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Taxing the invisible: Unobservable pollution and green transition in a multi-sector framework[☆]

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ABSTRACT

The rising demand for green products has encouraged firms to adopt sustainable practices, but it has also intensified the phenomenon of greenwashing. We develop a partial-equilibrium model of a multi-sector economy in which firms undertake both observable and unobservable abatement activities, creating incentives for greenwashing. The model incorporates the threat of nonpoint-source pollution taxes or taxes on individual emissions. Transition risk may induce some industries to sharply reduce, or even discontinue, production when expected environmental damages are high, yielding piecewise-smooth production dynamics of environmental degradation. We show that: (1) the threat of taxing observable pollution mitigates pollution stock; (2) very high expected tax may generate endogenous oscillations; (3) unobservable pollution amplifies this risk; and (4) when pollution is unobservable, consumer extra willingness to pay for green products does not eliminate zero-degradation equilibria.

1. Introduction

In recent years, environmental pollution has emerged as a critical issue, drawing increasing attention to sustainable consumption and the need for a green transition. Some clarifications, however, are warranted. While observable pollutants, such as plastic waste, have long attracted significant attention, less observable forms — such as the release of microplastics into the environment — have received comparatively less scrutiny and are difficult to control at the source (see Hoang, 2022 and the references therein for pollution from plastics). Regardless of the attention devoted to them and their degree of observability, all pollutants contribute to environmental degradation. Moreover, accumulation of pollution introduces an intertemporal (negative) spillover due to the environmental damage caused by past production activities, which increases the probability of tail events, as reported in Karydas and Xepapadeas (2022). These occurrences pose potential disruptions to the production process linked to extreme events and represent a significant risk to businesses (so-called *physical risk*), including those driven solely by profit motives.

To mitigate physical risk, effective climate policies are essential, particularly when markets fail to internalize environmental externalities. In this context, optimal taxation and incentive schemes have long been proposed in the literature as key regulatory instruments to reduce environmental damage (see Requate, 1993, Petrakis and Xepapadeas, 1999, Carlsson, 2000, Katsoulacos and Xepapadeas, 1996, Conrad and Wang, 1993, Poyago-Theotoky, 2007, Buccella et al., 2025, Agliardi and Lambertini, 2024, Arguedas et al., 2025). However, these policy recommendations are often disregarded as political incentives often prioritize immediate and

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visible issues, whereas long-term or uncertain risks — such as those posed by climate change — are frequently deferred.[1] The result is a policy environment where firms face regulatory ambiguity and where delayed action increases the likelihood of abrupt, costly transitions later on.[2] This delay gives rise to what is known as transition risk (see Nakhli et al., 2024, Ciola et al., 2023, Livieri et al., 2024, and Radi and Westerhoff, 2025). Transition risk can be defined as the uncertainty and potential disruption of economic activities arising from the threat of future events linked to pollution stock. Transition risk and physical risk are therefore closely related. Nevertheless, this study focuses on policy perspectives, specifically on transition risk, while physical risk is largely set aside and considered only to clarify the conceptual distinction between the two.

Recent studies have incorporated these sources of policy uncertainty and the associated forms of bounded rationality. La Torre et al. (2025), for example, develops a dynamic game model of transboundary pollution where the probability of environmental shocks depends on the pollution level. The study compares cooperative and noncooperative strategies between countries, showing that cooperation leads to lower long-run pollution. Moreover, it highlights the interaction between free riding and ecological uncertainty, and how the shape of long-term pollution outcomes depends on whether shock probabilities increase or decrease with pollution. Marsiglio and Tolotti (2024) develops a heterogeneous agents model of green and brown technology adopters to explore how uncertainty and complexity affect the success of low-carbon energy transitions. It shows that even under favorable conditions, transitions may fail, due to path-dependency and multistability. Public subsidies can help overcome these barriers, but their effectiveness is reduced when spatial interactions — such as social influence and transboundary pollution — are taken into account. A different approach is proposed in Zeppini (2015), where a behavioral model of technology adoption is proposed to examine transitions from polluting to sustainable technologies. Using a discrete choice framework with bounded rationality, social influence, and network effects, the model reveals how multiple equilibria and technological lock-ins can emerge.

Beyond uncertainties arising from structural and behavioral factors, the effectiveness of green policies may be constrained by negative externalities, informational asymmetries, or difficulties in monitoring pollution sources. The impact of environmental policies — such as pollution taxes and feed-in tariffs — depends on the magnitude of decision externalities and the pace of technological progress. While strong pollution taxes can accelerate the green transition, they may also induce economic instability. Moreover, the challenges posed by imperfect monitoring can hinder the shift toward a green economy. This issue has been addressed in Xepapadeas (1992), where a dynamic incentive scheme is proposed to regulate nonpoint-source pollution. By linking charges to deviations from desired pollution levels, this regulatory scheme guide emissions toward a socially optimal steady state. The analysis shows that static policies are inefficient in dynamic or uncertain environments, and that feedback-based strategies require stronger incentives due to strategic behavior and environmental variability. Moreover, policymakers tend to adjust the hazard rate for new green regulations in response to recently observed pollution levels. Thus, transition risk, like physical risk, introduces intertemporal dependencies between pollution and sustainable activities that are often overlooked in the existing literature. Another relevant aspect in green transition is the strategic use of transparency in technology adoption, which was explored in detail by Wu et al. (2020). Within green transition, environmental corporate social responsibility (CSR) is the main channel through which firms signal their commitment to cleaner production; under partial disclosure, such signaling can become strategic and give rise to greenwashing. Wu et al. (2020) developed a game-theoretic model to analyze how firms invest in CSR under conditions of partial transparency and distinguish between profit-maximizing firms and socially responsible firms. Limited observability of CSR activities can lead to *greenwashing*, that is, profit-driven firms mimic responsible behavior to gain consumer favor. The study reveals that greenwashing can have both negative and positive effects: while it may mislead consumers, it can also increase overall CSR investment. Interestingly, Wu et al. (2020) show that increasing transparency has a non-monotonic effect on social welfare—moderate transparency can improve outcomes, but excessive transparency may reduce incentives for CSR investment and lower welfare.

Incorporating the key aspects discussed so far — namely observable and unobservable pollutants, environmental externalities generating intertemporal negative spillovers, transition risk, greenwashing and behavioral factors — this paper contributes to the literature on green transition by analyzing the effect of the threat of a tax based solely on observed pollution, within a general economy composed by n production sectors. In this setting, monopolists with heterogeneous production technologies and abatement costs serve markets characterized by varying reservation prices and consumer willingness to pay for green products, interpreted as a willingness to support higher abatement efforts in production. The peculiarity of our setup lies in the endogenous updating of transition risk, which integrates intertemporal environmental externalities directly into the model. In particular, we develop a theoretical setup to examine the behavior of monopolies operating in sectors characterized by both observable and unobservable pollution. Firms are assumed to internalize the environmental damages they generate and engage in a two-stage optimization process: initially, they determine the optimal level of abatement effort, followed by the selection of output quantities that maximize profits. Firms solve this optimization problem by backward induction. To obtain analytical results while keeping the setup as simple as possible, we adopt an economic framework with linear demand functions and quadratic abatement cost functions, a common assumption in the literature, see Buccella et al. (2024) and Ulph (1996). It is important to note that firms are modeled as myopic agents who base their decisions solely on the current stock of pollution, without considering its future trajectory, due to the practical impossibility of obtaining a long-term forecast. Even though these myopic firms rely on observed information about environmental

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[1] A key source of uncertainty arises from political choices, as policymakers often refrain from introducing environmental taxes to preserve electoral support, thereby undermining the credibility and stability of future policy commitments. Moreover, transition risk is highly sensitive to prevailing political and social sentiments, making it difficult for firms to form reliable long-run expectations, which further motivates modeling agents as operating under limited rationality or as one-period-ahead maximizers.

[2] Moreover, the potential increase in taxation imposed by regulators as a preventive measure can take the form of an ambient charge, which is distributed across all market participants, regardless of their direct exposure to extreme events (see Karp, 2005).

damage, they use this information and anticipate the possible damage resulting from an intensification of the issues arising from high environmental damage (transition risk). In particular, firms may still choose to mitigate also unobservable emissions, which are not taxed but contribute to pollution stock and to an increase in transition risk. Firms may anticipate incurring costs associated with an upcoming catastrophic event related to high environmental damage, such as regulatory tightening, a shift in consumer preferences, or reputational damage. In our model, firms do not form forward-looking expectations about future regulation; instead, they evaluate their environmental performance based on past pollution levels, which guide their abatement decisions. This modeling choice is justified in the context of green policies, which are shaped by social sentiments as well as by considerations of rationality and efficiency. We also do not introduce heterogeneous consumers. Rather, following the distinction highlighted by Wu et al. (2020), we consider two model variants: one in which an extra willingness to pay (extra-WTP) for cleaner products is present, and one in which it is absent. Moreover, we allow for sector-specific differences in the emergence of extra-WTP without assuming general consumer heterogeneity. In this economic framework, we analyze the implications of the presence of both observable and unobservable pollution for environmental policy, particularly in terms of the design and efficacy of taxation mechanisms.

We begin the analysis for the case of fully observable pollution, where consumers are unwilling to pay for greener production. Not surprisingly, the environmental degradation equilibrium declines as taxation increases. More interestingly, high taxation can destabilize this equilibrium and may introduce unpredictability in environmental outcomes in line with Zeppini (2015). In other words, we detect a trade-off between the mitigating effect of high taxation and the stability of the equilibrium of environmental damage. This result extends to the general case of n independent and heterogeneous monopolistic sectors. In this case, a maximum taxation level that guarantees the existence of a viable stable equilibrium of environmental damage, that is, in which all economic sectors produce positive quantities and pollution stock stabilizes to a steady state.

When pollution is fully observable, and consumers demonstrate a willingness to pay a higher reservation price for greener products, the system's behavior changes in two important ways. First, the equilibrium characterized by environmental degradation remains stable for a wider range of environmental tax increases, showing greater resilience compared to the scenario in which consumers have no willingness to support greener production. Second, under these conditions, the model allows for a scenario with a globally stable zero environmental damage equilibrium. However, this outcome is only feasible when the cost of pollution abatement is sufficiently low. In any case, beyond a taxation threshold, complex dynamics emerge. Unobservable pollution makes it harder to achieve a stable, pollution-free equilibrium and tends to amplify fluctuations when instability occurs. These findings highlight the necessity of designing policy instruments that are robust to monitoring pollution and that account for the bounded rationality of firms. Increasing the taxation on pollution, interpreted as a stronger intensity of transition risk, appears to mitigate the adverse effects associated with partially observable pollution.

The dynamics of environmental degradation is modeled through piecewise-smooth maps, for which supporting theory and methods have been recently developed, see, in particular, Avrutin et al. (2019). We detect *border collision bifurcations*, a well-known phenomenon in piecewise-smooth maps that occurs when an invariant set (like a fixed point or a cycle) collides with a *border* where the system's definition changes. In our case, the change of definition of the map is due to transition risk, which implies an anticipated cutback/shutdown in production for mitigating its effects.

The paper proceeds as follows. Section 2 presents the theoretical model of production dynamics with degradation and transition risk. Section 3 focuses on the case in which pollution is observable, examining how this assumption affects the adoption of cleaner technologies and the design of environmental policies. Section 4 extends the framework to include both observable and unobservable pollution, highlighting the role of information asymmetries between firms and consumers and their implications for policy effectiveness. Some comparisons of the outcomes across the different scenarios are also presented. Finally, Section 5 concludes by summarizing the main findings and outlining directions for future research. All technical details are reported in the Appendices, where we also include a robustness check.

2. The model with transition risk

To keep the model as simple as possible, let us consider an economy consisting of n monopoly markets. Let q_i be the quantity of the goods produced by monopolist i. The production process of good i entails two kinds of emissions. In particular, a unit of production of q_i implies δ_i units of *observable* emissions, such as smog, oil spills, or plastic waste, and ζ_i units of *unobservable* emissions, such as airborne particulates, chemical contaminants, or microplastic (see Wu et al., 2020). Monopolist i can undertake abatement effort ω_i to reduce observable emissions and λ_i to reduce unobservable emissions. We assume that ω_i is publicly known, whereas λ_i is private to the firm. Depending on consumers' sensitivity to the environmental concern of firm i, let $\eta_i(\omega_i)$ be the extra willingness to pay for green products. The good i is sold in the market at a price:

$$P_i\left(q_i, \eta_i\left(\omega_i\right)\right) = \max\{a_i + \eta_i\left(\omega_i\right) - q_i, 0\}, \quad \text{with } i \in \{1, \dots, n\}, \tag{1}$$

where a_i is the reservation price of product i when $\omega_i = 0$, and $\eta_i\left(\omega_i\right) \geq 0$ is the extra willingness to pay due to pollution abatement. For now, we assume that $\eta_i\left(\omega_i\right)$ is continuous and almost everywhere differentiable with $\frac{\partial \eta_i(\omega_i)}{\partial \omega_i} \geq 0$. We then focus on the case of a linear extra willingness to pay.

Following Petrakis and Xepapadeas (1999), the cost function for the monopolist is assumed to be additively separable in production costs and environmental abatement costs, i.e.

$$C_i\left(q_i, \omega_i, \lambda_i\right) = c_i q_i + \gamma_i \frac{\omega_i^2}{2} + \beta_i \frac{\lambda_i^2}{2}, \tag{2}$$

where c_i is the constant marginal cost in production, while abatement costs are quadratic in observable (ω_i) and unobservable (λ_i) abatement efforts. Here γ_i and β_i are cost parameters for observable and unobservable abatement efforts, respectively. The additive form of abatement costs is justified when dealing with two distinct types of pollution — observable and unobservable — that require independent mitigation strategies. Observable pollution typically refers to emissions or discharges that can be directly monitored and measured (e.g., oil spills, particulate matter, wastewater effluents), while unobservable pollution may include diffuse or latent forms of environmental damage (e.g., groundwater or soil contamination, invisible air pollutants, greenhouse gases) that are harder to detect and quantify. Given this distinction, the technologies, procedures, and regulatory mechanisms required to mitigate each type of pollution are often entirely different. We assume that there are no cost interactions or externalities between the two abatement costs.

Pollution is a by-product of the total production process, with emissions released into the environment. The environmental degradation caused by total production is captured by the *damage function*, which we assume to be linear in net emissions for tractability, i.e.

$$D\left(q_1,\dots,q_n,\omega_1,\dots,\omega_n,\lambda_1,\dots,\lambda_n\right) = d\sum_{i=1}^{n}\left(\delta_i q_i - \omega_i\right) + s\sum_{i=1}^{n}\left(\zeta_i q_i - \lambda_i\right), \tag{3}$$

where $\delta_i q_i$ and $\zeta_i q_i$ denote, respectively, observable and unobservable gross emissions from the production of good i (after some appropriate choice of units), and d and s indicate the steepness of marginal damages concerning the two pollutants.[3] We assume that $\omega_i \in \left[0,\delta_i q_i\right]$ and $\lambda_i \in \left[0,\zeta_i q_i\right]$, that is monopolist i cannot abate beyond gross emissions of either kind. In other words, we rule out *negative emissions*. The additive form of the damage function is appropriate when emissions originate from distinct sources or activities — such as transportation and industrial production — each contributing independently to environmental harm. Whenever the environmental impacts of the two pollution sources are non-interacting, that is, the damage caused by one type of emission does not amplify or mitigate the damage caused by the other, it is reasonable to model total damage as the sum of separate damage functions. From the point of view of firm/market i, environmental abatement increases total costs but reduces emissions. In particular, for observable emissions, investing an amount of $\gamma_i \frac{\omega_i^2}{2}$ reduces its emissions by ω_i and may imply a higher extra willingness to pay by consumers. For unobservable emissions, investing an amount of $\beta_i \frac{\lambda_i^2}{2}$ reduces unobservable emissions by λ_i but does not entail a higher extra willingness to pay, as this activity is unobservable.

All emissions of both types contribute to defining a composite pollution index, which we assume is linear in net emissions and is dynamically updated to account for biodegradation and new pollution. We refer to this index as the level of *environmental degradation* at time t, denoted by N_t. To reflect natural regeneration and new emissions, the dynamics of environmental degradation in discrete time can be modeled by a map of the form

$$N_t = (1-\mu)\,N_{t-1} + D\left(q_{1,t},\dots,q_{n,t},\omega_{1,t},\dots,\omega_{n,t},\lambda_{1,t},\dots,\lambda_{n,t}\right), \tag{4}$$

where $q_{i,t}$, $\omega_{i,t}$, and $\lambda_{i,t}$ denote the levels of production, observable and unobservable abatement of monopolist i at time t. Parameter $\mu \in [0,1]$ represents the ecosystem's natural regenerative rate for a unit of time. Under the restrictions $\omega_i \in [0,\delta_i q_i]$ and $\lambda_i \in [0,\zeta_i q_i]$ for all i, the damage function is non-negative, which in turn implies $N_t \geq 0$ for all $t \in \mathbb{N}$.

Effective climate governance necessitates regulatory intervention to internalize environmental externalities. However, political choices often prioritize short-term concerns over long-term issues, thus leading to inertia. This deferral gives rise to *transition risk*, related to the uncertainty and potential disruption associated with anticipated but unrealized regulatory measures. Anyway, the *expectation* of future stronger regulatory intervention can influence actual corporate decision-making. Consequentially, we assume that monopolist i faces the risk of a tax τ for each unit of observable environmental degradation (transition risk). At each time t, the risk of being taxed by τ for each unit of emission is assumed to be given by a counting process with a hazard rate that is assumed to be proportional to environmental degradation N_{t-1} with a coefficient of proportionality ϕ, that is

$$\mathbb{E}_t\left[\tau\right] = \phi\tau N_{t-1}. \tag{5}$$

Let $T = \phi\tau$ denote the expected taxation coefficient ($T \geq 0$).

To reduce transition risk, the monopolist i can undertake abatement efforts ω_i and λ_i to reduce environmental damage, thus reducing the risk of an additional tax burden. Note that all abatement efforts entail costs and have beneficial effects on the profits of the firm. The effect of an investment in observable abatement ω_i is twofold, as it increases the consumers' extra willingness to pay and reduces the transition risk. On the other hand, the effect of an investment in unobservable abatement λ_i only reduces next-period transition risk. The effects of ω_i and λ_i on transition risk are qualitatively different. Specifically, an increment in ω_i reduces the firm's current emissions, thereby directly lowering its exposure to transition risk. In addition, by reducing observable emissions, an increment in ω_i also decreases the likelihood of future regulatory interventions, such as the introduction/increment of tax burden. In contrast, λ_i influences transition risk only indirectly: it affects the probability of future policy implementation (e.g., taxation).

Summing up, at time t the profit for the monopolist i can be expressed as follows

$$\pi_i\left(q_i,\omega_i,\lambda_i;T,N_{t-1}\right) := \left[A_i + \eta_i\left(\omega_i\right) - q_i\right]q_i - \gamma_i\frac{\omega_i^2}{2} - \beta_i\frac{\lambda_i^2}{2} - T\left(\delta_i q_i - \omega_i\right)N_{t-1}, \tag{6}$$

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[3] The marginal damages of pollutants d and s could, in principle, differ across sectors, reflecting sector-specific emissions, so that d_i and s_i denote these parameters for sector i. However, for the sake of simplicity, we assume them to be uniform across all sectors.

where $A_i := a_i - c_i$. Note that according to the definition of profit in (6), firm i does not internalize the damage that its production causes to the environment but only the past damage. That is, there is an intertemporal (negative) spillover due to the environmental damage caused by past productions that can be internalized in the profit function of all firms through a tax on emissions or the threat of a tax on emissions (transition risk). Moreover, the taxation is assumed to be on current net observable emissions. The intensity of the tax for a unit of net emission is proportional to the past level of overall environmental damage as observed by the regulator. Unobservable pollution is not taxed.

As overall pollution depends on the activities in all n sectors, we assume bounded rationality in decisions by the firms in choosing quantities and abatement efforts: firms condition on N_{t-1} and do not forecast future N_t. In particular, each monopolist optimizes decisions in two stages: given the observed environmental degradation, a firm has to set optimal (observable and unobservable) abatement and then optimal production to maximize its objective function. We assume here that all firms are profit maximizers.[4] At any discrete time t, each firm solves this two-stage game, which is different in each period because the level of environmental degradation dynamically responds to past production and abatement decisions. Firms then apply backward induction in each period.

The second and last stage of this game is the output selection. Taking as given the emission tax coefficient T and the environmental degradation N_{t-1}, the monopolist i chooses its output for time t to maximize its expected payoff function, that is:

$$q_i^*\left(\omega_i, \lambda_i; T, N_{t-1}\right) = \underset{q_i \in [0,+\infty)}{\arg\max}\, \pi_i\left(q_i, \omega_i, \lambda_i; T, N_{t-1}\right). \tag{7}$$

The first stage of the game involves selecting the optimal observable and unobservable abatement. Here, we assume that these two decisions are carried out simultaneously by maximizing the reduced-form profit function

$$\pi_i^*\left(\omega_i, \lambda_i; T, N_{t-1}\right) = \pi_i\left(q_i^*\left(\omega_i, \lambda_i; T, N_{t-1}\right), \omega_i, \lambda_i; T, N_{t-1}\right). \tag{8}$$

The optimal levels of abatement of observable and unobservable pollution at time t, that is ω_i^* and λ_i^* respectively, are such that

$$\left(\omega_i^*\left(T, N_{t-1}\right), \lambda_i^*\left(T, N_{t-1}\right)\right) = \underset{(\omega_i,\lambda_i)\in\mathcal{J}_t}{\arg\max}\, \pi_i^*\left(\omega_i, \lambda_i; T, N_{t-1}\right), \tag{9}$$

where $\mathcal{J}_t := \left\{ \left(\omega_i, \lambda_i\right) \middle| \, 0 \leq \omega_i \leq \delta_i q_i^*\left(\omega_i, \lambda_i; T, N_{t-1}\right) \text{ and } 0 \leq \lambda_i \leq \zeta_i q_i^*\left(\omega_i, \lambda_i; T, N_{t-1}\right) \right\}$ is the natural feasible region.[5]

Substituting the optimal levels of production and abatement of observable and unobservable pollution (for each firm i) in map (4), we obtain the following one-dimensional discrete-time dynamical system:

$$N_t = (1-\mu) N_{t-1} + D\left(q_{1,t}^*, \dots, q_{n,t}^*, \omega_{1,t}^*, \dots, \omega_{n,t}^*, \lambda_{1,t}^*, \dots, \lambda_{n,t}^*\right), \tag{11}$$

which describes how environmental degradation evolves over time in response to the policy parameter T. In (11), employed the short-hand notation $q_{i,t}^* = q_i^*\left(\omega_{i,t}^*, \lambda_{i,t}^*; T, N_{t-1}\right)$, $\omega_{i,t}^* = \omega_i^*\left(T, N_{t-1}\right)$ and $\lambda_{i,t}^* = \lambda_i^*\left(T, N_{t-1}\right)$. We shall use this notation in the following as long as no confusion arises.

The optimal levels of production and abatement of observable and unobservable pollution are functions of the previous period's level of environmental degradation, and their analytical expression depends on the parameter constellation of the model as shown in Appendix A, where the explicit form of the map (11) is provided.

In the following, we first analyze the dynamics of environmental degradation in (11) under the case of fully observable pollution and, only later, we consider the presence of partially observable pollution. The investigation focuses on the possible equilibria, their stability properties, and possible non-equilibrium dynamics that emerge in response to changes in the fiscal environmental policy, represented by the aggregated policy taxation coefficient T. Moreover, to keep the model as simple as possible, we assume that the extra willingness to pay for a greener product is linear in observable abatement, that is in (6) $\eta_i\left(\omega_i\right) = \eta_i \cdot \omega_i$. This functional form can be viewed as the limiting case ($\bar{\eta}_i = +\infty$) of a concave willingness-to-pay function featuring an upper bound given by

$$\eta_i\left(\omega_i\right) = \min\left\{\eta_i \cdot \omega_i; \bar{\eta}_i\right\}. \tag{12}$$

The effects of the upper bound are discussed in Appendix C.

Before starting the analysis of the model, Table 1 provides a summary of the model parameters, their admissible ranges, and their economic interpretation. We further note that the analysis will primarily focus on the impact of the policy parameter in our model, namely, the taxation coefficient, across various scenarios. The parametric configurations (scenarios) considered in this study are reported in Table 2, along with references to the sections where each scenario is analyzed in detail. Each scenario is examined under two market structures: one with n identical homogeneous monopolies and one with n heterogeneous monopolies. Before conducting the analysis aimed at studying the effects of the policy parameter, we note that we exclude cases in which $2\gamma_i < \eta_i^2$ and $2 < \eta_i \delta_i$ for at least one $i \in 1, \dots, n$, since these conditions would lead to an unbounded solution to the monopolist i's optimization problem (see Appendix A).

