Part III

DIVERGENCES IN INCOME DISTRIBUTION RANKING & POLICY IMPLICATIONS[1]

Abstract: This paper focuses on the differences between inequality and polarization measures in order to rank income distributions. Using two-spike distributions, our analysis shows the behaviour of inequality measures and polarization measures; in particular, we focus on monotonicity or non-monotonicity with respect to variations in the parameters of the distribution on the basis of the characteristics of each index. This paper also shows that the policy implications of theoretical and empirical models could diverge depending on the chosen measure.

Key Words: Economic Methodologies, Income Distribution, Public Good Provision

JEL Code: B4, D3, H4

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1 Introduction

There are many empirical papers and books on the policy effects of the distribution of incomes and there are many ways to rank distributions. Several indices referring to different concepts are used in the literature. Our purpose is to explore the relationships between the indices and we will also show that the policy implications of different ways to rank distributions could diverge, even if the measures we analyze refer to the same theoretical concept.

First of all, we explain the differences between the concepts of inequality and polarization given a generic income distribution,[2] then we will explore the different characteristics of the indices.

Second, we consider the case of the simplest possible non-uniform distribution of incomes, a two-spike distribution, calculating how variations in one of its parameters affects each index,[3] then we focus on similarities between measures referring to different concepts and differences between measures referring to the same concept.

In the end, we discuss how the policy implications of theoretical and empirical models could diverge depending upon the chosen measure.

We focus on different literatures. Theoretical works, like Gini (1939), Theil (1967), Atkinson (1970), Lam (1986), Wolfson (1994), Esteban and Ray (1994), Wang and Tsui (2000) and the handbook by Lambert (1993). Empirical works on the analysis of trends in polarization and inequality, like Esteban, Gradìn and Ray (2007), Wolfson (1997) and Zhang and Kanbur (2001). Empirical works on the policy implications of income distribution, like Persson and Tabellini (1994) on the link between income inequality and growth rate of the economy and Alesina and Perotti (1996) on the effects of income distribution on political stability of countries. We also focus on analyses on the effect of income inequality on public expenditure like Lindert (1996) and Milanovic (1999). Finally, the works by Corneo and Grüner (2002) and Ravallion and Lokshin (2000) give us interesting hints in order to study the policy implications of the indices.

This paper is organized as follows: Section 2 presents the indices; Section 3 describes the behaviour of the indices in case of two-spike distributions; Section 4 discusses the policy implications and Section 5 briefly concludes.

[2] We consider right-skewed income distributions; i.e., distributions where median income is lower than average income. Notice that there are no empirically observed income distributions worldwide where median income is higher than average income.

[3] In this paper we consider the case of non mean preserving spreads, in order to be able to study the effects of variations in one parameter of the distribution that is independent of variations in the other parameters.

2 Income distributions ranking

2.1 Inequality indices

A standard measure of inequality is a scalar representation of the interpersonal difference in income within a given population.

Frank A. Cowell

Gini Inequality Index

Gini Inequality Index is based on Lorenz curves method. In the sense of Gini (1939), inequality is the "difference" between Lorenz curve and equality line. The Index is defined as a ratio where the numerator is the area between equality line and Lorenz curve and the denominator is the area under uniform distribution line.

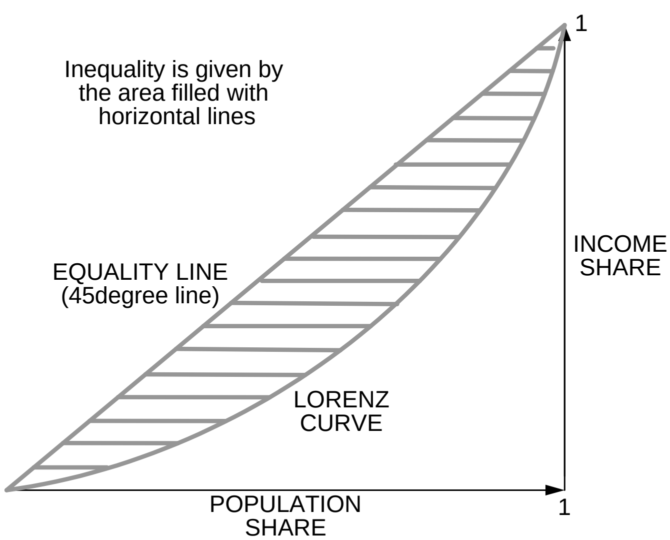

Figure 1: Gini Index

If Lorenz curve given distribution X can be represented by:

$$Lorenz\ curve = L_X(q)$$

where q represents income percentile(s), Gini Index is given by:

$$G = 1 - 2\int_0^1 L_X(q)dq$$

where $\int_0^1 L_X(q)dq$ is the area under the Lorenz curve.

Gini Coefficient takes values between 0 (perfect income equality) and 1 (perfect inequality; i.e., only one person holds richness).

Gini Coefficient gives more "weight" to the incomes around the mode and less "weight"to the ends of the distribution; it satisfies the weak principle of transfers, it is not decomposable and it is independent of population scale and income scale.

Theil's Entropy Inequality Index

Theil's definition of his own Index is: "[The Theil Index can be interpreted] as the expected information content of the indirect message which transforms the population shares as prior probability into the income shares as posterior probabilities" (1967). This index does not deal with the Lorenz curves method; it deals with the concept of entropy, which can be considered the "degree of disorder" of a system, as stated by Cowell (1995): in particular, if we refer to inequality measurement, entropy can be expressed as:

$$entropy = \sum_{i=1}^{n} p_i h\left(p_i\right) = -\sum_{i=1}^{n} p_i \ln\left(p_i\right)$$

where: n is the number of individuals and p_i is the share of person i in total income.

