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Value at Risk and Expected Shortfall based on Gram-Charlier-like expansions.

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Abstract

This paper offers a new approach to modeling the distribution of a portfolio composed of either asset returns or insurance losses. To capture the leptokurtosis, which is inherent in most financial series, data are modeled by using Gram–Charlier (GC) expansions. Since we are interested in operating with several series simultaneously, the distribution of the sum of GC random variables is derived. This latter turns out to be a tail-sensitive density, suitable for modeling the distribution of a portfolio return-losses and, accordingly, can be conveniently adopted for computing risk measures such as the value at risk and the expected shortfall as well as some performance measures based on its partial moments. The closed form expressions of these risk measures are derived for cases when the density of a portfolio is the sum of GC expansions, either with the same or different kurtosis. An empirical application of this approach to a portfolio of financial asset indexes provides evidence of the comparative effectiveness of this technique in computing risk measures, both in and out of the sample period.

JEL code: C1; G1

Keywords:

Gram-Charlier expansions; Value at Risk; Expected Shortfall; Heavy tailed distributions

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1. Introduction

In the last decades, financial, insurance markets and the development of financial engineering have highlighted the importance of an accurate evaluation of financial risk. In this regard, the choice of an appropriate distribution function underlying the measure of financial risks turns out to be a key problem for operators and analysts. The statistical models commonly in use rest on the assumption that asset returns are, by and large, normally distributed. However empirical evidence provides sound arguments against this hypothesis (see in this regard Mittnik et al. (2000) and Alles and Murray (2010)). In fact, it is well known that financial time series exhibit tails heavier than those of the normal distribution. This feature turns out to be of prominent importance in modeling volatility (Shuangzhe (2006); Curto et al. (2009)) and more generally in the evaluation of portfolio risk (Szegö (2004)). The presence of heavy tails is also a crucial topic for actuaries when modeling insurance loss data (Abu Bakar et al. (2015)). That's way, on one hand, alternative distributions like the Student t, the Pearson type VII, the inverse Gaussian and several stable distributions have become popular for modeling financial series and computing risk measures (see e.g., Mills and Markellos (2008); Rachev et al. (2010); Lee and Lin (2012)) and, on the other hand, some approaches have been developed to transform the Gaussian law to match the desired features (see Gallant and Tauchen (1989, 1993); Jondeau and Rockinger (2001); Zoia (2010)). This latter research line, which has the advantage of allowing greater flexibility in fitting empirical distributions, is the one we have used in this paper. Recently, Zoia (2010) and Bagnato et al. (2015) have proposed a method to account for the excess kurtosis of a density based on its polynomial transformation through its associated orthogonal polynomials. In the Gaussian case, these polynomials are the Hermite ones and the polynomially modified density is known as Gram-Charlier (GC) expansion. This approach is particularly interesting because it can be tailored on the specific features of the empirical distribution at hand and can be extended to other distributions besides the normal one (see Faliva et al. (2016)

for a detailed explanation of the use of GC in modeling asset-returns or insurance losses).

This paper expands this line of research so as to obtain the density of a sum of leptokurtic normal random variables. After adjusting normal laws using Hermite polynomials, the density function of their sum is obtained. This resulting density is a Gram-Charlier expansion (hereafter referred to as GCS) and proves to be a more tail-sensitive density than the Gaussian one. Consequently, it turns out to be more suitable for computing some risk measures such as the value at risk, VaR, and the expected shortfall, ES. The closed form expression of this latter risk measure is derived for the case when the density of the portfolio is the sum of GC expansions, either with the same or different kurtosis. This paper explores the potential of GCS expansions in computing both VaR and ES in the context of the new rules proposed by the Basel Committee on Banking Supervision (BCBS) in the Fundamental Review of Trading Book (FRTB) (see Basel Committee on Banking Supervision (2012, 2016)). Also the partial moments of a GCS density are obtained. It is proved that these moments, which can be expressed in terms of incomplete gamma functions, depend linearly on the excess kurtoses of the GC's involved in the sum. This result represents a generalization of that obtained by León and Moreno (2017) for a simple GC.

An empirical application to a portfolio of international financial indexes, with a data set window covering the period from January 2009 to December 2014, provides evidence of the effective performance of GCS densities. The structure of the paper is as follows. In section 2 a review of some standard risk-measures, typically used in the financial-insurance market, is provided. Section 3 explains how to obtain the distribution of GCS expansions. Section 4 provides closed-form expressions of the expected shortfall based on this distribution. Section 5 shows an application of this density to a portfolio of financial returns which provides evidence of the effectiveness of the proposed approach. Section 6 draws some conclusions. An Appendix completes the paper stating the essential notions regarding the sums of densities of normal random variables and GC expansions.

2. A glance at risk measures

As it is well known, different approaches are available to measure financial and/or insurance risks (see, for all, Albrecht (2004) and Dowd and Blake (2006), and the reference quoted therein). Descriptive measures based on the moments of a probability distribution give only a partial representation of the risk. To overcome this problem, a combination of these measures is often used, as happens for example with the mean and standard deviation in Markowitz portfolio theory or the skewness and kurtosis when symmetry and probability concentration in tails are of interest. Unfortunately, the estimation of the moments of a probability distribution may be quite sample sensitive and, when the moments are infinite, even impossible.

The standard theory for decision under risks, based on the expected utility approach, may be difficult to implement. In addition, it is sensitive to individual risk tolerance, due to the critical choice of the functional form of the utility function and the complex evaluation of the risk attitude parameter.

Measures of losses based on quantiles became very popular at the end of the 1980s, because of their implementation in determining the regulatory capital requirements of the US commercial banks. Value at risk based models were introduced in the Basel II agreement and later used for the calibration of the Solvency Capital Requirement, in the Solvency II agreement.

The Value at Risk (VaR) represents the minimum loss within a certain period of time for a given probability. By denoting with $F_X(x)$ the distribution function of a variable X representing the loss, VaR can be defined as

$$VaR_X(q) = inf\{x : F_X(x) \geq q\} = v_q = F_X^{-1}(q) \tag{1}$$

where $q \in (0,1)$. Since VaR is simply the threshold at a given probability q, it does not provide information about the size of any losses beyond this point of the distribution, although knowledge of the default size is crucial for shareholders, management and regulators. In addition VaR is a positively homogeneous but not subadditive and hence not a convex risk measure (see Föllmer and Schied

(2002)). Positive homogeneity assures the invariance of VaR with respect the change of currency, while failing of subadditivity means that the VaR of an aggregate position is not bounded by the sum of the individual $VaR's$. This, in turn, implies that this risk measure cannot satisfy the axiom of convexity according to which diversification of the positions held in a portfolio should not increase the risk. Failing to be convex, VaR cannot be a coherent risk measure (see Artzner et al. (1999)) unless losses are elliptically (e.g., normally) distributed. Otherwise, in case of losses/returns not normally distributed, VaR estimates may be incorrect and this shortcoming turns out to be very critical in the presence of fat tails. Furthermore, VaR based on discrete data series, can exhibit multiple local extrema (see Uryasev (2000)).

The interest of financial and insurance managers in tail risks clearly justifies the introduction of risk measures offering information on the magnitude of high risks. The Tail Conditional Expectation (TCE) provides the possible worst average loss and it is defined as

$$TCE_X(q) = \mathbb{E}[X|X \geq v_q] \tag{2}$$

where $\mathbb{E}$ denotes the expected value. The TCE is not generally a coherent measure of risk, because it can be not sub-additive. This drawback is evident when dealing with discontinuous distributions (for example with portfolios containing derivatives) because this measure may be very sensitive to small changes in the confidence level.

For real-valued finite-mean random variables X with absolutely continuous and strictly increasing distribution functions, the TCE coincides with the expected shortfall (ES) (see Acerbi and Tasche (2002)), which is a risk measure that respect the axioms of coherence. By denoting with $f(.)$ the density function of X, ES can be defined as

$$ES_X(q) = \frac{\int_{v_q}^{\infty} xf(x)dx}{\int_{v_q}^{\infty} f(x)dx}. \tag{3}$$

Another class of risk measures is based on the partial (lower/upper) moments of a density of stock returns. These risk measures, introduced by Fishburn (1977), can be used to compute some performance measures of the behaviour of a portfolio rankings, like Sortino ratio (see Sortino et al. (1999)), the Kappa ratio (see Sortino and Satchell (2005)) and Farinelli-Tibiletti ratio (see Farinelli and Tibiletti (2008)).
The lower and upper partial moments of order m of a density function $f(x)$, LPM_f and UPM_f respectively, computed with respect a minimal acceptable threshold τ, can be defined as follows

$$LPM_f(\tau, m) = \int_{-\infty}^{\tau} (\tau - x)^m f(x) dx \tag{4}$$

$$UPM_f(\tau, m) = \int_{\tau}^{\infty} (x - \tau)^m f(x) dx. \tag{5}$$

Setting $\tau = VaR_X(q)$ and noting that $LPM_f(\tau, 0) = F_X(\tau)$, the connection with $VaR_X(q)$ and $ES_X(q)$ is clear.

3. On the distribution of the sum of polynomially-modified Gaussian variables

In this section we tackle the issue of specifying the density function of the sum of polynomially-modified independent Gaussian variables (namely Gram-Charlier expansions). This density, being obtained by summing variables, whose kurtosis is tailored to that of the financial series of interest, turns out to be a tail-sensitive portfolio distribution. As such it can be usefully used to compute risk measures such as the value at risk and the expected shortfall.

