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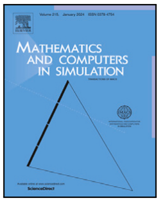

Original articles

Pricing natural-disasters and climate-change risks: Insights from CAPM with self and externally excited jumps[☆]

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ABSTRACT

We study the effect of climate change on market's risk premium and stock volatility through a general equilibrium asset pricing model with recursive preferences. We consider a financial market affected by environmental and macroeconomic shocks, both featuring time-varying intensities. Additionally, we incorporate the effects of green policies specifically targeting carbon-intensive assets. To capture the increasing frequency and clustering behavior of environmental risks, the model includes self-exciting and externally excited jump intensities for natural rare disasters. The representative agent's consumption–investment maximization problem is solved in closed form, and the analytical results suggest that consumption disasters with time-varying probability reduce the risk-free interest rates while increasing the market's premium and volatility of the stock market. The asymmetric impact of the transition risk on assets affects the optimal portfolio composition. Neglecting the clustering-like nature of environmental shocks lead to underestimate these effects.

1. Introduction

The financial risks related to climate change are caused by environmental events such as droughts, hurricanes, typhoons, floods, and extreme heat waves. These events affect productivity and worsen the credit quality of economic entities. These risks, known as physical risks, tend to have a large-scale economic impact. Mitigating these risks is therefore essential to preserve financial stability. The antidote to climate change offered by green policies expose, however, to extra costs and production constraints, i.e., to the so-called transition risk.

Transition risk asymmetrically impacts brown companies, using high carbon-emission technologies, and green firms, using low carbon-emission technologies. Physical risk, instead, belong to the family of environmental risks, which have the peculiarity of being driven by the clustering of rare natural disasters and are the combination of externally excited time-varying risks and self-excited

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time-varying risks. In case of an externally excited time-varying risk, the triggering event does not have economic consequences, so no direct financial distress, but it causes secondary disruptive events that have large economic impacts. An underwater earthquake, for example, do not have any direct impact on economic activities. However, it may generate a tsunami that hit the coasts of a country and may cause nuclear power plant meltdown. The economic effects are widespread damages, with national companies forced to temporarily shut down. Involved, there may be manufacturers who play a pivotal role in global supply chains, such as chip and electronics producers. In this case, the cascading events end up also affecting companies located in other countries, e.g., worldwide car manufacturing companies. This is what happened after the Tohoku earthquake and tsunami that struck the northeast coast of Japan on 11 March, 2011. Instead, self-excited time-varying risks refer to chains of related events, where also the triggering one causes financial distress. An example is a heat wave that leads to drought, which on its own exacerbates the risk of wildfires. The heat wave can cause power outages, either through overloaded circuits tripping or through equipment malfunctions. Droughts can reduce the water levels in reservoirs, impacting hydroelectric production and also affect the cooling systems of thermoelectric power plants, leading to reduced efficiency or shutdowns. Wildfires can damage power lines and substations.

Climate change contributes to increasing the frequency and intensity of these disruptive events, while the interconnected worldwide economy contributes to amplify the impact of their negative effects, see, e.g., [1,2]. Moreover, a world that warms up is likely to increase exposure to clustering of losses caused by chains of low-probability-high-consequences events, see, e.g., [2–4]. Natural scientists have always been aware of the perils of the clustering of rare events. As evidence of this, the literature on natural hazards coined the term *NaTech* (Natural-Hazard-Triggered Technological Disasters) to refer to accidents at industrial facilities or other infrastructure caused by natural events, see, e.g., [5,6], and a supranational institution, such as the European Commission, created the *eNatech* database to facilitate the monitoring and management of these risks, see [7].[1]

In environmental economics, rare disasters are typically modeled using deterministic damage functions that enter as costs in the policymaker's objective function. A standard modeling framework is provided by Integrated Assessment Models (IAMs), which allow for the computation of the Social Cost of Carbon (e.g., [9,10]). However, these models often neglect the risks associated with such events (see, e.g., [11]), which are instead more explicitly considered in the financial literature. In finance, indeed, rare disruptive events are a key source of risk which does not include only natural catastrophes, but also macroeconomic and geopolitical shocks. Modeling the impacts and probabilities of these rare events is of fundamental relevance to explain key features of the financial markets. Above all, compensation for the risk of rare disasters is the most recognized explanation for the equity premium and volatility puzzles. As shown in the empirical work of [12], large average equity return and small risk-free return, not consistent with classical general equilibrium models, the so-called equity premium puzzle, see, e.g., [13], are justified by rare disasters the probability of which is there assumed constant and calibrated to international data on large economic declines. A theoretical justification for both equity premium and volatility puzzle is instead offered in [14]. In the modeling framework of a general equilibrium asset pricing model, Wachter [14] considers time-varying probabilities of disasters and shows that it is responsible for a stock volatility that exceeds the one of the dividend process. Other features of the model are a representative agent with recursive preferences with unit elasticity of intertemporal substitution (EIS) and a parsimony that allows for a closed form solution. Disasters are represented by negative shocks with random size, that could be caused by changes in government policy, wars, financial crises, or also natural disasters. The shock intensity follows a square-root mean-reverting process.

The same recursive stochastic general equilibrium utility framework has been adopted by Karydas and Xepapadeas [15], but the emphasis is on modeling the risks due to climate change. Compared to [14], the sources of economic shocks are diversified to represent two types of risks: macroeconomic and environmental (physical). These risks homogeneously affect the portfolio, which contains a general (green) asset and a (brown) asset linked to carbon-intensive technologies. The modeling sophistication allows us to accommodate for transition risk, the intensity of which is assumed to be proportional to the one of climate events. This additional risk counts for the possible introduction of more stringent environmental regulations, which are responsible for left-tail losses since, as shown in [16], policy changes not motivated by economic objectives push stock prices down. It occurs for two main reasons. First, cash flows reduce because of the negative impact of the new policies, so-called political cost. Second, discount rates increase because of political uncertainty, that is, investors ask a premium for the uncertainty about political cost. To close the model, a Feller square-root process[2] is chosen for the two time-varying stochastic intensities for both macroeconomic and environmental risk, with the coefficients in the dynamics of environmental risk depending on (exogenously estimated) global anomalous temperatures taken as a proxy for climate change. Analytical results indicate that physical risk contributes to increase equity premium, stock market volatility, and reduce risk-free interest rate. In addition to that, transition risk impacts on the composition of the optimal portfolio by reducing the weight of the brown asset.

The flexible framework proposed in [15] can be enriched to accommodate a further key feature (more and more relevant as climate change progresses) of physical risk, that is the clustering of losses caused by cascades of rare natural disasters, which is one of the objectives of the current work. To reach this goal, we are inspired by Brachetta et al. [18], Santacroce and Trivellato [19], and for the dynamics of the intensity of physical risk we assume the model introduced in [20], which is a generalization of the Cox process with shot noise intensity and the Hawkes process; see, e.g., [21]. This model for rare events allows us to accommodate

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[1] NaTech are accidents that can involve releases of hazardous materials, fires, or explosions, and are considered a secondary effect of the natural disaster. A recent example was the 2017 Arkema chemical plant explosion and fire following from Hurricane Harvery flooding in the Houston region. The prevalence of such event is increasing, especially in the United States, see, e.g., [8].

[2] Feller square-root processes are mean-reverting square-root diffusion processes used for the first time in finance in the famous Cox–Ingersoll–Ross (CIR) model for the dynamics of the short interest rate, see, e.g. [17], and it is widely used in finance for modeling intensities of jump processes due to its peculiarities, such as analytical tractability and the zero-probability to breach the zero value.

the self- and externally excited time-varying nature of physical risk. We refer to [20] for a complete description of this model of rare events and applications to credit risk, and to [22] for some recent applications in insurance. The intensity of the transition risk is instead assumed to be constant to preserve analytical tractability and used as a policy parameter to conduct scenario analysis. Specifically, the policy shocks, that reduce the value of the brown investment only, are driven by a simple Poisson process. Except for these peculiarities, the model is as in [15].

