) & \text{if } \widehat{a} \leq a_{t-1} \leq \overline{a} \\ 0 & \text{if } a_{t-1} \geq \overline{a} \end{cases}$$

Since $0 \leq F(y^{**}(a), a) \leq \alpha$, $1 - \delta(a) \geq 1 - \alpha$ for any $a \in [\widehat{a}, \overline{a}]$. With these specifications, the coefficients β_t in (27)-(29) now become

$$\beta_0 = 1,\ \beta_1 = \beta \text{ and } \beta_t = \beta_t'/\beta_{t-1}' = \beta(1 - \delta(a_{t-2}))\ \forall t \geq 2.^{14}$$

As it will turn out, the most important case will be $a_t \leq \widehat{a}$ for all t and thus we shall have $\beta_t = \beta(1-\alpha)$ for all $t \geq 2$ which is independent of a_t.

3.3 Determination of $v'(a)$

To explore condition (29), we must next determine $v'(a)$. From (7) and by the envelope theorem,

$$v'(a) = \frac{\partial \Gamma(\widehat{y}(a), a)}{\partial a} - \lambda(a) \frac{\partial F(\widehat{y}(a), a)}{\partial a}$$

where $\lambda(a) \geq 0$ is the value of the Lagrange multiplier of the VaR constraint at the solution of the static problem. More precisely, we have, from (25) and using (15), (13) and (21),

[14] Note that for $\beta = 0$ the solution to (26) reduces to $\Gamma(\widehat{y}(a_0), a_0)$ and $a_t = 0$ for all $t \geq 1$. Thus the managers' objective can be considered a special case of that of the firm's shareholders which would arise for the latter if they were completely short-sighted.

$v'(a)$

$$= \begin{cases} \frac{\partial \mu}{\partial a}\left(\widehat{y}_I(a), a\right) + \frac{\partial \mu_1}{\partial a}\left(\widehat{y}_I(a), a\right) \\ -\lambda(a)\frac{\partial F}{\partial a}\left(\widehat{y}_I(a), a\right) - \lambda(a)\frac{\partial F}{\partial a}\left(\widehat{y}_I(a), a\right) & \text{if} \quad a < \widehat{a} \\ \\ \frac{\partial \mu}{\partial a}\left(y^{**}(a), a\right) + \frac{\partial \mu_1}{\partial a}\left(y^{**}(a), a\right) & \text{if} \quad \widehat{a} < a < \overline{a} \\ \\ \frac{\partial \mu}{\partial a}\left(y^{*}, a\right) & \text{if} \quad a > \overline{a} \end{cases}$$

$$= \begin{cases} R - \frac{R^2}{2\sigma\widehat{y}_I(a)}[w\Phi(\widehat{y}_I(a)) - a] + \frac{1-\sigma}{2\sigma}R \\ +\lambda(a)\frac{R}{2\sigma\widehat{y}_I(a)} & \text{if} \quad a < \widehat{a} \\ \\ R - \frac{R^2}{2\sigma y^{**}(a)}[w\Phi(y^{**}(a)) - a] + \frac{1-\sigma}{2\sigma}R & \text{if} \quad \widehat{a} < a < \overline{a} \\ \\ R & \text{if} \quad a > \overline{a} \end{cases} \tag{30}$$

Since $\widehat{y}_I(\widehat{a}) = y^{**}(\widehat{a})$ and $\lambda(\widehat{a}) = 0$, $v'^{-}(\widehat{a}) = v'^{+}(\widehat{a})$. Also, $y^{**}(\overline{a}) = y^*$ and $w\Phi(y^*) - \overline{a} = (1-\sigma)\, y^*/R$ imply

$$v'^{-}(\overline{a}) = R - \frac{R^2}{2\sigma y^*}(1-\sigma)\, y^*/R + \frac{1-\sigma}{2\sigma}R = R = v'^{+}(\overline{a}) \ .$$

Thus $v(a)$ is differentiable. Moreover, since $w\Phi(y^{**}(a)) - a > (1-\sigma)\, y^*/R$ for $a < \overline{a}$, $v'(a_1) < R = v'(a_2)$ for all $a_1 \in [\widehat{a}, \overline{a})$, $a_2 \geq \overline{a}$.

Consider now what happens to $\lambda(a)$ as $a \to 0$. Since the VaR-constraint is binding for $a < \widehat{a}$, $\lambda(a) > 0$ in that case. Moreover, from the Kuhn-Tucker conditions,

$$\frac{\partial \Gamma(\widehat{y}_I(a), a)}{\partial y} - \lambda(a)\frac{\partial F(\widehat{y}_I(a), a)}{\partial y} = 0$$

and thus

$$\lambda(a) = \frac{\frac{\partial \Gamma(\widehat{y}_I(a), a)}{\partial y}}{\frac{\partial F(\widehat{y}_I(a), a)}{\partial y}} \ .$$

From (13) and (21) we get

$$\frac{\partial\Gamma(y,a)}{\partial y}$$

$$\begin{aligned}
&= 1 - Rw\Phi'(y) \\
&\quad + \left(\frac{1}{4\sigma y}\right)^2 \Big(8\sigma y \{(1-\sigma) y - R[w\Phi(y) - a]\} \\
&\quad \big\{1 - \sigma - Rw\Phi'(y)\big\} - 4\sigma \{(1-\sigma) y - R[w\Phi(y) - a]\}^2\Big)
\end{aligned}$$

$$\begin{aligned}
&= 1 - Rw\Phi'(y) \\
&\quad + \frac{1}{2\sigma y} \{(1-\sigma) y - R[w\Phi(y) - a]\} \big\{1 - \sigma - Rw\Phi'(y)\big\} \\
&\quad - \frac{1}{4\sigma y^2} \{(1-\sigma) y - R[w\Phi(y) - a]\}^2 \ .
\end{aligned}$$