Theorem 1. *Consider n independent distributed random variables $X_1, \ldots, X_n$ identically distributed as a Gram-Charlier expansion defined as follows (see Definition 2 in Appendix)*

$$f_{X_i}(x_i; \beta) = \left(1 + \frac{\beta}{4!} p_4(x_i)\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{x_i^2}{2}} \tag{6}$$

where β is a positive parameter subject to $f_{X_i}(x_i;\beta)$ being a density. Then, the density function of the sum $Y = X_1 + \cdots + X_n$ is given by

$$f_Y(y;\beta) = \sum_{j=0}^{n} \binom{n}{j} \left(\frac{\beta}{4!}\right)^j \frac{1}{\sqrt{2n\pi}} \left(\frac{1}{\sqrt{n}}\right)^{4j} e^{-\frac{y^2}{2n}} p_{4j}\left(\frac{y}{\sqrt{n}}\right), \tag{7}$$

where p_{4j} is the $4j-th$ degree Hermite polynomial

$$p_{4j}(z) = z^{4j} + \sum_{i=1}^{2j} (-1)^i (2i-1)!! \binom{4j}{2i} z^{4j-2i} \tag{8}$$

and $i!!$ is the double factorial.

Proof. Hereafter, the notation $f(x) \leftrightarrow F(\omega)$ will be used to indicate that the functions $f(x)$ and $F(\omega)$ form a Fourier-transforms pair. Now, bearing in mind the following property of Fourier transforms,

$$\frac{d^n f(x)}{dx^n} \leftrightarrow (i\omega)^n F(\omega) \tag{9}$$

where $F(\omega)$ denotes the characteristic function of the variable X and taking into account the noteworthy property of the Gaussian law,

$$\frac{d^n \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}}{dx^n} = (-1)^n \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} p_n(x) \tag{10}$$

the following proves true (see formula (50) in Appendix)

$$(-1)^n \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} p_n(x) \leftrightarrow (i\omega)^n e^{-\frac{\omega^2}{2}}. \tag{11}$$

This entails that the Fourier transform associated to (6)- that is its characteristic function - is

$$F_X(\omega;\beta) = \left(1 + \frac{\beta}{4!}\omega^4\right) e^{-\frac{\omega^2}{2}}. \tag{12}$$

After the argument put forward in Lemma 1 in Appendix, the density of a sum of independent random variables is the convolution of the densities of the single variables and, accordingly, its characteristic function is the product of the characteristic functions of these same variables. Hence, the characteristic function of the sum of n Gram-Charlier expansions can be written as follows

$$F_Y(\omega;\beta) = \left(1 + \frac{\beta}{4!}\omega^4\right)^n e^{-\frac{n\omega^2}{2}} = \sum_{j=0}^{n} \binom{n}{j} \left(\frac{\beta}{4!}\right)^j \omega^{4j} e^{-\frac{n\omega^2}{2}}. \tag{13}$$

Now, thanks to the following property of Fourier transforms

$$|a|f(ay) \leftrightarrow F\left(\frac{\omega}{a}\right), \tag{14}$$

formula (9) can be conveniently generalized as follows

$$\frac{d^n|a|f_X(ax)}{dx^n} \leftrightarrow \left(i\frac{\omega}{a}\right)^n F\left(\frac{\omega}{a}\right) \tag{15}$$

and this, in light of (11), entails the following

$$\frac{|a|}{\sqrt{2\pi}}e^{-\frac{(ax)^2}{2}}p_{4j}(ax) \leftrightarrow \left(i\frac{\omega}{a}\right)^{4j} e^{-\frac{1}{2}\left(\frac{\omega}{a}\right)^2}. \tag{16}$$

Then, setting $a = \frac{1}{\sqrt{n}}$ in formula (16), yields

$$\left(\frac{1}{\sqrt{n}}\right)^{4j} \frac{1}{\sqrt{2n\pi}}e^{-\frac{y^2}{2n}}p_{4j}\left(\frac{y}{\sqrt{n}}\right) \leftrightarrow \omega^{4j}e^{-\frac{n\omega^2}{2}} \tag{17}$$

which, following the same argument advanced in (13), clears the way to eventually obtain (7). □

The density of the sum variable $Y = X_1 + \cdots + X_n$ given in (7) depends on the parameter β which plays the role of common excess kurtosis (with respect to the standard Gaussian law) of each variable X_i . In Zoia (2010) it is shown that the Gram-Charlier expansion (6) is a positive density if $0 \leq \beta \leq 4$ and unimodal if $0 \leq \beta \leq 2.4$. These constraints also hold in the case of the sum of n i.i.d variables, according to the Theorem 1.6 in Dharmadhikari (1988).
The graphs in Figure 1 depict the density functions of the sums of n Gram-Charlier expansions for different values of n ($n = 1$, $n = 2$ and $n = 3$) and β. In each graph β has been set equal to 0, 1, 2 and 4.

As a further extension of the Theorem 1, we prove the following corollary which covers the case of Gram-Charlier expansions of sums of variables characterized by different excess kurtosis $\beta' s$.

Corollary 1.1. *Let us consider n independent Gram-Charlier expansions of the random variables $X_1, \ldots, X_n$, characterized by excess kurtosis $\beta_1, \ldots, \beta_n$,*

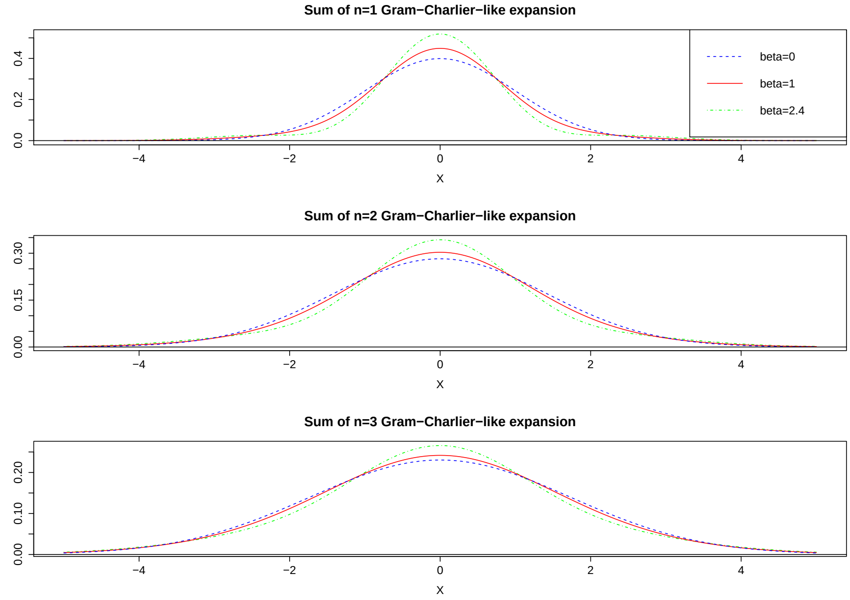

Figure 1: Densities of sums of Gram-Charlier expansions for $n = 1, 2, 3$, and $\beta = 0, 1, 2.4$.

respectively. Then, the density function of the sum $Y = X_1 + \cdots + X_n$ is given by

$$f_Y(y; \beta_1, \ldots, \beta_n) = \sum_{j=0}^{n} \left(\frac{b_{n,j}}{(4!)^j} \right) \frac{1}{\sqrt{2n\pi}} \left(\frac{1}{\sqrt{n}} \right)^{4j} e^{-\frac{y^2}{2n}} p_{4j} \left(\frac{y}{\sqrt{n}} \right) \tag{18}$$

where

$$b_{n,j} = \begin{cases} 1 & for \quad j = 0 \\ \sum_{i=1}^{n} \beta_i & for \quad j = 1 \\ \sum_{i_1=1}^{n-j+1} \sum_{i_2=1}^{i_1} \cdots \sum_{i_n=1}^{i_{n-1}} \beta_{i_1+n-1} \beta_{i_2+n-2} \ldots \beta_{i_n} & for \quad j = 2, 3, \ldots, n. \end{cases} \tag{19}$$

Proof. Following the same arguments put forward in Theorem 1, the characteristic function, $F_Y(\omega; \beta_1, \ldots, \beta_n)$, of the sum of n Gram-Charlier expansions with different excess kurtosis $\beta_j, j = 1, \ldots n$ is

$$\begin{aligned} F_Y(\omega; \beta_1, \ldots, \beta_n) = & \quad e^{-\frac{n\omega^2}{2}} \prod_{j=1}^{n} \left(1 + \frac{\beta_j}{4!} \omega^4 \right) = \\ = & \quad \sum_{j=0}^{n} \frac{b_{nj}}{(4!)^j} \omega^{4j} e^{-\frac{n\omega^2}{2}} \end{aligned} \tag{20}$$

where b_{nj} is as in (19) (see Nyblom (1999)).
Then, taking into account formulas (13) and (17) simple computations lead to (18). □

This approach can be extended to other densities, besides the normal law. However, when other distributions are taken into account, the density of the sum can be more conveniently obtained by convolution of the involved densities.

4. Expected Shortfall for sum of Gram-Charlier expansions

Gram-Charlier expansions are able to capture the excess of kurtosis and asymmetry of random variables better than the usual normal density. This property is true also for densities which are sums of Gram-Charlier expansions, GCS hereafter, with respect to densities of sums of simple Gaussian laws.
Hence, the next step is to use GCS to measure risks related to insurance or financial assets portfolios. In this section, following both the analysis of Landsman and Valdez (2003) and the studies of Acerbi and Tasche (2002), we show how to compute the expected shortfall to evaluate the right-tail risk of a sum of GC expansions. First we will consider the case of GC with same excess kurtosis, then the case of GC with different excess kurtosis.