Under mild technical conditions, we prove the existence of a solution to the recursive stochastic general equilibrium utility maximization problem defined in our framework. The associated value function is derived in closed form. Furthermore, we show that optimal consumption is a constant proportion of wealth, while the optimal portfolio composition is characterized as the solution to an equation that admits an explicit form in the absence of policy risk. The value function (dependent on the wealth and on the stochastic intensities of natural events and macroeconomic shocks) is described through three coefficients that are expressed as solution of a system of three ordinary differential equations. Analytical studies indicate that the composition of the optimal portfolio turns out to be not affected by macroeconomic and environmental risks, but only on the returns and volatility of the assets; while the presence of policy shocks increases the proportion of green investments in the portfolio, as expected. The risk-free interest rate and the instantaneous expected return on government debt are reduced, while the equity premium is increased, by the effects of the three (environmental, macroeconomic and policy) shocks. These effects are amplified by the representative agent's risk aversion, that increases propensity to save as a precautionary measure, and by the presence of self-excited and externally-excited jumps in the intensities of the environmental damages. Numerical studies are conducted to quantify the impact of the self-excited and externally excited time-varying probability of environmental disasters on the relevant economic quantities.

The paper is organized as follows. Section 2 describes the market model. Section 3 states the equilibrium recursive utility optimization problem, derives the closed-form solution, its main features and special cases. Section 4 presents the impact of environmental, macroeconomic, and policy shocks on the risk-free interest rate, the equity premium, and the government bond spread. A numerical illustration is provided in Section 5. Section 6 contains indications for future extensions and concludes. Appendix A contains the proofs of the theoretical results.

2. Market model

We build on the general equilibrium model of [17] by incorporating jumps in the economy's production possibilities, following [12,23,24], as well as time-varying risk as in [14]. Consistent with Karydas and Xepapadeas [15], we consider two sources of time-varying risk: pure macroeconomic disasters, also included in [14], and environmental disasters, including those driven by climate change. The damages caused by these natural disasters related to climate change are the source of what is commonly referred to as physical risk. A distinctive feature, further accentuated by climate change, is the clustering of these rare natural disasters. To capture this characteristic, we extend the framework in [15] by incorporating both self-exciting and exogenous jumps into the time-varying risk profile of natural disasters. Similar processes are used to model clustering of losses in insurance models; see, for example, [18,22]. In addition, climate change gives rise to another category of financial risk known as transition risk, which relates to the risks associated with stringent green economic policies that impose additional costs on production activities.

The production side of the economy is characterized by a single physical numeraire good, which can be allocated either to consumption or investment. For parsimony, its production is governed by two distinct linear technologies: a brown activity (B), which is exposed to the risk of stringent climate policy measures, and a green sector (G), which differs from B in that it remains unaffected by such regulatory interventions.

Specifically, on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F} = \{\mathcal{F}_t; t \geq 0\})$ satisfying the usual conditions, we consider a financial market consisting of two risky production opportunities with price dynamics:

$$\begin{aligned} \frac{dY_{G,t}}{Y_{G,t^-}} &= \mu_G dt + \sigma_G dW_{G,t} + \sum\nolimits_{j\in\{M,E\}} dQ_t^j \\ \frac{dY_{B,t}}{Y_{B,t^-}} &= \mu_B dt + \sigma_B dW_{B,t} + \sum\nolimits_{j\in\{M,E\}} dQ_t^j + dQ_t^X \end{aligned} \tag{1}$$

where μ_G, μ_B, σ_G and σ_B are constant and positive parameters, W_G and W_B are two correlated standard Wiener processes with a smaller than $\min\left\{\frac{\sigma_G}{\sigma_B}, \frac{\sigma_B}{\sigma_G}\right\}$ constant correlation,[3] and the components Q^j, with $j \in \{M, E, X\}$ are the jumps components.[4] $Q^X = \left(e^X - 1\right) N^X$ is written in terms of a negative constant X and a Poisson process N^X with constant intensity[5] $\rho_X > 0$, whereas Q^j, with $j \in \{M, E\}$, are (generalized) compound Poisson processes written in terms of two counting processes N^j with stochastic intensities λ^j. Specifically, $Q^j = \sum_{n=1}^{N^j}\left(e^{Z_n^j} - 1\right)$, where $\{Z_n^j\}_{n\geq 1}$ is a sequence of i.i.d. random variables with values in $(-\infty, 0)$

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[3] A correlation less than $\min\left\{\frac{\sigma_G}{\sigma_B}, \frac{\sigma_B}{\sigma_G}\right\}$ is a technical condition that is used in the following. Economically, it implies that while the green and brown assets may share positive macroeconomic trends, their risk profiles remain sufficiently distinct. They are not perfectly substitutable, meaning an investor cannot perfectly hedge one simply by short-selling the other, and must therefore hold positive quantities of both to construct the optimal portfolio.

[4] To lighten the notation we denote the processes $\varphi = \{\varphi_t, t \geq 0\}$ only by φ. From time to time φ will also denote φ_t, the value of the process at the generic time t.

[5] We assume a constant intensity for the policy shock, instead of being proportional to λ^E as in [15], in order to obtain a closed form solution for the Hamilton–Jacobi–Bellman equation related to the optimization problem. We conjecture that an extension with a time-varying, deterministic policy shock intensity, possibly driven by the expected (mean) environmental variables is viable.

and distribution function denoted by F_{Z^j}.[6] We assume that $\{Z_n^j\}_{n\geq 1}$ and N^j, with $j \in \{M, E\}$, are mutually independent and also independent of the Wiener processes W_G and W_B.

The diffusion term $\mu_i Y_{i,-} dt + \sigma_i Y_{i,-} dW_i$ represents the behavior of production process $i = \{G, B\}$ in normal times, and implies that when no disasters take place, log production growth over an interval Δt is normally distributed with mean $\left(\mu_i - \frac{\sigma_i^2}{2}\right)\Delta t$ and variance $\sigma_i^2 \Delta t$. Economic losses due to macroeconomic disasters are captured by the compound Poisson process Q^M, which allows for large instantaneous changes in both Y_G and Y_B. As in [14], the intensity of macroeconomic disaster is supposed to be time-varying and follows the process

$$d\lambda_t^M = k^M \left(\bar{\lambda}^M - \lambda_t^M\right) dt + \sigma_\lambda^M \sqrt{\lambda_t^M} dW_{\lambda,t}^M, \quad \lambda_0^M > 0, \tag{2}$$

where the mean-reversion speed k^M, the long-run mean $\bar{\lambda}^M$ and the volatility parameter σ_λ^M are positive real numbers, while W_λ^M is a Wiener process independent of all other processes. The Feller condition (see, [25]) $2k^M \bar{\lambda}^M > \left(\sigma_\lambda^M\right)^2$ is imposed to guarantee positive values of λ^M. As [17] discuss, the solution to (2) has a stationary distribution which is a Gamma distribution with shape parameter $2k^M \bar{\lambda}^M / \left(\sigma_\lambda^M\right)^2$ and scale parameter $\left(\sigma_\lambda^M\right)^2 / \left(2k^M\right)$. It is known that the stationary distribution of λ^M is highly skewed. The skewness arises from the square-root term multiplying the Brownian shock in (2): this square root term implies that the high realizations of λ^M make the process more volatile, and thus further high realizations more likely than they would be under a standard autoregressive process. The model therefore implies that there are times when rare macroeconomic disasters can occur with high probability, but these times are themselves unusual.