If $y = \widehat{y}_I(a)$, then by (11)

$$\begin{aligned}
\frac{\partial\Gamma(y,a)}{\partial y} &= 1 - Rw\Phi'(y) + \frac{1}{2\sigma y} \{-2\sigma\alpha y\} \big\{1 - \sigma - Rw\Phi'(y)\big\} \\
&\quad - \frac{1}{4\sigma y^2} \{-2\sigma\alpha y\}^2 \\
&= 1 - (1-\alpha) Rw\Phi'(y) - \sigma\alpha^2 - (1-\sigma)\alpha \ .
\end{aligned}$$

On the other hand, from (10) for $y = \widehat{y}_I(a)$

$$\frac{\partial F}{\partial y}(y,a)$$

$$\begin{aligned}
&= \frac{2\sigma y [Rw\Phi'(y) - (1-\sigma)] - 2\sigma \{R[w\Phi(y) - a] - (1-\sigma) y\}}{(2\sigma y)^2} \\
&= \frac{y [Rw\Phi'(y) - (1-\sigma)] - 2\sigma\alpha y}{2\sigma y^2} \\
&= \frac{Rw\Phi'(y) - 2\sigma\alpha - (1-\sigma)}{2\sigma y} \ .
\end{aligned}$$

Therefore

$$\lambda(a) = \frac{1 - (1-\alpha) Rw\Phi'(y) - \sigma\alpha^2 - (1-\sigma)\alpha}{Rw\Phi'(y) - 2\sigma\alpha - (1-\sigma)} 2\sigma y \ .$$

Inserting in (30) and using (11) yields

$$
\begin{aligned}
& v'(a) \\
= \;& R - \frac{R^2}{2\sigma\widehat{y}_I(a)}[w\Phi(\widehat{y}_I(a)) - a] + \frac{1-\sigma}{2\sigma}R \\
& + \frac{1-(1-\alpha)\,Rw\Phi'\left(\widehat{y}_I(a)\right) - \sigma\alpha^2 - (1-\sigma)\,\alpha}{Rw\Phi'(\widehat{y}_I(a)) - 2\sigma\alpha - (1-\sigma)}R \\
= \;& R\left[1 - \alpha + \frac{1-(1-\alpha)\,Rw\Phi'\left(\widehat{y}_I(a)\right) - \sigma\alpha^2 - (1-\sigma)\,\alpha}{Rw\Phi'(\widehat{y}_I(a)) - 2\sigma\alpha - (1-\sigma)}\right] \\
= \;& R\frac{(1-\alpha)\,(-2\sigma\alpha - (1-\sigma)) + 1 - \sigma\alpha^2 - (1-\sigma)\,\alpha}{Rw\Phi'(\widehat{y}_I(a)) - 2\sigma\alpha - (1-\sigma)} \\
= \;& R\frac{-2\sigma\alpha - (1-\sigma) + 2\sigma\alpha^2 + (1-\sigma)\,\alpha + 1 - \sigma\alpha^2 - (1-\sigma)\,\alpha}{Rw\Phi'(\widehat{y}_I(a)) - 2\sigma\alpha - (1-\sigma)} \\
= \;& R\frac{-2\sigma\alpha + \sigma + \sigma\alpha^2}{Rw\Phi'(\widehat{y}_I(a)) - 2\sigma\alpha - (1-\sigma)} \\
= \;& \frac{\sigma\,(1-\alpha)^2\,R}{Rw\Phi'(\widehat{y}_I(a)) - 2\sigma\alpha - (1-\sigma)}.
\end{aligned}
$$

Using that (11) implies

$$\widehat{y}'_I(a) = \frac{R}{Rw\Phi'(\widehat{y}_I(a)) - 2\sigma\alpha - (1-\sigma)} \tag{31}$$

we get

$$v'(a) = \sigma\,(1-\alpha)^2\,\widehat{y}'_I(a)\ . \tag{32}$$

By (11) and (31) we can rewrite $\widehat{y}'_I(a)$ as

$$\widehat{y}'_I(a) = \frac{1}{w\Phi'(\widehat{y}_I(a)) - \frac{w\Phi(\widehat{y}_I(a)) - a}{\widehat{y}_I(a)}}\ .$$

In particular,

$$\widehat{y}'_I(0) = \frac{1}{w\Phi'(\widehat{y}_I(0)) - \frac{w\Phi(\widehat{y}_I(0))}{\widehat{y}_I(0)}}\ . \tag{33}$$

Moreover, (11) yields

$$Rw\frac{\Phi(\widehat{y}_I(0))}{\widehat{y}_I(0)} = 2\sigma\alpha + 1 - \sigma \ . \tag{34}$$

3.4 Solution of the unconstrained dynamic problem

Let us now go back to the dynamic unconstrained optimality condition (29). We distinguish between:

- $a_t < \widehat{a}$, the VC region (VaR constrained),
- $\widehat{a} < a_t < \overline{a}$, the RN region (risk neutrality), and
- $a_t > \overline{a}$, the NB region (no bankruptcy).

Consider first the case that $a_t \in NB$. Then $\delta(a_t) = 0$, $\beta_t = \beta$ and hence, if there were an optimal policy (a_t), it should be constant. Since from (30) $v'(a_t) = R$, (29) becomes

$$-1 + \beta R = 0 \ .$$

Unless in the unlikely case that $\beta = 1/R$, this condition can obviously not be fulfilled. If $\beta R > 1$, then a_t should be increased without limit. If, on the contrary, $\beta R < 1$, then

$$u_2(a_{t-1}, a_t) + \beta u_1(a_t, a_{t+1}) = -1 + \beta R < 0$$

for all t, and hence it is always convenient to decrease a_t.[15] This leads the firm to exit the NB region. Thus we have the following result:

[15]It is common in the principal-agent literature to assume that the principal is less impatient than the agent (see e.g. Biais et al.(2010)). Since a loan at interest rate r transforms each unit of money into $R = 1+r$ units after one year, $1/R$ can be thought of as the bank's shareholders' discount factor. Thinking of them and the firm's shareholders as principal and agent, respectively, this means that $1/R > \beta$.