Theorem 2. *Let Y be the sum of n i.i.d GC expansions X_1, X_2,..., X_n and let its density function be defined as in (7). Then, the $ES_Y(q)$ of Y is*

$$ES_Y(q) = \frac{\sqrt{\frac{n}{2\pi}} e^{-\frac{v_q^2}{2n}} \left[1 + \sum_{j=1}^{n} \left(\frac{1}{\sqrt{n}}\right)^{4j} \binom{n}{j} \left(\frac{\beta}{4!}\right)^j \left(p_{4j}\left(\frac{v_q}{\sqrt{n}}\right) + 4j p_{4j-2}\left(\frac{v_q}{\sqrt{n}}\right)\right)\right]}{\frac{1}{2} erfc\left(\frac{v_q}{\sqrt{2n}}\right) + \frac{1}{\sqrt{2\pi}} e^{-\frac{v_q^2}{2n}} \left[\sum_{j=1}^{n} \left(\frac{1}{\sqrt{n}}\right)^{4j} \binom{n}{j} \left(\frac{\beta}{4!}\right)^j p_{4j-1}\left(\frac{v_q}{\sqrt{n}}\right)\right]}. \tag{21}$$

Proof. Let's consider the expected shortfall defined as in the right-hand side of formula (3) and let's denote the numerator and the denominator of the integral in this formula by A and B, respectively.
By replacing the density function $f_Y(y, \beta)$, defined as in (7), in the numerator

A we obtain

$$A = \int_{v_q}^{\infty} yf(y)dy = \sum_{j=0}^{n} \left(\frac{1}{\sqrt{n}}\right)^{4j} \binom{n}{j} \left(\frac{\beta}{4!}\right)^{j} \frac{1}{\sqrt{2\pi}} \int_{v_q}^{\infty} \frac{y}{\sqrt{n}} e^{-\frac{y^2}{2n}} p_{4j}\left(\frac{y}{\sqrt{n}}\right) dy =$$
$$= \frac{1}{\sqrt{2\pi}} \int_{v_q}^{\infty} \frac{y}{\sqrt{n}} e^{-\frac{y^2}{2n}} dy + \sum_{j=1}^{n} \left(\frac{1}{\sqrt{n}}\right)^{4j} \binom{n}{j} \left(\frac{\beta}{4!}\right)^{j} \frac{1}{\sqrt{2\pi}} \int_{v_q}^{\infty} \frac{y}{\sqrt{n}} e^{-\frac{y^2}{2n}} p_{4j}\left(\frac{y}{\sqrt{n}}\right) dy. \tag{22}$$

Now, let us denote with A_1 and A_2 the first and second term on the right-hand side of the above formula. As far as A_1 is concerned, setting $\frac{y}{\sqrt{n}} = t$ in this integral and bearing in mind that $p_1(t) = t$, yields

$$A_1 = \sqrt{\frac{n}{2\pi}} \int_{\frac{v_q}{\sqrt{n}}}^{\infty} te^{-\frac{t^2}{2}} dt = \sqrt{\frac{n}{2\pi}} \int_{\frac{v_q}{\sqrt{n}}}^{\infty} p_1(t) e^{-\frac{t^2}{2}} dt. \tag{23}$$

Now, in light of (10), the following

$$\frac{d}{dt}\left[\frac{d^l}{dt^l} e^{-\frac{t^2}{2}}\right] = \frac{d^{l+1}}{dt^{l+1}} e^{-\frac{t^2}{2}} = (-1)^{l+1} e^{-\frac{t^2}{2}} p_{l+1}(t) \tag{24}$$

holds true.

This entails that

$$\begin{aligned} \int (-1)^{l+1} e^{-\frac{t^2}{2}} p_{l+1}(t) dt = \int \frac{d^{l+1}}{dt^{l+1}} e^{-\frac{t^2}{2}} dt = \\ = \frac{d^l}{dt^l} e^{-\frac{t^2}{2}} = \\ = (-1)^l e^{-\frac{t^2}{2}} p_l(t). \end{aligned} \tag{25}$$

By using this result and bearing in mind that $p_0(t) = 1$, formula (23) becomes

$$\begin{aligned} A_1 &= -\sqrt{\frac{n}{2\pi}} e^{-\frac{t^2}{2}} \Big|_{\frac{v_q}{\sqrt{n}}}^{\infty} \\ &= \sqrt{\frac{n}{2\pi}} e^{-\frac{v_q^2}{2n}}. \end{aligned} \tag{26}$$

As far as A_2 is concerned, setting $t = \frac{y}{\sqrt{n}}$ in this integral yields

$$A_2 = \sum_{j=1}^{n} K_j \int_{\frac{v_q}{\sqrt{n}}}^{\infty} te^{-\frac{t^2}{2}} p_{4j}(t) dt \tag{27}$$

where $K_j = \sqrt{\frac{n}{2\pi}} \binom{n}{j} \left(\frac{1}{\sqrt{n}}\right)^{4j} \left(\frac{\beta}{4!}\right)^{j}$.

Now, in light of the following property of Hermite polynomials

$$p_{s+1}(t) = tp_s(t) - sp_{s-1}(t) \tag{28}$$

the integral (27) can be rewritten as:

$$A_2 = \sum_{j=1}^{n} K_j \int_{\frac{\upsilon_q}{\sqrt{n}}}^{\infty} \left[e^{-\frac{t^2}{2}} p_{4j+1}(t) + 4j e^{-\frac{t^2}{2}} p_{4j-1}(t) \right] dt \tag{29}$$

which, in light of (25) becomes

$$\begin{aligned} A_2 &= -\sum_{j=1}^{n} K_j \left[4j e^{-\frac{t^2}{2}} p_{4j-2}(t) + e^{-\frac{t^2}{2}} p_{4j}(t) \right] \Big|_{\frac{\upsilon_q}{\sqrt{n}}}^{\infty} = \\ &= \sum_{j=1}^{n} K_j e^{-\frac{\upsilon_q^2}{2n}} \left(p_{4j}\left(\frac{\upsilon_q}{\sqrt{n}}\right) + 4j p_{4j-2}\left(\frac{\upsilon_q}{\sqrt{n}}\right) \right). \end{aligned} \tag{30}$$

Accordingly the integral A turns out to be

$$A = \sqrt{\frac{n}{2\pi}} e^{-\frac{\upsilon_q^2}{2n}} + \sqrt{\frac{n}{2\pi}} e^{-\frac{\upsilon_q^2}{2n}} \sum_{j=1}^{n} \left(\frac{1}{\sqrt{n}}\right)^{4j} \binom{n}{j} \left(\frac{\beta}{4!}\right)^{j} \left(p_{4j}\left(\frac{\upsilon_q}{\sqrt{n}}\right) + 4j p_{4j-2}\left(\frac{\upsilon_q}{\sqrt{n}}\right) \right). \tag{31}$$

Similarly, after replacing $f_Y(y, \beta)$, defined as in (7), in the denominator of (3), we get

$$\begin{aligned} B &= \int_{\upsilon_q}^{\infty} f(y) dy = \sum_{j=0}^{n} \left(\frac{1}{\sqrt{n}}\right)^{4j} \binom{n}{j} \left(\frac{\beta}{4!}\right)^{j} \frac{1}{\sqrt{2\pi}} \int_{\upsilon_q}^{\infty} \frac{1}{\sqrt{n}} e^{-\frac{y^2}{2n}} p_{4j}\left(\frac{y}{\sqrt{n}}\right) dy = \\ &= \frac{1}{\sqrt{2\pi}} \int_{\upsilon_q}^{\infty} \frac{1}{\sqrt{n}} e^{-\frac{y^2}{2n}} dy + \sum_{j=1}^{n} \left(\frac{1}{\sqrt{n}}\right)^{4j} \binom{n}{j} \left(\frac{\beta}{4!}\right)^{j} \frac{1}{\sqrt{2\pi}} \int_{\upsilon_q}^{\infty} \frac{1}{\sqrt{n}} e^{-\frac{y^2}{2n}} p_{4j}\left(\frac{y}{\sqrt{n}}\right) dy. \end{aligned} \tag{32}$$

Now, let us denote with B_1 and B_2 the first and second term on the right-hand side of the above formula. As far as B_1 is concerned, setting $t = \frac{y}{\sqrt{2n}}$ in the integral yields

$$B_1 = \frac{1}{\sqrt{\pi}} \int_{\frac{\upsilon_q}{\sqrt{2n}}}^{\infty} e^{-t^2} dt = \frac{1}{2} \text{erfc}\left(\frac{\upsilon_q}{\sqrt{2n}}\right) \tag{33}$$

where $erfc$ is the complementary error function (see formula 7.1.2 in Abramowitz and Stegun (1964)).