Economic losses due to environmental disasters are captured by the compound Poisson process Q^E, which also allows for instantaneous changes in both Y_G and Y_B. Karydas and Xepapadeas [15] assumed a process similar to (2) for the arrival rate of natural disasters λ^E. As we consider environmental risks (not only physical risks), we enrich this modeling framework by adding the jump components in the dynamics of λ^E, which follows the process:

$$d\lambda_t^E = k^E \left(\bar{\lambda}^E(\Upsilon) - \lambda_{t^-}^E\right) dt + \sigma_\lambda^E \sqrt{\lambda_{t^-}^E} dW_{\lambda,t}^E + dQ_{\lambda,t}^{self} + dQ_{\lambda,t}^{ext}, \quad \lambda_0^E > 0, \tag{3}$$

where $k^E > 0$, $\bar{\lambda}^E(\Upsilon) \equiv \tilde{\lambda}^E + \xi\Upsilon$ and $\sigma_\lambda^E > 0$. In addition, the parameters $\tilde{\lambda}^E > 0$ and $\xi \geq 0$, express the dependence of the long-term mean of the intensities of environmental disasters on the change in global average temperature relative to a given time period Υ, taken as a proxy for climate change.

In (3), W_λ^E is a Wiener process independent of all other processes and Q_λ^{self}, Q_λ^{ext} are compound Poisson processes with intensities λ^E and the constant $\rho_\lambda > 0$, respectively. More specifically, the self-excited jumps in λ^E occur concurrently to environmental jumps in (1), that is $Q_\lambda^{self} = \sum_{n=1}^{N^E} S_n$, where $\{S_n\}_{n\geq 1}$ is a sequence of i.i.d. random variables with values in $(0, +\infty)$ and distribution function F_S. The externally excited part of λ^E is given by $Q_\lambda^{ext} = \sum_{n=1}^{N^{ext}} R_n$, where $\{R_n\}_{n\geq 1}$ is a sequence of i.i.d. random variables with values in $(0, +\infty)$ and distribution function F_R. We assume that $\{S_n\}_{n\geq 1}$, $\{R_n\}_{n\geq 1}$ and the counting process N^{ext} are mutually independent and independent of all other processes except N^E. Following [20,26], we suppose $2k^E \tilde{\lambda}^E > \left(\sigma_\lambda^E\right)^2$ to ensure the positiveness of the intensity process λ^E and $\mathbb{E}(S) = \int_0^\infty s\, F_S(ds) < k^E$ to guarantee its stationarity.

Finally, brown production is also exposed to a policy shock Q^X that reduces its log return by a fixed quantity $X < 0$.

The self-excited jumps represent the novelty in the model of λ^E, which captures the impact of environmental shocks on the probability of the occurrence of future environmental shocks.

In comparison to the standard square-root mean-reverting process, the presence of self-excited jumps implies a more likely persistence in high values of λ^E, while the presence of externally-excited jumps implies a higher frequency of observation of high values of λ^E. The combined effect of the two types of jump, together with the impact of a higher λ^E on the volatility, implies moments when a cluster of rare environmental disasters may occur with high probability, and these times may not be unusual.

3. The representative-agent problem

Let us consider a continuous-time analogue of the utility function defined by Epstein and Zin [27], Weil [28] that generalizes power utility to allow for preferences over the timing of the resolution of uncertainty. The continuous version is formulated by Duffie and Epstein [29]: we make use of a limiting case of their model that sets the parameter associated with the inter-temporal elasticity of substitution (EIS) equal to one. Hence, the representative agent maximizes the utility function U_t defined by the following recursion:

$$U_t = \mathbb{E}_t \int_t^\infty f\left(C_s, U_s\right) ds, \tag{4}$$

where

$$f(C, U) = \rho(1-\gamma) U \left(\log C - \frac{1}{1-\gamma} \log((1-\gamma)U)\right). \tag{5}$$

In (4), $\mathbb{E}_t$ denotes the expectation conditional on the information available at time t, $\mathcal{F}_t$, while C represents the consumption process. Therefore, U_t stands for the continuation utility, that is, utility associated to the future consumption stream. Existence and uniqueness

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[6] Z_n^j, as well as the other sequences of jump sizes, will be required to satisfy additional integrability conditions.

of the stochastic differential utility process U_t can be established under suitable integrability conditions; we refer to [30,31]. The parameter $\rho > 0$ is the (subjective) rate of time preference and, following a common practice, $\gamma > 0$ represents the agent's relative risk aversion. As γ approaches one, (5) can be shown to be ordinarily equivalent to logarithmic utility. In the following, we focus on the case $\gamma > 1$ as in [14,15], which implies a preference for early resolution of uncertainty [32].

3.1. Optimality conditions

We consider a representative agent who allocates her wealth in order to maximize her lifetime (infinite horizon) utility deciding among immediate consumption and investments in two risky production opportunities, a brown and a green risky asset, with price dynamics (1) and a riskless asset with instantaneous rate of return r. We denote by n_B and n_G the fractions of wealth A invested in the brown and green assets, respectively. Furthermore, we assume that the residual wealth is invested in the riskless asset. Then, we can write the dynamics of the wealth process A as

$$\begin{aligned} dA_t &= \left(\sum_{i=G,B} n_{i,t}\left(\mu_i - r_{t^-}\right) A_{t^-} + r_{t^-} A_{t^-} - C_{t^-} \right) dt + \sum_{i=G,B} n_{i,t}\sigma_i A_{t^-} dW_{i,t} \\ &\quad + \left(n_{G,t} + n_{B,t}\right) A_{t^-} \left(\sum_{j\in\{M,E\}} dQ_t^j \right) + n_{B,t} A_{t^-} dQ_t^X . \end{aligned} \tag{6}$$

The investor's aim is to maximize U_0, where U_t is defined in (4) and (5), over progressively measurable and strictly positive consumption processes C and portfolio predictable proportion strategies n_G and n_B such that the associated wealth process A is positive.

Let $V\left(A, \lambda^M, \lambda^E\right)$ be the value function (maximized utility) in states $\lambda^M \equiv \lambda^M_{t^-}$, $\lambda^E \equiv \lambda^E_{t^-}$ with wealth $A \equiv A_{t^-}$ and denote by $v = \left(C, n_G, n_B\right) \in \mathbb{R}^3_+$ the control variables at time t. Then, following [29] the optimal consumption expenditure and portfolio choices must satisfy the Hamilton–Jacobi–Bellman equation:

$$\sup_{v\in\mathbb{R}^3_+} \left\{ L^v\left(V\left(A, \lambda^M, \lambda^E\right)\right) + f\left(C, V\left(A, \lambda^M, \lambda^E\right)\right) \right\} = 0, \tag{7}$$