If we use Theil Index, overall inequality can be expressed through a weighted sum of the inequality values for every income subgroups:

$$T = \frac{1}{n} \sum_{i=1}^{n} \frac{y_i}{\bar{y}} \ln \frac{y_i}{\bar{y}}$$

where: n is the number of individuals; y_i is the income of individual i and $\bar{y}$ is average income.

The Theil's Entropy Inequality Index takes value 0 in case of perfect income equality and increases together with income inequality.

Theil's Entropy Index gives the same "weight" to every income group; it satisfies the strong principle of transfers, it is decomposable and it is independent of population and income scale.

Atkinson Inequality Index

Gini Coefficient and Theil's Entropy Index do not give more "weight" to the bottom end of the distribution. Atkinson Index, *de facto,* attaches different "weights" to different income levels depending on the parameter $e \in (0, +\infty)$ which represents inequality aversion; that is, the higher e, the higher the "weight" given to poor. If $e \longrightarrow 0^+$, inequality aversion is minimized, and Atkinson Index approaches zero for every income distribution; on the other hand, for every possible unequal distribution, the Atkinson Index increases together with e (*ceteris paribus*):

$$A(e) = 1 - \left[\frac{1}{n} \sum_{i=1}^{n} \left[\frac{y_i}{\bar{y}} \right]^{1-e} \right]^{\frac{1}{1-e}}$$

where: n is the number of individuals, y_i is the income of individual i and $\bar{y}$ is average income.

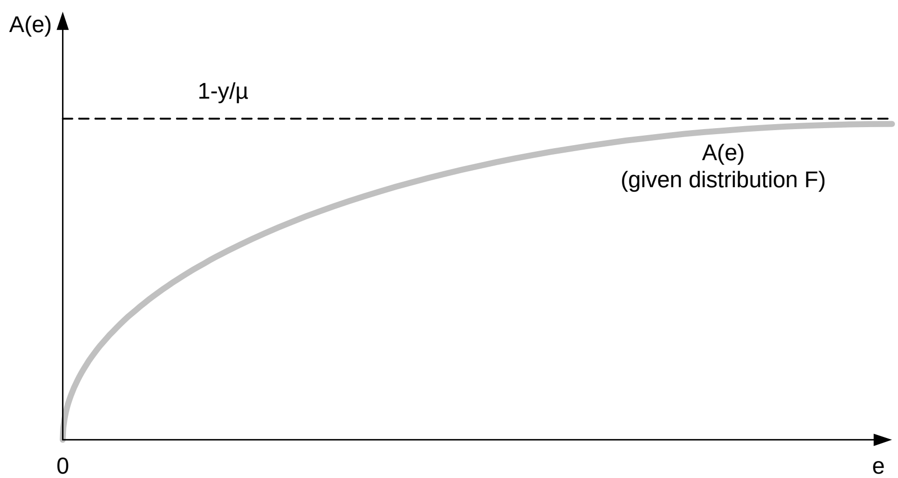

Figure 2: Atkinson Index given income distribution F

In case of extreme inequality aversion we have $e \longrightarrow +\infty$ and Atkinson Index reduces to:

$$A(e \longrightarrow +\infty) = 1 - \frac{y_L}{\bar{y}}$$

where: y_L is the lowest income within the jurisdiction and $\bar{y}$ is average income.

If inequality aversion is maximized all the "weight" is given to the poorest class and Atkinson Index is given by the distance between lowest and average income.

The Atkinson Inequality Index takes values between 0 (perfect income equality) and 1 (perfect income inequality); this range is valid for every $e > 0$.

As we've already pointed out, Atkinson Index gives different "weights" to different income groups depending on e; furthermore, it satisfies the weak principle of transfers, it is decomposable (if $e \neq +\infty$) and it is independent of population and income scale.

2.2 Polarization indices

> *Polarization places more emphasis on "clustering". Many phenomena, such as "the disappearing middle class", can be described as "polarization".*
>
> Xiaobo Zhang and Ravi Kanbur

Wolfson Polarization Index

Wolfson Index is derived from the method of the Lorenz curves.

Let's consider Figure 3: polarization is given by the area filled with vertical lines; this area is delimited at the bottom by the tangent to the Lorenz curve at the $50th$ percentile. From that area it is possible to derive the polarization curve: the higher is the curve, the more the distribution is spread away from the median value, the weaker is the middle class, the higher is income polarization.