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[4] This point can be relaxed by assuming the presence of non-profit motives. This analysis is not carried out in this work but is left for future research.

[5] As specified above, the constraints $\omega_i \in [0, \delta_i q_i]$ and $\lambda_i \in [0, \zeta_i q_i]$ for all i imply that $N_t \geq 0$ for all $t \in \mathbb{N}$. Moreover, $T \geq 0$ by assumption. Then, the additional constraints of the two-stage optimization problem — namely $q_i \geq 0$ and $(\omega_i, \lambda_i) \in \mathcal{J}_t$ — ensure that the optimal profit is positive. Indeed, independently of $\eta_i(\omega_i)$, it holds that

$$\pi\left(q_i^*(\omega_i), \omega_i, \lambda_i; T, N_{t-1}\right) = \left[q_i^*(\omega_i)\right]^2 - \gamma_i \frac{\omega_i^2}{2} - \beta_i \frac{\lambda_i^2}{2} + T\omega_i N_{t-1}. \tag{10}$$

Note that (10) is non-negative when $\omega_i = \lambda_i = 0$, since $\pi\left(q_i^*(0), 0, 0; T, N_{t-1}\right) = \left[q_i^*(0)\right]^2 \geq 0$. Because $(\omega_i^*, \lambda_i^*)$ denotes the optimal abatement levels in $\mathcal{J}_t$ and $(0,0) \in \mathcal{J}_t$, it follows that $\pi\left(q_i^*(\omega_i^*), \omega_i^*, \lambda_i^*; T, N_{t-1}\right) \geq \pi\left(q_i^*(0), 0, 0; T, N_{t-1}\right) \geq 0$. As a result, the optimal profit is always non-negative.

Table 1
Parameters of the model and economic meaning.

Parameter	Feasible range	Economic meaning
n	$\{1, 2, \ldots, \bar{n}\}, \bar{n} < \infty$	Number of monopoly markets
a_i	$(0, +\infty)$	Reservation price for good i
c_i	$(0, a_i)$	Constant marginal cost for good i
δ_i	$[0, +\infty)$	Units of observable emissions for unit of production of good i
ζ_i	$[0, +\infty)$	Units of unobservable emissions for unit of production of good i
γ_i	$(0, +\infty)$	Cost parameter for observable abatement effort for good i
β_i	$(0, +\infty)$	Cost parameter for unobservable abatement effort for good i
d	$[0, +\infty)$	Steepness of marginal damage caused by observable pollutant
s	$[0, +\infty)$	Steepness of marginal damage caused by unobservable pollutant
η_i	$[0, +\infty)$	Constant marginal extra willingness to pay for an additional unit of observable pollution abatement in sector i
$\bar{\eta}_i$	$[0, +\infty) \cup \{+\infty\}$	Upper bound on consumers' extra willingness to pay for observable pollution abatement in sector i
μ	$[0, 1]$	Ecosystem's natural regenerative rate per unit of time
$T = \phi\tau$	$[0, +\infty)$	Expected taxation coefficient

Table 2
Analysis of the different model scenarios.

Fully observable pollution ($\zeta_i = 0$)		
No extra willingness to pay ($\eta_i\bar{\eta}_i = 0$)	Positive extra willingness to pay ($\eta_i > 0$) without upper bound ($\bar{\eta}_i = +\infty$)	Positive extra willingness to pay ($\eta_i > 0$) with upper bound ($\bar{\eta}_i \in (0, +\infty)$)
Section 3.1	Section 3.2	Appendix C.1
Partially observable pollution ($\zeta_i > 0$)		
No extra willingness to pay ($\eta_i\bar{\eta}_i = 0$)	Positive extra willingness to pay ($\eta_i > 0$) without upper bound ($\bar{\eta}_i = +\infty$)	Positive extra willingness to pay ($\eta_i > 0$) with upper bound ($\bar{\eta}_i \in (0, +\infty)$)
Special case of Section 4	Section 4	Appendix C.2

3. Fully observable pollution

We begin the analysis by considering the special case in which production entails only fully observable pollution. We split this scenario by distinguishing between two subcases: (i) consumer unwillingness to pay for abatement ($\eta_i\bar{\eta}_i = 0$ for all $i = 1, \ldots, n$) and (ii) consumer extra willingness to pay for abatement ($\eta_i > 0$ and $\bar{\eta}_i = +\infty$ for all $i = 1, \ldots, n$). This distinction allows us to isolate the effect of the tax alone from the combined effect of the tax and consumers' extra willingness to pay for green products, and further, from the more complex interaction involving extra willingness to pay, taxation, and greenwashing behavior. In the next section, we will address the case of partially observable pollution and greenwashing behavior.

We begin by analyzing an economy with n homogeneous sectors. The simplicity of this case facilitates analytical tractability and serves as a benchmark for understanding the basic mechanisms of the model. We then extend the analysis to heterogeneous multisector economies, where differences in production technologies, pollution, and abatements among the different monopolists introduce sector-specific characteristics. This last scenario enables us to assess how the activity of all sectors collectively shapes environmental damage and influences the design of effective regulatory interventions.

3.1. *Consumer unwillingness to pay for abatement*

Let us start by considering the benchmark case in which consumers' extra willingness to pay for green products component is absent, i.e., we start considering the inverse demand function in (1) with $\eta_i(\omega_i) = 0$, that is $\eta_i\bar{\eta}_i = 0$. In this benchmark, consumers' extra willingness to pay is independent of firms' abatement efforts. Moreover, let us start by focusing on the benchmark case of homogeneous sectors.

The next proposition characterizes this case by specifying the unique environmental degradation equilibrium, its local stability conditions and the occurrence of bounded non-equilibrium dynamics. Let us start by imposing homogeneous sectors, that is $A_i = A$, $\gamma_i = \gamma$, $\delta_i = \delta$, for all $i = 1, \ldots, n$, and by defining the following degradation value:

$$\bar{N}^* := \frac{A}{\frac{2\mu}{n d \delta} + \left(\frac{2}{\gamma\delta} + \delta\right) T}, \tag{13}$$

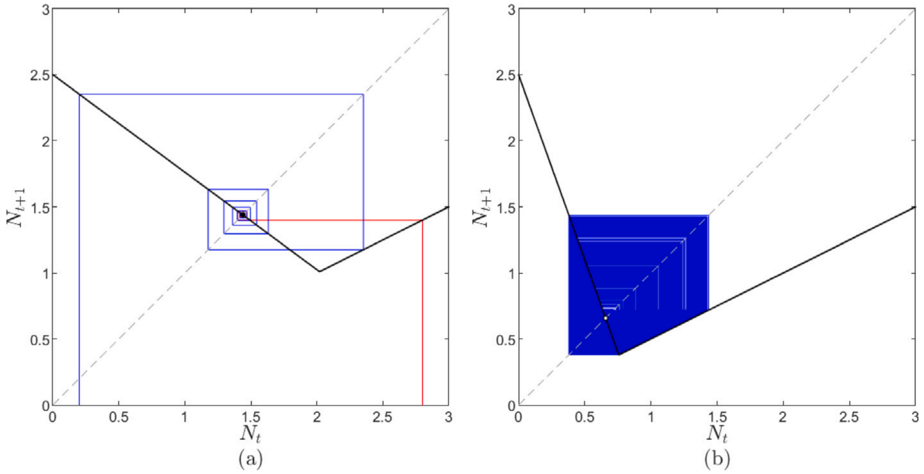


Fig. 1. Map (11) with $n = 2$ sectors: Panel (a) $T = 0.2$, corresponding to Proposition 1, Case (a); Panel (b) $T = 0.4$, corresponding to Proposition 1, Case (b). The dashed line represents current pollution N, while the solid black line shows the map (pollution in the next period). The staircase diagram of the trajectory starting at $N_0 = 2.8$, resp. at $N_0 = 0.2$, is shown in red, resp. in blue, in Panel (a). The staircase diagram of a long-run trajectory is shown in blue in Panel (b). Equilibria correspond to the intersection points between the map and the identity line; locally asymptotically stable equilibria are shown as black dots, while unstable ones are shown as empty dots. The remaining parameters are: $\mu = 0.5$, $A = 1$, $\gamma_1 = \gamma_2 = 1$, $\delta_1 = \delta_2 = 2.5$, $d = 1$, $n = 2$, and $\eta_1\bar{\eta}_1 = \eta_2\bar{\eta}_2 = \zeta_1 = \zeta_2 = s = 0$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

and the following taxation threshold:

$$T_b^* := \frac{2\gamma(2-\mu)}{\left(\gamma\delta^2+2\right)nd}. \tag{14}$$

Then, the following results hold (the proof of which is in Appendix B).

Proposition 1 (*Benchmark with n Homogeneous Sectors*). *Assume homogeneous sectors: $A_i = A$, $\gamma_i = \gamma$, $\delta_i = \delta$, for all $i = 1,\dots,n$. Furthermore, let $\zeta_i = 0$ and $\eta_i(\omega_i) = 0$, that is $\eta_i\bar{\eta}_i = 0$, for all $i = 1,\dots,n$. Under these assumptions, the map describing the dynamics of environmental degradation (11) reduces to the form given in (81) in Appendix A. For this map, the set*

$$\mathcal{C} := \left[0, (1-\mu)\frac{A}{\delta T} + \frac{nd\delta A}{2}\right] \tag{15}$$

is invariant and attracts all trajectories. Moreover, the unique equilibrium $\bar{N}^*$ in (13) satisfies the following properties:

(a) $\bar{N}^*$ is locally asymptotically stable if $T < T_b^*$;
(b) $\bar{N}^*$ is unstable and an attractor exists in $\mathcal{C}$ if $T > T_b^*$.

Proposition 1 presents a stability analysis for the unique equilibrium $\bar{N}^*$ under the monopolist's activity. Notably, when stability of the equilibrium is lost, all trajectories of environmental degradation will eventually enter and remain within the interval $\mathcal{C}$, which represents the bounded range of all possible environmental states under the monopolist's behavior. The results in Proposition 1 should represent the worst-case scenario for environmental degradation. In fact, taking a positive extra willingness to pay for clean production, that is $\eta_i > 0$ (and $\bar{\eta}_i = +\infty$), we should expect lower values of pollution. However, before testing this hypothesis in the following, we illustrate the global dynamics through numerical tests when the equilibrium loses stability. To this aim, we use staircase and one-dimensional bifurcation diagrams, where the bifurcation parameter is the policy parameter T, that is, the taxation coefficient for a unit of CO_2 emissions weighted.

The staircase diagrams of Figs. 1(a)–(b) show dynamics for scenarios (a) and (b) of Proposition 1, where the equilibrium degradation value $\bar{N}^*$ is, respectively, stable and unstable with a bounded attractor surrounding $\bar{N}^*$. The bifurcation diagram in Fig. 2(a) illustrates the trade-off induced by the environmental tax policy scheme: while higher taxation leads to a reduction in average pollution levels, it also introduces fluctuations in both production and pollution. From a policymaker perspective, oscillations raise concerns because they can turn an otherwise effective tax into a source of unpredictable production fluctuations, generating avoidable economic instability. In addition, irregular pollution patterns complicate regulatory planning and weaken the credibility and long-term effectiveness of environmental policy. The detected trade-off between low average degradation and its oscillations can be analytically grounded within the model's framework, as shown in the next corollary (the proof of which is omitted as trivially obtained from Proposition 1). The instability of an equilibrium as taxation reaches a certain threshold is in line with the results reported in Zeppini (2015).

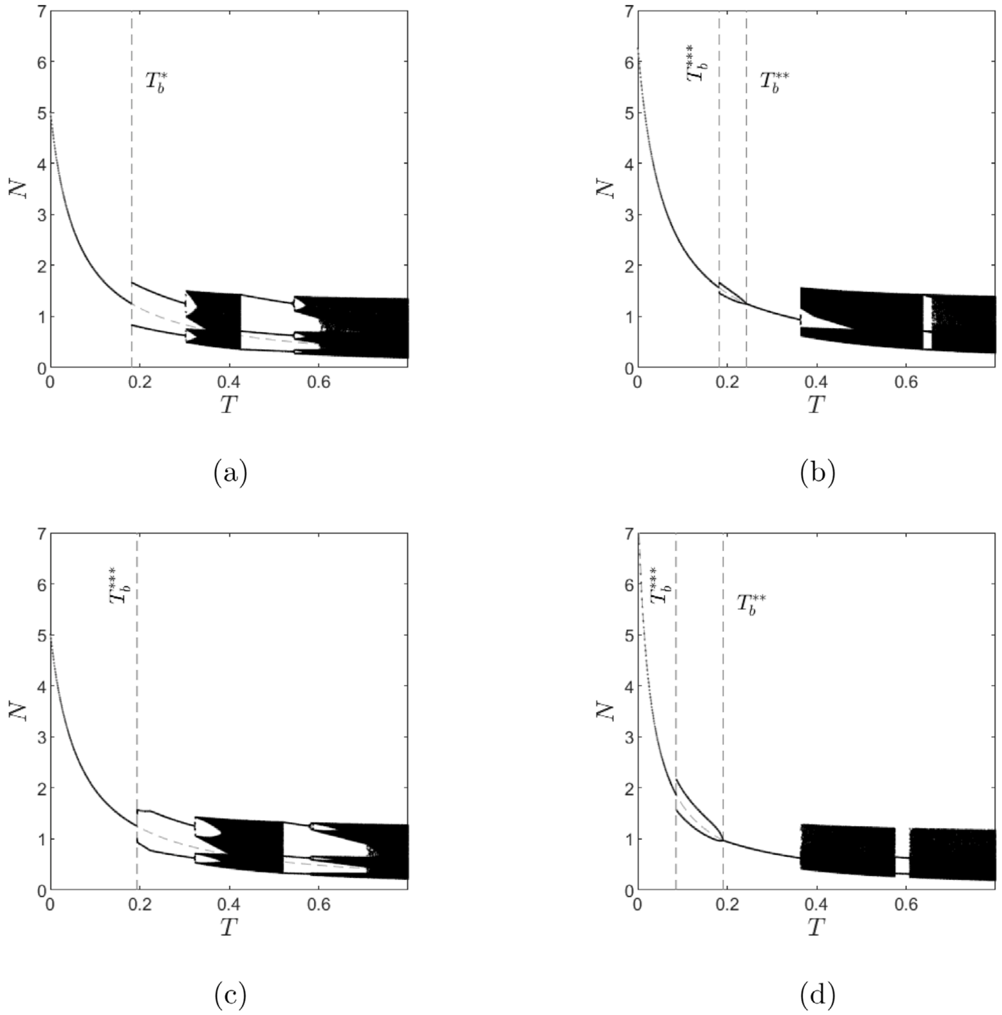


Fig. 2. Bifurcation diagrams with respect to the tax coefficient T for map (11) with $n = 2$ sectors. Panel (a): Homogeneous sectors. For $T < T_b^*$, Case (a) of Proposition 1 holds; for $T > T_b^*$, Case (b) applies. Parameters: $\mu = 0.5$, $A_i = A = 1$, $\gamma_i = \gamma = 1$, $\delta_i = \delta = 2.5$, $d = 1$, $s = 0$, and $\eta_i \bar{\eta}_i = \eta\bar{\eta} = \zeta_i = \zeta = 0$ for all $i \in \{1, 2\}$. Panel (b): Same as Panel (a), except $A_2 = 1.5$. Panel (c): Same as Panel (a), except $\gamma_2 = 2$. Panel (d): Same as Panel (a), except $\delta_2 = 5$.

Corollary 1 (*of Proposition* 1)**.** *Consider the same assumptions as in Proposition* 1*. An increase in the tax coefficient T reduces the pollution equilibrium $\bar{N}^*$ in (13) monotonically. Moreover, $\bar{N}^*$ converges to zero as T tends to infinity. However, when the equilibrium level of pollution $\bar{N}^*$ is below the threshold $\frac{A}{\frac{2\mu}{nd\delta}+2\gamma\left(\frac{2}{\gamma\delta}+\delta\right)\frac{2-\mu}{\left(\gamma\delta^2+2\right)nd}}$, then instability of the equilibrium $\bar{N}^*$ occurs with self-sustained oscillations in pollution levels. The oscillation can be confined inside the bounded interval $C^\infty := \left[0, \frac{nd\delta A}{2}\right]$ when the tax coefficient T goes to infinity.*

Before addressing the model with consumers' extra willingness to pay for green products, let us generalize the previous results to an economy with a generic number n of heterogeneous sectors, each of which served by a monopolist. We obtain the following results (the proof of which is in Appendix B).

Proposition 2 (*Benchmark with n Heterogeneous Sectors*)**.** *Assume $n \geq 1$ monopolists. Furthermore, let $\zeta_i = 0$ and $\eta_i(\omega_i) = 0$ (that is, $\eta_i\bar{\eta}_i = 0$) for all i. Consider sectors j and k, such that*

$$\frac{A_j}{\left(\frac{2}{\delta_j\gamma_j}+\delta_j\right)} \leq \frac{A_i}{\left(\frac{2}{\delta_i\gamma_i}+\delta_i\right)} \leq \frac{A_k}{\left(\frac{2}{\delta_k\gamma_k}+\delta_k\right)} \quad \forall i \in \{1, 2, \dots, n\}, \tag{16}$$

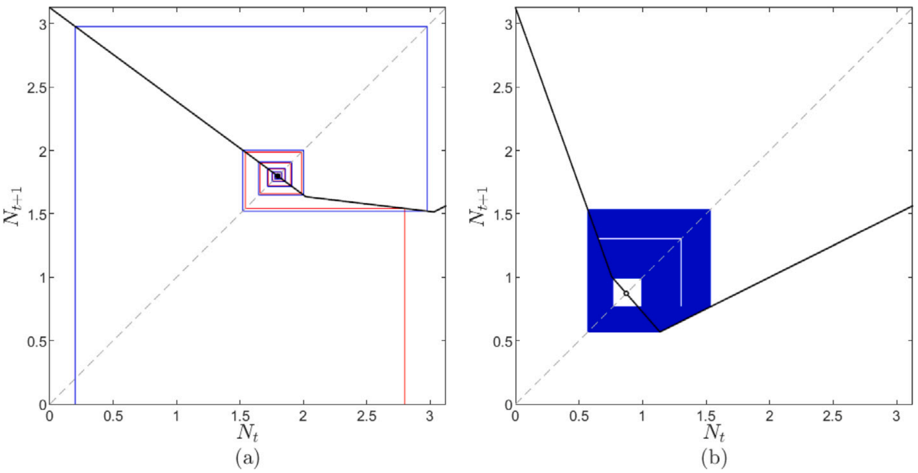


Fig. 3. Map (11) with $n = 2$ sectors: Panel (a) $T = 0.15$, corresponding to Proposition 2, case $T < T_b^{**}$ and $T < T_b^{***}$ (if homogeneous sectors, it corresponds to Proposition 1.(a)); Panel (b) $T = 0.4$, corresponding to Proposition 2, case $T_b^{***} < T < T_b^{**}$ (if homogeneous sectors, it corresponds to Proposition 1.(b)). The dashed line represents current pollution N, while the solid black line shows the map (pollution in the next period). In Panel (a), the staircase diagram of the trajectory starting at $N_0 = 2.8$ is shown in blue, while the one starting at $N_0 = 0.2$ is shown in red. In Panel (b), the staircase diagram of a long-run trajectory is shown in blue. Equilibria correspond to the intersection points between the map and the identity line; locally asymptotically stable equilibria are shown as black dots, while unstable ones are shown as empty dots. The remaining parameters are: $\mu = 0.5$, $A_1 = 1$, $A_2 = 1.5$, $\gamma_1 = \gamma_2 = 1$, $\delta_1 = \delta_2 = 2.5$, $d = 1$, and $\eta_1\bar{\eta}_1 = \eta_2\bar{\eta}_2 = \zeta_1 = \zeta_2 = s = 0$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

and define

$$N_b^u := \min\left\{(1-\mu)\frac{A_k}{\left(\frac{2}{\delta_k\gamma_k}+\delta_k\right)T}+\sum_{i=1}^{n}\frac{d\delta_i A_i}{2};\sum_{i=1}^{n}\frac{d\delta_i A_i}{2\mu}\right\}. \tag{17}$$

The set $C_n := \left[0, N_b^u\right]$ is invariant and attracts all trajectories and the viable (ensuring sustainability of production in all sectors) equilibrium with the lowest value of degradation is

$$\bar{N}^{**} := \frac{\sum_{i=1}^{n}\frac{d\delta_i}{2}A_i}{\mu + T\sum_{i=1}^{n}\left(\frac{d\delta_i}{2}\delta_i+\frac{d}{\gamma_i}\right)}. \tag{18}$$

Moreover, $\bar{N}^{**}$ exists when

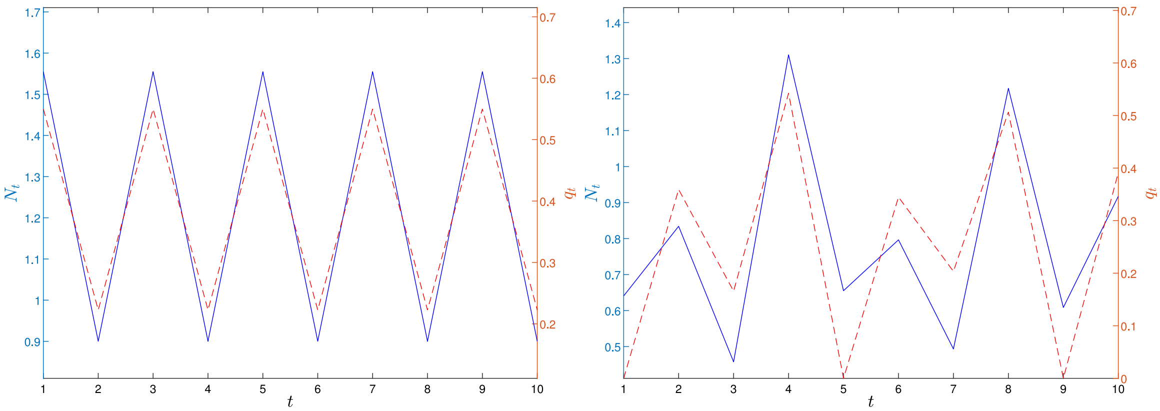
$$T < T_b^{**} := \frac{A_j\mu}{\sum_{i=1,i\neq j}^{n}\frac{d\delta_i}{2}\left[\left(\frac{2}{\delta_j\gamma_j}+\delta_j\right)A_i - A_j\left(\frac{2}{\gamma_i\delta_i}+\delta_i\right)\right]}, \tag{19}$$

and is locally asymptotically stable for

$$T < T_b^{***} := \frac{2-\mu}{d\sum_{i=1}^{n}\left(\frac{\delta_i}{2}\delta_i+\frac{1}{\gamma_i}\right)}. \tag{20}$$

Several policy implications emerge from Proposition 2. In a general economic setup with n heterogeneous production sectors, it is possible to derive the conditions under which the level of a pollution tax, adjusted for the probability of its enforcement, must be set to simultaneously achieve minimal pollution and maintain the economic viability of all sectors. These conditions are given in (19) and (20), assuming that the sector j that satisfies condition (16) is identified. Furthermore, through the analytical expression of the equilibrium level $\bar{N}^{**}$ of Proposition 2 in (18), one can quantify the minimum attainable pollution level under the credible threat of a pollution tax that ensures sectoral sustainability (positive production in all sectors).