Proposition 2. *In case the firm's shareholders' discount factor β is smaller than $1/R$, it is not dynamically optimal to stay in the region where there is no risk of bankruptcy.*

Next consider the case that $a_t \in RN$. Then, since $R[w\Phi(y^{**}(a)) - a] > (1-\sigma)\, y^{**}(a)$ for $a < \overline{a}$, (30) implies

$$-\frac{R^2}{2\sigma y^{**}(a)}[w\Phi(y^{**}(a)) - a] + \frac{1-\sigma}{2\sigma}R < 0 \text{ for } a < \overline{a}\ . \qquad (35)$$

Therefore we have $v'(a_t) < R$. Moreover, $\beta_t = \beta\,(1-\delta\,(a_{t-2})) < \beta$, and hence $-1+\beta_t v'(a_t) < 0$ if $\beta R < 1$. Thus also in this region no optimal a_t exists; it would be reduced until the region RN were left to enter region VC.

Finally consider region VC. Here $v'(a)$ varies between $v'(\widehat{a})$ and $v'(0)$. Since $\lambda(\widehat{a}) = 0$,

$$v'(\widehat{a}) = R - \frac{R^2}{2\widehat{y}_I(\widehat{a})}[w\Phi(\widehat{y}_I(\widehat{a})) - \widehat{a}] + \frac{1-\sigma}{2\sigma}R < R$$

by (35) as $\widehat{y}_I(\widehat{a}) = y^{**}(\widehat{a})$ and $\widehat{a} < \overline{a}$. Moreover, $\beta_t = \beta\,(1-\alpha) < \beta$. Thus $-1+\beta_t v'(\widehat{a}) < 0$ if $\beta\,(1-\alpha)\,R < 1$ and it is convenient to diminish a_t below $\widehat{a}$. This implies that a sufficient condition for the existence of a stationary positive stationary $\overline{a}^* \in VC$, i.e. such that $-1+\beta\,(1-\alpha)\,v'(\overline{a}^*) = 0$, is that

$$-1+\beta\,(1-\alpha)\,v'(0) > 0 \qquad (36)$$

$\Leftrightarrow$ (by (32))

$$-1+\beta\sigma\,(1-\alpha)^3\,\widehat{y}'_I(0) > 0$$

$\Leftrightarrow$ (by (33))

$$-1+\frac{\beta\sigma\,(1-\alpha)^3\,\widehat{y}_I(0)}{w\Phi'(\widehat{y}_I(0))\widehat{y}_I(0) - w\Phi(\widehat{y}_I(0))} > 0$$

$\Leftrightarrow$

$$\beta\sigma\,(1-\alpha)^3 > w\left[\Phi'(\widehat{y}_I(0)) - \frac{\Phi(\widehat{y}_I(0))}{\widehat{y}_I(0)}\right]$$

$\Leftrightarrow$

$$\Phi'(\widehat{y}_I(0)) - \frac{\Phi(\widehat{y}_I(0))}{\widehat{y}_I(0)} < \frac{\beta\sigma\,(1-\alpha)^3}{w}$$

$\Leftrightarrow$

$$\Phi'(\widehat{y}_I(0))\frac{\widehat{y}_I(0)}{\Phi(\widehat{y}_I(0))} - 1 < \frac{\beta\sigma\,(1-\alpha)^3}{w}\frac{\widehat{y}_I(0)}{\Phi(\widehat{y}_I(0))} \ . \qquad (37)$$

Note that we can express this inequality in terms of the *elasticity of scale* which, denoting the production function $y = f(n) := \Phi^{-1}(n)$, is

$$f'(n)\frac{n}{f(n)} = \frac{1}{\Phi'(y)}\frac{\Phi(y)}{y} =: \varepsilon_s(y) \ .$$

Thus (37) is equivalent to

$$\frac{1}{\varepsilon_s(\widehat{y}_I(0))} < 1 + \frac{\beta\sigma\,(1-\alpha)^3}{w}\frac{\widehat{y}_I(0)}{\Phi(\widehat{y}_I(0))}$$

which, using (34), yields

$$\varepsilon_s(\widehat{y}_I(0)) > \left[1 + \frac{\beta\sigma\,(1-\alpha)^3}{2\sigma\alpha + 1 - \sigma}R\right]^{-1} \ . \qquad (38)$$

Since Φ is strictly convex with $\Phi(0) = 0$, $\Phi'(y) > \Phi(y)/y$ for all y, which implies $\varepsilon_s(y) < 1$ for all y.

Example 2. *For $\Phi(y) = ky^2$ one obtains $\varepsilon_s(y) = 1/2$ for all y. Using $\alpha = 0.01$, $\beta = 0.9$ and $R = 1.1$, we get for the right hand side of (38)*

$$(1 - 0.98\sigma)\,/\,(1 - 0.019\sigma) =: \ \phi(\sigma)$$

which is smaller than $1/2$ whenever $\sigma > 0.515$. For example, $\phi(0.6) = 0.42$ and $\phi(1) = 0.02$. The larger is price riskiness, reflected by a larger σ, the stronger is the need to hold sufficient equity and the easier it is that the sufficient condition for $\overline{a}^* > 0$ be satisfied.