Similarly, setting $t = \frac{y}{\sqrt{n}}$ in the integral B_2 yields

$$B_2 = \sum_{j=1}^{n} \tilde{K}_j \int_{\frac{\upsilon_q}{\sqrt{n}}}^{\infty} e^{-\frac{t^2}{2}} p_{4j}(t) dt \tag{34}$$

where $\tilde{K}_j = \left(\frac{1}{\sqrt{n}}\right)^{4j} \binom{n}{j} \left(\frac{\beta}{4!}\right)^j \frac{1}{\sqrt{2\pi}}$.
Then, by using result (25), B_2 becomes

$$\begin{aligned} B_2 &= -\sum_{j=1}^{n} \tilde{K}_j \left[e^{-\frac{t^2}{2}} p_{4j-1}(t) \right] \Big|_{\frac{\upsilon_q}{\sqrt{n}}}^{\infty} \\ &= \sum_{j=1}^{n} \left(\frac{1}{\sqrt{n}}\right)^{4j} \binom{n}{j} \left(\frac{\beta}{4!}\right)^j p_{4j-1}\left(\frac{\upsilon_q}{\sqrt{n}}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{\upsilon_q^2}{2n}}. \end{aligned} \tag{35}$$

Accordingly, the integral B can be written as

$$B = \frac{1}{2}\text{erfc}\left(\frac{\upsilon_q}{\sqrt{2n}}\right) + \frac{1}{\sqrt{2\pi}} e^{-\frac{\upsilon_q^2}{2n}} \left[\sum_{j=1}^{n} \left(\frac{1}{\sqrt{n}}\right)^{4j} \binom{n}{j} \left(\frac{\beta}{4!}\right)^j p_{4j-1}\left(\frac{\upsilon_q}{\sqrt{n}}\right) \right]. \tag{36}$$

Finally, formula (21) is obtained by substituting the numerator and the denominator of formula (3) with A and B given in (31) and (36), respectively. □

This same procedure can be generalized to the case of n random variables with different extra-kurtosis parameters β_i.

Corollary 2.1. *Let Y be the sum of n independent GC expansions with different excess kurtosis and let $f_Y(.)$, defined as in (18), be its density function. Then, the $ES(q)$ of Y is*

$$ES_Y(q) = \frac{\sqrt{\frac{n}{2\pi}} e^{-\frac{\upsilon_q^2}{2n}} \left[1 + \sum_{j=1}^{n} \left(\frac{1}{\sqrt{n}}\right)^{4j} \binom{n}{j} \left(\frac{b_{n,j}}{(4!)^j}\right) \left(p_{4j}\left(\frac{\upsilon_q}{\sqrt{n}}\right) + 4j p_{4j-2}\left(\frac{\upsilon_q}{\sqrt{n}}\right) \right) \right]}{\frac{1}{2} erfc\left(\frac{\upsilon_q}{\sqrt{2n}}\right) + \frac{1}{\sqrt{2\pi}} e^{-\frac{\upsilon_q^2}{2n}} \left[\sum_{j=1}^{n} \left(\frac{1}{\sqrt{n}}\right)^{4j} \left(\frac{b_{n,j}}{(4!)^j}\right) p_{4j-1}\left(\frac{\upsilon_q}{\sqrt{n}}\right) \right]} \tag{37}$$

where $b_{n,j}$ is defined as in Corollary 1.1.

Proof. Formula (18), namely the density of a GCS built with GC variables with different excess kurtosis, differs from formula (7), which is the density of a GCS built with GC variables with the same excess kurtosis, only for the coefficients of the Hermite polynomials $p_{4j}\left(\frac{y}{\sqrt{n}}\right)$. Hence, replacing in (21) the coefficients of (7) with those of (18) yields formula (37). □

As a by-product of Theorem 2 and Corollary 2.1, we have the following.

Corollary 2.2. *Let Y be the sum of GC's specified either as in Theorem 2 or as in Corollary 2.1. Then, the VaR for this variable can be expressed as in formula (1) where*

$$F_Y(v_q) = 1 - \frac{1}{2} erfc\left(\frac{v_q}{\sqrt{2n}}\right) - \frac{1}{\sqrt{2\pi}} e^{\frac{v_q^2}{2n}} \left(\sum_{j=1}^{n} c_j p_{4j-1}\left(\frac{v_q}{\sqrt{n}}\right)\right). \tag{38}$$

Here $c_j = \left(\frac{1}{\sqrt{n}}\right)^{4j} \binom{n}{j} \left(\frac{\beta}{4!}\right)^j$ if the GC's are identically distributed or $c_j = \left(\frac{1}{\sqrt{n}}\right)^{4j} \left(\frac{b_{n,j}}{(4!)^j}\right)$ if the GC's have different kurtosis.

Proof. According to formula (1), $VaR_Y(q) = F_Y^{-1}(q)$, where $F_Y(v_q) = 1 - \int_{v_q}^{\infty} f_Y dy$, and f_Y, defined either as in Theorem 2 or as in Corollary 2.1, can be expressed as

$$f_Y = g_Y + \sum_{j=1}^{n} c_j p_{4j}\left(\frac{y}{\sqrt{n}}\right) g_Y \tag{39}$$

where $g_Y = \frac{1}{\sqrt{2n\pi}} e^{-\frac{y^2}{2n}}$ is the density of the sum of n independent standard Gaussian variables. Then, following the same lines developed in Theorem 2, the integral $\int_{v_q}^{\infty} f_Y dy$ can be worked out as in formula (38). □

Finally, let's now consider the upper partial moment (UPM) as defined in formula (5) for a GCS density $f_Y(y)$, specified as in (39). We will prove that the UPM for a GCS density turns out to be a linear function of the excess kurtosis(es) of the GC's involved in the sum as it happens for this risk measure computed by using a simple GC (see León and Moreno (2017)). In this connection we have the following.

Corollary 2.3. *The upper partial moment of order m for a GCS density, $UPM_f(\tau, m)$ hereafter, can be expressed as follows*

$$UPM_f(\tau, m) = \sum_{k=0}^{m} \zeta_{k,m} \Gamma\left(\frac{2k+1}{2}; \frac{\tau^2}{2n}\right) +$$
$$+ \sum_{j=1}^{n} c_j \sum_{k=0}^{m} \zeta_{k,m} \left(d_{0j}\Gamma\left(\frac{2k+1}{2}; \frac{\tau^2}{2n}\right) - d_{1j}\gamma^j \Gamma\left(\frac{2k+4j+1}{2}; \frac{\tau^2}{2n}\right) + \gamma^{2j}\Gamma\left(\frac{2k+8j+1}{2}; \frac{\tau^2}{2n}\right)\right) \tag{40}$$

where $\zeta_{k,m} = \frac{1}{2\sqrt{\pi}} \sum_{k=0}^{m} \gamma^{k/2}(-1)^k \binom{m}{k} \tau^{m-k}$, $\gamma^{k/2} = (2n)^{k/2}$.
$\Gamma(\alpha; x) = \int_x^\infty t^{\alpha-1} e^{-t} dt$ *is the incomplete gamma function (see e.g., Abramowitz and Stegun (1964)),* d_{ij}*,* $i = 0, 1$*, are the coefficients of the orthogonal polynomial of interest, that is*

$$p_{4j}(t) = \left(d_{0j} - d_{1j} t^{2j} + t^{4j}\right)$$

and the coefficients c_j *are defined as in Corollary 2.2. Note that the first term on the right-hand side of 40 is the upper partial moment of order m for the sum of n normally distributed random variables.*

Proof. Let $f_Y(y)$ be defined as in Corollary 2.2. Then, following León and Moreno (2017), the upper partial moment can be expressed as follows

$$UPM_f(\tau, m) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \tau^{m-k} \int_\tau^\infty \left(y^k g_Y(y) dy + \sum_{i=1}^{n} c_j y^k p_{4j}\left(\frac{y}{\sqrt{n}}\right) g_Y(y) dy \right).$$

Now, simple computations prove that

$$\int_\tau^\infty y^k g_Y(y) dy = \frac{1}{\sqrt{2n\pi}} \int_\tau^\infty y^k e^{-\frac{y^2}{2n}} dy = \frac{(\sqrt{n})^{k-1}}{\sqrt{2\pi}} \int_\tau^\infty \left(\frac{y}{\sqrt{n}}\right)^k e^{-\frac{1}{2}\left(\frac{y}{\sqrt{n}}\right)^2} dy$$
$$= \frac{n^{k/2}}{\sqrt{2\pi}} \int_{\tau/\sqrt{n}}^\infty r^k e^{-\frac{r^2}{2}} dr = \frac{(2n)^{k/2}}{2\sqrt{\pi}} \int_{\tau^2/2n}^\infty t^{(k-1)/2} e^{-t} dt = \frac{\gamma^{k/2}}{2\sqrt{\pi}} \Gamma\left(k + \frac{1}{2}; \frac{\tau^2}{2n}\right).$$

Accordingly, the integral $\int_\tau^\infty y^k p_{4j}\left(\frac{y}{\sqrt{n}}\right) g_Y(y) dy$ can be worked out as follows

$$\int_\tau^\infty y^k p_{4j}\left(\frac{y}{\sqrt{n}}\right) g_Y(y) dy = \int_\tau^\infty y^k \left(d_{0j} - d_{1j}\left(\frac{y}{\sqrt{n}}\right)^{2j} + \left(\frac{y}{\sqrt{n}}\right)^{4j} \right) \frac{1}{\sqrt{2n\pi}} e^{-\frac{y^2}{2n}} dy =$$
$$= \frac{1}{\sqrt{2\pi}} \left(d_{0j} \int_\tau^\infty n^{(k-1)/2} \left(\frac{y}{\sqrt{n}}\right)^k e^{-\frac{y^2}{2n}} dy + \right.$$
$$\left. - d_{1j} n^{(k+2j-1)/2} \int_\tau^\infty \left(\frac{y}{\sqrt{n}}\right)^{k+2j} e^{-\frac{y^2}{2n}} dy + n^{(k+4j-1)/2} \int_\tau^\infty \left(\frac{y}{\sqrt{n}}\right)^{k+4j} e^{-\frac{y^2}{2n}} dy \right) =$$
$$= \left(d_{0j} \frac{\gamma^{k/2}}{2\sqrt{\pi}} \Gamma\left(k + \frac{1}{2}; \frac{\tau^2}{2n}\right) - d_{ij} \frac{\gamma^{(k+2j)/2}}{2\sqrt{\pi}} \Gamma\left(k + 2j + \frac{1}{2}; \frac{\tau^2}{2n}\right) + \frac{\gamma^{(k+4j)/2}}{2\sqrt{\pi}} \Gamma\left(k + 4j + \frac{1}{2}; \frac{\tau^2}{2n}\right) \right)$$

which leads to the formula (40). □

5. An application to financial asset indexes

In order to evaluate the performance of the sum of Gram-Charlier expansions (GCS), we have carried out an application involving a set of four international

indexes, with different geographic locations and operational features. These are the Chinese stock exchange index (ˆHSI), a mining business index (GOLD), a telecommunication index (TIT.MI) and a pharmaceutical index (SXDP.Z).
As we are interested in measuring losses, data returns have been computed as minus the logarithm of the ratio between the prices at time t and $t-1$.
The preliminary statistics for these daily returns are reported in Table 1.