Comparing the bifurcation diagrams of Figs. 2(a), 2(b), 2(c) and 2(d), we can observe the effect of increasing the tax coefficient in case of an economy with two homogeneous production sectors (Fig. 2(a)), two sectors where one serves a market with a reservation price that is 1.5 times higher then the other one, (Fig. 2(b), with staircase diagrams in Fig. 3 illustrating the dynamics under both stable and unstable equilibria), two sectors where one has the growth rate of the marginal cost of abatement that is two times the other (Fig. 2(c)), two sectors where one has a pollution production rate for unit of output produced that is two times the other (Fig. 2(d)). In all these cases, we observe that an increase in the tax coefficient reduces the level of pollution, but, at the same time, it may lead to fluctuations. These nonlinearities are due to the negative intertemporal externalities in pollution abatement. Specifically,



(a) (b)

Fig. 4. Time series of map (11) with $n = 2$ sectors. Panel (a): $T = 0.2$, showing a two-period cycle. Panel (b): $T = 0.4$, showing non-periodic (possibly chaotic) dynamics. In both panels, the blue line represents the level of pollution, while the dashed red line represents the level of production. The remaining parameters are as in Fig. 2(a), namely: $\mu = 0.5$, $A_i = A = 1$, $\gamma_i = \gamma = 1$, $\delta_i = \delta = 2.5$, $d = 1$, $s = 0$ and $\eta_i\bar{\eta}_i = \eta\bar{\eta} = \zeta_i = \zeta = 0$ for all $i \in \{1, 2\}$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

there are periods in which it is convenient to do only partial abatement as part of the negative effects (costs) of polluting are transferred to the next period. However, in the next period, the high level of pollution makes the risk of taxation too high, preventing production. The production stop reduces the level of pollution due to the natural absorption of it, when the level of pollution is sufficiently low, production becomes convenient again, and the loop goes on. This effect can be observed in the time series of Fig. 4. In panel (a), the dynamics along a two-period cycle is depicted. As we can observe, the dynamics of the level of production (dashed line in red) shows ups and downs. During the downs, the resulting level of pollution, that is, the level of N, is lower. This lower level of environmental degradation reduces the risk of transition (risk of having a tax imposed on your pollution) in the next period. As a result, in the next period, the optimal level of production is higher. This implies a higher level of pollution generated and a higher value of environmental degradation. The implication of this is again a higher risk of taxation (transition risk increases), however, this risk increases only in the next period, not in the current one. As a consequence of this intertemporal negative externality, the optimal production in the next period will be lower. It follows that lower will also be the level of environmental degradation. The loop goes on. A similar mechanism is observed in panel (b), where, however, non-periodic dynamics takes place, that is, there are no regularities in the ups and downs of the production levels and in the level of environmental degradation.

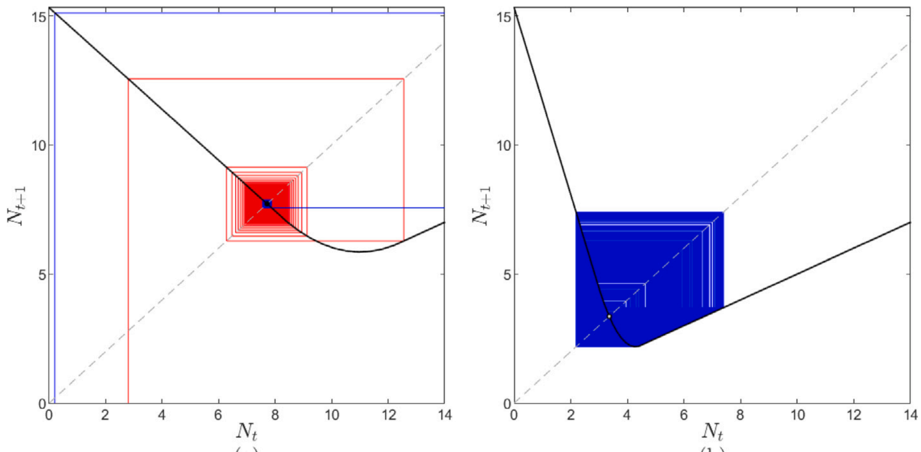
Going back to the bifurcation diagrams of Figs. 2(a), 2(b), 2(c) and 2(d), a relevant differences between the bifurcation diagram of Fig. 2(a) where the two sectors are homogeneous, and the bifurcation diagrams of Figs. 2(b) and 2(d) is the bifurcation sequence leading to oscillatory dynamics. If all sectors are homogeneous, there is a unique equilibrium and oscillatory dynamics occurs as the tax coefficient T increases. If sectors are heterogeneous, as in Figs. 2(b) and 2(d), a unique inner equilibrium loses stability when the tax coefficient increases; however, with a further increasing of the tax coefficient, the unstable equilibrium collides with a stable equilibrium belonging to a different branch of the map (border collision bifurcations, see Avrutin et al., 2019). After the collision, the stable equilibrium becomes real while the unstable one becomes virtual. As a consequence, the system shifts from oscillatory dynamics back to a stable equilibrium. Further increasing the tax coefficient, even this equilibrium loses stability, and oscillatory dynamics are back. In the case of the bifurcation diagram of Fig. 2(c), this effect of regaining stability is not observed as the equilibria that collide are both unstable.

Based on the results obtained so far, we can remark on the impact of a pollution tax coefficient T.

Remark. The tax coefficient T has two main effects: (1) it reduces pollution (environmental degradation), (2) it increases the intertemporal negative externality of the pollution process, which causes instability.

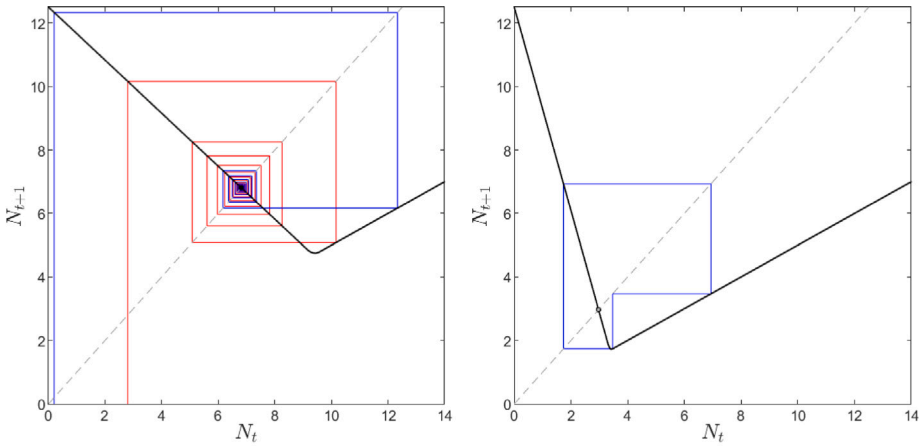
Increasing the number of heterogeneous sectors, the level of pollution at equilibria increases; however, the global dynamics is qualitatively similar to the case of one and two sectors. An example is shown in Fig. 5, with ten sectors differing for their reservation price. The equilibrium in Proposition 2, that is, the one which guarantees the lowest level of environmental degradation and operation of all production sectors, is stable for $T < T_b^{***} = 0.0\bar{3}6$ and exists for $T < T_b^{**} \approx 0.053872053872054$. This case is shown in Fig. 5(a). Increasing the taxation coefficient, the level of pollution decreases; however, as already observed for the case with two sectors, oscillatory dynamics arise as a side effect. This is illustrated in Fig. 5(b). As a remark on the shape of the map, in Fig. 5 ten sectors are present, so the map is piecewise-linear with several branches (see Avrutin et al., 2019). As a result, the map appears to be a smooth curve, but it is not. Nevertheless, this may suggest that the map may be approximated with a smooth function as the number of heterogeneous sectors increases.

Similar considerations hold in an economy with ten production sectors. Heterogeneity is introduced in the parameter γ_i, that is, in the growth rate of the marginal cost of reducing pollution levels, rather than on the reservation price A_i. This case is shown in



(a) (b)

Fig. 5. Map (11) with $n = 10$ sectors. Panel (a): $T = 0.036$; Panel (b): $T = 0.1$. The dashed red line represents current pollution N, while the solid black line shows the map, i.e., pollution in the next period. In Panel (a), the staircase diagram of the trajectory starting at $N_0 = 2.8$ is shown in blue, and the one starting at $N_0 = 0.2$ is shown in red. In Panel (b), the staircase diagram of a long-run trajectory is shown in blue. Equilibria correspond to the intersection points between the map and the identity line; locally asymptotically stable equilibria are shown as black dots, while unstable ones are shown as empty dots. The remaining parameters are: $\mu = 0.5$; $A_1 = 1$, $A_2 = 1.05$, $A_3 = 1.1$, $A_4 = 1.15$, $A_5 = 1.2$, $A_6 = 1.25$, $A_7 = 1.3$, $A_8 = 1.35$, $A_9 = 1.4$, $A_{10} = 1.45$; $d = 1$; $s = 0$ and, for all sectors $i \in \{1, 2, \ldots, 10\}$, $\gamma_i = 1$, $\delta_i = 2.5$, and $\bar{\eta}_i \eta_i = \zeta_i = 0$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)



(a) (b)

Fig. 6. Map (11) with $n = 10$ sectors. Panel (a): $T = 0.036$; Panel (b): $T = 0.1$. The dashed red line represents current pollution N, while the solid black line shows the map, i.e., pollution in the next period. In Panel (a), the staircase diagram of the trajectory starting at $N_0 = 2.8$ is shown in blue, and the one starting at $N_0 = 0.2$ is shown in red. In Panel (b), the staircase diagram of a long-run trajectory is shown in blue. Equilibria correspond to the intersection points between the map and the identity line; locally asymptotically stable equilibria are shown as black dots, while unstable ones are shown as empty dots. The remaining parameters are: $\mu = 0.5$; $d = 1$; $s = 0$ and for all sectors i, $A_i = 1$, $\delta_i = 2.5$, and $\bar{\eta}_i \eta_i = \zeta_i = 0$. The sector-specific values of γ_i are: $\gamma_1 = 1.5$, $\gamma_2 = 1.55$, $\gamma_3 = 1.6$, $\gamma_4 = 1.65$, $\gamma_5 = 1.7$, $\gamma_6 = 1.75$, $\gamma_7 = 1.8$, $\gamma_8 = 1.85$, $\gamma_9 = 1.9$, and $\gamma_{10} = 1.95$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Fig. 6, where in panel (a) we set $T < T_b^{**} \approx 0.603222288224882$ and $T < T_b^{***} \approx 0.040444583223222$. Thus, the equilibrium $\bar{N}^{**}$ in Proposition 2 exists and is stable. Increasing the tax coefficient T, the equilibrium first becomes unstable, as shown in Fig. 6(b), where a stable three-period cycle exists, and eventually disappears.

Similar considerations also hold in an economy with ten production sectors and heterogeneity is in the parameter δ_i, that is, on the pollution generated for the unit of production in sector i. Such a case is shown in Fig. 7, where in panel (a) we set $T < T_b^{**} \approx 0.203746870413537$ and $T < T_b^{***} \approx 0.034170186512268$. Thus, the equilibrium $\bar{N}^{**}$ in Proposition 2 exists and is stable. Increasing the tax coefficient T, the equilibrium becomes unstable, as shown in Fig. 7(b), where a stable three-period cycle exists, and eventually disappears.

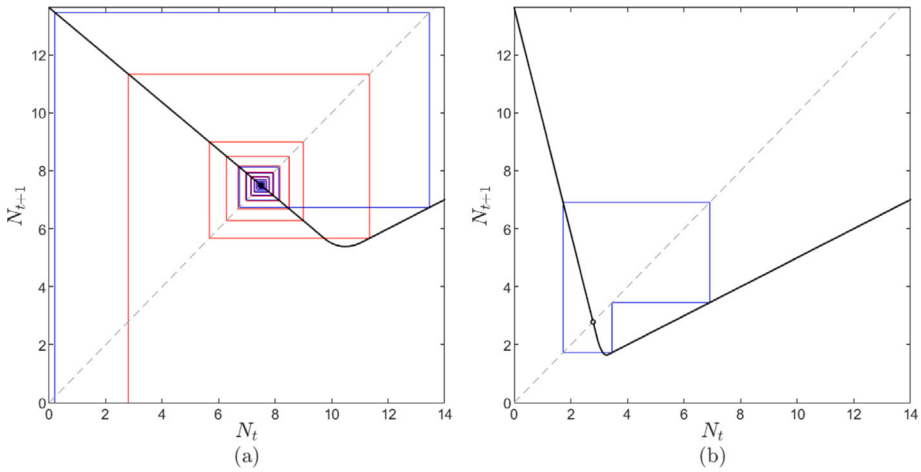


Fig. 7. Map (11) with $n = 10$ sectors: Panel (a) $T = 0.03$; Panel (b) $T = 0.1$. The dashed line represents current pollution N, and the solid black line is the map (pollution in the next period). Staircase diagram of the trajectory starting at $N_0 = 2.8$, resp. $N_0 = 0.2$, is in blue, resp. in red, in panel (a). Staircase diagram of a long-run trajectory is in blue in panel (b). Equilibria are the intersections between the growth curve and the dashed line and are marked with a black dot if locally asymptotically stable and an empty dot if unstable. Remaining parameters are $\mu = 0.5$; $\delta_1 = 2.95$; $\delta_2 = 2.90$; $\delta_3 = 2.85$; $\delta_4 = 2.80$; $\delta_5 = 2.75$; $\delta_6 = 2.70$; $\delta_7 = 2.65$; $\delta_8 = 2.60$; $\delta_9 = 2.55$; $\delta_{10} = 2.50$; $d = 1$; $s = 0$ and for all sector i we set $A_i = 1$; $\gamma_i = 1.5$ and $\eta_i = \zeta_i 0$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Observing the shape of the map (11) in the numerical examples, it is always convex, and the equilibrium is unique. This follows from the fact that a higher level of environmental degradation N, by increasing the transition risk, increases expected costs with fewer sectors that are profitable. It follows that the production is expected to decrease as N increases till the point at which it is not profitable to produce in any sector or just a few. From this point on, the map should start to be increasing in N.

Based on these considerations, we make the following conjecture.

Conjecture 1. *If $\bar{\eta}_i \eta_i = \zeta_i = 0$ for all i, map (11) is always convex and has at most one equilibrium, either stable or unstable.*

Note that, to prove that the map (11) is convex, it is sufficient to show that the damage functions D_i^m, i.e., the damage generated by production in market i under monopoly, are all convex, and to observe that the map (11) is given by the sum of these damage functions plus a linear term.

3.2. Consumer willingness to pay for abatement

Consumers may care about how goods are produced, in particular concerning their environmental impact. In this context, we assume consumers are willing to pay more for goods that are made with cleaner technologies. This means their reservation price, that is, the maximum they are willing to pay, increases when the good is associated with lower pollution. This behavior reflects a demand for abatement, as cleaner production becomes a quality aspect of the good itself. In (1), we now assume that $\eta_i\left(\omega_i\right) = \eta_i \cdot \omega_i$ with $\eta_i > 0$ (that is, $\bar{\eta}_i = +\infty$). We here investigate the effect on environmental degradation of an increasing consumers' extra willingness to pay for green products. In particular, it is worth understanding whether the introduction of this element can relieve the trade-off between market stability and pollution abatement that we detected in Section 3.1.

The next proposition clarifies the possible equilibria and their asymptotic stability in the case of an economy consisting only of n homogeneous monopoly markets. Similarly as before, consider $A_i = A$, $\gamma_i = \gamma$, $\delta_i = \delta$, and $\eta_i = \eta$ ($\bar{\eta}_i = \eta_i = +\infty$) for all $i = 1, \dots, n$, and define the degradation level:

$$\bar{N}^+ := \frac{nd A\left(\delta\gamma - \eta\right)}{\mu\left(2\gamma - \eta^2\right) - nd\left(2\left(\eta\delta - 1\right) - \delta^2\gamma\right)T}, \tag{21}$$

and the taxation threshold:

$$T_b^+ := \frac{(2-\mu)\left(2\gamma - \eta^2\right)}{nd\left(2 + \delta\left(\delta\gamma - 2\eta\right)\right)}. \tag{22}$$

Proposition 3 (*Extra Willingness to Pay for Green Products and n Homogeneous Sectors*). *Assume $\eta_i\left(\omega_i\right) = \eta_i \cdot \omega_i$ (i.e., $\bar{\eta}_i = +\infty$) and $\zeta_i = 0$ for all $i = 1, \dots, n$. Furthermore, assume homogeneous sectors: $A_i = A$, $\gamma_i = \gamma$, $\delta_i = \delta$ and $\eta_i = \eta$ for all $i = 1, \dots, n$. Then, for the dynamics of the map (11), the following results hold:*

1. *If either $2\gamma < \eta^2$ and $2 > \eta\delta$ or $2\gamma > \eta^2$, $\gamma\delta - \eta < 0$ and $2 > \delta(2\eta - \gamma\delta)$ are satisfied, then the unique equilibrium of the map (11) is $\bar{N}^0 = 0$, which is globally attractive.*
2. *If $2\gamma > \eta^2$, $\gamma\delta - \eta > 0$ and $2 > \delta(2\eta - \gamma\delta)$, then the unique equilibrium of the map (11) is given by $\bar{N}^+$ in (21). This equilibrium is locally stable for $T < T_b^+$, where T_b^+ is defined in (22); otherwise, a non-equilibrium attractor exists.*

Parameter configurations satisfying restrictions other than the ones in points 1. and 2. are not possible.

The last proposition identifies the conditions under which the model admits both a globally stable zero environmental degradation equilibrium and the conditions for having a stable equilibrium with positive pollution. The former case occurs when the cost of abatement for observable pollution is sufficiently low. Remarkably, a scenario with a zero environmental degradation equilibrium can occur assuming consumer extra willingness to pay for greener production ($\eta_i = \eta > 0$ and $\bar{\eta}_i = \bar{\eta} = +\infty$). In the analogous case with consumers' unwillingness to pay for greener production, this scenario is ruled out, as shown in Proposition 1. In the latter case of a stable equilibrium with positive pollution, pollution decreases as the environmental tax increases, reflecting the effectiveness of taxation in curbing emissions. However, it is also shown that this equilibrium loses stability when the tax coefficient becomes too high. As a result, while higher taxation initially reduces pollution and stabilizes the system, excessive taxation destabilizes the system, potentially leading to undesirable dynamic outcomes. These results for the positive equilibrium are in line with the case of consumer unwillingness to pay for green products ($\eta_i = \eta = 0$). In both scenarios, the dynamics exhibit a unique equilibrium with positive degradation whose stability depends on the value of the tax coefficient T, and a transition to more complex dynamics (e.g., the emergence of an attractor) occurs when a threshold is exceeded. The presence of consumer extra willingness to support environmental policies (captured by $\eta_i = \eta > 0$ and $\bar{\eta}_i = \bar{\eta} = +\infty$) tends to mitigate the intensity of the dynamic outcomes. This attenuation is particularly evident in the two-parameter bifurcation diagrams of Figs. 8(a), 8(b), 8(c) and 8(d), where the structure and extent of the attractors in the parameter space are visibly affected by the degree of consumer extra willingness to pay for green products, which has a stabilizing effect. Specifically, the loss of stability of the positive environmental degradation equilibrium occurs at a higher level of taxation. This implies that the system can sustain a stronger threat of taxation without being destabilized, provided that consumers are responsive to environmental quality. This effect is observable in the two-parameter bifurcation diagrams in Figs. 8(a) and 8(b), where the region of stability of the equilibrium expands with $\eta_i = \eta > 0$ and $\bar{\eta}_i = \bar{\eta} = +\infty$. For a given level of μ, an increment of taxation T entails the loss of stability of attractors with lower periodicity with a *period adding* structure, a typical feature of piecewise-smooth maps, see Avrutin et al. (2019).

Moreover, when natural abatement is low, see Fig. 8(c), the system exhibits a broader region of stability for the positive equilibrium. However, as the tax level increases, the system exhibits high-period cycles or even chaotic dynamics. Conversely, when natural abatement is high, see Fig. 8(d), the loss of stability occurs at lower tax levels. Nonetheless, in this case, a higher extra willingness to pay for sustainable products can lead to the emergence of low-period cycles, suggesting that consumer responsiveness can partially offset the destabilizing effects of taxation when natural abatement is high.

We now extend the baseline model to incorporate an economy composed of n distinct sectors, each characterized by the presence of just a monopoly. This generalization allows us to capture sectoral heterogeneity while preserving the main assumption of monopoly firms. Importantly, we continue to focus exclusively on observable pollution, maintaining the assumption that environmental externalities are measurable and directly attributable to sectoral outputs. The next proposition presents the main results in this case (the proof of which is in Appendix B).

Proposition 4 (*Extra Willingness to Pay for Green Products and n Heterogeneous Sectors*). *Assume $\eta_i(\omega_i) = \eta_i \cdot \omega_i$ (i.e., $\bar{\eta}_i = +\infty$), $\zeta_i = 0$, and $n \geq 1$. Let j denote a monopolist (production sector) such that*

$$\frac{(\gamma_j\delta_j - \eta_j)A_j}{\left(2 + \delta_j(\gamma_j\delta_j - 2\eta_j)\right)} \leq \frac{(\gamma_i\delta_i - \eta_i)A_i}{\left(2 + \delta_i(\gamma_i\delta_i - 2\eta_i)\right)} \quad \forall i \in \{1, 2, \dots, n\}. \tag{23}$$

Then, for the dynamics governed by the map (11), the following results hold:

1. *If for all i the following conditions are satisfied:*

$$2\gamma_i < \eta_i^2 \quad \text{and} \quad 2 > \eta_i\delta_i \quad \text{or} \quad 2\gamma_i > \eta_i^2, \quad \gamma_i\delta_i - \eta_i < 0, \quad \text{and} \quad 2 > \delta_i(2\eta_i - \gamma_i\delta_i), \tag{24}$$

then the unique equilibrium of the map (11) is $\bar{N}^0 = 0$, which is globally attractive.

2. *If for all i the following conditions hold:*

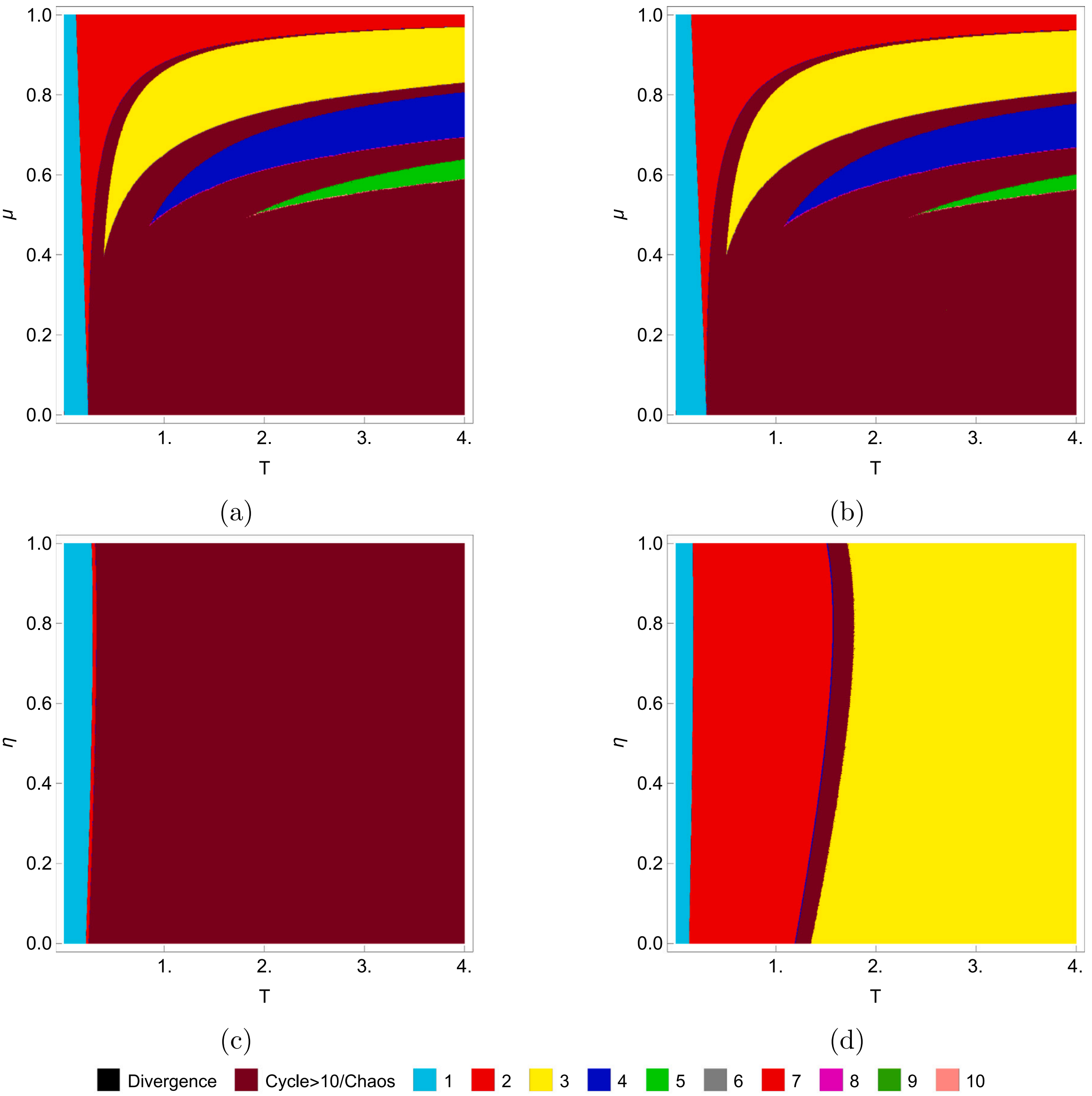
$$2\gamma_i > \eta_i^2, \quad \gamma_i\delta_i - \eta_i > 0, \quad \text{and} \quad 2 > \delta_i(2\eta_i - \gamma_i\delta_i), \tag{25}$$

then the map (11) admits a unique positive (non-hyperbolic) equilibrium in $\left[0, \frac{(\gamma_j\delta_j - \eta_j)A_j}{(2+\delta_j(\gamma_j\delta_j - 2\eta_j))T}\right)$ given by

$$\bar{N}^{++} = d\sum_{i=1}^{n} \frac{A_i(\delta_i\gamma_i - \eta_i)\prod_{k=1,\,k\neq i}^{n}(2\gamma_k - \eta_k^2)}{\mu\prod_{k=1}^{n}(2\gamma_k - \eta_k^2) + d\sum_{i=1}^{n}\left(\delta_i^2\gamma_i - 2(\eta_i\delta_i - 1)\right)\prod_{k=1,\,k\neq i}^{n}(2\gamma_k - \eta_k^2)T}, \tag{26}$$

which is feasible (real) for

$$T < T_b^{++} := \frac{\left(\gamma_j\delta_j - \eta_j\right)A_j\mu}{d\displaystyle\sum_{i=1,i\neq j}^{n}\frac{A_i\left(\delta_i\gamma_i - \eta_i\right)\left(2 + \delta_j\left(\gamma_j\delta_j - 2\eta_j\right)\right) - A_j\left(\gamma_j\delta_j - \eta_j\right)\left(2 + \delta_i\left(\gamma_i\delta_i - 2\eta_i\right)\right)}{2\gamma_i - \eta_i^2}}, \tag{27}$$



(a) (b) (c) (d)

Fig. 8. Two-dimensional bifurcation diagrams for map (11), with n homogeneous sectors ($A_i = A$, $\gamma_i = \gamma$, $\delta_i = \delta$, $\eta_i = \eta$, $\bar{\eta}_i = \bar{\eta}$ and $\zeta_i = \zeta$ for all i), where different colors distinguish different dynamical regimes: stable equilibria, periodic cycles and chaotic dynamics. Panel (a): Dynamics in the (T, μ) plane under consumer unwillingness to pay for abatement ($\eta\bar{\eta} = 0$). Panel (b): Dynamics in the (T, μ) plane under consumer extra willingness to pay for abatement ($\eta = 1$ and $\bar{\eta} = +\infty$). Panel (c): Dynamics in the (T, η) plane, with low natural regenerative rate ($\mu = 0.2$) and $\bar{\eta} = +\infty$. Panel (d): Dynamics in the (T, η) plane with high natural regenerative rate ($\mu = 0.9$) and $\bar{\eta} = +\infty$. Each panel illustrates the qualitative behavior of the system across parameter combinations. The remaining parameters are: $A = 1$, $\gamma = 1$, $\delta = 2.5$, $\zeta = 0$, $d = 1$, $s = 0$ and $n = 2$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

and locally attractive for

$$T < T_b^{+++} := \frac{2-\mu}{d\sum_{i=1}^{n}\frac{2(1-\eta_i\delta_i)+\delta_i^2\gamma_i}{2\gamma_i-\eta_i^2}}. \tag{28}$$

At $T = T_b^{+++}$, a degenerate flip bifurcation occurs.