Note that condition (38) can be verified without knowing the value of $\widehat{y}_I(0)$ if sufficient information is available about the elasticity function $\varepsilon_s(\cdot)$, for instance that it is constant. This is for example true if the labour requirement function is of the form $\Phi(y) = ky^\gamma$ in which case $\varepsilon_s = 1/\gamma$. Note also that (38) can never be fulfilled if $\beta = 0$, that is if shareholders do not care at all about any period other than the present one. In fact, it is clear that in this case $\overline{a}^* = 0$ since the shareholders choose $\eta_t = 1$ in any period t (as long as the firm exists). We summarize the results in the following proposition and illustrate it in Figure 2.

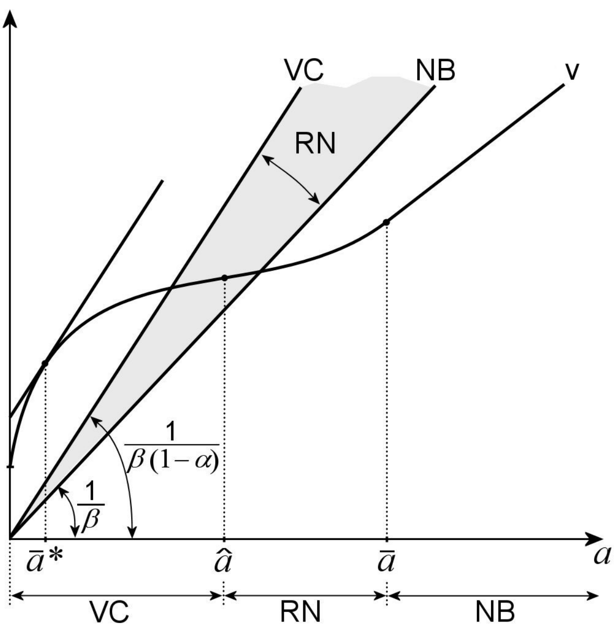

Figure 2: The function $v(a)$ and the graphical characterization of $\overline{a}^*$.

Proposition 3. *Assume a discount factor β for the firm's shareholders such that $\beta R < 1$ and $A_t = \mathbb{R}$ for all t. Then the optimal amount of equity is a constant value $\overline{a}^* \geq 0$ lying in region VC (i.e. the VaR constraint is binding) and satisfying $v'(\overline{a}^*) \leq [\beta(1-\alpha)]^{-1}$, with*

$$v'(\overline{a}^*) = \frac{1}{\beta(1-\alpha)} \text{ if } \overline{a}^* > 0 \ . \tag{39}$$

A sufficient condition for $\overline{a}^*$ to be positive is that the firm's elasticity of scale, $\varepsilon_s(y) = (1/\Phi'(y))(\Phi(y)/y)$, is not too small at $\widehat{y}_I(0)$, i.e.

$$\varepsilon_s(\widehat{y}_I(0)) > \left[1 + \frac{\beta\sigma(1-\alpha)^3}{2\sigma\alpha + 1 - \sigma} R\right]^{-1} \ . \tag{40}$$

Otherwise $\overline{a}^*$ may be zero, for example if β is close to zero. That would also be the outcome if the firm were not subjected to the VaR constraint.

3.5 Solution of the complete dynamic problem

Related to the unconstrained solution $\overline{a}^*$ is the optimal quantity of output $\overline{y}^* = \widehat{y}_I(\overline{a}^*)$. Since $\pi_t = p_t\overline{y}^* - R[w\Phi(\overline{y}^*) - \overline{a}^*]$ and p_t is random and hence varying, it may occur that $\overline{a}^* > \pi_t$ for some t. Then $\overline{a}^*$ is obviously not feasible and, since then $v'(a) > v'(\pi_t) > v'(\overline{a}^*)$ for all $a < \pi_t$, the best choice is $a^*_{t+1} = \pi_t$. The firm remains in region *VC*, retains all its period-t profit as new equity and does not pay any dividend. If in period $t+1$ it realizes a profit $\pi_{t+1} \geq \overline{a}^*$, it will choose $a^*_{t+2} = \overline{a}^*$ and pay the dividend $\pi_{t+1} - \overline{a}^*$. Otherwise it will set $a^*_{t+2} = \pi_{t+1}$, pay zero dividend, and so on, until for the first time $\pi_t < 0$ when it ceases activity. This, from the shareholders' point of view, is the best dividend policy. We can summarize it in the following way:

Corollary 1. *Under the same assumptions as in Proposition 3, except that now $A_{t+1} = [0, \max\{\pi_t, 0\}]$ for all t where π_t is the realized profit in period t, shareholders' best dividend policy is $d_t =$*

$\max\{\pi_t - \overline{a}^*, 0\}$ *for all* t. *If* $\pi_t < 0$, *which in each period occurs with probability* α, *the firm is bankrupt and ceases activity. In that case* $d_\tau = 0$ *for all* $\tau \geq t$.

Note that the lifespan of a firm is not influenced by the numerical values of the realizations of π_t as long as they are not negative. This is due to the fact that under the VaR-constraint the output quantity $\widehat{y}_I(a_t)$ is always chosen such that the probability of bankruptcy is equal to the confidence level α, independently of what a_t is.