Table 1: Summary statistics of losses

	SXDP.Z	ˆHSI	GOLD	TIT.MI
μ	-0.0123	-0.0465	-0.0688	0.0290
sd	0.3805	1.4820	1.1305	2.1717
sk	0.3342	0.0280	0.2547	-0.2194
K	4.8171	4.9397	4.7824	4.5609
ρ	1.0000	0.1734	0.0373	0.3630

Mean (μ), standard deviation (sd), skewness index (sk), kurtosis index (K) of each loss and correlation coefficient (ρ) between each loss and the pharmaceutical one.

Then, GCS for the three pairs of series (SXDP.Z-ˆHSI), (SXDP.Z-GOLD), (SXDP.Z-TIT.MI) have been considered. According to formula (18), the density of each GCS is given by

$$f_Y(y;\beta_1,\beta_2) = \left(1+\frac{1}{4}\left(\frac{\beta_1+\beta_2}{4!}\right)p_4\left(\frac{y}{\sqrt{2}}\right)+\frac{1}{16}\frac{\beta_1\beta_2}{(4!)^2}p_8\left(\frac{y}{\sqrt{2}}\right)\right)\frac{1}{\sqrt{4\pi}}e^{-\frac{y^2}{4}} \quad (41)$$

where $p_4(x)$ and $p_8(x)$ are defined as follows

$$p_4(x) = x^4 - \binom{4}{2}x^{4-2} + 3\binom{4}{4}x^{4-4} = x^4 - 6x^2 + 3 \quad (42)$$

$$p_8(x) = x^8 - \binom{8}{2}x^6 + 3\binom{8}{4}x^4 - 15\binom{8}{6}x^2 + 105\binom{8}{8} = x^8 - 28x^6 + 210x^4 - 420x^2 + 105 \quad (43)$$

and β_j denotes the excess kurtosis (with respect to the Normal law) of the $j-th$ loss. The estimated excess kurtosis of the returns, once each pair of series has been standardized and correlation removed, are shown in Table 2.

In order to assess the goodness of fit of these GCS to data, the Hellinger's entropy test S_ρ (Granger et al. (2004); Maasoumi and Racine (2002)) between the empirical and the estimated distributions for the mentioned couples of returns have been computed and the relative p-values have been reported in the last column of Table 2. The null hypothesis of the test, which assumes the coincidence of the two distributions, is confirmed for all the couples of returns, assuming a significance level at 1%.

Table 2: Estimates of the extrakurtoses'β and p-values of Hellinger's entropy test for losses in the first period (first 1000 days)

Loss 1	Loss 2	β_1	β_2	p-val(S_ρ)
SXDP.Z	ˆHSI	1.719407	1.94666	0.045045
SXDP.Z	GOLD	1.881584	1.80461	0.071071
SXDP.Z	TIT.MI	2.269109	1.60179	0.822822

Figure 2 shows the tails of the estimated GCS densities together with those of the corresponding empirical distributions. The graph highlights the good fit of GCS to data, especially in the tail areas which are the loci involved in the risk measure estimates.

Then, in order to evaluate the performance of the aforesaid GCS densities in and out of the sample, which goes from 01/01/2009 to 12/31/2014, data have been split into two periods. The data of the first period, running from 01/01/2009 to 09/17/2013, have been used to estimate the GCS densities and compute some risk measures, such as the value at risk (VaR) and the expected shortfall (ES). The data of the second period, running from 09/18/2013 to 12/31/2014, have been used to evaluate the out-of-sample performance of GCS densities in computing the aforementioned risk measures. In the following, both VaR and ES will be denoted by an apex indicating the sample period, first ($1p$) or second ($2p$), to which they refer to.

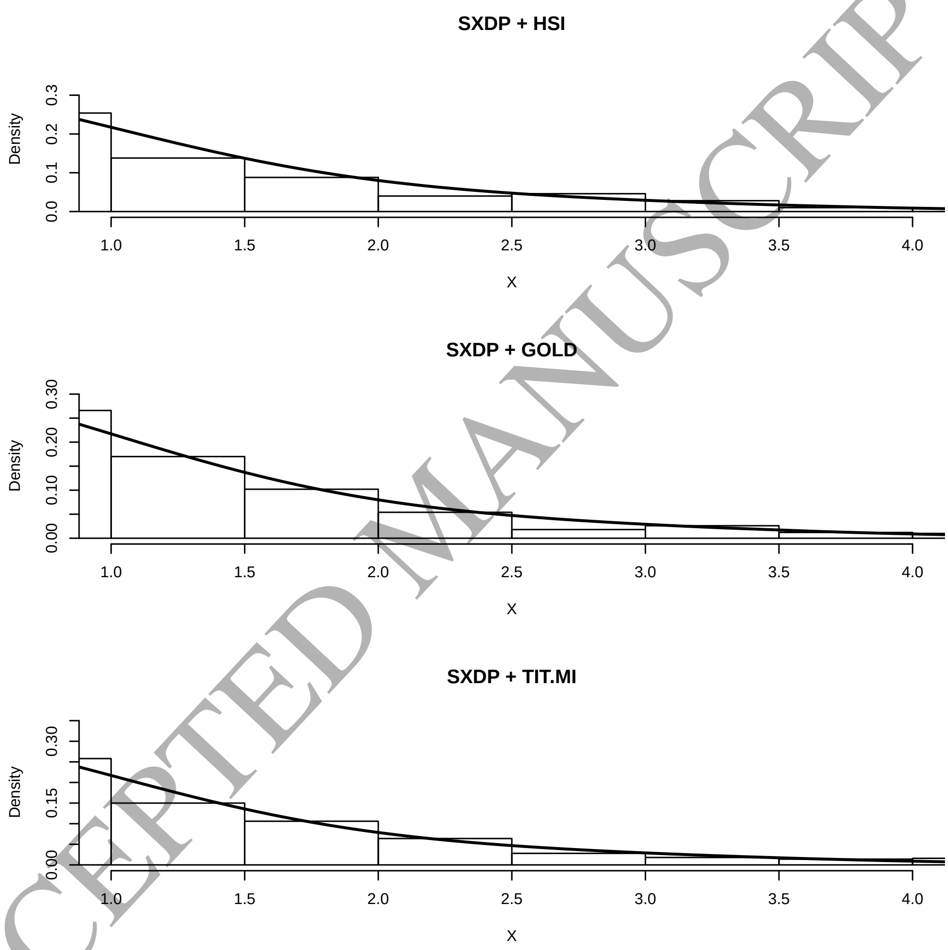

Figure 2: Histograms of the portfolio losses with the estimated GCS densities.

Table 3 compares $VaR^{1p}(\alpha)'s$ computed by using a $Normal$, a GCS and a t distribution, at different α level ($\alpha = 0.05$, $\alpha = 0.025$, $\alpha = 0.01$), with the corresponding empirical value at risk, $VaR^{1p}_{emp}(\alpha)$ (quantiles of the empirical distributions of the sum of couples of losses).
Looking at Table 3 we see that $VaR^{1p}(\alpha)$ computed assuming a $Normal$ distribution significantly underestimates the risk level so an adjustment is mandatory. Better estimates of $VaR^{1p}_{emp}(\alpha)$ are obtained by using leptokurtic distributions, like t and GCS densities, especially for $\alpha = 0.05$ and $\alpha = 0.025$. Hence, we can draw the conclusion that reference to the mere Gaussian law, leads to misleading results that may be dangerous for the risk management and in stark contrast to the regulatory philosophy.

Table 3: $VaR^{1p}(\alpha)$, $VaR^{1p}_{emp}(\alpha)$ and percentile bootstrap confidence intervals. The boldface font denotes values falling outside percentile confidence intervals.

				percentile C.I.		$VaR^{1p}(\alpha)$		
Loss 1	Loss 2	$1-\alpha$	$VaR^{1p}_{emp}(\alpha)$	l_p	L_p	$normal$	GCS	t
SXDP.Z	ˆHSI	0.95	2.3986	2.1737	2.5504	2.3262	2.3418	2.1757
SXDP.Z	ˆHSI	0.975	3.0089	2.7281	3.3551	2.7718	2.9377	2.8552
SXDP.Z	ˆHSI	0.99	4.0594	3.3557	4.7258	**3.29**	3.6165	3.8969
SXDP.Z	GOLD	0.95	2.3948	2.231	2.5501	2.3262	2.3423	**2.1422**
SXDP.Z	GOLD	0.975	3.0614	2.7068	3.4272	2.7718	2.9392	2.795
SXDP.Z	GOLD	0.99	4.0457	3.5272	4.6301	**3.29**	3.6179	3.7821
SXDP.Z	TIT.MI	0.95	2.3198	2.1681	2.5378	2.3262	2.3444	2.1736
SXDP.Z	TIT.MI	0.975	3.0087	2.7638	3.3424	2.7718	2.9501	2.8514
SXDP.Z	TIT.MI	0.99	3.7158	3.3328	4.5202	**3.29**	3.6332	3.8895

In Table 3 the upper and lower bounds, $(l_p - L_p)$, of the percentile bootstrap intervals, at confidence α, for $VaR^{1p}_{emp}(\alpha)$ have been also reported . These intervals, worked out by selecting 10000 bootstrap samples from the empirical distribution of each series, can be used to better investigate the behaviour of

$VaR^{1p}(\alpha)$ vs. $VaR^{1p}_{emp}(\alpha)$. The results shown in Table 3 show the validity of $VaR^{1p}(\alpha)$ provided by GCS. Looking at this table, we see that the estimates of this risk measure provided by GCS neither fall outside the percentile bootstrap intervals.