with $L^v[\cdot]$ a differential operator defined as

$$\begin{aligned} L^v\left[V\left(A, \lambda^M, \lambda^E\right)\right] &= V_A\left(A, \lambda^M, \lambda^E\right) \left(\sum_{i=G,B} n_i\left(\mu_i - r\right) A + rA - C \right) + \frac{1}{2} V_{AA}\left(A, \lambda^M, \lambda^E\right) \left(\sigma^A\left(n_G, n_B\right)\right)^2 A^2 \\ &\quad + \sum_{j=M,E} \left(V_{\lambda^j}\left(A, \lambda^M, \lambda^E\right) k^j \left(\bar{\lambda}^j - \lambda^j\right) + \frac{1}{2} V_{\lambda^j\lambda^j}\left(A, \lambda^M, \lambda^E\right) (\sigma^j_\lambda)^2 \lambda^j \right) \\ &\quad + \int_0^{+\infty} \int_{-\infty}^0 \left[V\left(A\left(1 - \left(n_G + n_B\right)\left(1 - e^{z^E}\right)\right), \lambda^M, \lambda^E + s \right) - V\left(A, \lambda^M, \lambda^E\right) \right] \lambda^E F_{Z^E}\left(dz^E\right) F_S(ds) \\ &\quad + \int_0^{+\infty} \left[V\left(A, \lambda^M, \lambda^E + \theta\right) - V\left(A, \lambda^M, \lambda^E\right) \right] \rho_\lambda F_R(d\theta) \\ &\quad + \int_{-\infty}^0 \left[V\left(A\left(1 - \left(n_G + n_B\right)\left(1 - e^{z^M}\right)\right), \lambda^M, \lambda^E \right) - V\left(A, \lambda^M, \lambda^E\right) \right] \lambda^M F_{Z^M}\left(dz^M\right) \\ &\quad + \left[V\left(A\left(1 - n_B\left(1 - e^X\right)\right), \lambda^M, \lambda^E\right) - V\left(A, \lambda^M, \lambda^E\right) \right] \rho_X , \end{aligned} \tag{8}$$

where the subscript of V denotes partial derivatives,[7] i.e., V_x is the partial derivative of V with respect to x and V_{xy} is the partial derivative of V with respect to x and y, for $x, y \in \left\{A, \lambda^M, \lambda^E\right\}$, and

$$\sigma^A\left(n_G, n_B\right) = \sqrt{n_B^2\sigma_B^2 + n_G^2\sigma_G^2 + 2n_B n_G \sigma_{GB}}, \quad \sigma_{GB} = \sigma_B\sigma_G corr\left[dW_G, dW_B\right]. \tag{9}$$

The first-order conditions with respect to C, n_G, n_B related to problem (7) give:

$$f_C\left(C, V\left(A, \lambda^M, \lambda^E\right)\right) = V_A\left(A, \lambda^M, \lambda^E\right), \tag{10}$$

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[7] Throughout the paper we adopt the convention that the t subscript on processes will be omitted when not essential for clarity. Moreover, the subscripts of a differentiable function $\phi(x, y)$ will denote partial derivatives, i.e., $\phi_x = \frac{\partial\phi}{\partial x}$, $\phi_{x,y} = \frac{\partial^2\phi}{\partial x\partial y}$.

$$
\begin{aligned}
r = \mu_G \quad & + \quad \frac{AV_{AA}\left(A,\lambda^M,\lambda^E\right)}{V_A\left(A,\lambda^M,\lambda^E\right)}\left(n_G\sigma_G^2 + n_B\sigma_{GB}\right) \\
& - \quad \int_0^{+\infty}\int_{-\infty}^0 \frac{V_A\left(A\left(1-\left(n_G+n_B\right)\left(1-e^{z^E}\right)\right),\lambda^M,\lambda^E+s\right)}{V_A\left(A,\lambda^M,\lambda^E\right)}\lambda^E\left(1-e^{z^E}\right)F_{Z^E}\left(dz^E\right)F_S\left(ds\right) \\
& - \quad \int_{-\infty}^0 \frac{V_A\left(A\left(1-\left(n_G+n_B\right)\left(1-e^{z^M}\right)\right),\lambda^M,\lambda^E\right)}{V_A\left(A,\lambda^M,\lambda^E\right)}\lambda^M\left(1-e^{z^M}\right)F_{Z^M}\left(dz^M\right),
\end{aligned}
\tag{11}
$$

and

$$
\begin{aligned}
r = \mu_B \quad & + \quad \frac{AV_{AA}\left(A,\lambda^M,\lambda^E\right)}{V_A\left(A,\lambda^M,\lambda^E\right)}\left(n_B\sigma_B^2 + n_G\sigma_{GB}\right) \\
& - \quad \int_0^{+\infty}\int_{-\infty}^0 \frac{V_A\left(A\left(1-\left(n_G+n_B\right)\left(1-e^{z^E}\right)\right),\lambda^M,\lambda^E+s\right)}{V_A\left(A,\lambda^M,\lambda^E\right)}\lambda^E\left(1-e^{z^E}\right)F_{Z^E}\left(dz^E\right)F_S\left(ds\right) \\
& - \quad \int_{-\infty}^0 \frac{V_A\left(A\left(1-\left(n_G+n_B\right)\left(1-e^{z^M}\right)\right),\lambda^M,\lambda^E\right)}{V_A\left(A,\lambda^M,\lambda^E\right)}\lambda^M\left(1-e^{z^M}\right)F_{Z^M}\left(dz^M\right) \\
& - \quad \frac{V_A\left(A\left(1-n_B\left(1-e^X\right)\right),\lambda^M,\lambda^E\right)}{V_A\left(A,\lambda^M,\lambda^E\right)}\rho_X\left(1-e^X\right).
\end{aligned}
\tag{12}
$$

The first equation in (10) is the usual envelope condition for the price of consumption. The system of Eqs. (11)–(12) solves, instead, the investor's risky portfolio allocation problem. In equilibrium it holds that the risk-free asset is in zero net supply such that $n_B+n_G=1$, see, e.g., [32]. Hence, we substitute n_G with $1-n_B$ and equating the right hand sides of (11) and (12), we obtain the no-arbitrage condition between risky assets; after adjusting for their relative risk, each risky asset should yield the same marginal expected return:

$$
\begin{aligned}
\mu_B + \frac{AV_{AA}\left(A,\lambda^M,\lambda^E\right)}{V_A\left(A,\lambda^M,\lambda^E\right)}n_B\left(\sigma_B^2-\sigma_{GB}\right) - \frac{V_A\left(A\left(1-n_B\left(1-e^X\right)\right),\lambda^M,\lambda^E\right)}{V_A\left(A,\lambda^M,\lambda^E\right)}\rho_X\left(1-e^X\right) = \\
\mu_G + \frac{AV_{AA}\left(A,\lambda^M,\lambda^E\right)}{V_A\left(A,\lambda^M,\lambda^E\right)}\left(1-n_B\right)\left(\sigma_G^2-\sigma_{GB}\right).
\end{aligned}
\tag{13}
$$

Eq. (13) will be used to calculate the optimal portfolio allocation n_B. Moreover, we can compare n_B with the allocation which minimizes portfolio volatility in times without disasters. In particular, since from the first equation in (9) portfolio volatility is a convex quadratic function, its minimum value is reached at

$$
n_{B,min} = \frac{\sigma_G^2-\sigma_{GB}}{\sigma_G^2+\sigma_B^2-2\sigma_{GB}}.
\tag{14}
$$

Note that, as in [33], there are two opposing effects relevant for portfolio composition which rely on the two volatilities of the diffusion components of the green and brown assets, and on their correlation. On the one hand both assets are needed in general for a diversified portfolio which will be perfectly balanced if the two dynamics have the same volatility; on the other hand, whenever σ_G^2 and σ_B^2 are different, the optimal allocation will penalize the asset which is more volatile and the magnitude will decrease with the absolute value of the correlation (or also of the covariance σ_{GB}) defined in (9). Now, taking in mind the benchmark $n_{B,min}$, we observe that (13) can be easily rewritten as

$$
n_B = n_{B,min} + \frac{1}{\hat{\sigma}_{GB}^2\,RRA}\left[\left(\mu_B-\mu_G\right)\underbrace{-\frac{V_A\left(A\left(1-n_B\left(1-e^X\right)\right),\lambda^M\lambda^E\right)}{V_A\left(A,\lambda^M,\lambda^E\right)}\rho_X\left(1-e^X\right)}_{<0}\right],
\tag{15}
$$

where we denoted $RRA = -\frac{AV_{AA}\left(A,\lambda^M,\lambda^E\right)}{V_A\left(A,\lambda^M,\lambda^E\right)}$ and $\hat{\sigma}_{GB}^2 = \sigma_G^2+\sigma_B^2-2\sigma_{GB}$. Therefore, compared to $n_{B,min}$, the optimal portfolio allocation n_B is determined by two additional components that act proportionally to a coefficient which takes into account risk aversion, volatilities and correlation. The first component, written in terms of the difference in the expected returns, favors the investment with the highest return, whereas the second is linked only to the policy risk. Indeed, since both assets are equally affected by extreme Poisson events of both types, the investor only reacts to the policy risk on climate-sensitive assets by lowering n_B.