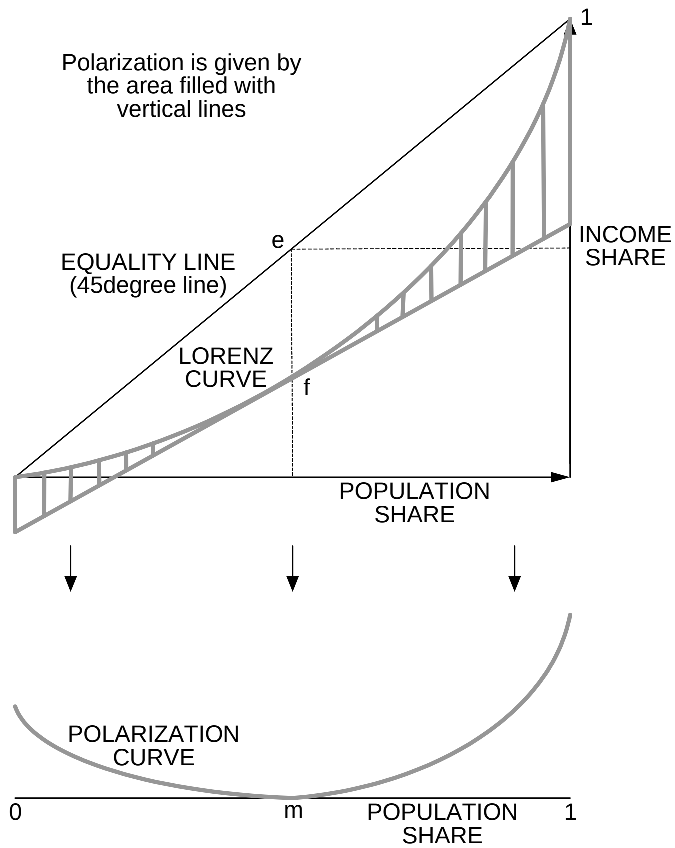

Figure 3: Wolfson Index

In order to show the difference between inequality and polarization, let us consider the case of a Pigou-Dalton Transfer. If the Transfer is from an individual above the median to an individual below the median (and nobody crosses the median because of the Transfer) both inequality and polarization decline: in such a case both the Lorenz curve and the tangent line at the $50th$ percentile move closer to uniform distribution line. On the other hand, if a Pigou-Dalton Transfer occurs between individuals on the same side with respect to the median, we observe that the Lorenz curve moves closer to uniform distribution line, whereas the tangent line at the $50th$ percentile is unaffected by the Transfer; in such a case inequality decreases as Lorenz curve moves closer to uniform distribution line and polarization increases as polarization curve goes up.

Formally, Wolfson Polarization Index is given by:

$$W = 2\frac{\bar{y}}{y_m}\left[2\left(0.5 - L(0.5)\right) - G\right]$$

where: $\bar{y}$ is average income; y_m is median income; $L(0.5)$ is the income share of the bottom half of the population and G is Gini Coefficient.

De facto, Wolfson Index measures the distance of a given distribution with respect to the one where all the population is concentrated at the median value.

The Wolfson Polarization Index take values between 0 (minimum income polarization) and 1 (maximum income polarization). It gives the same "weight" to every income group..

Esteban and Ray Polarization Index

The purpose of Esteban and Ray (1994) is to distinguish between inequality and polarization through examples from discrete distributions: in some cases, given a variation in the number of groups and/or in the distance between different groups, inequality goes up and polarization goes down, or vice versa. They also impose "reasonable" axioms to allowable measures of polarization; a distribution of individual attributes (natural logarithm of income) is polarized if: (i) there is a high degree of homogeneity within each group, (ii) there is a high degree of heterogeneity across groups, and/or (iii) there is a small number of significantly sized groups, given that small groups carry little "weight" in order to measure polarization.

Esteban and Ray introduce a continuous Identification Function:

$$I(\pi_{ci}) = \pi_{ci}^{s}$$

where: π_{ci} is the population share belonging to income class c of individual i and $s \in (0\ ,\ 1.6]$ is the polarization sensitivity parameter (sensitivity increases together with s).

The Identification Function is increasing in the population share π_i belonging to the same income class of individual i. For every individual, his sense of identification is increasing in the number of individuals with the same income level as him.

Furthermore, Esteban and Ray introduce a continuous Alienation Function:

$$a\left(\delta\left(\ln y_i, \ln y_j\right)\right) = \left|\ln y_i - \ln y_j\right|$$

which is non decreasing in the income distance between individual i and individual j. The Alienation Function characterizes the antagonism between individuals caused by income differences

Summarizing, the effective antagonism felt by y_i towards y_j is given by:

$$F(I, a)$$

and polarization in the sense of Esteban and Ray is given by the sum of all the antagonisms within population:

$$ER = \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_{ci} \pi_{cj} F\left(I(\pi_{ci}), a\left(\delta\left(\ln y_i, \ln y_j\right)\right)\right)$$

The Esteban and Ray Polarization Index is positive and increases together with income polarization. It gives the same "weight" to every income group; that is, it is "symmetric".[4]

"Asymmetric" Esteban and Ray Polarization Index

In the last part of their paper, Esteban and Ray (1994) discuss on the "symmetry" of their polarization measure: they argue that the alienation felt by poor with respect to rich is not the same of the one felt by rich with respect to poor. As a consequence, they consider the case where the Alienation Function registers positive values only for income values greater than that of the individual considered; that is, a case where different "weights" are given to different income groups.

Wang and Tsui Polarization Index

Following Wolfson (1994) and partially Esteban and Ray (1994), Wang and Tsui (2000) create a new class of polarization indices where the approach is "symmetric" and the focus is explicitly on the median income of the distribution. Accordingly with this Index, polarization is given by the average of a concave transformation of the distance with respect to median income.

Formally, Wang and Tsui Polarization Index is given by:

$$WT = \frac{\theta}{n} \sum_{i=1}^{n} \left(\left| \frac{y_i - y_m}{y_m} \right| \right)^r$$

[4] Notice that, even if the two indices refer to different theoretical concepts, Esteban and Ray (1994) compare their own index with Gini Coefficient. In the words of Esteban and Ray (1994, page 834): "Indeed barring the fact that we are using the logarithm of incomes, our measure would be the Gini if polarization sensitivity were equal to zero".

where: θ is a positive scalar; n is the number of individuals; y_i is the income of person i; y_m is median income and r is a coefficient between 0 and 1.