The bifurcation diagrams in Figs. 9(a), 9(b), 9(c) and 9(d), answer the two main points regarding the role of the extra willingness to pay for green products. Regarding an economy made of two homogeneous sectors, introducing an extra willingness to pay for green products increases the level of environmental degradation at the equilibrium. Compare the bifurcation diagram of Fig. 9(a),

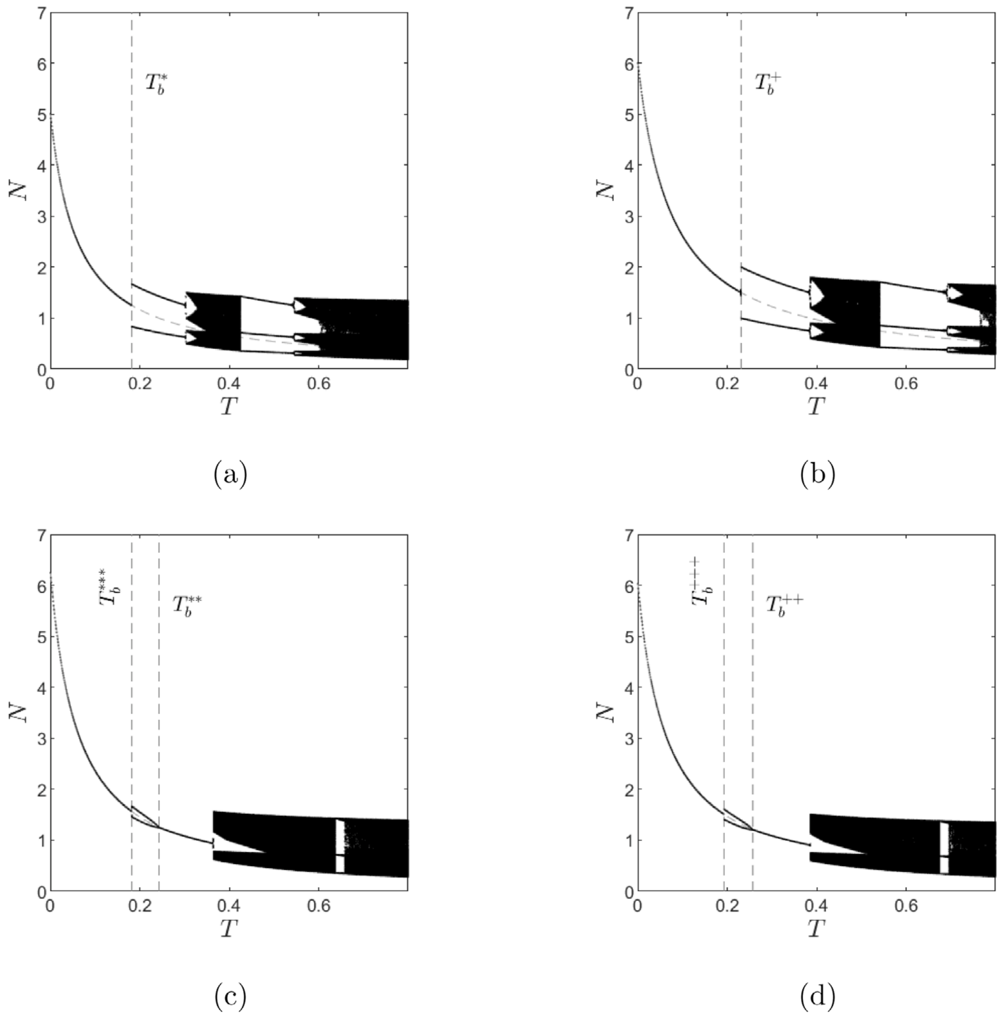


Fig. 9. Bifurcation diagrams with respect to the tax coefficient T for map (11) with $n = 2$ sectors. Panel (a): Homogeneous sectors, as in Fig. 2(a). Panel (b): Same as Panel (a), but with $\eta = 1$. Panel (c): Heterogeneous sectors, as in Fig. 2(b). Panel (d): Same as Panel (c), but with $\eta = 0.1$.

where there is no extra willingness to pay for green products, and Fig. 9(b), where the only difference with the previous case is a positive extra willingness to pay for green products. As we can see, the extra willingness to pay has two size effects: on the one hand, it increases the level of environmental degradation at the equilibrium (when it exists and is stable). On the other hand, it increases the stability region of the equilibrium, that is, it prevents the appearance of oscillatory dynamics. This can be explained by the fact that a lack of extra willingness to pay for green products reduces the profitability of environmentally friendly production, thereby lowering optimal production levels. At the same time, numerical simulations confirm that this discourages oscillatory dynamics and reduces their amplitude, as there is less overshooting—meaning the optimal production level becomes less sensitive to current levels of environmental degradation. In case of heterogeneous sectors, similar results hold, compare the bifurcation diagrams in Figs. 9(c) and 9(d).

In summary, a higher extra willingness to pay for green products leads to increased environmental degradation by incentivizing higher production levels. However, it also expands the stability region of the environmental degradation equilibrium. Despite this, it does not eliminate the oscillatory dynamics caused by the intertemporal negative externalities of pollution activities.

We are now ready to address the most relevant case of our model, in which the by-product of production comprises both observable and unobservable emissions, and both types of emissions define the total environmental degradation.

4. Partially observable pollution

In this section, we extend the model by incorporating unobservable pollutants that, while not monitored or taxed, contribute to total environmental degradation and, as such, may be of interest to corporate strategies for transition risk reduction. In particular, consumers' extra willingness to pay is related exclusively to observable pollution abatement activities. For the sake of brevity, we focus here on the case of positive extra willingness to pay, noting that the case of zero willingness arises as a special instance of this setup. In this context, we interpret *greenwashing* as a strategy in which the firm captures extra willingness to pay from consumers based on observable pollution abatement, that is, as a function of ω_i, while undertaking no abatement λ_i on unobservable pollution $\zeta_i q_i$. Again, we begin with n homogeneous sectors and subsequently extend the analysis to heterogeneous sectors.

Consider the following homogeneous setup: $A_i = A$, $\gamma_i = \gamma$, $\delta_i = \delta$, $\eta_i = \eta$, $\bar{\eta}_i = \bar{\eta}$ and $\zeta_i = \zeta$ for all $i = 1, \dots, n$. Furthermore, let us define the following degradation levels:

$$\bar{N}^1 := \frac{ns\zeta A}{\mu(2-\eta\delta) + ns\zeta T\delta} \qquad \bar{N}^2 := \frac{(s\zeta\gamma + d(\gamma\delta-\eta))nA}{\mu\left(2\gamma-\eta^2\right) + (s\zeta(\gamma\delta-\eta) + d(2+\delta(\gamma\delta-2\eta)))nT}, \tag{29}$$

and the following taxation thresholds:

$$T_b^1 := \frac{(2-\mu)(2-\eta\delta)}{ns\zeta\delta} \qquad T_b^3 := \frac{(\gamma\delta-\eta)\mu(2-\eta\delta)}{ns\zeta(2-\eta\delta)} \qquad T_b^4 := \frac{(\mu-2)\left(2\gamma-\eta^2\right)}{n\left(s\zeta(\eta-\gamma\delta) + d\left(2(\eta\delta-1)-\gamma\delta^2\right)\right)}. \tag{30}$$

Then the following results hold (the proof of which is in Appendix B).

Proposition 5 (*Extra Willingness to Pay for Green Products, Partially Observable Pollution and n Homogeneous Sectors*). *Assume $\eta_i(\omega_i) = \eta_i \cdot \omega_i$ (i.e., $\bar{\eta}_i = +\infty$), with $\zeta_i > 0$ for all $i = 1, \dots, n$. Moreover, assume $A_i = A$, $\gamma_i = \gamma$, $\delta_i = \delta$, $\eta_i = \eta$ and $\zeta_i = \zeta$ for all $i = 1, \dots, n$. Consider $\bar{N}^1$ and $\bar{N}^2$ in (29), T_b^1 in (30), and T_b^3 and T_b^4 in (30). Then, for the dynamics of the map (11), the following results hold:*

(1) If either $2\gamma < \eta^2$ and $2 > \eta\delta$ or $2\gamma > \eta^2$, $\gamma\delta - \eta < 0$, and $2 > \delta(2\eta - \gamma\delta)$, then the map (11) admits a unique equilibrium $\bar{N}^1$. This equilibrium is locally asymptotically stable for $T < T_b^1$. For $T \geq T_b^1$, the equilibrium becomes unstable, and an attractor exists.

(2) If $2\gamma > \eta^2$, $\gamma\delta - \eta > 0$, and $2 > \delta(2\eta - \gamma\delta)$, then $\bar{N}^2$ is an equilibrium when $T < T_b^3$ and it is locally asymptotically stable when $T < T_b^4$, $\bar{N}^1$ is an equilibrium when $T > T_b^3$ and it is locally asymptotically stable when $T < T_b^1$.

Parameter constellations satisfying neither of the cases above are not possible.

Comparing the results of Proposition 5, where it is $\zeta_i = \zeta > 0$ for all i, with those of Proposition 3, where it is $\zeta_i = \zeta = 0$ for all i, we can observe the following. Let us start by considering *Case 1* of Proposition 3, which describes a scenario characterized by an equilibrium with no environmental damage, that is, the equilibrium is at 0 and is globally stable. Under the same parameter conditions but with $\zeta_i = \zeta > 0$, implying that pollution is unobservable, *Case 1* of Proposition 5 applies. This case indicates that the presence of unobservable pollution may either have no effect, render the zero-damage equilibrium only locally stable, or lead to a shift in the equilibrium. In the latter case, there exists an equilibrium with a positive level of environmental damage, denoted by $\bar{N}_1$, instead of the zero-damage equilibrium. Moreover, as the tax coefficient increases, the equilibrium $\bar{N}_1$ approaches the zero-damage equilibrium, but it loses stability when $T > T_b^1$. This effect is illustrated in the bifurcation diagram in Fig. 10(a).

The remaining *Case 2* of Proposition 3 must be compared with the remaining *Case 2* of Proposition 5. The analytical results indicate that the two cases are qualitatively similar, both featuring an equilibrium with positive environmental damage that loses stability as the tax coefficient T increases. If $\zeta_i = \zeta > 0$, at a certain point this equilibrium may collide with another equilibrium. As a result, stability can be regained as T increases. Despite this similarity, the most interesting differences between the two cases are quantitative. This information can be obtained by comparing the bifurcation diagrams in Fig. 10(b), where $\zeta_i = \zeta = 0$, and Fig. 10(c), where $\zeta_i = \zeta > 0$. As shown, unobservable pollution $\zeta > 0$ increases the level of environmental damage both in terms of the equilibrium value and the average value during oscillatory behavior. These are side effects of greenwashing and highlight the need for strategies to control unobservable pollution. In particular, we observe that the models with and without unobservable pollution exhibit similar dynamics as the tax coefficient T increases. Addressing an environmental tax (increasing transition risk) can represent an antidote to the additional environmental damage caused by unobservable sources of pollution.

As for the case of fully observable pollution, we now address the case with n heterogeneous monopolists operating, that is, the case with n heterogeneous production sectors. The following proposition collects the main results of this case (the proof of which is in Appendix B).

Proposition 6 (*Extra Willingness to Pay for Green Products, Partially Observable Pollution, n Heterogeneous Sectors*). *Assume $\eta_i(\omega_i) = \eta_i\omega_i$ (i.e., $\bar{\eta}_i = +\infty$) and $n \geq 1$. Assume j is a monopolist (production sector) such that*

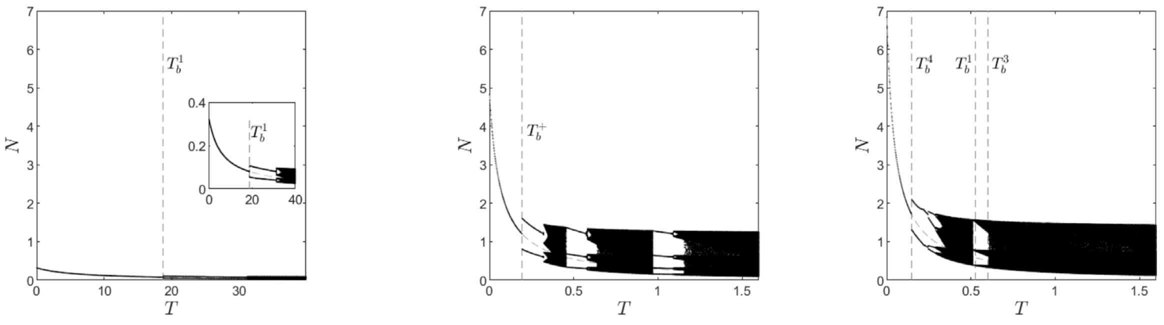
$$\frac{\left(\gamma_j\delta_j - \eta_j\right)A_j}{\left(2 + \delta_j\left(\gamma_j\delta_j - 2\eta_j\right)\right)} \leq \frac{\left(\gamma_i\delta_i - \eta_i\right)A_i}{\left(2 + \delta_i\left(\gamma_i\delta_i - 2\eta_i\right)\right)} \quad \forall i \in \{1, 2, \dots, n\}, \tag{31}$$

and assume h is a monopolist (production sector) such that

$$\frac{A_h}{\delta_h} \leq \frac{A_i}{\delta_i} \quad \forall i \in \{1, 2, \dots, n\}. \tag{32}$$

Define the following degradation levels:

$$\bar{N}^3 := \frac{\sum_{i=1}^n \frac{s\zeta_i A_i}{2-\eta_i\delta_i}}{\mu + \sum_{i=1}^n \frac{s\zeta_i T\delta_i}{2-\eta_i\delta_i}} \qquad \bar{N}^4 := \frac{\sum_{i=1}^n \frac{dA_i\left(\gamma_i\delta_i-\eta_i\right)+s\zeta_i\gamma_i A_i}{2\gamma_i-\eta_i^2}}{\mu + T\sum_{i=1}^n \frac{d\left(2-\delta_i\left(2\eta_i-\gamma_i\delta_i\right)\right)+s\zeta_i\left(\gamma_i\delta_i-\eta_i\right)}{2\gamma_i-\eta_i^2}} \tag{33}$$



(a) (b) (c)

Fig. 10. Bifurcation diagrams with respect to the tax coefficient T for map (11) with two sectors $n = 2$. Panel (a): Homogeneous sectors, corresponding to Case 1 of Proposition 5. For $T < T_b^1$, the equilibrium $\bar{N}^1$ is locally stable; for $T > T_b^1$, it becomes unstable. Parameters: $\delta_i = \delta = 0.5$, $\eta_i = \eta = 1.5$, and $\zeta_i = \zeta = 0.1$. Panel (b): Homogeneous sectors, corresponding to Case 2 of Proposition 3. Parameters: $\delta_i = \delta = 2.5$, $\eta_i = \eta = 0.1$, and $\zeta_i = \zeta = 0$. Panel (c): Homogeneous sectors, corresponding to Case 3 of Proposition 5. Parameters: $\delta_i = \delta = 2.5$, $\eta_i = \eta = 0.1$, and $\zeta_i = \zeta = 1$. In all panels, the remaining parameters are: $\mu = 0.5$, $A_i = A = 1$, $\gamma_i = \gamma = 1$, $d = 1$, and $s = 1$, $i \in \{1, 2\}$.

and the following taxation thresholds:

$$T_b^5 := \frac{A_h \mu}{s \sum_{i=1}^n \zeta_i \frac{\delta_h A_i - \delta_i A_h}{2 - \eta_i \delta_i}} \qquad T_b^6 := \frac{2 - \mu}{\sum_{i=1}^n \frac{s \zeta_i \delta_i}{2 - \eta_i \delta_i}} \qquad T_b^9 := \frac{2 - \mu}{\sum_{i=1}^n \frac{d(2 - \delta_i(2\eta_i - \gamma_i \delta_i)) + s\zeta_i(\delta_i \gamma_i - \eta_i)}{2\gamma_i - \eta_i^2}} \tag{34}$$

$$T_b^8 := \frac{\frac{(\gamma_j \delta_j - \eta_j) A_j}{(2 - \delta_j(2\eta_j - \gamma_j \delta_j))} \mu}{\sum_{i=1}^n \left[\frac{d(2 - \delta_i(2\eta_i - \gamma_i \delta_i))}{2\gamma_i - \eta_i^2} \left(\frac{(\gamma_i \delta_i - \eta_i) A_i}{(2 - \delta_i(2\eta_i - \gamma_i \delta_i))} - \frac{(\gamma_j \delta_j - \eta_j) A_j}{(2 - \delta_j(2\eta_j - \gamma_j \delta_j))} \right) + \frac{s\zeta_i(\gamma_i \delta_i - \eta_i)}{2\gamma_i - \eta_i^2} \left(\frac{\gamma_i A_i}{(\gamma_i \delta_i - \eta_i)} - \frac{(\gamma_j \delta_j - \eta_j) A_j}{(2 - \delta_j(2\eta_j - \gamma_j \delta_j))} \right) \right]}. \tag{35}$$

For the dynamics of the map (11) the following results hold:

1. *If either $2\gamma_i < \eta_i^2$ and $2 > \eta_i \delta_i$ or $2\gamma_i > \eta_i^2$; $\gamma_i \delta_i - \eta_i < 0$ and $2 > \delta_i(2\eta_i - \gamma_i \delta_i)$ for all $i = 1, \ldots, n$, the equilibrium with the lowest level of pollution and at which production is positive in all sectors is $\bar{N}^3$, which is feasible and it is the only equilibrium in $\left[0, \frac{A_h}{T\delta_h}\right)$ for $T < T_b^5$ and it is locally asymptotically stable for $T < T_b^6$;*
2. *If $2\gamma_i > \eta_i^2$; $\gamma_i \delta_i - \eta_i > 0$ and $2 > \delta_i(2\eta_i - \gamma_i \delta_i)$ for all $i = 1, \ldots, n$, the equilibrium with the lowest level of pollution and at which production is positive in all sectors is $\bar{N}^4$, which is feasible and it is the only equilibrium in $\left[0, \frac{(\gamma_j \delta_j - \eta_j) A_j}{(2 + \delta_i(\gamma_j \delta_j - 2\eta_j))T}\right)$ for $T < T_b^8$, and is locally asymptotically stable for $T < T_b^9$.*

Staircase diagrams are used to illustrate the dynamics of the model in the two scenarios described in Proposition 6; see Fig. 11. In panel (a), *Case 1* holds, and the trajectories converge to the equilibrium $\bar{N}^3$. In panel (b), *Case 2* occurs, and the equilibrium $\bar{N}^4$ is globally stable, i.e., it attracts all trajectories. All cases refer to an economy with two sectors.

5. Conclusions

We proposed a novel dynamic framework to explore in a multi-sector economy the role of pollution taxation on the green transition and its economic consequences when pollution is either fully observable or partially unobservable. By incorporating transition risk as an adaptive regulatory strategy, we show how firms, modeled as myopic agents, may respond to environmental policies in complex ways. In particular, we find that while taxation on observable emissions can reduce environmental degradation, high tax threats may destabilize production and lead to cyclical or even chaotic dynamics of production and environmental degradation. The presence of unobservable pollution, which is neither taxed nor visible to consumers, contributes to increasing the risk of taxation on observable pollution in the next period. Moreover, it may further increase the instability of the system, reducing the effectiveness of traditional policy instruments. Our model also addresses the role of consumer preferences: when consumers are willing to pay more for environmentally friendly products, firms are incentivized to invest in abatement. However, if only fully observable pollution is considered, this may lead to firms' opportunistic behaviors such as greenwashing, where firms tackle visible environmental improvements but neglect hidden environmental harms and provide misleading communication. We also show that sectoral heterogeneity may add more complexity in the model. From a methodological perspective, we modeled regime shifts induced by transition risk through piecewise-smooth dynamical systems, see Avrutin et al. (2019).