Example 3. *As in the previous examples assume* $\Phi(y) = ky^2$, $k > 0$. *Then (22) implies*

$$\widehat{y}_I'(a) = \frac{R}{2}\left(\alpha_\sigma^2 + R^2wka\right)^{-1/2}$$

and (32)

$$v'(a) = \sigma(1-\alpha)^2 \frac{R}{2}\left(\alpha_\sigma^2 + R^2wka\right)^{-1/2} \quad .$$

Thus

$$\begin{aligned} v'(a) &= \frac{1}{\beta(1-\alpha)} \\ &\Leftrightarrow \quad \beta\sigma(1-\alpha)^3 R = 2\left(\alpha_\sigma^2 + R^2wka\right)^{1/2} \\ &\Leftrightarrow \quad 4R^2wka = \beta^2\sigma^2(1-\alpha)^6 R^2 - 4\alpha_\sigma^2 \end{aligned}$$

$\Leftrightarrow$

$$\overline{a}^* = \frac{(1-\alpha)^6 R^2\beta^2\sigma^2 - 4\alpha_\sigma^2}{4R^2wk} \quad . \tag{41}$$

With $\alpha = 0.01$, $\beta = 0.9$, $R = 1.1$, $w = 1$ *and* $k = 0.01$ *this yields* $\overline{a}^*$ *as an increasing function of* σ, *with* $\overline{a}^* > 0$ *whenever* $\sigma > 0.515$. *In particular, for* $\sigma = 1$ *the following numerical values result:* $\overline{a}^* = 19.06$ *and, from (22),* $\overline{y}^* = \widehat{y}_I(\overline{a}^*) = 44.58 < 45.46 = y^*$. *Then* $\pi_t = p_t\overline{y}^* - R[w\Phi(\overline{y}^*) - \overline{a}^*] \geq \overline{a}^*$ *if and only if* $p_t \geq 0.45$. *By (A1) the probability that this occurs is approximately* 78%.

Alternatively, to have at least a 50% probability that $p_t\widehat{y}_I(a_t) - R\left[w\Phi\left(\widehat{y}_I(a_t)\right) - a_t\right] \geq \overline{a}^*$*, it is sufficient that* $a_t \geq 3.43$ *which is 18% of* $\overline{a}^*$*. Finally, a* 50% *probability of survival until period t is taken on when* $t = 69$*.*

3.6 Comparative statics

From the above discussion it is clear that $a_t^* \leq \overline{a}^*$ for all t where the case $a_t^* < \overline{a}^*$ is forced upon shareholders only if the realization π_{t-1} was too small. We therefore henceforth call $\overline{a}^*$ the dynamically *desired* level to distingiush it from the dynamically optimal sequence $(a_t^*)_{t\geq 1}$. The comparative statics effect of a change in the confidence level α is captured by how $\overline{a}^*$ and the corresponding output level $\overline{y}^* := \widehat{y}_I(\overline{a}^*, \alpha)$ react to it, where $\widehat{y}_I(a, \alpha)$ is implicitly defined by (11). We thus proceed now in this sense.

By (32) and (39) we have

$$\frac{\partial \widehat{y}_I}{\partial a}(\overline{a}^*, \alpha) = \frac{1}{\beta\sigma\left(1-\alpha\right)^3} \text{ if } \overline{a}^* > 0\ . \tag{42}$$

Since from (11)

$$\frac{\partial \widehat{y}_I}{\partial a}(a, \alpha) = \frac{1}{w\Phi'\left(\widehat{y}_I(a, \alpha)\right) - \left(2\sigma\alpha + 1 - \sigma\right)/R}$$

(42) implies for $\overline{y}^*$

$$w\Phi'\left(\overline{y}^*\right) - \frac{2\sigma\alpha + 1 - \sigma}{R} = \beta\sigma\left(1-\alpha\right)^3\ . \tag{43}$$

This yields the following comparative statics results.

Proposition 4. *The dynamically desired values of output* $\overline{y}^*$ *and equity base* $\overline{a}^*$ *react to a change in the confidence level* α *as follows:*

$$\frac{\partial \overline{y}^*}{\partial \alpha}\begin{cases} < 0 \text{ for } \beta > \underline{\beta}(\alpha) \\ > 0 \text{ for } \beta < \underline{\beta}(\alpha) \end{cases} \tag{44}$$

where

$$\underline{\beta}(\alpha) := \frac{2}{3R\left(1-\alpha\right)^2}$$

and $\underline{\beta}(\alpha) < 1$ *for* $\alpha < 1 - \sqrt{2/3} \approx 0.184$. *A sufficient condition for*

$$\frac{\partial \overline{a}^*}{\partial \alpha} < 0$$

is $\beta > \underline{\beta}(\alpha)$.

Proof. Differentiating (43) implicitly yields

$$\frac{\partial \overline{y}^*}{\partial \alpha} = -\frac{-2\sigma/R + 3\beta(1-\alpha)^2}{w\Phi''(\overline{y}^*)} = \sigma\frac{2/R - 3\beta(1-\alpha)^2}{w\Phi''(\overline{y}^*)}$$

which is negative iff

$$\beta > \frac{2}{3R(1-\alpha)^2} = \underline{\beta}\ (\alpha)\,.$$

Since $R > 1$, $\alpha < 1 - \sqrt{2/3}$ implies $\underline{\beta}\ (\alpha) < 1$.

From (11) we obtain

$$\begin{aligned} \frac{\partial \overline{a}^*}{\partial \alpha} &= \left[w\Phi'(\overline{y}^*) - \frac{2\sigma\alpha + 1 - \sigma}{R}\right]\frac{\partial \overline{y}^*}{\partial \alpha} - \frac{2\sigma}{R}\overline{y}^* \\ &= \sigma\left\{\left[\beta(1-\alpha)^3\right]\frac{\partial \overline{y}^*}{\partial \alpha} - \frac{2}{R}\overline{y}^*\right\} \qquad (45) \end{aligned}$$

by (43). (44) implies the claim. □

An increase in the confidence level has different effects on output depending on how farsighted the firm's shareholders are. A larger α decreases the right hand side in the optimality condition (43) and, thus, the left hand side must be decreased as well. Since Φ is convex, Φ' is increasing in y, and therefore decreasing $\overline{y}^*$ favours to rebalance the equation. But an increase in α decreases the left hand side anyway, and so it is not always necessary to reduce $\overline{y}^*$. That depends on the size of the change on the right hand side which depends on β. If β is large, so is the decrease of the right hand side and thus the left hand side decrease must be large, too, which then may require a decrease of $\overline{y}^*$. Therefore farsighted shareholders ($\beta > \underline{\beta}\,(\alpha)$, e.g. with $R = 1.1$, $\underline{\beta}\,(0.01) = 0.62$) prefer to decrease output while shortsighted ones would opt for the opposite.