3.2. The value function

In the following theorem, we find the value function, the optimal consumption (in closed form), together with the optimal portfolio allocation. The proof is in Appendix A.

Theorem 1 (*Value Function*)**.** *For preferences defined by (4) and (5), the value function that solves (7) is*

$$V\left(A,\lambda^M,\lambda^E\right)=\frac{A^{1-\gamma}}{1-\gamma}e^{d+b^M\lambda^M+b^E\lambda^E}, \tag{16}$$

where

$$\begin{aligned} d={}&(1-\gamma)(\log\rho-1)+\frac{1-\gamma}{\rho}\left(\sum_{i=G,B}n_i\mu_i-\frac{1}{2}\gamma\left(\sigma^A\left(n_G,n_B\right)\right)^2\right)\\ &+\sum_{j\in\{M,E\}}b^j\frac{k^j\bar{\lambda}^j}{\rho}+\frac{\rho_\lambda}{\rho}\left(\mathbb{E}\left[e^{b^E R}\right]-1\right)+\frac{\rho_X}{\rho}\left[\left(1-n_B\left(1-e^X\right)\right)^{1-\gamma}-1\right] \end{aligned} \tag{17}$$

$$b^M=\frac{k^M+\rho}{\left(\sigma_\lambda^M\right)^2}-\sqrt{\left(\frac{k^M+\rho}{\left(\sigma_\lambda^M\right)^2}\right)^2-2\,\frac{\mathbb{E}\left[e^{(1-\gamma)Z^M}\right]-1}{\left(\sigma_\lambda^M\right)^2}} \tag{18}$$

and b^E solves

$$\frac{1}{2}\left(\sigma_\lambda^E\right)^2\left(b^E\right)^2-\left(\rho+k^E\right)b^E+\mathbb{E}\left[e^{b^E S}\right]\mathbb{E}\left[e^{(1-\gamma)Z^E}\right]-1=0 \tag{19}$$

provided that the mean values in (17)–(19) are finite and the square root in (18) is well defined. The optimal consumption is $C=\rho A$ and the optimal portfolio allocation n_B solves

$$\left(\sigma_G^2+\sigma_B^2-2\sigma_{GB}\right)\gamma\,n_B+\left(1-n_B\left(1-e^X\right)\right)^{-\gamma}\rho_X\left(1-e^X\right)=\mu_B-\mu_G+\gamma\left(\sigma_G^2-\sigma_{GB}\right). \tag{20}$$

The existence and uniqueness of n_B in $(0,1)$ is guaranteed for

$$-\gamma(\sigma_B^2-\sigma_{GB})<\mu_G-\mu_B+\rho_X(1-e^X)<\gamma(\sigma_G^2-\sigma_{GB}). \tag{21}$$

Regarding the optimal portfolio allocation, similarly to (15), we can rewrite (20) as

$$n_B=n_{B,min}+\frac{\mu_B-\mu_G}{\hat{\sigma}_{GB}^2\gamma}-\frac{\left(1-n_B\left(1-e^X\right)\right)^{-\gamma}\rho_X\left(1-e^X\right)}{\hat{\sigma}_{GB}^2\gamma} \tag{22}$$

noting that $\gamma=RRA=-\dfrac{AV_{AA}\left(A,\lambda^M,\lambda^E\right)}{V_A\left(A,\lambda^M,\lambda^E\right)}$.

For a graphic inspection, in Fig. 1 the optimal value of n_B is plotted in black as a function of the risk aversion coefficient γ, for $X=-2$, $\rho_X=0.1$, $\sigma_B=\sigma_G=0.4$, $\sigma_{GB}=0.016$, $\mu_G=1$ and for different values of μ_B. Specifically, $\mu_B=1.5$ in panel(a), $\mu_B=\mu_G=1$ in panel (b) and $\mu_B=0.8$ in panel (c). The value of n_B that minimizes portfolio volatility, $n_{B,min}$, is instead reported in red. Note that in panel (b), since $\mu_B=\mu_G=1$, the difference between $n_{B,min}$ and the optimal value of n_B is entirely due to the presence of transition risk. The negative value of this difference indicates that transition risk reduces the weight of brown assets in the portfolio regardless of the value of the risk aversion coefficient. However, this difference does not have a monotonic trend with respect to γ. In particular, we observe that in absolute value this difference first decreases and then increases as γ increases. In panel (c), where $\mu_G>\mu_B$ (green premium) we have a similar qualitative behavior of the optimal n_B, however, from a quantitative point of view we observe a higher weight of green asset in the optimal portfolio. Instead, in panel (a), where $\mu_G<\mu_B$ (brown premium), the optimal value of n_B is monotonically decreasing with respect to the risk adverse coefficient γ and, for γ close to one we observe that the optimal n_B is even higher than $n_{B,min}$, that is, the brown premium component prevails on transition risk.

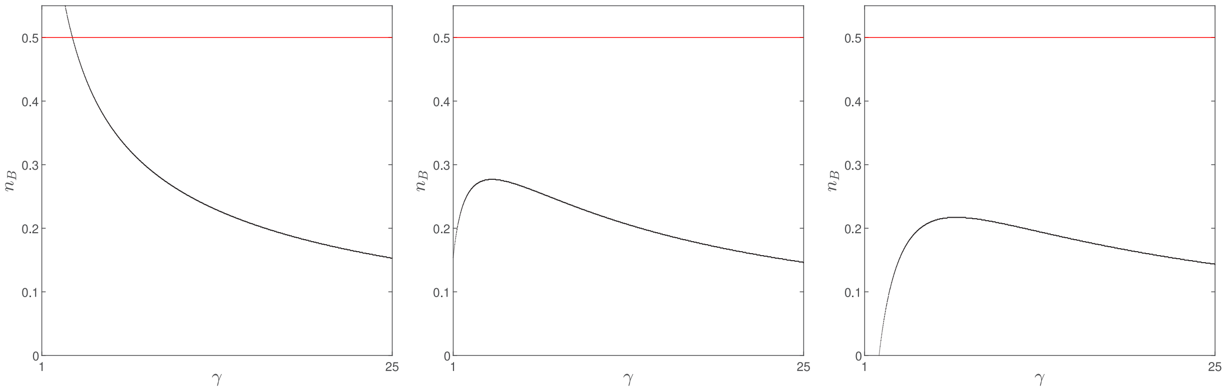
Regarding the value function, we comment on Eqs. (17)–(20). Specifically, from (17), we observe that d depends on the solutions of (18), (19) and (20), whose details will be discussed later, as well as on the policy X, which influences the value function solely through its effect on d. In particular, from (22), we see that the presence of X leads to a decrease in the quota of portfolio invested in the brown technology n_B, increasing d, which reaches its minimum in case of no transition risk ($X=0$). Hence, together with the contribution of the last term of (17), d increases. As a result, the value function in (16) decreases since $1-\gamma<0$.