Several policy implications emerge from our findings. Policymakers should consider regulatory instruments that combine direct taxation on observable emissions with indirect mechanisms targeting unobservable pollution, such as ambient charges,

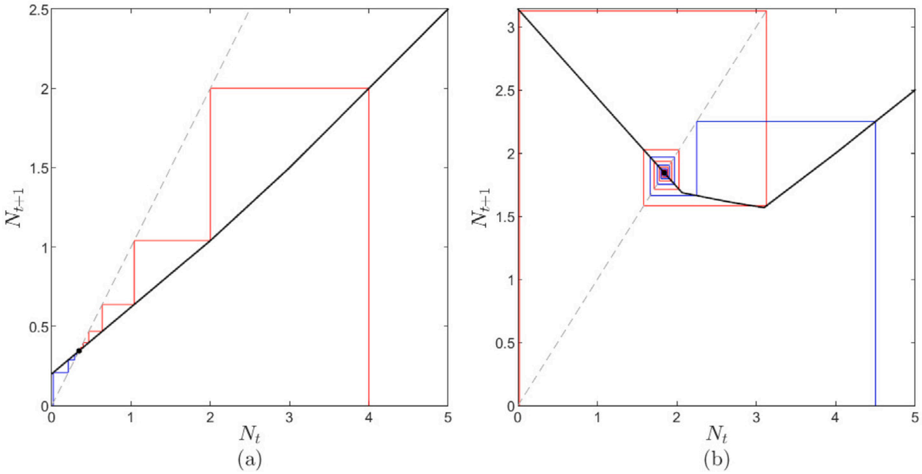


Fig. 11. Map (11) with two sectors: Panel (a), *Case 1* of Proposition 6. Parameters: $\mu = 0.5$, $A_1 = 1$, $A_2 = 1.5$ $\gamma_1 = \gamma_2 = 1$, $\delta_1 = \delta_2 = 0.5$, $d = 1$, $s = 1$, $n = 2$, $\eta_1 = \eta_2 = 1.5$, $\bar{\eta}_1 = \bar{\eta}_2 = +\infty$ and $\zeta_1 = \zeta_2 = 0.1$. All trajectories converge to the equilibrium $\bar{N}^3$. Inequalities $T = 1 < T_b^8 = 18.75$ and $T = 1 < T_b^7 = 25$ are satisfied. Red and blue curves show the staircase diagrams of trajectories starting at $N_0 = 4$ and $N_0 = 0.02$, respectively. Panel (b), *Case 2* of Proposition 6. Parameters: $\mu = 0.5$, $A_1 = 1$, $A_2 = 1.5$ $\gamma_1 = \gamma_2 = 1$, $\delta_1 = \delta_2 = 1.5$, $d = 1$, $s = 1$, $n = 2$, $\eta_1 = \eta_2 = 1.5$, $\bar{\eta}_1 = \bar{\eta}_2 = +\infty$ and $\zeta_1 = \zeta_2 = 0.1$. Trajectories converge to the equilibrium $\bar{N}^4$. The parameter values satisfy $T = 0.15 < T_b^9 \approx 0.2368$ and $T = 0.15 < T_b^{10} \approx 0.1868$. Red and blue curves show the staircase diagrams of trajectories starting at $N_0 = 0.02$ and $N_0 = 4.5$, respectively. Equilibria are the intersections between the growth curve and the dashed line and are marked with a black dot if locally asymptotically stable and an empty dot if unstable. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

transparency requirements, or reputational incentives. Taxation should be carefully adjusted to avoid production collapses or instability, particularly in sectors with high pollution or limited abatement capacity. Enhancing monitoring capabilities and enforcing the disclosure of environmental performance that reduces information asymmetries and limits greenwashing behaviors increases the effectiveness of a pollution tax. Finally, promoting consumer awareness through environmental labeling and certification schemes could reinforce market-based incentives for economic sustainability.

Future research could extend this framework in several directions. The first extension is to introduce both physical and transition risks that evolve endogenously, with firms exhibiting different sensitivities to each source of risk. A second extension would explore the possibility that each firm communicates an abatement level that differs from its actual effort, potentially at some cost. This would allow strategic information provision to shape outcomes, going beyond the baseline framework in which abatement is perfectly observed. Another direction for future investigation would be to enrich the framework with additional behavioral assumptions, for instance by linking consumers' extra willingness to pay directly to the current level of observable pollution or environmental degradation, thereby endogenizing preferences with respect to environmental quality.[6] Another development would be to consider sectors served by oligopolistic firms (Lambertini, 2013). In this case, it would be interesting to incorporate behavioral heterogeneity among firms, such as the presence of firms that also pursue Corporate Social Responsibility (CSR) objectives in addition to profit maximization, see Kopel and Lamantia (2018). These firms may undertake stronger abatement efforts, even in the absence of direct regulatory enforcement, because of reputational concerns or stakeholder expectations. Capturing such motivations would provide a deeper understanding of the conditions under which voluntary environmental action can reinforce the regulatory activity.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this article.

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[6] An alternative specification for the extra willingness to pay, as suggested by one referee, could be $\eta(\omega) = \max\{0, \bar{\eta} - \eta(\delta q - \omega)\}$. For positive abatement levels, this is equivalent to a re-parameterization of the reservation price and the slope of the demand function when the upper bound of $\bar{\eta}$ is sufficiently high; otherwise, it corresponds to a kink in the demand curve.

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Appendix A. Map derivation

Let us focus on the (general) case of partially observable pollution, that is $\zeta_i > 0$. Then, let us note that the damage function (3) is additive with respect to the monopolists, and the damage function of each monopolist does not depend on the action taken by the other monopolists, that is:

$$D\left(q_1, \dots, q_n, \omega_1, \dots, \omega_n, \lambda_1, \dots, \lambda_n\right) = \sum_{i=1}^{n} D_i^m\left(q_i, \omega_i, \lambda_i\right). \tag{36}$$

It follows that to derive the analytical expression of map (11), it is sufficient to derive the damage function for a single monopolist i, denoted by D_i^m.

To this aim, consider the profit function in (6) for a generic monopolist i (production sector i) and the optimization problem (7) in the case of unobservable or nontaxable pollution, that is $\zeta_i > 0$. Looking for an inner solution, the first-order condition for profit maximization with respect to q_i becomes:

$$A_i + \eta_i\left(\omega_i\right) - 2q_i - \delta_i T N_{t-1} = 0, \tag{37}$$

where $A_i = a_i - c_i$. Solving the first-order condition (37) for production level and imposing the non-negativity constraint, we obtain

$$q_i\left(\omega_i; T, N_{t-1}\right) = \max\left\{\frac{1}{2}\left(A_i + \eta_i\left(\omega_i\right) - \delta_i T N_{t-1}\right), 0\right\}. \tag{38}$$

Given the optimal production level, the monopolist i chooses its abatement effort $\left(\omega_i, \lambda_i\right)$ in order to maximize

$$\pi_i^*\left(\omega_i, \lambda_i; T, N_{t-1}\right) = \left[q_i\left(\omega_i; T, N_{t-1}\right)\right]^2 + \omega_i T N_{t-1} - \frac{1}{2}\gamma_i\omega_i^2 - \frac{1}{2}\beta_i\lambda_i^2 \tag{39}$$

in the feasibility region $\left\{\left(\omega_i, \lambda_i\right) \middle| 0 \le \omega_i \le \delta_i q_i\left(\omega_i; T, N_{t-1}\right), 0 \le \lambda_i \le \zeta_i q_i\left(\omega_i; T, N_{t-1}\right)\right\}$. Note that,

$$\frac{\partial \pi_i^*\left(\omega_i, \lambda_i; T, N_{t-1}\right)}{\partial \lambda_i} = -\beta_i\lambda_i < 0 \qquad \forall \lambda_i \in \left[0, \zeta_i q_i\left(\omega_i; T, N_{t-1}\right)\right]. \tag{40}$$

The optimal level of λ_i at any time t is zero and is denoted λ_i^*. The final optimization problem faced by monopolist i is:

$$\arg\max_{\omega_i \in \mathcal{J}_{i,t}^{\omega}} \pi_i^*\left(\omega_i, 0; T, N_{t-1}\right), \tag{41}$$

where $\mathcal{J}_{i,t}^{\omega} := \left\{\omega_i \middle| 0 \le \omega_i \le \delta_i q_i\left(\omega_i; T, N_{t-1}\right)\right\}$. Note that, $q_i\left(\omega_i; T, N_{t-1}\right) > 0$ for all $\omega_i \in \operatorname{int}\left(\mathcal{J}_{i,t}^{\omega}\right)$. Hence, the objective function is differentiable everywhere on $\operatorname{int}\left(\mathcal{J}_{i,t}^{\omega}\right)$, and any candidate solution to the optimization problem in this set must satisfy the first-order condition:

$$\frac{\partial \pi_i^*\left(\omega_i, 0; T, N_{t-1}\right)}{\partial \omega_i} = 0 \Leftrightarrow \omega_i = \frac{T N_{t-1} + \frac{1}{2}\left(A_i + \eta_i\left(\omega_i\right) - \delta_i T N_{t-1}\right)\frac{\partial \eta_i\left(\omega_i\right)}{\partial \omega_i}}{\gamma_i}. \tag{42}$$

In case the consumer offers a constant propensity to pay for green production ($\eta_i\left(\omega_i\right) = \bar{\eta}_i$), the profit function (39) is strictly concave in ω_i (we assume $\gamma_i > 0$) and the optimal production in (38) becomes $q\left(\omega_i; T, N_{t-1}\right) = \max\left\{\frac{1}{2}\left(A_i + \bar{\eta}_i - T N_{t-1}\delta_i\right), 0\right\}$, that is, it does not depend on ω_i and we denote it by $q^*\left(T, N_{t-1}\right)$. Moreover, $\omega_i = \frac{T N_{t-1}}{\gamma_i}$ is the unique solution of the first-order condition (42). Hence, as long as $q^*\left(T, N_{t-1}\right) > 0$ and $\delta_i q^*\left(T, N_{t-1}\right) - \frac{T N_{t-1}}{\gamma_i} > 0$ (the second condition implies the first one), the optimal abatement for monopolist i is proportional to the (risk of a) tax and inversely related to the decreasing return to scale parameter γ_i. More generally, the optimal abatement for monopolist i in case of constant propensity to pay for abatement is

$$\omega_{i,t}^{*,c} = \begin{cases} \frac{T N_{t-1}}{\gamma_i} & \text{if} \quad A_i + \bar{\eta}_i \ge \left(\frac{2}{\gamma_i\delta_i} + \delta_i\right) T N_{t-1} \\ \delta_i q^*\left(T, N_{t-1}\right) & \text{if} \quad \left(\frac{2}{\gamma_i\delta_i} + \delta_i\right) T N_{t-1} > A + \bar{\eta}_i > \delta_i T N_{t-1} \\ 0 & \text{if} \quad \delta_i T N_{t-1} > A + \bar{\eta}_i \end{cases} \tag{43}$$

Thus,

$$D_i^m\left(q_{i,t}^{*,c},\omega_{i,t}^{*,c},\lambda_i^*\right)=\begin{cases}\frac{d\delta_i+s\zeta_i}{2}\left(A_i+\bar{\eta}_i-\delta_i T N_{t-1}\right)-\frac{dT}{\gamma_i}N_{t-1} & \text{if}\quad \frac{A_i+\bar{\eta}_i}{\left(\frac{2}{\gamma_i\delta_i}+\delta_i\right)T}>N_{t-1}\geq 0\\ \frac{s\zeta_i}{2}\left(A_i+\bar{\eta}_i-\delta_i T N_{t-1}\right) & \text{if}\quad \frac{A+\bar{\eta}_i}{\delta_i T}>N_{t-1}\geq\frac{A_i+\bar{\eta}_i}{\left(\frac{2}{\gamma_i\delta_i}+\delta_i\right)T}\\ 0 & \text{if}\quad N_{t-1}\geq\frac{A+\bar{\eta}_i}{\delta_i T}\end{cases}, \tag{44}$$

where the short-hand notation $q_{i,t}^{*,c}$ is used instead of $q^*\left(T,N_{t-1}\right)$. Assume n homogeneous sectors (monopolists), that is, $A_i=A$, $\gamma_i=\gamma$, $\delta_i=\delta$, $\bar{\eta}_i=\bar{\eta}$ and $\zeta_i=\zeta$ for all $i=1,\dots,n$. Using (36), we observe that the damage function is given by $nD^m\left(q_t^{*,c},\omega_t^{*,c},\lambda^*\right)$, where $D^m\left(q_t^{*,c},\omega_t^{*,c},\lambda^*\right)$ is obtained from (44) by removing the subscript. We obtain

$$N_t=\begin{cases}(1-\mu)N_{t-1}+\frac{n(d\delta+s\zeta)}{2}\left(A+\bar{\eta}-\delta T N_{t-1}\right)-\frac{ndT}{\gamma}N_{t-1} & \text{if}\quad \frac{A+\bar{\eta}}{\left(\frac{2}{\gamma\delta}+\delta\right)T}>N_{t-1}\geq 0\\ (1-\mu)N_{t-1}+\frac{ns\zeta}{2}\left(A+\bar{\eta}-\delta T N_{t-1}\right) & \text{if}\quad \frac{A+\bar{\eta}}{\delta T}>N_{t-1}\geq\frac{A+\bar{\eta}}{\left(\frac{2}{\gamma\delta}+\delta\right)T}\\ (1-\mu)N_{t-1} & \text{if}\quad N_{t-1}\geq\frac{A+\bar{\eta}}{\delta T}\end{cases}, \tag{45}$$

In case the consumer offers a propensity to pay for green production that is proportional to abatement ($\eta_i\left(\omega_i\right)=\eta_i\omega_i$, i.e., $\bar{\eta}_i=+\infty$), the optimal abatement itself changes. Here, we derive the level of pollution related to the optimal abatement for all possible parameter configurations.

Lemma 1 (*Monopolist i's Pollution when $\eta_i>0$, $\bar{\eta}_i=+\infty$ and $\zeta_i>0$*). *Assume $\eta_i>0$, $\bar{\eta}_i=+\infty$ and $\zeta_i>0$. Regarding the optimal abatement, we obtain*

$$\omega_{i,t}^{*,d}=\begin{cases}0 & \textit{if}\quad \left(A_i,\eta_i,\gamma_i,\delta_i,T,N_{t-1}\right)\in\mathcal{P}_{0,t}\\ \frac{\eta_i A_i+\left(2-\eta_i\delta_i\right)TN_{t-1}}{2\gamma_i-\eta_i^2} & \textit{if}\quad \left(A_i,\eta_i,\gamma_i,\delta_i,T,N_{t-1}\right)\in\mathcal{P}_{1,t}\\ \delta_i\frac{A_i-\delta_i T N_{t-1}}{2-\eta_i\delta_i} & \textit{if}\quad \left(A_i,\eta_i,\gamma_i,\delta_i,T,N_{t-1}\right)\in\mathcal{P}_{2,t}\\ +\infty & \textit{if}\quad \left(A_i,\eta_i,\gamma_i,\delta_i,T,N_{t-1}\right)\in\mathcal{P}_{3,t}\end{cases}, \tag{46}$$

where $+\infty$ is used to indicate no bounded solution and

$$\begin{aligned}\mathcal{P}_{0,t}:=\Big\{\left(A_i,\eta_i,\gamma_i,\delta_i,T,N_{t-1}\right)\,\Big|\,& 2>\eta_i\delta_i \textit{ and } \delta_i T N_{t-1}>A_i \textit{ OR}\\ & (\eta_i-\gamma_i\delta_i)A_i>(2\eta_i\delta_i-2-\gamma_i\delta_i^2)TN_{t-1}\\ & \textit{and } (2-\gamma_i\delta_i^2)A_i>(2\eta_i\delta_i-2-\gamma_i\delta_i^2)\delta_i T N_{t-1}\\ & \textit{and } A_i<\delta_i T N_{t-1} \textit{ and } 2<\eta_i\delta_i \textit{ and } 2\gamma_i>\eta_i^2\Big\},\end{aligned} \tag{47}$$

$$\begin{aligned}\mathcal{P}_{1,t}:=\Big\{\left(A_i,\eta_i,\gamma_i,\delta_i,T,N_{t-1}\right)\,\Big|\,& (\eta_i-\gamma_i\delta_i)A_i<(2\eta_i\delta_i-2-\gamma_i\delta_i^2)TN_{t-1}\\ & \textit{and } A_i>\delta_i T N_{t-1} \textit{ and } 2>\eta_i\delta_i \textit{ and } 2\gamma_i>\eta_i^2 \textit{ OR}\\ & A_i>\delta_i T N_{t-1} \textit{ and } 2<\eta_i\delta_i \textit{ and } 2\gamma_i>\eta_i^2 \textit{ OR}\\ & (\eta_i-\gamma_i\delta_i)A_i<(2\eta_i\delta_i-2-\gamma_i\delta_i^2)TN_{t-1}\\ & \textit{and } A_i<\delta_i T N_{t-1} \textit{ and } 2<\eta_i\delta_i \textit{ and } 2\gamma_i>\eta_i^2\Big\},\end{aligned} \tag{48}$$

$$\begin{aligned}\mathcal{P}_{2,t}:=\Big\{\left(A_i,\eta_i,\gamma_i,\delta_i,T,N_{t-1}\right)\,\Big|\,& (\eta_i-\gamma_i\delta_i)A_i>(2\eta_i\delta_i-2-\gamma_i\delta_i^2)TN_{t-1}\\ & \textit{and } A_i>\delta_i T N_{t-1} \textit{ and } 2>\eta_i\delta_i \textit{ and } 2\gamma_i>\eta_i^2 \textit{ OR}\\ & (\eta_i-\gamma_i\delta_i)A_i>(2\eta_i\delta_i-2-\gamma_i\delta_i^2)TN_{t-1}\\ & \textit{and } (2-\gamma_i\delta_i^2)A_i<(2\eta_i\delta_i-2-\gamma_i\delta_i^2)\delta_i T N_{t-1}\\ & \textit{and } A_i<\delta_i T N_{t-1} \textit{ and } 2<\eta_i\delta_i \textit{ and } 2\gamma_i>\eta_i^2 \textit{ OR}\\ & A_i>\delta_i T N_{t-1} \textit{ and } 2>\eta_i\delta_i \textit{ and } 2\gamma_i<\eta_i^2\Big\},\end{aligned} \tag{49}$$

$$\mathcal{P}_{3,t}:=\left\{\left(A_i,\eta_i,\gamma_i,\delta_i,T,N_{t-1}\right)\,\middle|\,2<\eta_i\delta_i \textit{ and } 2\gamma_i<\eta_i^2\right\}. \tag{50}$$

Regarding the level of pollution related to the optimal abatement in (46), two cases occur:

- *If $2\gamma_i>\eta_i^2$,*

  - *and* $2 > \eta_i\delta_i$*, one has:*

$$\delta_i q_{i,t}^{*,d} - \omega_{i,t}^{*,d} = \begin{cases} \frac{2(\eta_i\delta_i-1)-\gamma_i\delta_i^2}{2\gamma_i-\eta_i^2} T N_{t-1} + \frac{A_i(\gamma_i\delta_i-\eta_i)}{2\gamma_i-\eta_i^2} & \text{if} \quad A_i > \delta_i T N_{t-1} \\ & \text{and} \ \left(\eta_i - \gamma_i\delta_i\right) A_i < \left(2\eta_i\delta_i - 2 - \gamma_i\delta_i^2\right) T N_{t-1} \\ 0 & \text{otherwise} \end{cases}, \tag{51}$$

and

$$\zeta_i q_{i,t}^{*,d} - \lambda_i^* = \begin{cases} \zeta_i \frac{\gamma_i A_i + (\eta_i - \gamma_i\delta_i) T N_{t-1}}{2\gamma_i-\eta_i^2} & \text{if} \quad A_i > \delta_i T N_{t-1} \\ & \text{and} \ \left(\eta_i - \gamma_i\delta_i\right) A_i < \left(2\eta_i\delta_i - 2 - \gamma_i\delta_i^2\right) T N_{t-1} \\ \zeta_i \frac{A_i - \delta_i T N_{t-1}}{2-\eta_i\delta_i} & \text{if} \quad A_i > \delta_i T N_{t-1} \\ & \text{and} \ \left(\eta_i - \gamma_i\delta_i\right) A_i > \left(2\eta_i\delta_i - 2 - \gamma_i\delta_i^2\right) T N_{t-1} \\ 0 & \text{otherwise} \end{cases}, \tag{52}$$

  - *and* $2 < \eta_i\delta_i$*, one has:*

$$\delta_i q_{i,t}^{*,d} - \omega_{i,t}^{*,d} = \begin{cases} \frac{2(\eta_i\delta_i-1)-\gamma_i\delta_i^2}{2\gamma_i-\eta_i^2} T N_{t-1} + \frac{A_i(\gamma_i\delta_i-\eta_i)}{2\gamma_i-\eta_i^2} & \text{if} \quad A_i > \delta_i T N_{t-1} \ \text{or} \ A_i < \delta_i T N_{t-1} \\ & \text{and} \ \left(\eta_i - \gamma_i\delta_i\right) A_i < \left(2\eta_i\delta_i - 2 - \gamma_i\delta_i^2\right) T N_{t-1} \\ 0 & \text{otherwise} \end{cases} \tag{53}$$

and

$$\zeta_i q_{i,t}^{*,d} - \lambda_i^* = \begin{cases} \zeta_i \frac{\gamma_i A_i + (\eta_i - \gamma_i\delta_i) T N_{t-1}}{2\gamma_i-\eta_i^2} & \text{if} \quad A_i > \delta_i T N_{t-1} \ \text{or} \ A_i < \delta_i T N_{t-1} \\ & \text{and} \ \left(\eta_i - \gamma_i\delta_i\right) A_i < \left(2\eta_i\delta_i - 2 - \gamma_i\delta_i^2\right) T N_{t-1} \\ \zeta_i \frac{A_i - \delta_i T N_{t-1}}{2-\eta_i\delta_i} & \text{if} \quad A_i < \delta_i T N_{t-1} \\ & \text{and} \ \left(\eta_i - \gamma_i\delta_i\right) A_i > \left(2\delta_i\eta_i - 2 - \gamma_i\delta_i^2\right) T N_{t-1} \\ & \text{and} \ \left(2 - \gamma_i\delta_i^2\right) A_i < \left(2\eta_i\delta_i - 2 - \gamma_i\delta_i^2\right) \delta_i T N_{t-1} \\ 0 & \text{otherwise} \end{cases} \tag{54}$$

- *If* $2\gamma_i < \eta_i^2$,
  - *and* $2 > \eta_i\delta_i$*, one has:*

$$\delta_i q_{i,t}^{*,d} - \omega_{i,t}^{*,d} = 0 \tag{55}$$

and

$$\zeta_i q_{i,t}^{*,d} - \lambda_i^* = \begin{cases} \zeta_i \frac{A_i - \delta_i T N_{t-1}}{2-\eta_i\delta_i} & \text{if } N_{t-1} < \frac{A_i}{T\delta_i} \\ 0 & \text{otherwise} \end{cases} \tag{56}$$

  - *for* $2 < \eta_i\delta_i$*, unbounded solutions arise.*

where $q_{i,t}^{*,d}$ *is used as a short-hand notation for* $q_i\left(\omega_{i,t}^{*,d}; T, N_{t-1}\right)$.