Figure 3 compares $VaR^{1p}_{emp}(\alpha)$ with $VaR^{1p}(\alpha)$ estimated via the GCS distributions. The graph highlights the good performance of GCS in computing this risk measure.

Also the less debatable expected shortfall has been computed as risk measure. According to (37), the ES of a couple of losses with GCS as a parent law is given by

$$ES_Y(q) = \\ = \frac{\frac{1}{\sqrt{\pi}} e^{-\frac{\upsilon_q^2}{4}} \left[1 + \frac{1}{2}\left(\frac{\beta_1+\beta_2}{4!}\right)\left(p_4\left(\frac{\upsilon_q}{\sqrt{2}}\right) + 4p_2\left(\frac{\upsilon_q}{\sqrt{2}}\right)\right) + \frac{1}{16}\left(\frac{\beta_1\beta_2}{(4!)^2}\right)\left(p_8\left(\frac{\upsilon_q}{\sqrt{2}}\right) + 8p_6\left(\frac{\upsilon_q}{\sqrt{2}}\right)\right)\right]}{\frac{1}{2}\mathrm{erfc}\left(\frac{\upsilon_q}{2}\right) + \frac{1}{\sqrt{2\pi}} e^{-\frac{\upsilon_q^2}{4}} \left[\frac{1}{2}\left(\frac{\beta_1+\beta_2}{4!}\right) p_3\left(\frac{\upsilon_q}{\sqrt{2}}\right) + \frac{1}{16}\left(\frac{\beta_1\beta_2}{(4!)^2}\right) p_7\left(\frac{\upsilon_q}{\sqrt{2}}\right)\right]}. \tag{44}$$

$ES^{1p}(\alpha)$ has been computed according to formula (44) by using $VaR^{1p}(\alpha)$, estimated by using a $Normal$, a GCS and a t distribution. The estimates of $ES^{1p}(\alpha)$ are reported in Table 4, for different α level, together with the values of the empirical shortfall, $ES^{1p}_{emp}(\alpha)$, computed by using the empirical density. Looking at these results, we draw the conclusion that, on one hand, GCS distributions fit the empirical series better than the $Normal$ law and, on the other hand, they maintain the prudential attitude that emerges in risk values computed with the t-distribution.

To assess the goodness of the $ES^{1p}(\alpha)$ estimates, the lower and upper bounds, $(l_p - L_p)$, of percentile bootstrap intervals, at confidence α, for $ES^{1p}_{emp}(\alpha)$ have been computed by using 10000 bootstrap samples from the empirical distribution of each series. The results, shown in Table 4 confirm the validity of these estimates obtained via GCS.

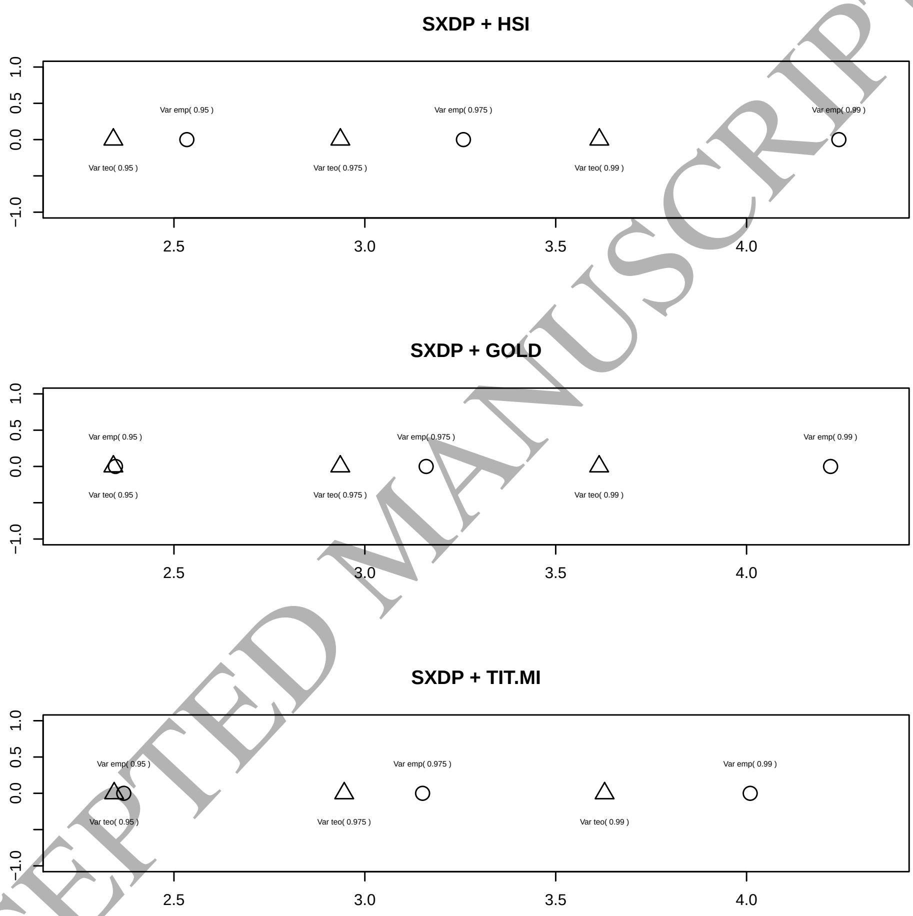

Figure 3: Empirical VaR^{1p} vs VaR^{1p}_{emp} estimated by GCS. Circles denote $VaR^{1p}_{emp}(\alpha)$; while triangles denote $VaR^{1p}(\alpha)$ with $\alpha \in (0.05, 0.025, 0.01)$.

Table 4: $ES^{1p}(\alpha)$, $ES^{1p}_{emp}(\alpha)$ and percentile confidence intervals. Boldface font denotes values falling outside percentile intervals

				percentile C.I.		$ES^{1p}(\alpha)$		
Loss 1	Loss 2	$1-\alpha$	$ES^{1p}_{emp}(\alpha)$	l_p	L_p	*normal*	*GCS*	*t*
SXDP.Z	ˆHSI	0.95	3.3367	3.111	3.6326	**2.9171**	3.2451	3.3273
SXDP.Z	ˆHSI	0.975	4.0114	3.5324	4.4706	**3.3062**	3.6608	4.1843
SXDP.Z	ˆHSI	0.99	4.8736	4.052	5.4001	**3.7692**	4.2647	**5.538**
SXDP.Z	GOLD	0.95	3.5524	3.2028	3.8247	**2.9171**	3.2459	3.2319
SXDP.Z	GOLD	0.975	4.057	3.6328	4.7549	**3.3062**	3.661	4.038
SXDP.Z	GOLD	0.99	4.8978	4.2233	5.6887	**3.7692**	4.2658	5.2942
SXDP.Z	TIT.MI	0.95	3.2409	2.9815	3.566	**2.9171**	3.2582	3.3212
SXDP.Z	TIT.MI	0.975	3.8534	3.6029	4.1423	**3.3062**	3.6717	**4.1749**
SXDP.Z	TIT.MI	0.99	4.3754	3.9028	5.4514	**3.7692**	4.2833	**5.5222**

To evaluate the out-of-sample performance of GCS densities, the estimates of $VaR^{1p}(\alpha)$, computed by using different distributions, have been used to evaluate some punctual measures of losses in the second part of the sample, that is in the last 480 days.

Reference is made to the $ABLF$ (average binary loss function), the $AQLF$ (average quadratic loss function) and the UL (unexpected loss), which evaluate the number of returns exceeding VaR according to a specific loss.

The binary loss function (BL) gives a penalty of one to each exception of VaR, without concern to its magnitude

$$\text{BL} = \begin{cases} 1 & \text{if} \quad r_t > VaR \\ 0 & \text{if} \quad r_t \leq VaR. \end{cases}$$

The quadratic loss function (QL) penalizes the exceptions with a different rule and pays attention to their magnitude.

$$\text{QL} = \begin{cases} 1 + (r_t - VaR)^2 & \text{if} \quad r_t > VaR \\ 0 & \text{if} \quad r_t \leq VaR. \end{cases}$$

Finally, the unexpected loss (UL) is the average magnitude of the violation, where the magnitude of the exception is defined as follows

$$\mathrm{L} = \begin{cases} r_t - VaR & \text{if} \quad r_t > VaR \\ 0 & \text{if} \quad r_t \leq VaR. \end{cases}$$

The values of these losses are displayed in Table 5. Looking at this table we can see that the estimates of the losses in the first part of the sample (first 1000 days), obtained by using GCS distributions are the lowest. This result proves that the GCS distributions are to be preferred not only in the sample but also out of the sample, especially for $\alpha = 0.05$ and $\alpha = 0.025$.