The Index is positive and increases together with income polarization.

Wang and Tsui Polarization Index gives the same "weight" to every income group.

3 Two-spike distributions

Most of the authors dealing with the theoretical definitions of the measures underlines the basic concepts of the indices in an informal way, through the examples of multiple spike or multiple densities. In particular, Wolfson (1994) and Esteban and Ray (1994, 2005) focused on the differences between inequality and polarization through the description of the effects of shifts of population mass or squeezes of densities. They found that sometimes the variations in the measured inequality and in the measured polarization diverge.

Divergences between inequality and polarization emerge also in empirical works. Wolfson (1997) calculated Gini Index and Wolfson Polarization Index using data on the distribution of incomes in Canada from the sixties to the nineties; he found that inequality diverges with respect to polarization in 20% of observations. Zhang and Kanbur (2001) calculated polarization indices (Wolfson, Esteban and Ray, Wang and Tsui) in 28 Chinese provinces from 1983 to 1995; their analysis showed that in general polarization grows up but at different rates depending on the chosen measure.

In order to check if such results are given to data and/or to the differences between the measures, we use two-spike income distributions. Such distributions have been already used in the literature on income distribution; see, for example, the works on income inequality by Lam (1986) and Fields (1993) on income inequality in dual economy models and the paper by Burger (2001) on the effects of inequality aversion in the Atkinson Index. The basic features of the two-spike distributions are similar to the ones of the multiple-spike distributions we find in Esteban and Ray (1994) on income polarization; "spiked" distributions show perfect homogeneity within each group.[5] We find support for the choice of two-spike distributions also from the empirical analysis by Esteban, Gradìn and Ray (2007) on five OECD countries: they showed that the results for different polarization measures

[5] As we have already pointed out, perfect homogeneity within groups is one of the basic features of polarization (Esteban and Ray 1994, page 824).

are similar if population is divided in two, three or four groups; in particular, for higher values of polarization sensitivity parameter, two-groups representation turns out to yield higher levels of polarization.

We assume that population has mass equal to 1. Given a two-spike income distribution, individuals are divided in two groups, call them "poor" and "rich", and there is no income heterogeneity within groups. $y_P = y$ is the income of poor individuals and $y_R = ky$ is the income of rich individuals, where $k > 1$ measures income differential between income groups. The income distribution is right-skewed, then the share of poor individuals α belongs to $(0.5, 1)$ and $1 - \alpha$ is the share of rich individuals.

In order to summarize, we have:

$$y_P = y_m = y$$

$$y_R = ky$$

$$\bar{y} = \alpha y + (1 - \alpha)ky$$

If we use two-spike distributions and such distributions are assumed to be right skewed, we are not able to distinguish between low and middle class as a consequence: the median income equals the lowest one. This is the main problem with two-spike distributions, given that in particular the concept of polarization is strongly linked with the "weight" of the middle with respect to the ends of the distribution of incomes.

As we have already pointed out (footnote 3, page 62), in our paper we consider the effects of variations in α or in k. These variations are "non mean-preserving". In such a case we are not able to distinguish between inequality and income effects but we can consider variations in α that does not affect k and variations in k that does not affect α. Also Burger (2001), for example, explicitly focused on non mean-preserving variations in income distribution in his analysis on the Atkinson Index.

3.1 Inequality indices

Gini Inequality Index

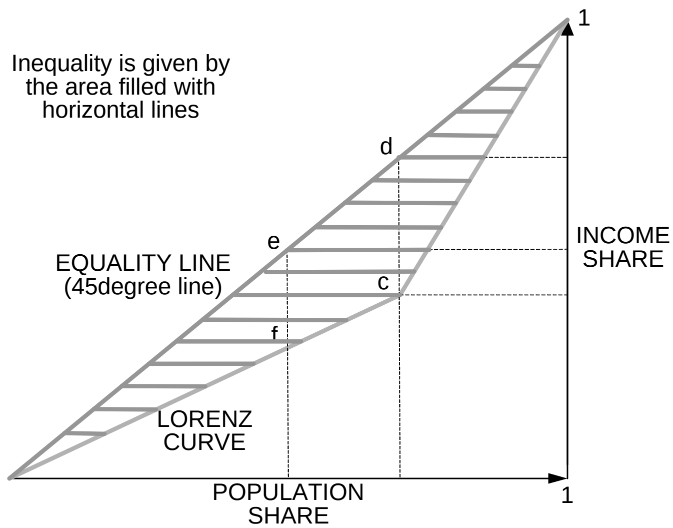

Figure 4: Gini Index (two-spike distributions)

Gini Index is given by the area between equality line and Lorenz curve, therefore we have:[6]

$$G = 2\left[\frac{\alpha\left(\alpha - \frac{\alpha}{\alpha+(1-\alpha)k}\right)}{2} + \frac{(1-\alpha)\left(\alpha - \frac{\alpha}{\alpha+(1-\alpha)k}\right)}{2}\right]$$

After algebraic manipulation, we obtain:

$$G = \alpha\left[1 - \frac{1}{\alpha + (1-\alpha)k}\right]$$

The effects of income differential and percentage of poor on Gini Index are the following:

[6] In Figure 4 we have:
$c = \left(\alpha\ ,\ \frac{\alpha}{\alpha+(1-\alpha)k}\right)$
$d = (\alpha\ ,\ \alpha)$
$e = (0.5\ ,\ 0.5)$
$f = \left(0.5\ ,\ \frac{0.5}{\alpha+(1-\alpha)k}\right)$

$$\frac{\partial G}{\partial k} = \frac{\alpha(1-\alpha)}{[\alpha + (1-\alpha)k]^2} > 0 \tag{1}$$

$$\frac{\partial G}{\partial \alpha} = 1 - \frac{k}{[\alpha + (1-\alpha)k]^2} \gtreqless 0 \tag{2}$$

Gini Index is (strictly) monotonically increasing in the income differential between rich and poor and non monotone in the percentage of poor.