With respect to b^M, the fact that the quantities under the square root of (18) have to be non-negative places a joint restriction on the severity of disasters, the risk aversion, the rate of time preferences and the volatility of disasters; see [34]. Under our standing assumptions $k^M>0$, $\rho>0$ and $\gamma>1$, it is $b^M>0$ when we impose some restrictions on the severity of macroeconomic disasters Z^M, that is,

$$\mathbb{E}\left[e^{(1-\gamma)Z^M}\right]\leq 1+\frac{\left(k^M+\rho\right)^2}{2\left(\sigma_\lambda^M\right)^2}. \tag{23}$$

We remark that b^M in (18) comes from the quadratic equation

$$\frac{1}{2}\left(\sigma_\lambda^M\right)^2\left(b^M\right)^2-\left(\rho+k^M\right)b^M+\mathbb{E}\left[e^{(1-\gamma)Z^M}\right]-1=0, \tag{24}$$



(a) (b) (c)

Fig. 1. The optimal n_B compared with $n_{B,min}$ (horizontal line), both represented as a function of the risk aversion parameter γ in the range $[1,4]$. Panel (a): $\mu_B = 1.5$; Panel (b) $\mu_B = 1$; Panel (c) $\mu_B = 0.8$. Remaining parameters: $X = -2$; $\rho_X = 0.1$; $\sigma_G = 0.4$; $\sigma_B = 0.4$; $\sigma_{GB} = 0.016$ and $\mu_G = 1$.

which has as solutions

$$b^M_{1,2} = \frac{k^M+\rho}{\left(\sigma^M_\lambda\right)^2} \pm \sqrt{\left(\frac{k^M+\rho}{\left(\sigma^M_\lambda\right)^2}\right)^2 - 2\frac{\mathbb{E}\left[e^{(1-\gamma)Z^M}-1\right]}{\left(\sigma^M_\lambda\right)^2}}. \tag{25}$$

Although the presence of two roots in (25) suggests multiple possible solutions, as suggested by Wachter [14], only the smaller solution, that is the solution with the negative sign in front of the square root, displays reasonable economic properties, which is also consistent with the intrinsic meaning of the value function as a utility maximizer. Indeed, in case of Z^M identically equal to zero, we have that the Poisson process N^M has positive realizations, but these have no economic consequence. There are no disasters in this case and the value function should reduce to its counterpart under the model with no macroeconomic disasters. However, the choice of the positive root in (25) implies that the representative agent's utility is reduced by an increased likelihood of these inconsequential Poisson realizations. The choice of the negative root does not suffer from this defect.

In order to obtain the value function (16), it is still needed to solve (19). In Proposition 1 we give conditions that guarantee the existence of a meaningful economic solution of (19), which, differently from Wachter [14], is not straightforward. In Proposition 2, the determination of an explicit solution of (19) is provided under the assumption that self excited jumps are exponentially distributed.

Proposition 1. *Let S be a positive random variable such that $\mathbb{E}\left[e^{xS}\right] < +\infty$ and $\mathbb{E}\left[Se^{xS}\right] < +\infty$ for $x \in \left[0, \frac{\rho+k^E}{\left(\sigma^E_\lambda\right)^2}\right]$, and $\mathbb{E}(S)\mathbb{E}\left[e^{(1-\gamma)Z^E}\right] < \rho + k^E$. Define*

$$\Phi(x;S) = \frac{1}{2}\left(\sigma^E_\lambda\right)^2 (x)^2 - \left(\rho + k^E\right)x + \mathbb{E}\left[e^{xS}\right]\mathbb{E}\left[e^{(1-\gamma)Z^E}\right] - 1. \tag{26}$$

- *(i) If $\Phi(\tilde{x};S) \le 0$ at the stationary point $\tilde{x} \in \left(0, \frac{\rho+k^E}{\left(\sigma^E_\lambda\right)^2}\right)$, then there exists a unique solution x_S of $\Phi(x;S) = 0$ in $[0,\tilde{x}]$;*
- *(ii) Consider (26) with $S = 0$. If $\Phi(\tilde{x};0) \le 0$ at the stationary point $\tilde{x} = \frac{\rho+k^E}{\left(\sigma^E_\lambda\right)^2}$ or, equivalently, if $\mathbb{E}\left[e^{(1-\gamma)Z^E}\right] \le 1 + \frac{\left(k^E+\rho\right)^2}{2\left(\sigma^E_\lambda\right)^2}$, then the unique solution of $\Phi(x;0) = 0$ in $[0,\tilde{x}]$ is given by*

$$x_0 = \frac{k^E+\rho}{\left(\sigma^E_\lambda\right)^2} - \sqrt{\left(\frac{k^E+\rho}{\left(\sigma^E_\lambda\right)^2}\right)^2 - 2\frac{\mathbb{E}\left[e^{(1-\gamma)Z^E}-1\right]}{\left(\sigma^E_\lambda\right)^2}}; \tag{27}$$

- *(iii) Consider (26) with the second-order approximation $e^{xS} \simeq 1 + xS + \frac{1}{2}x^2S^2$, and denote*

$$\begin{aligned}\Phi_{ap}(x;S) = \frac{1}{2}\left(\left(\sigma^E_\lambda\right)^2 + \mathbb{E}(S^2)\mathbb{E}\left[e^{(1-\gamma)Z^E}\right]\right)x^2 - \left(\rho + k^E - \mathbb{E}(S)\mathbb{E}\left[e^{(1-\gamma)Z^E}\right]\right)x \\ + \mathbb{E}\left[e^{(1-\gamma)Z^E}\right] - 1.\end{aligned} \tag{28}$$

If $\Phi_{ap}(\tilde{x}; S) \leq 0$ *at the stationary point*

$$\tilde{x} = \frac{\rho + k^E - \mathbb{E}(S)\mathbb{E}\left[e^{(1-\gamma)Z^E}\right]}{\left(\sigma_\lambda^E\right)^2 + \mathbb{E}(S^2)\mathbb{E}\left[e^{(1-\gamma)Z^E}\right]}$$

or, equivalently, if $\mathbb{E}\left[e^{(1-\gamma)Z^E}\right] \leq 1 + \frac{\left(\rho+k^E-\mathbb{E}(S)\mathbb{E}\left[e^{(1-\gamma)Z^E}\right]\right)^2}{2\left(\left(\sigma_\lambda^E\right)^2+\mathbb{E}(S^2)\mathbb{E}\left[e^{(1-\gamma)Z^E}\right]\right)}$, *then the unique solution of* $\Phi_{ap}(x; S) = 0$ *in* $[0, \tilde{x}]$ *is given by*

$$x_{ap} = \frac{\rho + k^E - \mathbb{E}(S)\mathbb{E}\left[e^{(1-\gamma)Z^E}\right]}{\left(\sigma_\lambda^E\right)^2 + \mathbb{E}(S^2)\mathbb{E}\left[e^{(1-\gamma)Z^E}\right]} - \sqrt{\left(\frac{\rho + k^E - \mathbb{E}(S)\mathbb{E}\left[e^{(1-\gamma)Z^E}\right]}{\left(\sigma_\lambda^E\right)^2 + \mathbb{E}(S^2)\mathbb{E}\left[e^{(1-\gamma)Z^E}\right]}\right)^2 - 2\frac{\mathbb{E}\left[e^{(1-\gamma)Z^E} - 1\right]}{\left(\sigma_\lambda^E\right)^2 + \mathbb{E}(S^2)\mathbb{E}\left[e^{(1-\gamma)Z^E}\right]}}; \tag{29}$$

(iv) The following inequalities hold:

$$x_0 < x_{ap} < x_S. \tag{30}$$

Using assertion (iv) of the above proposition, we deduce the following meaningful result.