A similar argument holds for $\overline{a}^*$. A more severe VaR constraint (smaller α) induces the firm to hold a larger capital buffer as a prudential measure in case shareholders have a more farsighted attitude. When they are very shortsighted, however, a reduction in $\overline{a}^*$ cannot be excluded, this depending by (45) on the sign and the size of the change in output.

3.7 Moral hazard

A result of the static model was that for values of the equity base a in the interval $(\underline{a}, \overline{a})$ there arises moral hazard since for those values the output $\widehat{y}\,(a)$, taking the firm advantage of limited liability, is larger than y^*, the output under unlimited liability. This implies a larger probability of bankruptcy than without limited liability. Since in the dynamic context it is the firm which chooses its equity, i.e. $\overline{a}^*$, for no moral hazard to occur it is therefore sufficient to show that $\overline{y}^* = \widehat{y}\,(\overline{a}^*) \leq y^*$. This is made precise in the following theorem and illustrated in Figure 3.

Theorem 1. *Assume $\beta R < 1$ and $\alpha \leq \left(3 - \sqrt{5}\right)/2 \approx 0.382$. Then, whenever the dynamically desired value of capital $\overline{a}^*$ is positive there is no moral hazard. If $\overline{a}^* = 0$, no moral hazard occurs whenever $\sigma\,(1 - 2\alpha) \geq \underline{\sigma}$.*

Proof. If $\overline{a}^* > 0$, (43) yields

$$\Phi'\,(\overline{y}^*) = \frac{2\sigma\alpha + 1 - \sigma}{Rw} + \frac{\beta\sigma\,(1-\alpha)^3}{w}\ .$$

By (16) it follows that $\overline{y}^* \leq y^*$ iff

$$(\Phi')^{-1}\left(\frac{2\sigma\alpha + 1 - \sigma}{Rw} + \frac{\beta\sigma\,(1-\alpha)^3}{w}\right) \leq (\Phi')^{-1}\left(\frac{1}{Rw}\right)$$

which, by strict convexity of Φ, is equivalent to

$$\begin{aligned}\frac{2\sigma\alpha+1-\sigma}{Rw}+\frac{\beta\sigma\left(1-\alpha\right)^{3}}{w} &\leq \frac{1}{Rw}\Leftrightarrow 2\alpha-1+\beta\left(1-\alpha\right)^{3}R\leq 0\\ &\Leftrightarrow \beta R\leq\frac{1-2\alpha}{\left(1-\alpha\right)^{3}}\ .\end{aligned}$$

It is elementary to show that the term on the right-hand side is larger than or equal to one for $\alpha\in[0,1]$ if and only if $0\leq\alpha\leq\left(3-\sqrt{5}\right)/2$.

Next consider the case $\overline{a}^{*}=0$. From Lemma 1 we know that for $0<\alpha<1/2$ $\widehat{y}_{I}(0)\leq y^{*}$ if and only if (σ,α) is such that $\sigma\left(1-2\alpha\right)\geq\underline{\sigma}$. Since then $\overline{y}^{*}=\widehat{y}_{I}(0)$, and as $\left(3-\sqrt{5}\right)/2<1/2$, this proves the claim. □

Recall that the solution of the complete dynamic problem as discussed in Section 3.5 takes account of the possibility that $a_{t}=\pi_{t-1}<\overline{a}^{*}$ for some t. Since the region of no moral hazard reaches from zero to $\underline{a}\geq\overline{a}^{*}$ it is clear that the above theorem extends to all these possible cases.

Note that condition (40) for $\overline{a}^{*}>0$ can always be satisfied, for a given firm's elasticity of scale ε_{s}, by assuming price riskiness sufficiently large - i.e. σ sufficiently close to one - and the VaR constraint sufficiently tight - i.e. α sufficiently close to zero, thus avoiding moral hazard. But even if $\overline{a}^{*}=0$, the same is true with regard to satisfying the condition $\sigma\left(1-2\alpha\right)\geq\underline{\sigma}$. Moreover, this condition implies that, whenever in the case $\overline{a}^{*}=0$ moral hazard might become a problem, for given price riskiness $\sigma>\underline{\sigma}$ there always exists $\alpha>0$ small enough so as to exclude it.

Note also that, although moral hazard has been overcome, limited liability has been preserved: the cost of bankruptcy is still zero. What the VaR constraint does is *not* to eliminate limited liability; rather, it limits the firm's choice such that the probability of bankruptcy is not larger than α. Moreover, that probability *at the chosen point* $(\overline{a}^{*},\overline{y}^{*})$ is smaller than it would be under unlimited liability and without the VaR constraint , i.e. at $(\overline{a}^{*},y^{*})$. Hence there is no moral hazard.