In Figure 5, we show the behaviour of the derivative of Gini Index (and Theil Index) with respect to the percentage of poor.

Theil's Entropy Inequality Index

Theil Inequality Index is given by:

$$T = \frac{\alpha}{\alpha + (1-\alpha)k} \ln \frac{1}{\alpha + (1-\alpha)k} + \frac{(1-\alpha)k}{\alpha + (1-\alpha)k} \ln \frac{k}{\alpha + (1-\alpha)k}$$

After algebraic manipulations, we obtain:

$$T = \frac{(1-\alpha)k}{\alpha + (1-\alpha)k} \ln k - \ln [\alpha + (1-\alpha)k]$$

The effects of income differential and percentage of poor within population are the following:

$$\frac{\partial T}{\partial k} = \frac{\alpha(1-\alpha)}{[\alpha + (1-\alpha)k]^2} \ln k > 0 \tag{3}$$

$$\frac{\partial T}{\partial \alpha} = \frac{k-1}{\alpha + (1-\alpha)k} - \frac{k}{[\alpha + (1-\alpha)k]^2} \ln k \gtreqless 0 \tag{4}$$

In Figure 5, we show the behaviour of the derivative of Theil Index (and Gini Index) with respect to the percentage of poor.

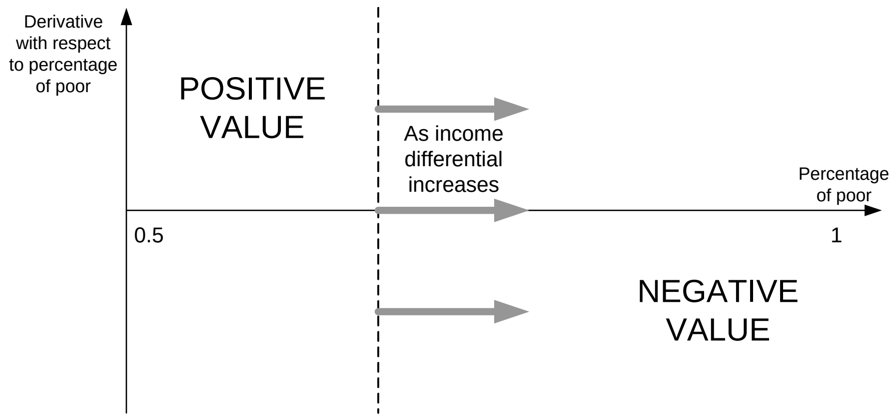

Figure 5: Derivative of Gini and Theil w.r.t. poor

Theil Index is (strictly) monotonically increasing in the income differential between rich and poor and non monotone in the percentage of poor.

Atkinson Inequality Index $(e \longrightarrow +\infty)$

Atkinson Inequality Index $(e \longrightarrow +\infty)$ is given by:

$$A = 1 - \frac{1}{\alpha + (1-\alpha)k}$$

[7]

The effects of income differential and percentage of poor on Atkinson Index $(e \longrightarrow +\infty)$ are the following:

$$\frac{\partial A}{\partial k} = \frac{1-\alpha}{[\alpha + (1-\alpha)k]^2} > 0 \quad (5)$$

$$\frac{\partial A}{\partial \alpha} = \frac{1-k}{[\alpha + (1-\alpha)k]^2} < 0 \quad (6)$$

Atkinson Index $(e \longrightarrow +\infty)$ is (strictly) monotonically increasing in the income differential between rich and poor and (strictly) monotonically decreasing in the percentage of poor.

[7] Notice that if we compare Atkinson Index $(e \longrightarrow +\infty)$ with Gini Index in case of two-spike distributions we have: $G = \alpha A$.

3.2 Polarization indices

Wolfson Polarization Index

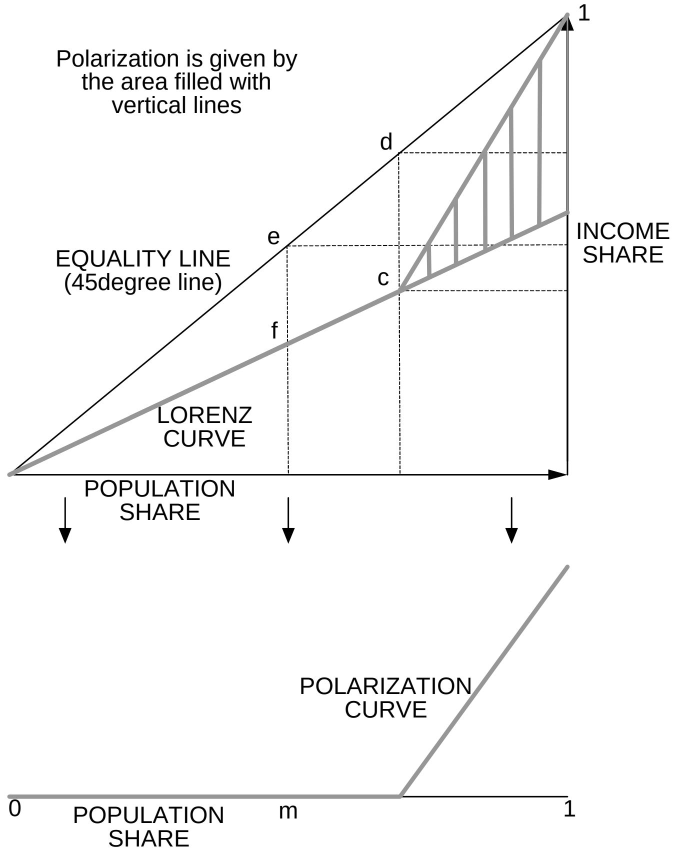

Figure 6: Wolfson Index (two-spike distributions)