Proof of Lemma 1. Knowing that the optimal abatement of unobservable pollution is always 0 and is indicated by λ_i^*, let us derive the optimal abatement of observable pollution at time t, when $\eta_i\left(\omega_i\right) = \eta_i\omega_i$ (i.e., $\bar{\eta}_i = +\infty$), denoted $\omega_{i,t}^{*,d}$. Immediately, we obtain that the set of feasible values of ω_i is:

$$\mathcal{J}_{i,t}^{\omega} := \left\{ \omega_i \middle| 0 \le \omega_i \le \delta_i q_i\left(\omega_i; T, N_{t-1}\right) \right\} = \begin{cases} \left[0, \delta_i \frac{A_i - \delta_i T N_{t-1}}{2-\eta_i\delta_i}\right] & \text{if} \quad A_i > \delta_i T N_{t-1} \text{ and } 2 > \eta_i\delta_i \\ \{0\} & \text{if} \quad A_i < \delta_i T N_{t-1} \text{ and } 2 > \eta_i\delta_i \\ [0, +\infty) & \text{if} \quad A_i > \delta_i T N_{t-1} \text{ and } 2 < \eta_i\delta_i \\ \{0\} \cup \left[\delta_i \frac{A_i - \delta_i T N_{t-1}}{2-\eta_i\delta_i}, +\infty\right) & \text{if} \quad A_i < \delta_i T N_{t-1} \text{ and } 2 < \eta_i\delta_i \end{cases}. \tag{57}$$

Then, if $A_i < \delta_i T N_{t-1}$ and $2 > \eta_i\delta_i$, the optimal abatement is $\omega_{i,t}^{*,d} = 0$. In all other cases, the optimal abatement, if it exists, is either a boundary solution or an interior solution. Note that the objective function is smooth in the interior of $\mathcal{J}_{i,t}^{\omega}$. Indeed, non-differentiability arises only when production is equal to zero, which implies that abatement ω_i is also zero, that is a boundary point of $\mathcal{J}^{\omega}_{i,t}$. It follows that an interior solution has to satisfy the first-order condition and the objective function has to be strictly concave, otherwise only a boundary solution can exist. Computing and studying the sign of its second derivative, the objective function $\pi^*_i\left(\omega_i, 0; T, N_{t-1}\right)$ is strictly concave in any ω_i in the interior of $\mathcal{J}^{\omega}_{i,t}$ for $2\gamma_i > \eta_i^2$ and it is strictly convex for $2\gamma_i < \eta_i^2$. In the first case, an interior solution can exist, in the second case only boundary solutions. Let us consider the first case: $2\gamma_i > \eta_i^2$. The unique value of ω_i that solves the first order condition in (42) with $\eta_i\left(\omega_i\right) = \eta_i\omega_i$ is

$$\omega^{+}_{i,t} := \frac{\eta_i A_i + \left(2 - \eta_i\delta_i\right) T N_{t-1}}{2\gamma_i - \eta_i^2}, \tag{58}$$

which is the optimal value of abatement for $\pi^*_i\left(\omega_i, 0; T, N_{t-1}\right)$ in $\mathbb{R}$, instead of zero, when $\omega^{+}_{i,t}$ as well as

$$q_i\left(\omega^{+}_{i,t}; T, N_{t-1}\right) = \max\left\{\frac{\gamma_i A_i + \left(\eta_i - \delta_i\gamma_i\right) T N_{t-1}}{2\gamma_i - \eta_i^2}; 0\right\} \tag{59}$$

are positive. As $2\gamma_i > \eta_i^2$, note that $\omega^{+}_{i,t} \leq 0$ implies $q_i\left(\omega^{+}_{i,t}; T, N_{t-1}\right) = 0$. Let us now consider the optimal solution in $\mathcal{J}^{\omega}_{i,t} \subset \mathbb{R}$. Consider $A_i > \delta_i T N_{t-1}$ and $2 > \eta_i\delta_i$. Then, the feasible set $\mathcal{J}^{\omega}_{i,t}$ is $\left[0, \delta_i q_i\left(\omega^{*,d}_{i,t}; T, N_{t-1}\right)\right]$ as shown (57). Imposing the feasibility condition, the optimal abatement is: (1) $\omega^{*,d}_{i,t} = 0$ when $\omega^{+}_{i,t} < 0$ (hence, $q_i\left(\omega^{+}_{i,t}; T, N_{t-1}\right) = 0$), that is, when

$$\eta_i A_i < \left(\eta_i\delta_i - 2\right) T N_{t-1}, \text{ implying } \eta_i\delta_i > 2 \text{ and } A_i < \delta_i T N_{t-1}; \tag{60}$$

(2) $\omega^{+}_{i,t}$ when $0 < \omega^{+}_{i,t} < \delta_i q_i\left(\omega^{+}_{i,t}; T, N_{t-1}\right)$, that is, when

$$\eta_i A_i > \left(\eta_i\delta_i - 2\right) T N_{t-1} \text{ and } \left(\gamma_i\delta_i - \eta_i\right) A_i > \left(2 + \delta_i\left(\gamma_i\delta_i - 2\eta_i\right)\right) T N_{t-1}; \tag{61}$$

(3) the positive solution of $\omega_i = \delta_i q_i\left(\omega_i; T, N_{t-1}\right)$ when $\omega^{+}_{i,t} > \delta_i q_i\left(\omega^{+}_{i,t}; T, N_{t-1}\right)$ and $\omega^{+}_{i,t} > 0$, that is

$$\omega^{*,d}_{i,t} = \delta_i \frac{A_i - T N_{t-1}\delta_i}{2 - \eta_i\delta_i} \text{ when } \eta_i A_i > \left(\eta_i\delta_i - 2\right) T N_{t-1} \text{ and } \left(\gamma_i\delta_i - \eta_i\right) A_i < \left(2 + \delta_i\left(\gamma_i\delta_i - 2\eta_i\right)\right) T N_{t-1}, \tag{62}$$

noting that $\omega^{*,d}_{i,t}$ in (62) is positive for $A_i - \delta_i T N_{t-1} > 0$ and $2 - \eta_i\delta_i > 0$. Hence,

$$\omega^{*,d}_{i,t} = \begin{cases} \frac{\eta_i A_i + \left(2-\eta_i\delta_i\right) T N_{t-1}}{2\gamma_i - \eta_i^2} & \text{if} \quad \gamma_i\delta_i A_i + \delta_i\left(\eta_i - \gamma_i\delta_i\right) T N_{t-1} > \eta_i A_i - \left(\eta_i\delta_i - 2\right) T N_{t-1} > 0 \\ & \quad\quad \text{and } A_i > \delta_i T N_{t-1} \text{ and } 2 > \eta_i\delta_i \text{ and } 2\gamma_i > \eta_i^2 \\ \delta_i \frac{A_i - \delta_i T N_{t-1}}{2 - \eta_i\delta_i} & \text{if} \quad \eta_i A_i - \left(\eta_i\delta_i - 2\right) T N_{t-1} > \gamma_i\delta_i A_i + \delta_i\left(\eta_i - \gamma_i\delta_i\right) T N_{t-1} \\ & \quad\quad \text{and } A_i > \delta_i T N_{t-1} \text{ and } 2 > \eta_i\delta_i \text{ and } 2\gamma_i > \eta_i^2 \end{cases}. \tag{63}$$

Consider $A_i > \delta_i T N_{t-1}$ and $2 < \eta_i\delta_i$. Then, the feasible set $\mathcal{J}^{\omega}_{i,t}$ is $[0, +\infty)$ as shown (57). Consider again the abatement value $\omega^{+}_{i,t}$ in (58) that solves the first-order condition. Imposing the feasibility condition that abatement cannot be outside the range $[0, +\infty)$, the optimal abatement $\omega^{*,d}_{i,t}$ is equal to zero when $\omega^{+}_{i,t} < 0$, which cannot be for $A_i > \delta_i T N_{t-1}$, and is $\omega^{+}_{i,t}$ otherwise. Thus,

$$\omega^{*,d}_{i,t} = \left\{ \frac{\eta_i A_i + \left(2-\eta_i\delta_i\right) T N_{t-1}}{2\gamma_i - \eta_i^2} \quad \text{if} \quad A_i > \delta_i T N_{t-1} \text{ and } 2 < \eta_i\delta_i \text{ and } 2\gamma_i > \eta_i^2 \right. . \tag{64}$$

Consider $A_i < \delta_i T N_{t-1}$ and $2 < \eta_i\delta_i$. Then, $\mathcal{J}^{\omega}_{i,t} = \{0\} \cup \left[\delta_i \frac{A_i - \delta_i T N_{t-1}}{2 - \eta_i\delta_i}, +\infty\right)$ is not a convex set and the abatement value $\omega^{+}_{i,t}$ in (58) that solves the first-order condition is the optimal abatement level when is an interior solution, that is, when $\omega^{+}_{i,t} > \delta_i \frac{A_i - \delta_i T N_{t-1}}{2 - \eta_i\delta_i}$. Otherwise, the optimal abatement is a boundary solution either 0 or $\delta_i \frac{A_i - \delta_i T N_{t-1}}{2 - \eta_i\delta_i}$. Note that the optimal production in $\omega_i = 0$ is zero, so is profit, when $A_i < \delta_i T N_{t-1}$ and $2 < \eta_i\delta_i$. Then, the boundary solution $\delta_i \frac{A_i - \delta_i T N_{t-1}}{2 - \eta_i\delta_i}$ is preferable to the other boundary solution 0, if and only if $\pi^*_i\left(\delta_i \frac{A_i - \delta_i T N_{t-1}}{2 - \eta_i\delta_i}, 0; T, N_{t-1}, 0\right) > 0$, which holds if and only if $\left(2 - \gamma_i\delta_i^2\right) A < \left(2\eta_i\delta_i - 2 - \gamma_i\delta_i^2\right)\delta_i T N$. Hence,

$$\omega^{*,d}_{i,t} = \begin{cases} 0 & \text{if} \quad \left(\eta_i - \gamma_i\delta_i\right) A_i > \left(2\delta_i\eta_i - 2 - \gamma_i\delta_i^2\right) T N_{t-1} \\ & \quad\quad \text{and } \left(2 - \gamma_i\delta_i^2\right) A_i > \left(2\eta_i\delta_i - 2 - \gamma_i\delta_i^2\right)\delta_i T N \\ & \quad\quad \text{and } A_i < \delta_i T N_{t-1} \text{ and } 2 < \eta_i\delta_i \text{ and } 2\gamma_i > \eta_i^2 \\ \delta_i \frac{A_i - \delta_i T N_{t-1}}{\delta_i\left(2 - \eta_i\delta_i\right)} & \text{if} \quad \left(\eta_i - \gamma_i\delta_i\right) A_i > \left(2\delta_i\eta_i - 2 - \gamma_i\delta_i^2\right) T N_{t-1} \\ & \quad\quad \text{and } \left(2 - \gamma_i\delta_i^2\right) A_i < \left(2\eta_i\delta_i - 2 - \gamma_i\delta_i^2\right)\delta_i T N \\ & \quad\quad \text{and } A_i < \delta_i T N_{t-1} \text{ and } 2 < \eta_i\delta_i \text{ and } 2\gamma_i > \eta_i^2 \\ \frac{\eta_i A_i + \left(2-\eta_i\delta_i\right) T N_{t-1}}{2\gamma_i - \eta_i^2} & \text{if} \quad \left(\eta_i - \gamma_i\delta_i\right) A_i < \left(2\delta_i\eta_i - 2 - \gamma_i\delta_i^2\right) T N_{t-1} \\ & \quad\quad \text{and } A_i < \delta_i T N_{t-1} \text{ and } 2 < \eta_i\delta_i \text{ and } 2\gamma_i > \eta_i^2 \end{cases}. \tag{65}$$

Consider $2\gamma_i < \eta_i^2$. Then $\pi^*_i\left(\omega_i, 0; T, N_{t-1}\right)$ is convex in ω_i and the optimal abatement is attained on the boundary of the feasible set $\mathcal{J}^{\omega}_{i,t}$. As shown in (57), $\mathcal{J}^{\omega}_{i,t}$ is unbounded for $2 - \eta_i\delta_i < 0$, and $\pi^*_i\left(\omega_i, 0; T, N_{t-1}\right) \to +\infty$ as $\omega_i \to +\infty$. It follows that the optimal

Table 3
Combinations of the signs of the relevant quantities, depending on the model parameters, used to construct the map. The last column reports feasibility: P = possible, NP = not possible, NA = no by assumption.

	$2\gamma_i - \eta_i^2$	$2 - \eta_i\delta_i$	$\gamma_i\delta_i - \eta_i$	$2 - \delta_i\left(2\eta_i - \gamma_i\delta_i\right)$	
CASE 1	–	+	/	/	P
CASE 2	–	–	/	/	NA
CASE 3	+	+	+	+	P
CASE 4	+	–	+	+	P
CASE 5	+	–	–	+	NP
CASE 6	+	–	–	–	NP
CASE 7	+	+	–	–	NP
CASE 8	+	+	+	–	NP
CASE 9	+	+	–	+	P
CASE 10	+	–	+	–	NP

solution does not exist and we set $+\infty$ as the optimal solution by convention. Instead, $\mathcal{J}_{i,t}^{\omega}$ is a bounded set for $2-\eta_i\delta_i > 0$, that is, is the singleton $\{0\}$ when $A_i < \delta_i T N_{t-1}$, whence 0 is the optimal ω_i, and is the interval $\left[0, \delta_i \frac{A_i - \delta_i T N_{t-1}}{2-\eta_i\delta_i}\right]$ otherwise, see again (57). Note that $q_i\left(0; T, N_{t-1}\right) \geq 0$, which implies

$$\frac{\partial \pi_i^*\left(0, 0; T, N_{t-1}\right)}{\partial \omega_i} := \eta_i q_i\left(0; T, N_{t-1}\right) + T N_{t-1} > 0. \tag{66}$$

It follows that the optimal abatement is $\delta_i \frac{A_i - \delta_i T N_{t-1}}{2-\eta_i\delta_i}$. Thus,

$$\omega_{i,t}^{*,d} = \begin{cases} \delta_i \frac{A_i - \delta_i T N_{t-1}}{2-\eta_i\delta_i} & \text{if} \quad A_i > \delta_i T N_{t-1} \text{ and } 2 > \eta_i\delta_i \text{ and } 2\gamma_i < \eta_i^2 \\ 0 & \text{if} \quad A_i < \delta_i T N_{t-1} \text{ and } 2 > \eta_i\delta_i \text{ and } 2\gamma_i < \eta_i^2 \\ +\infty & \text{if} \quad 2 < \eta_i\delta_i \text{ and } 2\gamma_i < \eta_i^2 \end{cases}, \tag{67}$$

and $\omega_{i,t}^{*,d}/\delta_i$ is the optimal level of production. Hence, (46) follows. Knowing that optimal abatement of unobservable pollution is always $\lambda_i^* = 0$, one has that $\zeta_i q_i\left(\omega_{i,t}^{*,d}; T, N_{t-1}\right) - \lambda_i^* = \zeta_i q_i\left(\omega_{i,t}^{*,d}; T, N_{t-1}\right)$. ☐

The level of pollution in Lemma 1, resulting from the optimal levels of production and abatement, can be used to derive the explicit form of map (4) when $\zeta_i > 0$. This form depends on the sign of the following quantities: (1) $2\gamma_i - \eta_i^2$; (2) $2 - \eta_i\delta_i$; (3) $\gamma_i\delta_i - \eta_i$; (4) $2 - \delta_i\left(2\eta_i - \gamma_i\delta_i\right)$. The relevant combinations of the signs of these quantities are summarized in Table 3, which reports the ten possible cases analyzed below. We start with CASE 1. If $2\gamma_i < \eta_i^2$ and $2 > \eta_i\delta_i$, by Lemma 1, we have $\delta_i q_{i,t}^{*,d} - \omega_{i,t}^{*,d} = 0$ and $\zeta_i q_i\left(\omega_{i,t}^{*,d}; T, N_{t-1}\right) - \lambda_i^*$ as in (56). Then, assuming n homogeneous sectors (monopolists), that is, $A_i = A$, $\gamma_i = \gamma$, $\delta_i = \delta$, $\eta_i = \eta$ and $\zeta_i = \zeta$ for all $i = 1, \ldots, n$ and using the additivity property with respect to single sector (monopolist) of the damage function, one has

$$N_t = \begin{cases} \left(1-\mu\right) N_{t-1} + ns\zeta \frac{A - \delta T N_{t-1}}{2-\eta\delta} & \text{if } N_{t-1} < \frac{A}{\delta T} \\ \left(1-\mu\right) N_{t-1} & \text{otherwise} \end{cases}, \tag{68}$$

which the analytical expression of the map for CASE 1. If $2\gamma_i < \eta_i^2$ and $2 < \eta_i\delta_i$, by Lemma 1, we do not have a solution to the optimization problem and this parameter configuration, denoted CASE 2, is excluded by assumption. If $2\gamma_i > \eta_i^2$, eight possible sub-cases occur. CASE 3: (1) $2\gamma_i > \eta_i^2$; (2) $2 > \eta_i\delta_i$; (3) $\gamma_i\delta_i - \eta_i > 0$; (4) $2 > \delta_i\left(2\eta_i - \gamma_i\delta_i\right)$. Thus,

$$0 < \frac{\left(\gamma_i\delta_i - \eta_i\right) A_i}{\left(2 + \delta_i\left(\gamma_i\delta_i - 2\eta_i\right)\right) T} < \frac{A_i}{\delta_i T}. \tag{69}$$

From (51) it holds that

$$\delta_i q_{i,t}^{*,d} - \omega_{i,t}^{*,d} = \begin{cases} \frac{A_i\left(\delta_i\gamma_i - \eta_i\right)}{2\gamma_i - \eta_i^2} + \frac{2\left(\eta_i\delta_i - 1\right) - \delta_i^2\gamma_i}{2\gamma_i - \eta_i^2} T N_{t-1} & \text{if} \quad 0 < N_{t-1} < \frac{\left(\gamma_i\delta_i - \eta_i\right) A_i}{\left(2 + \delta_i\left(\gamma_i\delta_i - 2\eta_i\right)\right) T} \\ 0 & \text{if} \quad \frac{\left(\gamma_i\delta_i - \eta_i\right) A_i}{\left(2 + \delta_i\left(\gamma_i\delta_i - 2\eta_i\right)\right) T} < N_{t-1} \end{cases}, \tag{70}$$

and from (52) one has that

$$\zeta_i q_{i,t}^{*,d} - \lambda_i^* = \begin{cases} \zeta_i \frac{\gamma_i A_i + (\eta_i - \delta_i \gamma_i) T N_{t-1}}{2\gamma_i - \eta_i^2} & \text{if} \quad 0 < N_{t-1} < \frac{(\gamma_i \delta_i - \eta_i) A_i}{(2 + \delta_i (\gamma_i \delta_i - 2\eta_i)) T} \\ \zeta_i \frac{A_i - \delta_i T N_{t-1}}{2 - \eta_i \delta_i} & \text{if} \quad \frac{(\gamma_i \delta_i - \eta_i) A_i}{(2 + \delta_i (\gamma_i \delta_i - 2\eta_i)) T} < N_{t-1} < \frac{A_i}{\delta_i T} \\ 0 & \text{if} \quad \frac{A_i}{\delta_i T} < N_{t-1} \end{cases}. \tag{71}$$

Moreover, assume n homogeneous sectors (monopolists), that is, $A_i = A$, $\gamma_i = \gamma$, $\delta_i = \delta$, $\eta_i = \eta$ and $\zeta_i = \zeta$ for all $i = 1, \ldots, n$. Using (36), the damage function is given by $nD^m \left(q_t^{*,d}, \omega_t^{*,d}, \lambda^* \right)$, where $D^m \left(q_t^{*,d}, \omega_t^{*,d}, \lambda^* \right)$ is obtained by summing up (70) and (71) once subscript i is removed. Thus, we conclude that

$$N_t = \begin{cases} N_{t-1}(1-\mu) + nd \left(\frac{A(\gamma\delta - \eta)}{2\gamma - \eta^2} + \frac{2(\eta\delta - 1) - \gamma\delta^2}{2\gamma - \eta^2} T N_{t-1} \right) & \\ \qquad + ns\zeta \frac{\gamma A + (\eta - \gamma\delta) T N_{t-1}}{2\gamma - \eta^2} & \text{if} \quad 0 < N_{t-1} < \frac{(\gamma\delta - \eta) A}{(2 + \delta(\gamma\delta - 2\eta)) T} \\ N_{t-1}(1-\mu) + ns\zeta \frac{A - \delta T N_{t-1}}{2 - \eta\delta} & \text{if} \quad \frac{(\gamma\delta - \eta) A}{(2 + \delta(\gamma\delta - 2\eta)) T} < N_{t-1} < \frac{A}{\delta T} \\ N_{t-1}(1-\mu) & \text{if} \quad \frac{A}{\delta T} < N_{t-1} \end{cases}. \tag{72}$$

CASE 4: (1) $2\gamma_i > \eta_i^2$; (2) $2 < \eta_i \delta_i$; (3) $\gamma_i \delta_i - \eta_i > 0$; (4) $2 > \delta_i \left(2\eta_i - \gamma_i \delta_i \right)$. Hence, $2 < \gamma_i \delta_i^2$ (as implied by $2 < \eta_i \delta_i$ and $2\gamma_i > \eta_i^2$) and

$$0 < \frac{A_i}{\delta_i T} < \frac{\left(\gamma_i \delta_i - \eta_i \right) A_i}{\left(2 + \delta_i \left(\gamma_i \delta_i - 2\eta_i \right) \right) T} < \frac{\left(\gamma_i \delta_i^2 - 2 \right) A_i}{\left(2 + \delta_i \left(\gamma_i \delta_i - 2\eta_i \right) \right) \delta_i T}. \tag{73}$$

From (53) we conclude that

$$\delta_i q_{i,t}^{*,d} - \omega_{i,t}^{*,d} = \begin{cases} \frac{A_i (\delta_i \gamma_i - \eta_i)}{2\gamma_i - \eta_i^2} + \frac{2(\eta_i \delta_i - 1) - \delta_i^2 \gamma_i}{2\gamma_i - \eta_i^2} T N_{t-1} & \text{if} \quad 0 \le N_{t-1} < \frac{(\gamma_i \delta_i - \eta_i) A_i}{(2 + \delta_i (\gamma_i \delta_i - 2\eta_i)) T} \\ 0 & \text{if} \quad \frac{(\gamma_i \delta_i - \eta_i) A_i}{(2 + \delta_i (\gamma_i \delta_i - 2\eta_i)) T} < N_{t-1} \end{cases}, \tag{74}$$

and from (54), that

$$\zeta_i q_{i,t}^{*,d} - \lambda_i^* = \begin{cases} \zeta_i \frac{\gamma_i A_i + (\eta_i - \delta_i \gamma_i) T N_{t-1}}{2\gamma_i - \eta_i^2} & \text{if} \quad 0 \le N_{t-1} < \frac{(\gamma_i \delta_i - \eta_i) A_i}{(2 + \delta_i (\gamma_i \delta_i - 2\eta_i)) T} \\ \zeta_i \frac{A_i - \delta_i T N_{t-1}}{2 - \eta_i \delta_i} & \text{if} \quad \frac{(\gamma_i \delta_i - \eta_i) A_i}{(2 + \delta_i (\gamma_i \delta_i - 2\eta_i)) T} \le N_{t-1} < \frac{\left(\gamma_i \delta_i^2 - 2 \right) A_i}{(2 + \delta_i (\gamma_i \delta_i - 2\eta_i)) \delta_i T} \\ 0 & \text{if} \quad \frac{\left(\gamma_i \delta_i^2 - 2 \right) A_i}{(2 + \delta_i (\gamma_i \delta_i - 2\eta_i)) \delta_i T} \le N_{t-1} \end{cases}. \tag{75}$$

Moreover, assume n homogeneous sectors (monopolists), that is, $A_i = A$, $\gamma_i = \gamma$, $\delta_i = \delta$, $\eta_i = \eta$ and $\zeta_i = \zeta$ for all $i = 1, \ldots, n$. Using (36), the damage function is given by $nD^m \left(q_t^{*,d}, \omega_t^{*,d}, \lambda^* \right)$, where $D^m \left(q_t^{*,d}, \omega_t^{*,d}, \lambda^* \right)$ is obtained by summing up (74) and (75) once subscript i is removed. Thus, we conclude that

$$N_t = \begin{cases} N_{t-1}(1-\mu) + nd \left(\frac{A(\gamma\delta - \eta)}{2\gamma - \eta^2} + \frac{2(\eta\delta - 1) - \gamma\delta^2}{2\gamma - \eta^2} T N_{t-1} \right) & \\ \qquad + ns\zeta \frac{\gamma A + (\eta - \gamma\delta) T N_{t-1}}{2\gamma - \eta^2} & \text{if} \quad 0 \le N_{t-1} < \frac{(\gamma\delta - \eta) A}{(2 + \delta(\gamma\delta - 2\eta)) T} \\ N_{t-1}(1-\mu) + ns\zeta \frac{A - \delta T N_{t-1}}{2 - \eta\delta} & \text{if} \quad \frac{(\gamma\delta - \eta) A}{(2 + \delta(\gamma\delta - 2\eta)) T} \le N_{t-1} < \frac{(\gamma\delta^2 - 2) A}{(2 + \delta(\gamma\delta_i - 2\eta)) \delta T} \\ N_{t-1}(1-\mu) & \text{if} \quad \frac{(\gamma\delta^2 - 2) A}{(2 + \delta(\gamma\delta_i - 2\eta)) \delta T} \le N_{t-1} \end{cases}. \tag{76}$$

CASE 5: (1) $2\gamma_i > \eta_i^2$; (2) $2 < \eta_i \delta_i$; (3) $\gamma_i \delta_i - \eta_i < 0$; (4) $2 > \delta_i \left(2\eta_i - \gamma_i \delta_i \right)$. This case is not possible, indeed, (1) $2\gamma_i > \eta_i^2$; (2) $2 < \eta_i \delta_i$ implies $\gamma_i \delta_i^2 > 2$. At the same time, (1) $2\gamma_i > \eta_i^2$ and (3) $\gamma_i \delta_i - \eta_i < 0$, implies $2 > \delta_i^2 \gamma_i$. CASE 6: (1) $2\gamma_i > \eta_i^2$; (2) $2 < \eta_i \delta_i$; (3) $\gamma_i \delta_i - \eta_i < 0$; (4) $2 < \delta_i \left(2\eta_i - \gamma_i \delta_i \right)$. It is impossible for the same reasons as it is impossible CASE 5. CASE 7: (1) $2\gamma_i > \eta_i^2$; (2) $2 > \eta_i \delta_i$; (3) $\gamma_i \delta_i - \eta_i < 0$; (4) $2 < \delta_i \left(2\eta_i - \gamma_i \delta_i \right)$. Note that the right-hand side of (4), that is of $2 < \delta_i \left(2\eta_i - \gamma_i \delta_i \right)$, has its maximum in $\delta_i = \frac{\eta_i}{\gamma_i}$, that is $2 < \delta_i \left(2\eta_i - \gamma_i \delta_i \right) < \frac{\eta_i^2}{\gamma_i}$. This contradicts (2) $2\gamma_i > \eta_i^2$. Hence, this case is not possible. CASE 8: (1) $2\gamma_i > \eta_i^2$; (2) $2 > \eta_i \delta_i$; (3) $\gamma_i \delta_i - \eta_i > 0$; (4) $2 < \delta_i \left(2\eta_i - \gamma_i \delta_i \right)$. Note that (4) $2 < \delta_i \left(2\eta_i - \gamma_i \delta_i \right)$ and (3) $\gamma_i \delta_i > \eta_i$ implies $2 < \eta_i \delta_i$ which contradicts (2) $2 > \eta_i \delta_i$. So it is not possible. CASE 9: (1) $2\gamma_i > \eta_i^2$; (2) $2 > \eta_i\delta_i$; (3) $\gamma_i\delta_i - \eta_i < 0$; (4) $2 > \delta_i\left(2\eta_i - \gamma_i\delta_i\right)$. Thus,

$$0 > \frac{\left(\gamma_i\delta_i - \eta_i\right)A_i}{\left(2 + \delta_i\left(\gamma_i\delta_i - 2\eta_i\right)\right)T}, \tag{77}$$

and $\left(\gamma_i\delta_i - \eta_i\right)A_i > \left(2 + \delta_i\left(\gamma_i\delta_i - 2\eta_i\right)\right)TN_{t-1}$ is never satisfied. From (51) it holds that

$$\delta_i q_{i,t}^{*,d} - \omega_{i,t}^{*,d} \quad = \quad 0 \ , \tag{78}$$

and from (52) that

$$\zeta_i q_{i,t}^{*,d} - \lambda_i^* = \begin{cases} \zeta_i \frac{A_i - \delta_i T N_{t-1}}{2 - \eta_i\delta_i} & \text{if} \quad N_{t-1} < \frac{A_i}{\delta_i T} \\ 0 & \text{if} \quad N_{t-1} > \frac{A_i}{\delta_i T} \end{cases} . \tag{79}$$

Moreover, assume n homogeneous sectors (monopolists), that is, $A_i = A$, $\gamma_i = \gamma$, $\delta_i = \delta$, $\eta_i = \eta$ and $\zeta_i = \zeta$ for all $i = 1, \dots, n$. Using (36), the damage function is given by $nD^m\left(q_t^{*,d}, \omega_t^{*,d}, \lambda^*\right)$, where $D^m\left(q_t^{*,d}, \omega_t^{*,d}, \lambda^*\right)$ is obtained by summing up (78) and (79) once subscript i is removed. Thus, we conclude that

$$N_t = \begin{cases} N_{t-1}\left(1-\mu\right) + ns\zeta\frac{A - \delta T N_{t-1}}{2 - \eta\delta} & \text{if} \quad N_{t-1} < \frac{A}{\delta T} \\ N_{t-1}\left(1-\mu\right) & \text{if} \quad N_{t-1} > \frac{A}{\delta T} \end{cases} . \tag{80}$$

CASE 10: (1) $2\gamma_i > \eta_i^2$; (2) $2 < \eta_i\delta_i$; (3) $\gamma_i\delta_i - \eta_i > 0$; (4) $2 < \delta_i\left(2\eta_i - \gamma_i\delta_i\right)$. Note that (4) $2 < \delta_i\left(2\eta_i - \gamma_i\delta_i\right) < \frac{\eta_i^2}{\gamma_i}$, that is, (4) implies $2\gamma_i < \eta_i^2$, which contradicts $2\gamma_i > \eta_i^2$. Thus, this case is not possible. Finally, note that to derive the maps when pollution is fully observable, it is sufficient to set $\zeta_i = 0$ in the above steps.