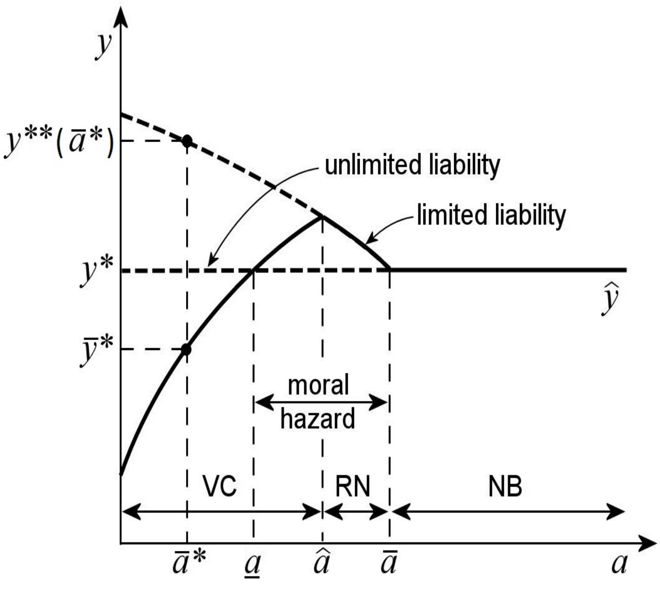

Figure 3: The dynamically optimal choice of the firm and its shareholders.

Example 4. *Under the same assumptions as in the previous examples, i.e. in particular* $\alpha = 0.01$, *it can be calculated that* $\sigma(1-2\alpha) < \underline{\sigma} < \sigma$ *and hence* $\overline{a}^* = 0$ *with* $\overline{y}^* > y^*$ *if* $0.5 < \sigma < 0.51$; $\sigma(1-2\alpha) \geq \underline{\sigma}$ *and* $\overline{a}^* = 0$ *with* $\overline{y}^* \leq y^*$ *if* $0.51 \leq \sigma \leq 0.515$; *and* $\overline{a}^* > 0$ *(with* $\overline{y}^* < y^*$*) if* $0.515 < \sigma \leq 1$. *In the latter case, price uncertainty is large (and* α *small) enough to induce the firm to hold a positive amount of equity to meet its VaR constraint. When* $\sigma < 0.515$, *the firm would like to reduce equity holding, but it cannot do this as the non-negativity constraint on* $\overline{a}^*$ *is binding. Thus the firm produces the maximum quantity it is permitted under the VaR constraint with zero equity holding. As long as* $\sigma(1-2\alpha) \geq \underline{\sigma}$, *the corresponding quantity remains below* y^*. *But when* $\sigma(1-2\alpha) < \underline{\sigma}$, *price riskiness is so small and/or the confidence level so large that, even with a binding VaR constraint, the firm is allowed to produce a quantity larger than* y^*. *Thus there is moral hazard for* $0.5 < \sigma < 0.51$ *and no moral hazard for* $0.51 \leq \sigma \leq 1$. *(For* $\sigma \leq 0.5$ *there is no risk of bankruptcy and no moral hazard since the expected profit maximizing firm chooses* y^*.*)*

In particular, for $\sigma = 1$ *we get* $\overline{a}^* = 19.06 < \underline{a} = 19.84 < \widehat{a} = 20.24 < \overline{a} = 20.66$ *and* $\overline{y}^* = 44.58 < y^* = 45.46 < \widehat{y}_I(\widehat{a}) = 45.91$. *Moreover, using (2) and (A1) to calculate the probability of bankruptcy*

$$\begin{aligned} prob\,(\Pi(y,a) < 0) &= prob\left(p < \frac{R\left[w\Phi(y) - a\right]}{y}\right) \\ &= \frac{1}{2}\frac{R\left[w\Phi(y) - a\right]}{y} = \frac{1.1\left(0.01y^2 - a\right)}{2y}, \end{aligned}$$

at $(\overline{a}^*, y^*)$ *it is* $0.019 > 0.01 = \alpha$, *the latter being the probability of bankruptcy at* $(\overline{a}^*, \overline{y}^*)$. *Without the VaR constraint but with limited liability, i.e. at* $(\overline{a}^*, y^{**}(\overline{a}^*)) = (19.06, 47.12)$ *(from (24)), the corresponding number is* 0.037. *Thus with limited liability, the introduction of the VaR constraint reduces the probability of bankruptcy by* $(0.037 - 0.01)/0.037 = 72.8\%$,*requiring a reduction of* $(0.471 - 0.446)/0.471 = 5.4\%$ *of output only! These values con-*

firm Theorem 1 and underline quite strikingly the effectiveness of the VaR constraint.

4 Concluding remarks

In this paper we have investigated the behaviour of a firm which is subject to a Value-at-Risk constraint. The rationale for doing this is to discipline the firm in its choices under uncertainty - bearing the risk of bankruptcy - and limited liability. The latter is at the heart of the functioning of the capitalist system. Tightening or even abolishing it would clearly have a negative effect on economic activity.[16] These circumstances create a moral-hazard problem as the firm can shift a part of the cost of risk-taking to its creditors and may distort its incentives towards behaving in a gambling way. A VaR constraint limits this distortion in that it induces the firm to abandon a risk-neutral attitude in case the risk of bankruptcy is about to exceed a certain predefined probability, namely, the confidence level. Thus the firm comes to behave as if it were kind of risk averse, albeit not risk averse in the conventional sense.

In a static set-up, when the capital endowment or equity base of the firm is given, the type of the firm's behavior - either "VaR constraint risk averse" or risk neutral - varies according to the size of the equity base. The different regimes create a non-monotonicity in the firm's output decision with respect to the capital endowment.

In a dynamic framework capital can be chosen in each period by selecting the corresponding dividend payment to shareholders. The model implies that the desired amount of capital to be retained in each period is constant over time and lies in a subset of the regime where the VaR constraint is binding and where the moral-hazard problem does not arise, even though limited liability is preserved. Thus Value-at-Risk achieves to reconcile two apparently conflicting goals, namely, to encourage entrepreneurial

[16]See e.g. Berkowitz and White (2004) and Fan and White (2003).

activity by means of limited liability and to avoid irresponsible gambling due to the incentives provided by it.