Wolfson Index in case of two-spike distribution is given by:

$$W = 2\left[\alpha + (1-\alpha)k\right]\left[1 - \frac{1}{\alpha + (1-\alpha)k} - \left(\alpha - \frac{\alpha}{\alpha + (1-\alpha)k}\right)\right]$$

After algebraic manipulation, we obtain:

$$W = 2(1-\alpha)^2(k-1)$$

The effects of income differential and percentage of poor on Wolfson Index are the following:

$$\frac{\partial W}{\partial k} = 2(1-\alpha)^2 > 0 \tag{7}$$

$$\frac{\partial W}{\partial \alpha} = -(1-\alpha)(k-1) < 0 \tag{8}$$

Wolfson Index is (strictly) monotonically increasing in the income differential between rich and poor and (strictly) monotonically decreasing in the percentage of poor.

Esteban and Ray Polarization Index

Esteban and Ray Polarization Index is given by:

$$ER = \left[\alpha^{1+s}(1-\alpha) + (1-\alpha)^{1+s}\alpha\right] \ln k$$

where $s \in (0\ ,\ 1.6]$ is the polarization sensitivity parameter and alienation felt by poor individuals with respect to rich ones equals alienation felt by rich individuals with respect to poor ones.

The effects of income differential and percentage of poor on Esteban and Ray Polarization Index are the following:

$$\frac{\partial ER}{\partial k} = \frac{\alpha^{1+s}(1-\alpha) + (1-\alpha)^{1+s}\alpha}{k} > 0 \tag{9}$$

$$\frac{\partial ER}{\partial \alpha} = \left\{(1+s)\left[\alpha^{s}(1-\alpha) - (1-\alpha)^{s}\alpha\right] + (1-\alpha)^{1+s} - \alpha^{1+s}\right\} \ln k < 0 \tag{10}$$

Esteban/Ray Polarization Index is (strictly) monotonically increasing in the income differential between rich and poor and (strictly) monotonically decreasing in the percentage of poor.

"Asymmetric" Esteban and Ray Polarization Index

Given that poor individuals feel alienation with respect to rich ones, it may be argued that such alienation is greater than the one felt by rich individuals with respect to poor ones (*ceteris paribus*). As we have already pointed out before, this argument has been discussed by Esteban and Ray (1994, 2005); they consider the extreme case in which individuals simply do not feel alienation with respect to poorer ones.

If alienation is felt only by poor individuals with respect to rich ones, Esteban and Ray "Asymmetric"Index is given by:

$$ER(A) = \left[\alpha^{1+s}(1-\alpha)\right] \ln k$$

The effects of income differential and percentage of poor on the Index are the following:

$$\frac{\partial ER(A)}{\partial k} = \frac{\alpha^{1+s}(1-\alpha)}{k} > 0 \quad (11)$$

$$\frac{\partial ER(A)}{\partial \alpha} = \left[(1+s)\alpha^{s}(1-\alpha) - \alpha^{1+s}\right] \ln k \gtreqless 0 \quad (12)$$

In Figure 7, we show the behaviour of the derivative of Asymmetric Esteban and Ray Index with respect to the percentage of poor.

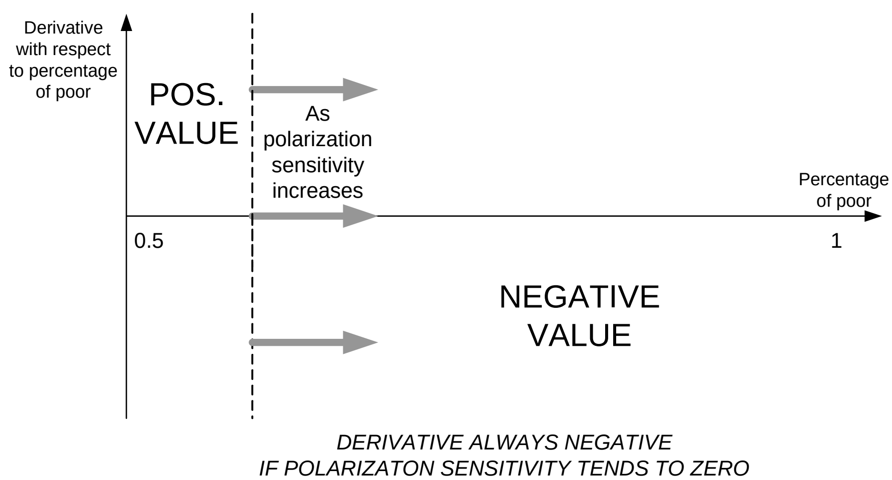

Figure 7: Derivative of ER(A) w.r.t. poor

Esteban / Ray Asymmetric Polarization Index is (strictly) monotonically increasing in the income differential between rich and poor and non monotone in the percentage of poor.