Appendix B. Technical proofs

This appendix contains proofs of propositions, lemmas, and corollaries excluding those in Appendix A.

Proof of Proposition 1. As shown in Appendix A, map (45) is obtained from (11) assuming $\eta_i\left(\omega_i\right) = \bar{\eta}$ for all i and n homogeneous sectors. Further assuming $\zeta = 0$ and $\bar{\eta} = 0$, map (45) becomes:

$$N_t = \begin{cases} \left(1-\mu\right)N_{t-1} + \frac{nd\delta}{2}\left(A - \delta T N_{t-1}\right) - \frac{ndT}{\gamma}N_{t-1} & \text{if} \quad \frac{A}{\left(\frac{2}{\gamma\delta}+\delta\right)T} > N_{t-1} \geq 0 \\ \left(1-\mu\right)N_{t-1} & \text{if} \quad N_{t-1} \geq \frac{A}{\left(\frac{2}{\gamma\delta}+\delta\right)T} \end{cases} . \tag{81}$$

Note that map (81) is piecewise-continuous with the state space divided in two regions inside which the map is linear, that is $\mathcal{A}_1 := \left\{ N : \frac{A}{\left(\frac{2}{\gamma\delta}+\delta\right)T} > N \right\}$ and $\mathcal{A}_2 := \left\{ N : N \geq \frac{A}{\left(\frac{2}{\gamma\delta}+\delta\right)T} \right\}$. Due to linearity, the map has at most one hyperbolic equilibrium inside each of the two regions. Impose the equilibrium condition $\bar{N} = N_t = N_{t-1}$ in the two regions. In $\mathcal{A}_1$, we obtain $\bar{N} = \bar{N}^*$, with $\bar{N}^*$ defined as in (13). Note that $\bar{N}^* \in \mathcal{A}_1$, that is, $\bar{N}^*$ is always an equilibrium of the model. In $\mathcal{A}_2$, we obtain $\bar{N} = 0 \notin \mathcal{A}_2$. Hence, no equilibria exist in $\mathcal{A}_2$. Regarding the local asymptotic stability of $\bar{N}^*$, compute the slope of the branch of map (81) that applies in $\mathcal{A}_1$ and impose the stability condition, that is, impose that the slope is between -1 and 1. Then, the equilibrium $\bar{N}^*$ is locally asymptotically stable if and only if $T < T_b^*$, where T_b^* is defined in (14). To complete the proof, consider the set $\mathcal{C}$ and note that map (81) is never negative in such a set and its maximum is lower than $\left(1-\mu\right)\frac{A}{\delta T} + \frac{nd\delta A}{2}$. That is, $\mathcal{C}$ is mapped to itself by (81). Moreover, for all $N_{t-1} > \left(1-\mu\right)\frac{A}{\delta T} + \frac{nd\delta A}{2}$ we have $N_t = \left(1-\mu\right)N_{t-1} < N_{t-1}$, while for all $N_{t-1} < 0$, we have $N_t = \left(1-\mu\right)N_{t-1} > N_{t-1}$. Hence, $\mathcal{C}$ is an attracting invariant set and it is an asymptotic attractor or a subset of it is an asymptotic attractor.[7] □

Proof of Proposition 2. As discussed in Appendix A, map (11) can be rewritten as

$$N_t = \left(1-\mu\right)N_{t-1} + \sum_{i=1}^{n} D_i^m\left(q_{i,t}^{*,c}, \omega_{i,t}^{*,c}, \lambda_i^*\right), \tag{82}$$

where D_i^m are defined as in (44) since $\zeta_i = 0$ and $\eta_i(\omega_i) = \bar{\eta}_i$ with $\bar{\eta}_i = 0$ by assumption. Hence (by construction), map (82) with $\bar{\eta}_i = \zeta_i = 0$ is such that $N_t \geq 0$ and $N_t \leq \left(1-\mu\right)N_{t-1} + \sum_{i=1}^{n}\frac{d\delta_i A_i}{2}$ for all N_{t-1}. Moreover, $N_t \left(= \left(1-\mu\right)N_{t-1}\right) < N_{t-1}$ for all $N_{t-1} > \frac{A_k}{\left(\frac{2}{\delta_k\gamma_k}+\delta_k\right)T}$, being k the sector (monopolist) that satisfies condition (16). Hence, 0, resp. N_b^u, is both the minimum, resp. maximum, that the map can reach in C_n and the lower, resp. upper, boundary of C_n, thus proving that C_n is invariant. Consider $N_b^u > \frac{A_k}{\left(\frac{2}{\delta_k\gamma_k}+\delta_k\right)T}$, then $N_t = \left(1-\mu\right)N_{t-1}$ for all $N_{t-1} \notin C_n$. It follows that C_n is globally attracting for the map. Consider

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[7] Here, $\mathcal{A}$ is an invariant set for the map f if $f\left(\mathcal{A}\right) \subseteq \mathcal{A}$.

$N_b^u \leq \frac{A_k}{\left(\frac{2}{\delta_k \gamma_k}+\delta_k\right)T}$, then all point of $f\left(N_{t-1}\right) := (1-\mu)\, N_{t-1} + \sum_{i=1}^{n} \frac{d\delta_i A_i}{2}$ that are outside C_n are attracted in C_n. Since $N_t \leq f\left(N_{t-1}\right)$ for all N_{t-1}, the same holds for map (82) with $\bar{\eta}_i = \zeta_i = 0$. It follows that C_n is an invariant attracting region for map (82) with $\bar{\eta}_i = \zeta_i = 0$. If j is the production sector such that (16) is satisfied, then by construction of the map (map (82) with D_i as in (44) and $\bar{\eta}_i = \zeta_i = 0$), for N_t in

$$C_n^0 = \left[0, \frac{A_j}{\left(\frac{2}{\delta_j \gamma_j} + \delta_j\right) T}\right] m \tag{83}$$

it holds that

$$N_t = (1-\mu)\, N_{t-1} + \sum_{i=1}^{n} \frac{d\delta_i}{2}\left(A_i - \delta_i T N_{t-1}\right) - \sum_{i=1}^{n} \frac{dT}{\gamma_i} N_{t-1}. \tag{84}$$

Imposing the equilibrium condition to map (84), we obtain $\bar{N}^{**}$ in (18). Imposing feasibility condition, that is, $\bar{N} \in C_n^0$, we obtain

$$\frac{\sum_{i=1}^{n} \frac{d\delta_i}{2} A_i}{\mu + T \sum_{i=1}^{n}\left(\frac{d\delta_i}{2}\delta_i + \frac{d}{\gamma_i}\right)} < \frac{A_j}{\left(\frac{2}{\delta_j\gamma_j} + \delta_j\right) T}, \tag{85}$$

which is equivalent to

$$T \sum_{i=1}^{n} \frac{d\delta_i}{2}\left[\left(\frac{2}{\delta_j \gamma_j} + \delta_j\right) A_i - A_j \left(\frac{2}{\gamma_i \delta_i} + \delta_i\right)\right] < A_j \mu, \tag{86}$$

from which (19) follows. Imposing that the derivative of map (84) with respect to N_{t-1} is between −1 and 1, we obtain the condition for local asymptotic stability, that is condition (20). □

Proof of Proposition 3. By assumption $\eta_i\left(\omega_i\right) = \eta_i \cdot \omega_i$, $A_i = A$, $\gamma_i = \gamma$, $\delta_i = \delta$, $\eta_i = \eta$, $\zeta_i = 0$ for all $i = 1, \dots, n$. Consider the parameter space of map (11), partitioned into two regions: Region 1, where the conditions in 1. are satisfied, and Region 2, where the conditions in 2. are satisfied. Consider Region 1 and note that $2 > \delta\,(2\eta - \gamma\delta) = \eta\delta + \delta\,(\eta - \gamma\delta)$ and $\gamma\delta - \eta < 0$ imply $2 > \eta\delta$. Then, in Region 1 either CASE 1 or CASE 9 as defined in Appendix A occur and the map (11) reduces to $N_t = (1-\mu)\, N_{t-1}$. See Appendix A. Since $\mu \in (0,1)$ by assumption, Point 1. follows from the classical results on homogeneous linear maps. Consider Region 2, where either CASE 3 or CASE 4 as defined in Appendix A occur. Then, the map (11) reduces to

$$N_t \;=\; \begin{cases} N_{t-1}\,(1-\mu) + nd\left(\frac{A(\gamma\delta-\eta)}{2\gamma-\eta^2} + \frac{2(\eta\delta-1)-\gamma\delta^2}{2\gamma-\eta^2} T N_{t-1}\right) & \text{if} \quad 0 < N_{t-1} < \frac{(\gamma\delta-\eta)A}{(2+\delta(\gamma\delta-2\eta))T} \\ N_{t-1}\,(1-\mu) & \frac{(\gamma\delta-\eta)A}{(2+\delta(\gamma\delta-2\eta))T} < N_{t-1} \end{cases}, \tag{87}$$

as $\zeta_i = 0$ for all $i = 1, \dots, n$. See again Appendix A. Note that the slope of the map is negative as $2\,(\eta\delta - 1) - \gamma\delta^2 > 0$ and $2\gamma - \eta^2 > 0$ and $A\,(\gamma\delta - \eta) > 0$. Imposing the equilibrium condition $\bar{N} = N_t = N_{t-1}$ in the region

$$\left[0, \frac{(\gamma\delta - \eta)\, A}{(2 + \delta\,(\gamma\delta - 2\eta))\, T}\right], \tag{88}$$

we obtain $\bar{N}^{+}$ as in (21). Note that $\bar{N}^{+}$ is always within that region; as a consequence, it is always feasible. Instead, outside of that region, the possible equilibrium is 0, which, however, is never feasible. By standard results, $\bar{N}^{+}$ local asymptotic stability requires and is ensured when the slope of the map in $\bar{N}^{+}$ is between −1 and 1, that is, for

$$-1 < \frac{(1-\mu)\left(2\gamma - \eta^2\right) + nd\left(2\,(\eta\delta - 1) - \delta^2\gamma\right) T}{2\gamma - \eta^2} < 1, \tag{89}$$

which holds if and only if

$$\frac{(\mu - 2)\left(2\gamma - \eta^2\right)}{nd\left(2\,(\eta\delta - 1) - \gamma\delta^2\right)} > T > \underbrace{\frac{\mu\left(2\gamma - \eta^2\right)}{nd\left(2\,(\eta\delta - 1) - \gamma\delta^2\right)}}_{<0}. \tag{90}$$

Since $T > 0$ by assumption, (local) stability condition $T < T_b^{+}$ (T_b^{+} as in (22)) follows. Note that

$$\left[0, \max\left\{1 - \mu + nd\frac{2\,(\eta\delta - 1) - \gamma\delta^2}{2\gamma - \eta^2} T; 1\right\} \frac{(\gamma\delta - \eta)\, A}{(2 + \delta\,(\gamma\delta - 2\eta))\, T} + \frac{nd A\,(\gamma\delta - \eta)}{2\gamma - \eta^2}\right] \tag{91}$$

is a region where the map is positive and always lower than the upper bound of the set. Hence, the set is invariant. Moreover, outside this region, the map is such that $N_t = (1-\mu)\, N_{t-1}$. It follows that all the points outside this region are mapped inside in a finite number of iterations. Thus, the region is invariant and attracting. Hence, either it is an attractor or an attractor exists inside it. This proves the existence of an attractor even when the unique equilibrium of the map is unstable. The proof is completed by the results in Appendix A, showing that parameter configurations not satisfying either conditions in 1. or in 2. are not possible. □

Proof of Proposition 4. Consider the parameter space of map (11), partitioned into two regions: Region 1, where the conditions in 1. are satisfied, and Region 2, where the conditions in 2. are satisfied. Since $2 > \delta_i(2\eta_i - \gamma_i\delta_i) = \eta_i\delta_i + \delta_i(\eta_i - \gamma_i\delta_i)$ and $\gamma_i\delta_i - \eta_i < 0$ imply $2 > \delta_i\eta_i$, for all $i = 1, \dots, n$, Region 1 is the union of CASE 1 and CASE 9 as defined in Appendix A. Region 2 is instead portioned in CASES 3 and CASE 4 as defined in Appendix A. Using the additivity of the damage function with respect to monopolists (production sectors), that is, $D = \sum_{i=1}^{n} D_i^m$ and the results in Appendix A, that is $D_i^m = 0$ when parameters are as in Region 1, the map reduces to $N_t = (1-\mu) N_{t-1}$. Hence, Point 1 follows from the classical results on homogeneous linear maps. Consider Case 2. The results in Appendix A indicate that

$$D_i^m \left(q_{i,t}^{*,d}, \omega_{i,t}^{*,d}, \lambda_i^*\right) = \begin{cases} d\left(\frac{A_i(\delta_i\gamma_i - \eta_i)}{2\gamma_i - \eta_i^2} + \frac{2(\eta_i\delta_i - 1) - \delta_i^2\gamma_i}{2\gamma_i - \eta_i^2} T N_{t-1}\right) & \text{if} \quad 0 < N_{t-1} < \frac{(\gamma_i\delta_i - \eta_i)A_i}{(2+\delta_i(\gamma_i\delta_i - 2\eta_i))T} \\ 0 & \frac{(\gamma_i\delta_i - \eta_i)A_i}{(2+\delta_i(\gamma_i\delta_i - 2\eta_i))T} < N_{t-1} \end{cases}, \tag{92}$$

where $q_{i,t}^{*,d}$, $\omega_{i,t}^{*,d}$, λ_i^* are optimal production, optimal observable abatement, optimal unobservable abatement, respectively, and

$$\frac{\left(\gamma_j\delta_j - \eta_j\right)A_j}{\left(2 + \delta_j\left(\gamma_j\delta_j - 2\eta_j\right)\right)T} \le \frac{\left(\gamma_i\delta_i - \eta_i\right)A_i}{\left(2 + \delta_i\left(\gamma_i\delta_i - 2\eta_i\right)\right)T} \quad \forall i \in \{1, 2, \dots, n\}, \tag{93}$$

by assumption. Recall the additivity property of the demand function with respect to monopolists (production sectors), that is

$$D\left(q_1, \dots, q_n, \omega_1, \dots, \omega_n, \lambda_1, \dots, \lambda_n\right) = \sum_{i=1}^{n} D_i^m\left(q_i, \omega_i, \lambda_i\right). \tag{94}$$

Hence, in $\left[0, \frac{(\gamma_j\delta_j - \eta_j)A_j}{(2+\delta_j(\gamma_j\delta_j - 2\eta_j))T}\right)$ we conclude that

$$N_t = N_{t-1}(1-\mu) + d\sum_{i=1}^{n}\left(\frac{A_i\left(\delta_i\gamma_i - \eta_i\right)}{2\gamma_i - \eta_i^2} + \frac{2\left(\eta_i\delta_i - 1\right) - \delta_i^2\gamma_i}{2\gamma_i - \eta_i^2} T N_{t-1}\right). \tag{95}$$

Imposing the equilibrium condition $N_t = N_{t-1} = \bar{N}$, we obtain $\bar{N} = \bar{N}^{++}$, with $\bar{N}^{++}$ defined in (26). Note that $\bar{N}^{++} > 0$ by parameter assumptions. Hence, the existence condition reduces to $\bar{N}^{++} < \frac{(\gamma_j\delta_j - \eta_j)A_j}{(2+\delta_j(\gamma_j\delta_j - 2\eta_j))T}$, that is

$$d\sum_{i=1}^{n} \frac{A_i \frac{\delta_i\gamma_i - \eta_i}{2\gamma_i - \eta_i^2}}{\mu + d\sum_{i=1}^{n} \frac{\delta_i^2\gamma_i - 2(\eta_i\delta_i - 1)}{2\gamma_i - \eta_i^2} T} < \frac{\left(\gamma_j\delta_j - \eta_j\right)A_j}{\left(2 + \delta_j\left(\gamma_j\delta_j - 2\eta_j\right)\right)T}. \tag{96}$$

By straightforward algebra, such condition can be rewritten as

$$Td\sum_{i=1}^{n} \frac{\left(2 + \delta_j\left(\gamma_j\delta_j - 2\eta_j\right)\right)\left(2 + \delta_i\left(\gamma_i\delta_i - 2\eta_i\right)\right)}{2\gamma_i - \eta_i^2}\left[\frac{A_i\left(\delta_i\gamma_i - \eta_i\right)}{\left(2 + \delta_i\left(\gamma_i\delta_i - 2\eta_i\right)\right)} - \frac{A_j\left(\gamma_j\delta_j - \eta_j\right)}{\left(2 + \delta_j\left(\gamma_j\delta_j - 2\eta_j\right)\right)}\right] < \left(\gamma_j\delta_j - \eta_j\right)A_j\mu. \tag{97}$$

Note that

$$\left[\frac{A_i\left(\delta_i\gamma_i - \eta_i\right)}{\left(2 + \delta_i\left(\gamma_i\delta_i - 2\eta_i\right)\right)} - \frac{A_j\left(\gamma_j\delta_j - \eta_j\right)}{\left(2 + \delta_j\left(\gamma_j\delta_j - 2\eta_j\right)\right)}\right] > 0 \tag{98}$$

by assumptions. Hence, $T < T_b^{++}$ where T_b^{++} is defined as in (27). Since the map inside $\left[0, \frac{(\gamma_j\delta_j - \eta_j)A_j}{(2+\delta_j(\gamma_j\delta_j - 2\eta_j))T}\right]$ is smooth, by standard arguments $\bar{N}^{++}$ is locally asymptotically stable when

$$-1 < 1 - \mu + d\sum_{i=1}^{n} \frac{2\left(\eta_i\delta_i - 1\right) - \delta_i^2\gamma_i}{2\gamma_i - \eta_i^2} T < 1. \tag{99}$$

Note that $2\left(\eta_i\delta_i - 1\right) - \delta_i^2\gamma_i < 0$ and $2\gamma_i - \eta_i^2 > 0$ in Case 2. Hence, the stability condition reduces to

$$d\sum_{i=1}^{n} \frac{2\left(1 - \eta_i\delta_i\right) + \delta_i^2\gamma_i}{2\gamma_i - \eta_i^2} T < (2 - \mu), \tag{100}$$

from which stability condition (28) follows. This completes the proof. □

Proof of Proposition 5. By assumption: $\eta_i\left(\omega_i\right) = \eta_i \cdot \omega_i$, $A_i = A$, $\gamma_i = \gamma$, $\delta_i = \delta$, $\eta_i = \eta$, $\zeta_i = \zeta$ for all $i = 1, \dots, n$. Moreover, the parameter space of map (11) can be partitioned into two regions: Region 1, where the conditions in 1. are satisfied, and Region 2, where the conditions in 2. are satisfied. Since $2 > \delta\left(2\eta - \gamma\delta\right) = \eta\delta + \delta\left(\eta - \gamma\delta\right)$ and $\gamma\delta - \eta < 0$ imply $2 > \eta\delta$, Region 1 of the parameter space is partitioned in CASE 1 and CASE 9 in Appendix A. Hence, map (11) reduces to map (68), that is

$$N_t = \begin{cases} (1-\mu)N_{t-1} + ns\zeta\frac{A - \delta T N_{t-1}}{2 - \eta\delta} & \text{if } N_{t-1} < \frac{A}{\delta T} \\ (1-\mu)N_{t-1} & \text{otherwise} \end{cases}. \tag{101}$$

Imposing the equilibrium condition $\bar{N} = N_t = N_{t-1}$ in the region $\left[0, \frac{A}{\delta T}\right)$, we obtain $\bar{N}^1$ as in (29). Note that $\bar{N}^1$ is always inside that region, therefore it is always feasible. Instead, outside that region, the possible equilibrium is 0, which, however, is never feasible. By standard results, $\bar{N}^1$ local asymptotic stable requires and is ensured when the slope of the map in $\bar{N}$ is between -1 and 1, which is ensured for $T < T_b^1$, with T_b^1 as in (30). Consider Region 2 of the parameter space and let us distinguish two subregions, that is, Region 2(I): $2\gamma > \eta^2$, $2 > \eta\delta$, $\gamma\delta - \eta > 0$, and $2 > \delta(2\eta - \gamma\delta)$; and Region 2(II): $2\gamma > \eta^2$, $2 < \eta\delta$, $\gamma\delta - \eta > 0$, and $2 > \delta(2\eta - \gamma\delta)$. Region 2(I) corresponds to CASE 3 in Appendix A. Hence map (11) reduces to map (72):

$$N_t = \begin{cases} N_{t-1}(1-\mu) + nd\left(\frac{A(\gamma\delta-\eta)}{2\gamma-\eta^2} + \frac{2(\eta\delta-1)-\gamma\delta^2}{2\gamma-\eta^2} T N_{t-1}\right) & \\ \qquad + ns\zeta \frac{\gamma A + (\eta - \delta\gamma) T N_{t-1}}{2\gamma-\eta^2} & \text{if} \quad 0 < N_{t-1} < \frac{(\gamma\delta-\eta)A}{(2+\delta(\gamma\delta-2\eta))T} \\ N_{t-1}(1-\mu) + ns\zeta \frac{A - \delta T N_{t-1}}{2-\eta\delta} & \text{if} \quad \frac{(\gamma\delta-\eta)A}{(2+\delta(\gamma\delta-2\eta))T} < N_{t-1} < \frac{A}{\delta T} \\ N_{t-1}(1-\mu) & \text{if} \quad \frac{A}{T\delta} < N_{t-1} \end{cases} \tag{102}$$