The comparative statics analysis of the dynamically desired values of retained capital and output with respect to the confidence level α reveals that the reaction of the desired output level is reversed in the case shareholders are farsighted as opposed to the one in which they are shortsighted. Moreover, a reduction of α - a strengthening of the VaR constraint - induces firms, with shareholders sufficiently farsighted, to increase their capital holding. This result underlines the potential of the VaR constraint as a possible policy instrument of prudential regulation of risk-taking, which should be welcome also in light of the recent financial and economic crisis where apparently one of its reasons was that banks were too lenient providing loans. In future research we intend to take up this issue and look more closely at practical applications of our theoretical model.

Appendix

Lemma 2. *$\widehat{y}_I(a)$ is a strictly increasing function.*

Proof. From $F(\widehat{y}_I(a), a) = \alpha$ for all a differentiating implicitly yields $\widehat{y}_I'(a) = -\frac{\partial F}{\partial a}(y,a)/\frac{\partial F}{\partial y}(y,a)$ where $y = \widehat{y}_I(a)$. Using (10), $\frac{\partial F}{\partial a}(y,a) = -R/(2\sigma y) < 0$ whereas

$$\begin{aligned}
\frac{\partial F}{\partial y}(y,a) &= \frac{1}{(2\sigma y)^2}\left\{2\sigma y\left[Rw\Phi'(y) - (1-\sigma)\right]\right. \\
&\quad \left. - \left\{R[w\Phi(y) - a] - (1-\sigma)y\right\}2\sigma\right\} \\
&= \frac{1}{2\sigma y^2}\left\{y\left[Rw\Phi'(y)\right] - \left\{R[w\Phi(y) - a]\right\}\right\} \\
&= \frac{1}{2\sigma y}\left\{Rw\left[\Phi'(y) - \frac{\Phi(y)}{y}\right] + \frac{Ra}{y}\right\} > 0
\end{aligned}$$

since by (A2) $\Phi'(y) - \Phi(y)/y > 0$ for any $y > 0$. □

To show that $y^{**}(a)$ is a strictly decreasing function we shall apply the following

Lemma 3. *Let $h(x,a)$ be a twice differentiable function such that $x^*(a) := \arg\max_x h(x,a)$ exists for all a and*

$$\frac{\partial^2 h(x,a)}{\partial x \partial a} < 0 \tag{46}$$

for all x and a. Then $x^*(a)$ is strictly decreasing.

Proof. Let $a_1 > a_0$ and assume to the contrary that there exists $\overline{x} \geq x^*(a_0)$ such that $h(\overline{x}, a_1) \geq h(x, a_1)$ for all x. Then $h(\overline{x}, a_1) \geq h(x^*(a_0), a_1)$ and

$$\int_{x^*(a_0)}^{\overline{x}} \frac{\partial h}{\partial t}(t, a_1)\, dt = h(\overline{x}, a_1) - h(x^*(a_0), a_1) \geq 0.$$

Thus by (46)

$$\int_{x^*(a_0)}^{\overline{x}} \frac{\partial h}{\partial t}(t, a_0)\, dt > \int_{x^*(a_0)}^{\overline{x}} \frac{\partial h}{\partial t}(t, a_1)\, dt \geq 0$$

and therefore

$$h(\overline{x}, a_0) = \int_{x^*(a_0)}^{\overline{x}} \frac{\partial h}{\partial t}(t, a_0)\, dt + h(x^*(a_0), a_0) > h(x^*(a_0), a_0)$$

which is a contradiction. □

Lemma 4. *$y^{**}(a)$ is a strictly decreasing function.*

Proof. We seek to sign $\partial^2 \Gamma(y,a) / \partial y \partial a$ where $\Gamma(y,a) = \mu(y,a) + \mu_1(y,a)$. By (13) $\partial^2 \mu(y,a)/\partial y \partial a = 0$ whereas, by (14),

$$\begin{aligned}
&\frac{\partial \mu_1(y,a)}{\partial y} \\
&= -\frac{1}{4\sigma y^2}\left\{(1-\sigma)y - R[w\Phi(y) - a]\right\}^2 \\
&\quad + \frac{1}{2\sigma y}\left\{(1-\sigma)y - R[w\Phi(y) - a]\right\}\left\{1 - \sigma - Rw\Phi'(y)\right\} \\
&= \frac{1}{2\sigma y}\left\{(1-\sigma)y - R[w\Phi(y) - a]\right\} \\
&\quad \times \left\{1 - \sigma - Rw\Phi'(y) - \frac{1}{2y}\left\{(1-\sigma)y - R[w\Phi(y) - a]\right\}\right\}
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{\partial^2 \mu_1(y,a)}{\partial y \partial a} &= \frac{1}{2\sigma y}\left\{R\left\{1-\sigma-Rw\Phi'(y)\right\}\right. \\
&\left.-\frac{1}{2y}\left\{(1-\sigma)y-R\left[w\Phi(y)-a\right]\right\}\right\} \\
&-\frac{1}{2\sigma y}\left\{(1-\sigma)y-R\left[w\Phi(y)-a\right]\frac{R}{2y}\right\} \\
&= \frac{R}{2\sigma y}\left\{1-\sigma-Rw\Phi'(y)\right. \\
&\left.-\frac{1}{y}\left\{(1-\sigma)y-R\left[w\Phi(y)-a\right]\right\}\right\} \\
&= \frac{R}{2\sigma y}\left\{-Rw\Phi'(y)+\frac{Rw\Phi(y)}{y}-\frac{Ra}{y}\right\}<0
\end{aligned}$$

by strict convexity of Φ and $\Phi(0)=0$. This implies $\partial^2\Gamma(y,a)/\partial y\partial a<0$. Hence we can apply the previous Lemma to conclude that $y^{**}(a)$ is decreasing for $a<\overline{a}$. □

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