Wang and Tsui Polarization Index

Wang and Tsui Index is given by:

$$WT = \theta(1-\alpha)(k-1)^{r}$$

The effects of the parameters of the distributions on the Index are the following:

$$\frac{\partial WT}{\partial k} = \theta(1-\alpha)r(k-1)^{r-1} > 0 \tag{13}$$

$$\frac{\partial WT}{\partial \alpha} = -\theta(k-1)^{r} < 0 \tag{14}$$

Wang/Tsui Polarization Index is (strictly) monotonically increasing in the income differential between rich and poor and (strictly) monotonically decreasing in the percentage of poor

3.3 Income inequality and income polarization

	Inequality			Polarization			
	G	T	A_p	W	ER	$ER(A)_p$	WT
k increases	$+$	$+$	$+$	$+$	$+$	$+$	$+$
α increases	$+/-$	$+/-$	$-$	$-$	$-$	$+/-$	$-$

(p means "more weight" to the bottom end of the distribution)

All the indices are (strictly) monotonically increasing in the income differential between rich and poor: the more the incomes of rich and poor differ, the more there is inequality and polarization.

Different is the case of the percentage of poor: G, T and $ER(A)$ are non monotone in the percentage of poor; A, W, ER and WT are (strictly) monotonically decreasing in the percentage of poor.

Let us focus on inequality indices. The index A gives all the "weight" to the lowest income class and inequality is strictly monotonically decreasing in the percentage of poor. On the other hand, inequality indices that does not "overweight" the bottom end of the distribution, G and T, are non-monotone in the percentage of poor.

Let us focus now on polarization indices. The indices W, WT and ER are "symmetric" and they all are strictly monotonically decreasing in the percentage of poor. The index $ER(A)$ is non-monotone with respect to the percentage of poor and it gives different "weights" to different income groups; that is, it is an "asymmetric" index.[8]

[8] The question of the symmetry/asymmetry of the alienation between rich and poor is for sure an interesting topic, but the concepts of polarization is by definition symmetric: following the paper by Esteban and Ray (1994), the less the size of the groups differs, the higher polarization (ceteris paribus). As a consequence $ER(A)$, in our opinion, cannot be properly considered as a measure of polarization.

PROPOSITION 1a **Given a right-skewed two-spike income distribution, inequality indices are (strictly) monotone in the percentage of poor only if they assign all the "weight" to the poorest income class. They are non-monotone otherwise.**

PROPOSITION 1b **Given a right-skewed two-spike income distribution, polarization indices are non-monotone in the percentage of poor only if they give asymmetric "weights" to income groups. They are strictly monotone otherwise.**

Our analysis confirms the results of the empirical works by Wolfson (1997) and Zhang and Kanbur (2001): polarization and inequality sometimes diverge and there are differences between different inequality measures and between different polarization measures.

4 Policy implications

4.1 What does this Index measure?

Given that inequality and polarization refer to different aspects of a distribution, in the literature there are cases where the same measure is linked with different concepts and cases where different measures are linked with the same concept.

Mean/Median Ratio, in many writing on "inequality and growth", like Persson and Tabellini (1994), is used as an approximation of income inequality; on the other hand, Wolfson (1994) refers to Mean/Median Ratio as a polarization-related statistic, even if he calls it "a measure of income skewness".[9]

The econometric analysis by Alesina and Perotti (1996) showed that inequality increases instability. They measured income inequality through the income share of the third and the fourth quintile of the population: other

[9] Income skewness is given by the distance between average and median income: the wider is the distance, the higher is income skewness. The Index simply compares this two incomes and can be calculated even if do not observe the whole distribution. Graphically, the Index represent the inverse of the slope of the tangent of the Lorenz curve at the $50th$ percentile. Formally, Mean/Median Skewness Index is given by:

$$S = \frac{\bar{y}}{y_m}$$

Where $\bar{y}$ is average income and y_m is median income. The Index equals 1 in case of egalitarian distribution of incomes and increases together with income skewness.

authors would consider it as a polarization-related measure, given that the weakening of the middle class is at the basis of the concept of polarization.

Lindert (1996) analyzed the determinants of public spending in 19 OECD countries from 1960 to 1992: following his definitions of the variables income inequality and income skewness,[10] Lindert shows contrasting results. An increase in income skewness raises social public expenditure and lowers non-social public expenditure; on the other hand, an increase in income inequality lowers total public expenditure as share of GDP. The anti-spending effect of greater income inequality is in contrast with theories predicting that greater income inequality raises public expenditure, like Meltzer and Richard (1981): they considered Mean/Median Ratio as the determinant of the spending effect, but it refers to income skewness, not to income inequality. Milanovic (1999) analyzed public spending in 24 countries from the 1970s to the 1990s; he found support on the fact that higher income inequality, measured through Gini Index, raises redistribution.

4.2 Two-spike distributions & the real world

4.2.1 Social Rivalry Effect

In the econometric analysis by Corneo and Grüner (2002) on International Social Survey Programme data (1992), it is shown that an increase in Social Rivalry Effect (SRE) makes the individuals less likely to support redistribution. If we associate a social value to each income class, where v_c is the social value associated to income class c, SRE_c is given by downward value differential minus upward value differential with respect to the two neighboring classes of c:

$$SRE_c = v_c - v_{c-1} - (v_{c+1} - v_c)$$

where:

$$SRE = \sum_{c=1}^{C} |SRE_c|$$

SRE does not depend on group size and increases as income differential between neighboring classes increases: we can consider either income inequality or income polarization: both of them go up as SRE increases.