Corollary 2. *Under the assumptions of Proposition 1, the presence of self-excited jumps reduces the value function.*

With respect to the externally-excited jumps, we observe that they impact only on d. Specifically, d increases of the amount $\frac{\rho_\lambda}{\rho}\left(\mathbb{E}\left[e^{b^E R}\right] - 1\right) > 0$ because of externally-excited jumps. Since we assume $\gamma > 1$, this implies that an increase of environmental disaster risk due to externally-excited jumps reduces utility of the representative agent. Moreover, the amplitude of this reduction depends positively on b^E, which solves (19) and so depends on self-excited jumps. In particular, by (30) in Proposition 1 we see that b^E increases because of self-excited jumps. So, self-excited jumps amplifies the effect of externally-excited jumps as we might expect since the increased risk of environmental disaster caused by an externally excited jump is amplified by the presence of self-excited jumps.

Since we assume $\gamma > 1$, and $b^M > 0$ ($b^E > 0$), by (16), an increase in macroeconomic (environmental) disaster risk reduces the utility of the representative agent. Specifically, a higher λ^M (λ^E) reduces the indirect utility of the risk-averse representative agent — and thus increases the marginal utility, i.e. $V_{\lambda^M} < 0$ ($V_{\lambda^E} < 0$) and $V_{A\lambda^M} > 0$ ($V_{A\lambda^E} > 0$). Equity premia arise from the comovement of the marginal utility with the price process of the underlying asset, such that increases in marginal utility should be compensated by higher premia when market prices drop in light of such risk. At the same time, as we will see later, the risk-free rate decreases as the risk of macroeconomic (environmental) disaster increases. This also positively impacts the equity premium.[8]

With regard to the solution $b^E > 0$ of (19), which in general is not analytically available, we make the assumption that S is exponentially distributed and find the following result; see the proof in Appendix A.

Proposition 2. *Let us consider Eq. (19) for* b^E *and assume* S *exponentially distributed,* $S \sim Exp(\alpha)$, *with* $\alpha > 0$ *such that* $\mathbb{E}\left[e^{(1-\gamma)Z^E}\right] < \alpha\left(\rho + k^E\right)$ *and* $\frac{\rho+k^E}{\left(\sigma_\lambda^E\right)^2} < \alpha$.

(i) Define the constants

$$B_3 = -\frac{1}{2}\left(\sigma_\lambda^E\right)^2 \langle 0, \quad B_2 = -\alpha B_3 + \rho + k^E \rangle 0, \quad B_1 = 1 - \alpha\left(\rho + k^E\right) < 0, \quad B_0 = \alpha\left(\mathbb{E}\left[e^{(1-\gamma)Z^E}\right] - 1\right) > 0 \tag{31}$$

and the polynomial in b^E

$$\mathcal{P}(b^E) = B_3\left(b^E\right)^3 + B_2\left(b^E\right)^2 + B_1 b^E + B_0. \tag{32}$$

If

$$\mathcal{P}\left(\frac{-B_2 + \sqrt{B_2^2 - 3B_3B_1}}{3B_3}\right) < 0 \tag{33}$$

then b^E *in Theorem 1 is given by*

$$b^E = \min_{j=0,1,2}\left\{2\sqrt{-\frac{p_1}{3}}\cos\left(\frac{\theta - 2\pi j}{3}\right) - \frac{B_2}{3B_3}\right\} \in (0, \alpha), \tag{34}$$

---

[8] The conditions for the sign of the climate risk premium are discussed in [35,36].

where

$$p_1 = \frac{B_1}{B_3} - \frac{B_2^2}{3B_3^2}, \quad \theta = \mathit{arcccos}\left(-\frac{p_2}{2p_3}\right), \quad p_2 = \frac{B_0}{B_3} - \frac{B_2 B_1}{3B_3^2} + \frac{2B_2^3}{27B_3^3}, \quad p_3 = \sqrt{-\frac{\left(p_1\right)^3}{27}}. \tag{35}$$

Moreover, if

$$\mathbb{E}\left[e^{(1-\gamma)Z^E}\right] < 1 + \frac{\left(k^E + \rho\right)^2}{2\left(\sigma_\lambda^E\right)^2},$$

condition (33) is satisfied for α sufficiently large.

(ii) *Consider the second-order approximation $e^{b^E S} \simeq 1 + b^E S + \frac{1}{2}\left(b^E\right)^2 S^2$. An approximated solution from below of (19) is given by*

$$b_{ap}^E = \alpha\left(\frac{\alpha\left(\rho + k^E\right) - \mathbb{E}\left[e^{(1-\gamma)Z^E}\right]}{\alpha^2\left(\sigma_\lambda^E\right)^2 + \mathbb{E}\left[e^{(1-\gamma)Z^E}\right]} - \sqrt{\left(\frac{\alpha\left(\rho + k^E\right) - \mathbb{E}\left[e^{(1-\gamma)Z^E}\right]}{\alpha^2\left(\sigma_\lambda^E\right)^2 + \mathbb{E}\left[e^{(1-\gamma)Z^E}\right]}\right)^2 - 2\frac{\mathbb{E}\left[e^{(1-\gamma)Z^E}\right] - 1}{\alpha^2\left(\sigma_\lambda^E\right)^2 + \mathbb{E}\left[e^{(1-\gamma)Z^E}\right]}}\right).$$

4. Main economic results

This section is devoted to study the risk-free interest rate, the equity premium, and the government bond spread, with a particular focus on how these variables are affected by natural, macroeconomic, and policy shocks. Compared to previous studies, we emphasize the role played by the introduction of self- and externally-excited rare disasters in shaping these effects.

4.1. The risk-free interest rate

Regarding the risk-free rate implied by the model, we have the following result, the proof of which is given in Appendix A.

Proposition 3 (*Risk-free Rate*). *The risk-free rate reads as:*

$$\begin{aligned} r \quad = \quad & \underbrace{\rho + g\left(n_G, n_B\right) - \gamma\left(\sigma^A(n_G, n_B)\right)^2}_{\text{Standard Model}} - \underbrace{\mathbb{E}\left[e^{-\gamma Z^M}\left(1 - e^{Z^M}\right)\right]\lambda^M}_{\text{Macroeconomic Risk}} \\ & - \underbrace{\mathbb{E}\left[e^{-\gamma Z^E}\left(1 - e^{Z^E}\right)\right]\lambda^E}_{\text{Environmental Risk}} \quad \underbrace{\mathbb{E}\left[e^{b^E S}\right]}_{\text{Extra Risk from self-excitement}} \quad \underbrace{- n_B\left(1 - e^X\right)\left(1 - n_B\left(1 - e^X\right)\right)^{-\gamma}\rho_X}_{\text{Policy Risk}}, \end{aligned} \tag{36}$$

where $n_G + n_B = 1$ and n_B is at its optimum level from (20), $g\left(n_G, n_B\right) = \sum_{i=G,B} n_i \mu_i - \rho$ is the consumption growth in a deterministic setting, and $b^E > 0$ is defined in Theorem 1.

The term above the first underbrace in (36) is the same as in the standard model without disaster risk; ρ represents the role of discounting, $g\left(n_G, n_B\right)$ intertemporal smoothing, and γ precautionary savings. The term multiplying λ^M in (36) arises from the risk of a negative macroeconomic event. Because $e^{Z^M} < 1$, the risk-free rate is decreasing in λ^M. Hence, an increase in the probability of a rare disaster increases the representative agent's desire to save, and thus lowers the risk-free rate. The greater is risk aversion γ, the greater is this effect. The same applies to λ^E. Hence, higher future temperatures that increase the probability of natural disasters λ^E, will also reduce risk-free rate. At the same time, since $\mathbb{E}\left[e^{b^E S}\right] > 1$, the self-excited nature of the environmental disaster probability λ^E implies an extra reduction of the interest rate. This is especially relevant in a low interest rate environment, as it constrains central banks' ability to target inflation effectively via the standard Taylor rule [37].