Table 5: Descriptive analysis of VaR in the second part of the sample.

			Normal			*GCS*			*t*		
Loss 1	Loss 2	$1-\alpha$	*ABLF*	*AQLF*	*UL*	*ABLF*	*AQLF*	*UL*	*ABLF*	*AQLF*	*UL*
SXDP.Z	ˆHSI	0.95	**0.0542**	0.1755	0.0638	**0.0542**	**0.1737**	**0.063**	0.0604	0.2023	0.0724
SXDP.Z	ˆHSI	0.975	0.0438	0.1179	0.0429	**0.0333**	**0.0944**	**0.0366**	0.0375	0.1048	0.0396
SXDP.Z	ˆHSI	0.99	0.025	0.0639	0.0267	0.0229	0.0471	0.0186	**0.0188**	**0.0342**	**0.0125**
SXDP.Z	GOLD	0.95	**0.0479**	0.2481	0.0597	**0.0479**	**0.2463**	**0.0589**	0.0583	0.2822	0.0693
SXDP.Z	GOLD	0.975	0.0271	0.1822	0.0431	**0.0229**	**0.1644**	**0.0389**	0.0271	0.1802	0.0425
SXDP.Z	GOLD	0.99	0.0167	0.1334	0.0319	0.0125	0.11	0.0275	**0.0104**	**0.099**	**0.0257**
SXDP.Z	TIT.MI	0.95	**0.0479**	0.1492	0.0521	**0.0479**	**0.1474**	**0.0513**	0.0562	0.1747	0.0603
SXDP.Z	TIT.MI	0.975	0.0375	0.1011	0.0331	**0.025**	**0.0779**	**0.0276**	0.0312	0.0898	0.0304
SXDP.Z	TIT.MI	0.99	0.0208	0.0578	0.0193	0.0146	0.0406	0.0133	**0.0083**	**0.0284**	**0.0101**

For each couple of indexes (first two columns), at level α,(third column) the table displays the indexes ABLF, AQLF and UL computed by using $Normal$ (fourth-sixth columns), GCS (seventh-ninth columns) and t (eleventh- thirteenth columns) distributions. The boldface font denotes the lowest values of the indexes.

The out-of-sample performance of GCS in estimating $VaR(\alpha)$, for a given significance level, has been also evaluated by implementing two tests: the likelihood-ratio test and the binomial two-sided test. The null hypothesis of both tests assumes consistency between the percentage of losses which in the second part in the sample exceed $VaR^{1p}(\alpha)$ with the expected loss frequency for a given confidence level. The percentage of losses have been estimated by using the aforementioned distributions. A p-value lower (or equal) than the significance level α can be interpreted as evidence against the null hypothesis (for more de-

tails see Kupiec (1995); Christoffersen et al. (1998)).
The results of both these tests are shown in Table 6 and lead to the non rejection of the null hypothesis for GCS densities.
Furthermore, a reading of the likelihood-ratio test of the $VaR(\alpha)$, inspired by the "traffic light" approach suggested by the Basel Committee (see Basel Committee on Banking Supervision (2016)), places GCS results in the "green zone".

Table 6: Likelihood-ratio and binomial tests

			Normal		*GCS*		*t*	
Loss 1	Loss 2	$1-\alpha$	p-val(LRuc)	p-val(VaR)	p-val(LRuc)	p-val(VaR)	p-val(LRuc)	p-val(VaR)
SXDP.Z	ˆHSI	0.95	0.6792	0.6745	0.6792	0.6745	0.3099	0.2938
SXDP.Z	ˆHSI	0.975	0.0172	0.0177	0.2654	0.2396	0.102	0.1045
SXDP.Z	ˆHSI	0.99	0.0055	0.0038	0.0149	0.01	0.0858	0.0629
SXDP.Z	GOLD	0.95	0.833	0.9167	0.833	0.9167	0.4139	0.4009
SXDP.Z	GOLD	0.975	0.7729	0.7686	0.7669	0.8842	0.7729	0.7686
SXDP.Z	GOLD	0.99	0.1803	0.1593	0.5962	0.49	0.9274	0.8173
SXDP.Z	TIT.MI	0.95	0.833	0.9167	0.833	0.9167	0.5376	0.5287
SXDP.Z	TIT.MI	0.975	0.102	0.1045	1	1	0.3983	0.3774
SXDP.Z	TIT.MI	0.99	0.0373	0.0325	0.3448	0.3496	0.7055	1

For each couple of indexes (first two columns), at level α (third column), the table displays the p-values of both the likelihood ratio and binomial test computed by using $Normal$ (fourth-fifth columns), GCS (sixth-eighth columns) and t (ninth-tenth columns) distributions.

As far as the expected shortfall is concerned, the out-of-sample performance of GCS densities in estimating this risk measure has been evaluated by comparing the empirical expected shortfall, computed by using data of the second part of the sample, $ES^{2p}_{emp}(\alpha)$ from now on, with $ES^{1p}(\alpha)$. Table 7 shows these estimates together with the lower and upper bounds, $(l_p - L_p)$, of the percentile bootstrap intervals at confidence, α, $(l_p - L_p)$, for $ES^{2p}_{emp}(\alpha)$. These latter have been obtained by selecting 10000 bootstrap samples from the empirical distribution of each series in the second part of the sample. The results shown in Table 7 provide evidence of the out-of-sample stability of this risk measure estimated via GCS.

The goodness of ES estimates has also been assessed by implementing two

Table 7: $ES^{1p}(\alpha)$, $ES^{2p}_{emp}(\alpha)$ and percentile confidence intervals. The boldface font denotes values falling outside percentile confidence intervals

				percentile C.I.		$ES^{1p}(\alpha)$		
Loss 1	Loss 2	$1-\alpha$	ES^{2p}_{emp}	l_p	L_p	*normal*	*GCS*	*t*
SXDP.Z	ˆHSI	0.95	3.5469	3.212	3.9624	**2.9171**	3.2451	3.3273
SXDP.Z	ˆHSI	0.975	4.1028	3.5019	4.5892	**3.3062**	3.6608	4.1843
SXDP.Z	ˆHSI	0.99	4.5613	3.945	5.1881	**3.7692**	4.2647	**5.538**
SXDP.Z	GOLD	0.95	3.5685	2.9828	4.2283	**2.9171**	3.2459	3.2319
SXDP.Z	GOLD	0.975	4.36	3.4056	6.278	**3.3062**	3.661	4.038
SXDP.Z	GOLD	0.99	6.2411	4.1449	8.0854	**3.7692**	4.2658	5.2942
SXDP.Z	TIT.MI	0.95	3.4116	3.0762	3.8683	**2.9171**	3.2582	3.3212
SXDP.Z	TIT.MI	0.975	4.0494	3.4892	4.6991	**3.3062**	3.6717	4.1749
SXDP.Z	TIT.MI	0.99	5.1049	3.7837	6.1575	**3.7692**	4.2833	5.5222

For each couple of indexes (first two columns) at each level α (third column) there are displayed the empirical ES^{2p}_{emp} evaluated on the second sample, ES^{1p}_{α} for the specific distribution, the p-values for the Z_1 and Z_2 tests computed by using $Normal$ (fourth-seventh columns), GCS (eighth-eleventh columns) and t (twelfth-fifteenth columns) distributions. The significance level is fixed at 1%.

tests based on a Monte Carlo (MC) procedure. The null hypothesis of both of them assumes that the distribution used to evaluate ES tallies with the empirical one. Accordingly, under the null hypothesis, $ES^{1p}(\alpha)$ should provide a good estimate of the empirical expected shortfall computed from data of the second period by using $VaR^{1p}_{emp}(\alpha)$. This expected shortfall will be denoted by $ES^{2p}_{emp,out}(\alpha)$ from now on. Under the alternative, this is not the case and $ES^{1p}(\alpha)$ systematically underestimates the effective losses mean, $ES^{2p}_{emp,out}(\alpha)$, thus implying a great damage .

The first test, proposed by McNeil and Frey (2000), is based on the statistic

$$Z_1 = \frac{1}{N}\sum_{t=1}^{N}\frac{X_t I_{X_t>VaR(\alpha)}}{ES(\alpha)} - 1 \tag{45}$$

where N is the number of losses X_t that in the second part of the sample (the last 480 days) lie over the VaR_α and $I_{X_t > VaR(\alpha)}$ is an indicator variable which assumes value equal to 1 if $X_t > VaR(\alpha)$ and 0 otherwise.
The second test, proposed by Acerbi and Szekely (2014), is quite similar to the previous one. The test statistic is

$$Z_2 = \frac{1}{T} \sum_{t=1}^{T} \frac{X_t I_{X_t > VaR(\alpha)}}{\alpha ES(\alpha)} - 1 \tag{46}$$

where T denotes the sample size (480 in the case under exam).
Both testing procedures have been performed by implementing a bootstrap simulation. In both cases, 999 samples have been extracted form the distribution under test, namely the $Normal$, the $t - Student$ and a GCS distribution, one at a time. Then the statistics Z_1 and Z_2 have been computed by using these samples, whose size is the same as the out-of-sample data set (480 days), and the p-values of both tests have been calculated as percentages of the Z_1 and Z_2 statistics, computed from simulated samples, which exceed the corresponding statistics Z_1 and Z_2 respectively computed by using data of the second part of the sample (last 480 days). Looking at these p-values, reported in Table 8, we can conclude that the out-of-sample performance of the GCS densities proves quite good in most cases.