[10] Lindert (1996) considers: (i) natural logarithm of the ratio between first and third income quintile ("upper income gap") and (ii) natural logarithm of the ratio between third and fifth income quintile, named ("lower income gap"). His inequality index is given by (i) plus (ii); his skewness index is given by (i) minus (ii).

Given that for Corneo and Grüner (2002) an increase in Social Rivalry Effect makes the individuals less likely to support redistribution, it follows that an increase in inequality or polarization due to an increase in income differential should lower public expenditure, given our two-spike distributions.

4.2.2 Income trajectories

It is possible to "test" another result of the econometric analysis by Corneo and Grüner (2002); they found that a rising-income trajectory inhibits demand for redistribution. Such finding is confirmed by the empirical analysis on the "tunnel-effect" in Russia by Ravallion and Lokshin (2000).

Given a two-spike distribution of incomes, income-trajectories go up if: (i) the income differential increases (income of poor individuals and percentage of poor unchanged), (ii) the income of poor individuals increases (income differential and percentage of poor unchanged) and/or (iii) the percentage of poor decreases (income of poor individuals and income differential unchanged).

The comparison between these works and our analysis of two-spike distributions shows that there are no contradictions if we focus on changes in income differential.

If we focus on changes in the income of poor individuals, we see that our distributions are neutral to changes in wealth affecting the whole population given that such changes do not affect inequality nor polarization.

If we focus on a decrease in the percentage of poor, we have already analyzed the monotonicity or non-monotonicity of the indices; in particular, we observe that for two-spike distributions the effects on public expenditure depend upon the indices we use to rank distributions. If we consider measures of inequality, A is monotone in the percentage of poor, then public expenditure should decrease; on the other hand, G and T are non-monotone, then policy implications in terms of public spending could diverge. If we consider measures of polarization, W, ER and WT are monotone in the percentage of poor, then public expenditure should decrease; on the other hand, $ER(A)$ is non monotone, then policy implications in terms of public spending could diverge.

A rising-income trajectory that follows a decrease in the percentage of poor can make G, T and $ER(A)$ whether increase or decrease, depending on the percentage of poor within population and other variables.[11] It follows

[11] In Figure 5 and Figure 7, we see that the sign of the derivative with respect to percentage of poor depends upon income differential for Gini Index and Theil Index; on

that the policy implications could diverge, even if in the empirical works by Corneo and Grüner (2002) and Ravallion and Lokshin (2000) it is shown that a rising-income trajectory should mean less redistribution.

4.3 Furthermore...

If we refer to two-spike (and right-skewed) distributions there is no differences between the lowest income and the median one, as we've already pointed out in Section 3.

If lowest income equals median income, that's a problem for measures depending on the difference between the given distribution and the one where the income of all the individuals equals median income, for example WT. Furthermore, Lindert (1996) calculates his indices assuming that a wider "lower income gap"[10] implies less social spending: using two-spike distributions, lower income gap could equal zero.[12] Other works refer to redistribution and public spending processes as a battle between the ends and the middle of the distribution; in a two-spike distribution we have a "great middle" which also includes the bottom end.

An analysis on the divergences in policy implications in case of multiple-spike or more complex distributions is the main question left open.

5 Conclusion

There are different ways to rank distributions; they refer to different concepts: inequality, polarization (or skewness); they can diverge even if they refer to the same concept; they can have similar behaviour even if they refer to different concepts.

The choice of one measure instead of another one is not neutral: each measure refers to particular aspect of the distributions; each measure has its own characteristics: it gives, for example, different "weights" to different income groups or the same "weight" to every income group. We have shown how policy implications could diverge depending on the chosen measure in case of two-spike income distributions. Our results hold for two-spike distributions, but they can reasonably be supposed to hold also for more complex ones.

the other hand, it depends upon polarization sensitivity for Esteban and Ray Asymmetric Index.

[12] In particular, "lower income gap" in the sense of Lindert equals zero if poor individuals are more than 60% of the population, that is, if $\alpha > 0.6$.

Glossary

y_i	income of individual i
$\bar{y}$	average income
y_m	median income
$\alpha \in (0.5, 1)$	share of poor individuals
$k \in (1, +\infty)$	income differential between rich and poor individuals
$y_P = y$	income of poor individuals
$y_R = ky$	income of rich individuals
n	number of individuals
$L_X(q)$	Lorenz curve given distribution X
q	income percentile(s)
$h(.)$	function of income shares
p_i	share of individual i in total income of country/region
$e \in (0, +\infty)$	inequality aversion
$z \in (1, +\infty)$	coefficient
$L(0.5)$	income share of the bottom half of the population
$ID(.)$	identification function
$AL(.)$	alienation function
$\delta(.)$	function of income differential
$F(.)$	function of antagonism between individuals
π_c	population share (belonging to income class c)
$ps \in (0, 1.6]$	polarization sensitivity parameter
$\omega \in (0, +\infty)$	coefficient
$\theta \in (0, +\infty)$	positive scalar
$r \in (0, 1)$	coefficient
v_c	social value (associated with income class c)
SK	Skewness Index (mean/median ratio)
	SUPERSCRIPTS
$-$	average
	SUBSCRIPTS
i, j, c	individual, individual, individual in an income class
m	median
C	income class
P, R, L	poor individual, rich individual, lowest income
p	"more weight" to the bottom end of the distribution

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