By imposing $\bar{N} = N_t = N_{t-1}$ in each of the three subregions that partition the state space, we obtain the possible equilibria. Specifically, the equilibrium of the map in $\left[0, \frac{(\gamma\delta-\eta)A}{(2+\delta(\gamma\delta-2\eta))T}\right)$ is $\bar{N}^2$ in (29). Based on the parameter restrictions of Case 3(I), one has $\bar{N}^2 > 0$, while imposing

$$\frac{n(s\zeta\gamma + d(\gamma\delta-\eta))A}{\mu\left(2\gamma-\eta^2\right) + n(s\zeta(\eta-\gamma\delta) + d(2+\delta(\gamma\delta-2\eta)))T} < \frac{(\gamma\delta-\eta)A}{(2+\delta(\delta\gamma-2\eta))T}, \tag{103}$$

which is equivalent to $T < T_b^3$. Moreover, when feasible, it is locally asymptotically stable for

$$-1 < \frac{(1-\mu)\left(2\gamma-\eta^2\right) + n\left(s\zeta(\eta-\delta\gamma) + d\left(2(\eta\delta-1)-\delta^2\gamma\right)\right)T}{2\gamma-\eta^2} < 1. \tag{104}$$

Under the parameter restrictions of Case 3(I), it holds that $ns\zeta(\eta-\gamma\delta) + nd\left(2(\eta\delta-1)-\delta^2\gamma\right) < 0$ and $2\gamma-\eta^2 > 0$. Hence, (104) can be rewritten as

$$\underbrace{\frac{(\mu-2)\left(2\gamma-\eta^2\right)}{\left(s\zeta(\eta-\delta\gamma) + d\left(2(\eta\delta-1)-\delta^2\gamma\right)\right)n}}_{>0} > T > \underbrace{\frac{\mu\left(2\gamma-\eta^2\right)}{\left(s\zeta(\eta-\delta\gamma) + d\left(2(\eta\delta-1)-\delta^2\gamma\right)\right)n}}_{<0}. \tag{105}$$

Hence, the equilibrium is locally asymptotically stable for $T < T_b^4$, where T_b^4 as in (30). The equilibrium in $\left[\frac{(\gamma\delta-\eta)A}{(2+\delta(\gamma\delta-2\eta))T}, \frac{A}{\delta T}\right]$ is $\bar{N}^1$ in (29). Since $2 > \eta\delta$ in Case 3(I), we conclude that

$$\bar{N}^1 < \frac{A}{\delta T}, \tag{106}$$

while

$$\bar{N}^1 > \frac{(\gamma\delta-\eta)A}{(2+\delta(\gamma\delta-2\eta))T} \tag{107}$$

if and only if $T > T_b^3$, with T_b^3 defined as in (30). The local asymptotic stability of $\bar{N}^1$ is discussed in Case 1. In $\left[\frac{A}{\delta T}, +\infty\right)$, the unique equilibrium is 0, which is clearly unfeasible (virtual). Consider Region 2(II), which corresponds to CASE 4 in Appendix A. Map (11) reduces to map (76), that is

$$N_t = \begin{cases} N_{t-1}(1-\mu) + nd\left(\frac{A(\gamma\delta-\eta)}{2\gamma-\eta^2} + \frac{2(\eta\delta-1)-\gamma\delta^2}{2\gamma-\eta^2} T N_{t-1}\right) & \\ \qquad + ns\zeta \frac{\gamma A + (\eta-\gamma\delta) T N_{t-1}}{2\gamma-\eta^2} & \text{if} \quad 0 \le N_{t-1} < \frac{(\gamma\delta-\eta)A}{(2+\delta(\gamma\delta-2\eta))T} \\ N_{t-1}(1-\mu) + ns\zeta \frac{A-\delta T N_{t-1}}{2-\eta\delta} & \text{if} \quad \frac{(\gamma\delta-\eta)A}{(2+\delta(\gamma\delta-2\eta))T} \le N_{t-1} < \frac{\left(\gamma\delta^2-2\right)A}{\left(2+\delta\left(\gamma\delta_i-2\eta\right)\right)\delta T} \\ N_{t-1}(1-\mu) & \text{if} \quad \frac{\left(\gamma\delta^2-2\right)A}{\left(2+\delta\left(\gamma\delta_i-2\eta\right)\right)\delta T} \le N_{t-1} \end{cases} \tag{108}$$

In $\left[0, \frac{A}{\delta T}\right)$, the map (108) is as in Case 2(I). In the region $\left[\frac{A}{\delta T}, +\infty\right)$, the map (108) can only have the equilibria $\bar{N}^0$ and $\bar{N}^1$. However, $\bar{N}^0, \bar{N}^1 < \frac{A}{\delta T}$ always hold. This completes the proof, since no other cases are possible (see Appendix A). ☐

Proof of Proposition 6. Consider the parameter space of the map (11), under the restriction $\bar{\eta}_i = +\infty$ for all $i = 1, \ldots, n$, partitioned into the following two regions: Region 1, where the conditions in 1. are satisfied, and Region 2, where the conditions in 2. are satisfied. Since $2 > \delta_i(2\eta_i - \gamma_i\delta_i) = \eta_i\delta_i + \delta_i(\eta_i - \gamma_i\delta_i)$ and $\gamma_i\delta_i - \eta_i < 0$ imply $2 > \delta_i\eta_i$, for all $i = 1, \ldots, n$, Region 1 is the union of CASE

1 and CASE 9 as defined in Appendix A. Region 2 is instead portioned in CASES 3 and CASE 4 as defined in Appendix A. Consider Region 1. The results in Appendix A indicate that $\forall i \in \{1, 2, \dots, n\}$

$$D_i\left(q_{i,t}^{*,d}, \omega_{i,t}^{*,d}, \lambda_i^*\right) = s\zeta_i \frac{A_i - \delta_i T N_{t-1}}{2 - \eta_i \delta_i} \quad \text{when} \quad N_{t-1} < \frac{A_h}{\delta_h T}, \tag{109}$$

where

$$\frac{A_h}{\delta_h T} \le \frac{A_i}{\delta_i T} \quad \forall i \in \{1, 2, \dots, n\} \tag{110}$$

by assumption. Recall the additivity property of the demand function with respect to monopolists (production sectors), that is

$$D\left(q_1, \dots, q_n, \omega_1, \dots, \omega_n, \lambda_1, \dots, \lambda_n\right) = \sum_{i=1}^{n} D_i^m\left(q_i, \omega_i, \lambda_i\right). \tag{111}$$

Hence, in $\left[0, \frac{A_h}{\delta_h T}\right)$ we conclude that

$$N_t = (1 - \mu) N_{t-1} + \sum_{i=1}^{n} s\zeta_i \frac{A_i - \delta_i T N_{t-1}}{2 - \eta_i \delta_i}. \tag{112}$$

Imposing the equilibrium condition $N_t = N_{t-1} = \bar{N}$, we obtain

$$\bar{N} = \frac{\displaystyle\sum_{i=1}^{n} s\zeta_i A_i \prod_{k=1,k\neq i}^{n} \left(2 - \delta_k \eta_k\right)}{\displaystyle\mu \prod_{k=1}^{n} \left(2 - \delta_k \eta_k\right) + \sum_{i=1}^{n} s\zeta_i \delta_i T \prod_{k=1,k\neq i}^{n} \left(2 - \delta_k \eta_k\right)} =: \bar{N}^3. \tag{113}$$

Feasibility implies that $\bar{N}^3 < \frac{A_h}{\delta_h T}$, which holds if and only if

$$sT \sum_{i=1}^{n} \zeta_i \left(\delta_h A_i - \delta_i A_h\right) \prod_{k=1,k\neq i}^{n} \left(2 - \delta_k \eta_k\right) < A_h \mu \prod_{k=1}^{n} \left(2 - \delta_k \eta_k\right), \tag{114}$$

from which we obtain

$$T < \frac{\displaystyle A_h \mu \prod_{k=1}^{n} \left(2 - \delta_k \eta_k\right)}{\displaystyle s \sum_{i=1}^{n} \zeta_i \left(\delta_h A_i - \delta_i A_h\right) \prod_{k=1,k\neq i}^{n} \left(2 - \delta_k \eta_k\right)} =: T_b^5. \tag{115}$$

Local asymptotic stability implies that

$$1 - \mu - \sum_{i=1}^{n} \frac{s\zeta_i \delta_i T}{2 - \eta_i \delta_i} \in (-1, 1). \tag{116}$$

The quantity is always smaller than 1; let us hence impose that it is also greater than -1. This implies $T < T_b^6$. Consider Region 2. The results in Appendix A indicate that

$$D_i\left(q_{i,t}^{*,d}, \omega_{i,t}^{*,d}, \lambda_i^*\right) = d\left(\frac{A_i\left(\delta_i \gamma_i - \eta_i\right)}{2\gamma_i - \eta_i^2} + \frac{2\left(\eta_i \delta_i - 1\right) - \delta_i^2 \gamma_i}{2\gamma_i - \eta_i^2} T N_{t-1}\right) + s\zeta_i \frac{\gamma_i A_i + \left(\eta_i - \delta_i \gamma_i\right) T N_{t-1}}{2\gamma_i - \eta_i^2} \quad \forall i \in \{1, 2, \dots, n\}, \tag{117}$$

when

$$0 < N_{t-1} < \frac{\left(\gamma_j \delta_j - \eta_j\right) A_j}{\left(2 + \delta_i\left(\gamma_j \delta_j - 2\eta_j\right)\right) T}, \tag{118}$$

where

$$\frac{\left(\gamma_j \delta_j - \eta_j\right) A_j}{\left(2 + \delta_j\left(\gamma_j \delta_j - 2\eta_j\right)\right) T} \le \frac{\left(\gamma_i \delta_i - \eta_i\right) A_i}{\left(2 + \delta_i\left(\gamma_i \delta_i - 2\eta_i\right)\right) T} \quad \forall i \in \{1, 2, \dots, n\} \tag{119}$$

by assumption. Recall the additivity property of the demand function with respect to monopolists (production sectors), that is

$$D\left(q_1, \dots, q_n, \omega_1, \dots, \omega_n, \lambda_1, \dots, \lambda_n\right) = \sum_{i=1}^{n} D_i^m\left(q_i, \omega_i, \lambda_i\right). \tag{120}$$

Hence, in $\left[0, \frac{\left(\gamma_j \delta_j - \eta_j\right) A_j}{\left(2 + \delta_j\left(\gamma_j \delta_j - 2\eta_j\right)\right) T}\right)$ we obtain

$$N_t = N_{t-1}(1 - \mu) + \sum_{i=1}^{n} \left[d\left(\frac{A_i\left(\delta_i \gamma_i - \eta_i\right)}{2\gamma_i - \eta_i^2} + \frac{2\left(\eta_i \delta_i - 1\right) - \delta_i^2 \gamma_i}{2\gamma_i - \eta_i^2} T N_{t-1}\right) + s\zeta_i \frac{\gamma_i A_i + \left(\eta_i - \delta_i \gamma_i\right) T N_{t-1}}{2\gamma_i - \eta_i^2}\right]. \tag{121}$$

Imposing the equilibrium condition $N_t = N_{t-1} = \bar{N}$, by straightforward algebra we obtain $\bar{N} = \bar{N}^4$. Based on the parameter constraints that identify Region 2, $\bar{N}^4 > 0$ always holds. Hence, for its feasibility we need to impose $\bar{N}^4 < \frac{(\gamma_j\delta_j-\eta_j)A_j}{(2+\delta_j(\gamma_j\delta_j-2\eta_j))T}$, which is equivalent to

$$T\sum_{i=1}^{n}\left[\frac{d\left(2-\delta_i\left(2\eta_i-\gamma_i\delta_i\right)\right)}{2\gamma_i-\eta_i^2}\left(\frac{\left(\gamma_i\delta_i-\eta_i\right)A_i}{\left(2-\delta_i\left(2\eta_i-\gamma_i\delta_i\right)\right)}-\frac{\left(\gamma_j\delta_j-\eta_j\right)A_j}{\left(2-\delta_j\left(2\eta_j-\gamma_j\delta_j\right)\right)}\right)+s\zeta_i\left(\gamma_i\delta_i-\eta_i\right)\frac{\frac{\gamma_iA_i}{(\gamma_i\delta_i-\eta_i)}-\frac{(\gamma_j\delta_j-\eta_j)A_j}{(2-\delta_j(2\eta_j-\gamma_j\delta_j))}}{2\gamma_i-\eta_i^2}\right]<\frac{\left(\gamma_j\delta_j-\eta_j\right)A_j}{\left(2-\delta_j\left(2\eta_j-\gamma_j\delta_j\right)\right)}\mu. \tag{122}$$

Note that

$$\frac{\gamma_iA_i}{\left(\gamma_i\delta_i-\eta_i\right)}>\frac{\left(\gamma_i\delta_i-\eta_i\right)A_i}{\left(2-\delta_i\left(2\eta_i-\gamma_i\delta_i\right)\right)} \tag{123}$$

when $2\gamma_i > \eta_i^2$ and

$$\frac{\left(\gamma_i\delta_i-\eta_i\right)A_i}{\left(2-\delta_i\left(2\eta_i-\gamma_i\delta_i\right)\right)}>\frac{\left(\gamma_j\delta_j-\eta_j\right)A_j}{\left(2-\delta_j\left(2\eta_j-\gamma_j\delta_j\right)\right)}\quad\forall i\in\{1,2,\dots,n\}, i\neq j. \tag{124}$$

Hence, the quantity that multiplies T in (122) is positive under the parameter conditions identifying Region 2, and the equilibrium is feasible when $T < T_b^8$. Its locally asymptotical stability implies instead that the slope of the map at the equilibrium is such that

$$-1<1-\mu-T\sum_{i=1}^{n}\frac{d\left(2-\delta_i\left(2\eta_i-\gamma_i\delta_i\right)\right)+s\zeta_i\left(\delta_i\gamma_i-\eta_i\right)}{2\gamma_i-\eta_i^2}<1. \tag{125}$$

Based on the parameter conditions that define Region 2, the quantity in the middle is always smaller than 1. Hence, we have to impose

$$-2<-\mu-T\sum_{i=1}^{n}\frac{d\left(2-\delta_i\left(2\eta_i-\gamma_i\delta_i\right)\right)+s\zeta_i\left(\delta_i\gamma_i-\eta_i\right)}{2\gamma_i-\eta_i^2}, \tag{126}$$

that is $T < T_b^9$. This completes the proof. □

Appendix C. The effect of the upper bound on extra willingness to pay for observable abatement

In this Appendix, we consider the map (11) when $\bar{\eta}_i$ is finite. In this case, consumers' extra willingness to pay for abatement is a concave function with an upper bound. The results in Appendix A can be used to derive the map, although some additional considerations are required, as detailed below.

We focus on the case where $2\gamma_i - \eta_i^2 > 0$ for all $i = 1,\dots,n$. In this case, the reduced profit function $\pi_i^*\left(\omega_i,\lambda_i;T,N_{t-1}\right)$ in (8) is concave in ω_i, and the optimal λ_i remains zero, as the arguments in Appendix A still apply.

The optimal value of ω_i differs from the case without an upper bound. Indeed, the objective function is defined piecewise as follows:

$$\pi_i^*\left(\omega_i,0;T,N_{t-1}\right)=\begin{cases}\pi_i^{*,1}\left(\omega_i,0;T,N_{t-1}\right) & \text{if}\quad \eta_i\omega_i\leq\bar{\eta}_i\\ \pi_i^{*,2}\left(\omega_i,0;T,N_{t-1}\right) & \text{if}\quad \eta_i\omega_i>\bar{\eta}_i\end{cases} \tag{127}$$

where $\pi_i^{*,1}\left(\omega_i,0;T,N_{t-1}\right)$ is obtained as in Appendix A using $\eta_i\left(\omega_i\right) = \eta_i\omega_i$, while $\pi_i^{*,2}\left(\omega_i,0;T,N_{t-1}\right)$ is obtained as shown in Appendix A using $\eta_i\left(\omega_i\right) = \bar{\eta}_i$.

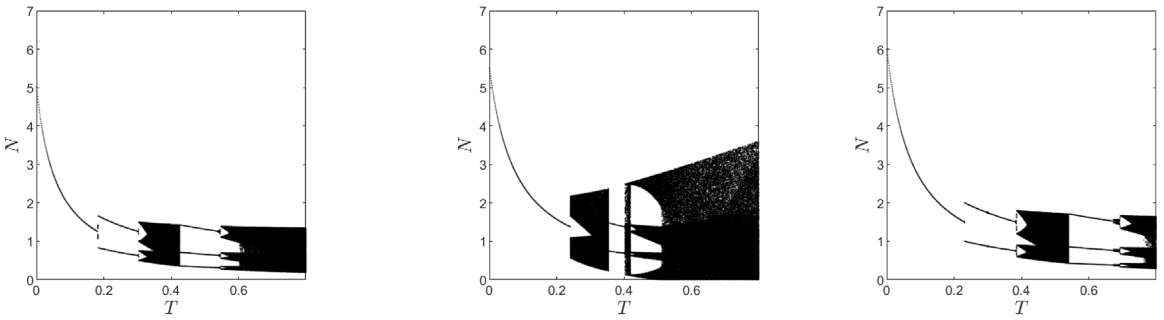
Under the parameter restrictions imposed in this Appendix, both $\pi_i^{*,1}\left(\omega_i,0;T,N_{t-1}\right)$ and $\pi_i^{*,2}\left(\omega_i,0;T,N_{t-1}\right)$ are strictly concave functions. Moreover, $\pi^{*,1}i\left(\omega_i,0;T,N_{t-1}\right) \leq \pi_i^{*,2}\left(\omega*i,0;T,N_{t-1}\right)$ for $\eta_i\omega_i < \bar{\eta}_i$, and the inequality is reversed for $\eta_i\omega_i > \bar{\eta}_i$. Hence, $\pi_i^*\left(\omega_i,0;T,N_{t-1}\right)$ can be expressed as the minimum of two strictly concave functions, which is itself strictly concave by standard results. Therefore, the solution to the optimization problem is unique.

To find the solution, note that $\omega_{i,t}^{*,d}$, defined as in (46) in Appendix A, is the optimal value for $\pi_i^{*,1}\left(\omega_i,0;T,N_{t-1}\right)$, while $\omega_{i,t}^{*,c}$, defined as in (43) in Appendix A, is the optimal value for $\pi^{*,2}*i\left(\omega_i,0;T,N_{t-1}\right)$. Taking into account the feasibility conditions, namely $\eta_i\omega_{i,t}^{*,d} < \bar{\eta}_i$ and $\eta_i\omega_{i,t}^{*,c} > \bar{\eta}_i$, the optimal value of abatement in (127), denoted by $\omega_{i,t}^{*,p}$, is selected from the triplet $\left\{\omega_{i,t}^{*,c},\omega_{i,t}^{*,d},\bar{\eta}_i\right\}$.

Then, by plugging the optimal abatement into the damage function of each monopolist — that is, by computing $D_i^m\left(\omega_{i,t}^{*,p},0;T,N_{t-1}\right)$ and summing over all $i\in 1,\dots,n$ — we obtain the aggregate damage function D, which can be used to derive the map (11). In the following two subsections of this appendix, we numerically investigate the global dynamics of this map for different values of $\bar{\eta}_i$.

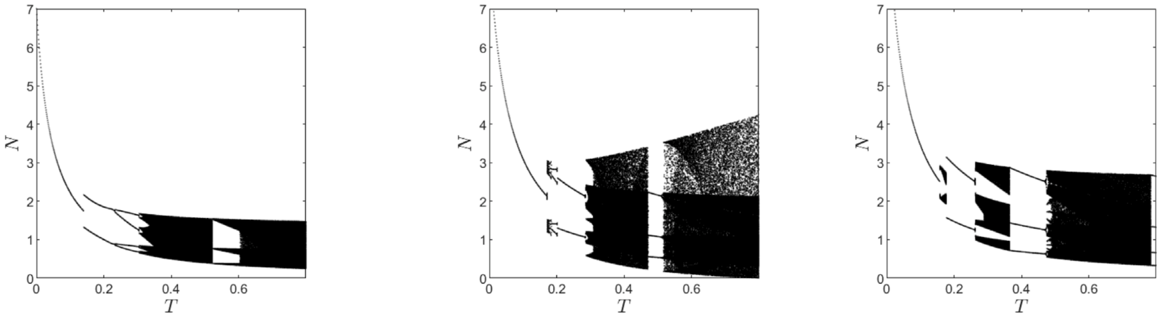
C.1. *Fully observable pollution*

Consider a homogeneous setting with two monopolistic markets and fully observable pollution, that is $\zeta_i = 0$ for all $i = 1,\dots,n$. The one-dimensional bifurcation diagram in Fig. 12(a) shows the long run dynamics of the model as the policy parameter T varies.



(a) (b) (c)

Fig. 12. Bifurcation diagrams with respect to the tax coefficient T for map (11) with two sectors ($n = 2$). Homogeneous sectors. Panel (a): $\bar{\eta} = 0$. Panel (b): $\bar{\eta} = 0.5$. Panel (c): $\bar{\eta} = 1.5$. In all panels, the remaining parameters are: $\mu = 0.5$, $d = 1$, $s = 0$, $A_i = A = 1$, $\gamma_i = \gamma = 1$, $\delta_i = \delta = 2.5$, $\eta_i = \eta = 1$, and $\zeta_i = \zeta = 0$, $i \in \{1, 2\}$.



(a) (b) (c)

Fig. 13. Bifurcation diagrams with respect to the tax coefficient T for map (11) with two sectors ($n = 2$). Homogeneous sectors. Panel (a): $\bar{\eta} = 0$. Panel (b): $\bar{\eta} = 0.5$. Panel (c): $\bar{\eta} = 1.5$. In all panels, the remaining parameters are: $\mu = 0.5$, $d = 1$, $s = 1$, $A_i = A = 1$, $\gamma_i = \gamma = 1$, $\delta_i = \delta = 2.5$, $\eta_i = \eta = 1$, and $\zeta_i = \zeta = 1$, $i \in \{1, 2\}$.

The extra willingness to pay is set to zero, that is $\bar{\eta}_i = \bar{\eta} = 0$. As we can observe, increasing the tax coefficient T reduces the average level of environmental degradation, although it induces fluctuations. Consider a positive extra willingness to pay for abatement, with $\eta_i = \eta = 1$ and $\bar{\eta}_i = \bar{\eta} = 0.5$. The one-dimensional bifurcation diagram in Fig. 12(b) illustrates the long-run dynamics of the model as the policy parameter T varies. Compared with the case of zero extra willingness to pay (Fig. 12(a)), a higher extra willingness to pay for abatement induces firms to increase production. Although pollution per unit of output declines due to additional abatement, total pollution may rise and the increase in production offsets this effect. Consequently, environmental degradation becomes more volatile, oscillating between high values and levels close to zero.

Overall, the effect of the upper bound on the extra willingness to pay is nonlinear. When increasing the upper bound to $\bar{\eta}_i = \bar{\eta} = 1.5$, the corresponding bifurcation diagram of the long-run dynamics as the policy parameter T varies is reported in Fig. 12(c). We observe that the amplitude of the fluctuations decreases. Nevertheless, a low upper bound of the extra willingness to pay mitigates the potential adverse effects associated with it, namely higher environmental degradation driven by overproduction.

C.2. Partially observable pollution

Consider a setting with two monopolists, as in the previous section of this Appendix, but assuming unobservable pollution, i.e., $\zeta_i = \zeta = 1$ and $s = 1$. The qualitative conclusions remain unchanged, although the quantitative effects are stronger. Specifically, starting from a situation with no extra willingness to pay for abatement (bifurcation diagram in Fig. 13(a)), moving to a scenario in which extra willingness to pay is present but limited to $\bar{\eta}_i = \bar{\eta} = 0.5$ (Fig. 13(b)), and finally to a case in which extra willingness to pay is present with an upper bound of $\bar{\eta}_i = \bar{\eta} = 1.5$, which is almost never binding (Fig. 13(c)), we observe a gradual increase in the level of environmental degradation and the emergence of fluctuations for progressively lower values of the tax coefficient T.

At the same time, the amplitude of fluctuations increases when the upper bound on extra willingness to pay is relatively low, but begins to decline as this upper bound is further relaxed. These results confirm the robustness of our findings.

Data availability

No data was used for the research described in the article.

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