Table 8: Out-of-sample ES performance

			$Normal$				GCS				t			
Loss 1	Loss 2	$1-\alpha$	$ES^{2p}_{emp,out}(\alpha)$	$ES(\alpha)$	p-val(Z1)	p-val(Z2)	$ES^{2p}_{emp,out}(\alpha)$	$ES(\alpha)$	p-val(Z1)	p-val(Z2)	$ES^{2p}_{emp,out}(\alpha)$	$ES(\alpha)$	p-val(Z1)	p-val(Z2)
SXDP.Z	ˆHSI	0.95	3.5046	2.9171	0.003	0.011	3.5046	3.2451	0.4825	0.3604	3.3744	3.3273	0.4044	0.1011
SXDP.Z	ˆHSI	0.975	3.7546	3.3062	0.015	0	4.0349	3.6608	0.3644	0.1261	3.9122	4.1843	0.7027	0.044
SXDP.Z	ˆHSI	0.99	4.3582	3.7692	0.019	0.003	4.429	4.2647	0.7347	0.0501	4.5661	5.538	0.8909	0.1011
SXDP.Z	GOLD	0.95	3.5722	2.9171	0.001	0.0891	3.5722	3.2459	0.3714	0.5786	3.3304	3.2319	0.3423	0.1071
SXDP.Z	GOLD	0.975	4.3646	3.3062	0.001	0.036	4.6346	3.661	0.0581	0.3804	4.3646	4.038	0.2072	0.2422
SXDP.Z	GOLD	0.99	5.2045	3.7692	0.001	0.003	5.8154	4.2658	0.1201	0.2733	6.2477	5.2942	0.1171	0.2923
SXDP.Z	TIT.MI	0.95	3.4152	2.9171	0.002	0.1912	3.4152	3.2582	0.6446	0.6927	3.2469	3.3212	0.5676	0.2643
SXDP.Z	TIT.MI	0.975	3.6547	3.3062	0.043	0.001	4.0536	3.6717	0.3193	0.4705	3.8252	4.1749	0.7698	0.2793
SXDP.Z	TIT.MI	0.99	4.2193	3.7692	0.049	0.003	4.5434	4.2833	0.7477	0.4054	5.1102	5.5222	0.5816	0.7087

All the analyses have been carried out by using software R (R Core Team (2015)). In particular, basic financial operations have been worked out by using *tseries* package (Trapletti and Hornik (2015)). Computations involving Hermite polynomials with *EQL* package (Thorn Thaler (2009)) and tests for the evaluation of goodness of fitting have been implemented by using *np* package (Hayfield and Racine (2008)) .

6. Conclusion

In this paper, we have devised a method to specify the distribution of sums of leptokurtic Gaussian variables. This approach rests on the polynomial transformation of Gaussian variables by means of their associated Hermite polynomials and the resulting Gram-Charlier (GC) expansions. The sum of Gram Charlier expansions (GCS) turns out to be a tail sensitive density and as such can be effectively used to represent the distribution of a portfolio return-losses. It can thus be conveniently used to compute some risk measures such as the Value at Risk and the expected shortfall. In particular its partial moments, which can be used to compute some performance measures of a portfolio of stock returns, are proved to be linear functions of the excess kurtoses of the GC's involved in the sum. An empirical application to a portfolio of a set of financial asset indexes provides evidence of the effectiveness of the proposed technique as it shows the goodness of GCS performance in VaR and expected shortfall estimation, both in and out of the sample period. We can therefore conclude that the results provided by GCS are more than satisfactory in according to both the current standard approach of risk measurement, based on VaR, and the new direction of the research based on more suitable risk measures, such as ES.

Appendix

In the following we run through the classic procedure to obtain the density of sum of independent standard-normal random variables. The following result, although well known in literature (see e.g. Ch.6 in Freund (1971)), is worth stating as its proof is a useful starting point for further results we are primarily interested in. The same procedure applies to sums of Gram-Charlier expansions with due computations as shown in Section 3, (see also e.g. Johnson and Kotz

(1970); Stuart and Ord (2004)).

In this connection let us first state the following.

Lemma 1. *Let* $Y = X_1 + X_2$*, be the sum of two i.i.d. normal random variables. Then the density of* Y *is*

$$f_Y(y) = (4\pi)^{-1/2} e^{-\frac{y^2}{4}}. \tag{47}$$

Proof. As it is well known, the density of Y is

$$f_Y(y) = f_X(x_1) * f_X(x_2) \tag{48}$$

where the symbol $*$ denotes convolution. Further, the characteristic function $F_Y(\omega)$ of Y is the product of the characteristic functions of the X_1 and X_2, that is

$$F_Y(\omega) = F_{X_1}(\omega) F_{X_2}(\omega) = F_X^2(\omega). \tag{49}$$

Now, bearing in mind the Fourier-transform pair

$$\sqrt{\frac{a}{\pi}} e^{-at^2} \leftrightarrow e^{-\frac{\omega^2}{4a}} \tag{50}$$

and setting $a = \frac{1}{2}$, yields

$$F_X(\omega) = e^{-\frac{\omega^2}{2}} \tag{51}$$

which is the characteristic function of the standard normal variable.

According to (49), the characteristic function of the sum of two i.i.d. standard normal is

$$F_Y(\omega) = e^{-\omega^2}. \tag{52}$$

In turn, by setting $a = 1/4$ in (50), the density function of the sum $f_Y(y)$ proves to be as in (47). $\square$

The same procedure applies to obtain the density function of the sum of two Gram-Charlier expansions as in Theorem 1 of Section 3.

In this connection let us introduce the following.

Definition 1. *Orthogonal polynomials.*

Given a density $f(x)$ with finite moments m_j, we can determine a system of polynomials $p_n(x) = \sum_j \delta_j x^j$ such that

$$\int_{-\infty}^{\infty} p_n(x)p_m(x)f(x)dx = \begin{cases} \gamma_n & for \quad m = n \\ 0 & for \quad m \neq n \end{cases} \tag{53}$$

the condition (53) determines $p_n(x)$ up to a constant factor and the coefficients δ_j turn out to be algebraic function of the moments m_j

$$m_j = \int_{-\infty}^{+\infty} x^j f(x)dx \tag{54}$$

(see Faliva et al. (2016) for details).

When the density $f(x)$ is even, $p_n(x)$ is either even or odd depending on n being even or odd, respectively.

Should $f(x)$ be the standard Gaussian law, then $\{p_n(x)\}$ would correspond to the well known Hermite polynomials, that is

$$p_j(x) = (-1)^j e^{\frac{x^2}{2}} \frac{d^j}{dx^j} e^{-\frac{x^2}{2}}. \tag{55}$$

and their squared norms γ_j, defined in (53), turn out to be equal to $j!$.

The first four Hermite polynomials are

$p_0(x) = 1$

$p_1(x) = x$

$p_2(x) = x^2 - 1$

$p_3(x) = x^3 - 3x$

$p_4(x) = x^4 - 6x^2 + 3.$

Orthogonal polynomials can be used to modify the moments of the parent density via Gram-Charlier expansions. In this connection we have the following.

Lemma 2. *Gram-Charlier expansions*

Let

$$q(x, \beta) = 1 + \frac{\beta}{\gamma_j} p_j(x) \tag{56}$$

where $p_j(x)$ is the orthogonal polynomial of degree j associated with a standard Gaussian density $f(x)$, β is a positive parameter and γ_j is defined as in (53). Then,

$$\varphi(x,\beta) = q(x,\beta)f(x) \tag{57}$$

subject to $q(x,\beta)$ being positive, is a density - known as Gram-Charlier expansion- whose lower order moments, μ_j, are related to those of $f(x)$, m_j, as follows

$$\mu_j = \begin{cases} m_i & for \quad i = 1,2,3,...j-1 \\ m_i + \beta & for \quad i = j. \end{cases} \tag{58}$$

Higher moments of $\varphi(x,\beta)$ turn out to be algebraic functions of the moments of $f(x)$.

Proof. The function $\varphi(x,\beta)$ is a density because the product (57) is positive and

$$\int_{\infty}^{\infty} \varphi(x,\beta)dx = \int_{\infty}^{\infty} [1 + \frac{\beta}{\gamma_j} p_j(x)]f(x)dx = \int_{\infty}^{\infty} f(x)dx = 1 \tag{59}$$

as

$$\int_{\infty}^{\infty} p_j(x)f(x)dx = \int_{\infty}^{\infty} p_0(x)p_j(x)f(x)dx = 0 \quad for \quad j \neq 0 \tag{60}$$

bearing in mind that $p_0(x) = 1$. Further, the $l-th$ moment, μ_l of $\varphi(x,\beta)$ is given by

$$\mu_l = \int_{\infty}^{\infty} x^l \varphi(x,\beta)dx = \int_{\infty}^{\infty} x^l f(x)dx + \frac{\beta}{\gamma_j}\int_{\infty}^{\infty} x^l p_j(x)f(x)dx. \tag{61}$$

Now, taking into account the following relationship among powers of x and orthogonal polynomials

$$x^l = p_l(x) + \eta_{l-1}p_{l-1}(x) + ... + \eta_0 p_0 \tag{62}$$

as well as the property (53), it is easy to draw the conclusion that

$$\frac{\beta}{\gamma_j}\int_{\infty}^{\infty} x^l p_j(x)f(x)dx = \begin{cases} 0 & \text{for} \quad l < j \\ \beta & \text{for} \quad l = j \end{cases} \tag{63}$$

which proves (58). □

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