All in all, the formula (36) suggests that higher risk induces precautionary savings, which reduces the risk-free rate; the greater the risk aversion γ the greater this effect. Moreover, the greater the amplitude of self-excited jumps, the greater this effect.

Regarding the externally-excited jumps, they contribute to increase the average probability of natural disasters λ^E. Therefore they have an indirect impact on the interest rate; the greater the amplitude and the frequency of the externally-excited jumps the higher the value of λ^E, therefore the lower the risk-free interest rate.

Eq. (36) can be readily used to infer possible evolutions of the risk-free rate for various temperature scenarios (that effectively change the distribution of the probability of environmental disasters λ^E), portfolio composition n_B (which may, or may not, be set at its optimal value from (20) given (16)) and distributions $Z^{(j)}$ of disaster magnitudes.

4.2. State-price density

To study the impact of environmental risks on the equity premium, the price of claims to future dividends must be determined. A convenient approach is to price claims to aggregate dividends by first deriving the state-price density, which captures the equilibrium compensation investors require for bearing different sources of risk in the economy. Let m denote the state-price density (or pricing kernel) which, due to the recursive nature of the utility function, depends on the value function. In particular, as shown in [38] the state-price density for recursive preferences (4)–(5) is given by

$$m_t = f_C\left(C_t, U_t\right) \exp\left(\int_0^t f_U\left(C_s, U_s\right) ds\right). \tag{37}$$

Let P denote the price of the claim to future dividends. The absence of arbitrage then implies that P is the integral of future dividends, discounted using the state-price density.

$$P_t = \mathbb{E}_t\left[\int_t^\infty \frac{m_s}{m_t} D_s ds\right] \tag{38}$$

which represents the usual asset pricing equation.

In Appendix A, we prove the following proposition.

Proposition 4 (*State-price Density*). *Let $m_t \equiv m$ and $m_{t^-} \equiv m_-$. Let $b^M > 0$ and $b^E > 0$ be defined as in Theorem 1. The state-price density has the following dynamics:*

$$\frac{dm}{m_-} = \mu_{m-}\, dt - \gamma \sum_{i=G,B} n_i \sigma_i dW_i + \sum_{j=M,E} b^j \sigma_\lambda^j \sqrt{\lambda_-^j}\, dW_\lambda^j + d\tilde{Q}^M + d\tilde{Q}^E + d\tilde{Q}^{ext} + d\tilde{Q}^X, \tag{39}$$

where

$$\begin{aligned} \mu_m \quad = \quad & -r - \mathbb{E}\left[e^{-\gamma Z^M} - 1\right]\lambda^M - \left(\mathbb{E}\left[e^{-\gamma Z^E}\right]\mathbb{E}\left[e^{b^E S}\right] - 1\right)\lambda^E \\ & - \left(\mathbb{E}\left[e^{b^E R}\right] - 1\right)\rho_\lambda - \left[\left(1 - n_B\left(1 - e^X\right)\right)^{-\gamma} - 1\right]\rho_X, \end{aligned} \tag{40}$$

and

$$\tilde{Q}_t^M = \sum_{n=1}^{N_t^M}\left(e^{-\gamma Z_n^M} - 1\right), \quad \tilde{Q}_t^E = \sum_{n=1}^{N_t^E}\left(e^{-\gamma Z_n^E + b^E S_n} - 1\right), \quad \tilde{Q}_t^{ext} = \sum_{n=1}^{N_t^{ext}}\left(e^{b^E R_n} - 1\right) \tag{41}$$

$$\tilde{Q}_t^X = N_t^X\left[\left(1 - n_B\left(1 - e^X\right)\right)^{-\gamma} - 1\right]. \tag{42}$$

The second element of (39) implies that the standard diffusion risk in consumption is priced. Specifically, the marginal utility (as represented by the state-price density) changes in response to the diffusion risk of both brown and green assets, proportionally to their fraction in the portfolio. More interestingly, changes in the intensities of macroeconomic and environmental disasters, that is λ^M and λ^E, are also priced as reflected by the first (see (40)) and third element of (39). Moreover, in the event of a macroeconomic or environmental disaster, marginal utility jumps upward, as the (positive) jump components in (39) and (41) show. These upward jumps represent the fact that investors require compensation for bearing macroeconomic and environmental disaster risks. In particular, in (41) jumps due to environmental factors (last two terms) are emphasized by their clustering-like nature generated by the self and externally excited intensity and this explains the behavior of investors which require further compensation due to the related extra damages.

The last term in (39) represents the jump component due to a green-policy shock, for which marginal utility increases. In fact, this jump is zero when the representative agent invests only in green assets, that is when the fraction of the brown asset in the portfolio n_B is null, and increases as n_B increases.

4.3. Pricing green and brown equity claims

In order to price climate change risks for long-lived assets we follow [14,39,40]. We assume that the dividend the aggregate market pays is modeled as leveraged consumption, that is, $D = C^\eta$. We suppose $\eta > 1$, so that dividends fall more than consumption in the event of a negative shock [41]. Equity premia arise from the co-movement of marginal utility of the risk-averse investor with the price of the underlying asset or portfolio, both in normal times and times of disasters.

The dynamics of $D = C^\eta$ is

$$\frac{dD}{D_-} = \mu_D dt + \eta \sum_{i=G,B} n_i \sigma_i dW_i + \sum_{j=M,E}\left(e^{\eta Z^j} - 1\right)\Delta N^j + \left[\left(1 - n_B\left(e^X - 1\right)\right)^\eta - 1\right]\Delta N^X, \tag{43}$$

where

$$\mu_D = \eta\left(g(n_G, n_B) + \frac{1}{2}(\eta - 1)\left(\sigma^A(n_G, n_B)\right)^2\right) \tag{44}$$

(see (A.22) in the proof of Proposition 4 with η instead of $-\gamma$).

The dynamics of dividend is used to compute P defined as in (38) and the aggregate market's risk-premium, the semi-closed formulas of which are provided in the following propositions. The proofs go along the lines of [14], but are more involved due to the presence of different types of jumps.

Proposition 5 (*Aggregate Price*). *The price–dividend ratio for the aggregate market $G \equiv \dfrac{P}{D}$ is given by*

$$G\left(\lambda^M, \lambda^E\right) = \int_0^\infty e^{d_\eta(\tau) + b_\eta^M(\tau)\lambda^M + b_\eta^E(\tau)\lambda^E} d\tau, \tag{45}$$

where $d_\eta(\tau)$, $b_\eta^j(\tau)$ satisfy the following system of ordinary differential equations:

$$\begin{aligned}
\left(d_\eta\right)'(\tau) &= \mu_D - g(n_G,n_B) - \rho + (1-\eta)\gamma\left(\sigma^A(n_G,n_B)\right)^2 + \rho_\lambda \mathbb{E}\left[e^{b^E R}\left(e^{b_\eta^E(\tau)R}-1\right)\right] + \textstyle\sum_{j=M,E} b_\eta^j(\tau)k^j\bar{\lambda}^j \\
&\quad + \rho_X\left[\left(1-n_B\left(1-e^X\right)\right)^{\eta-\gamma} - \left(1-n_B\left(1-e^X\right)\right)^{1-\gamma}\right] \\
\left(b_\eta^M\right)'(\tau) &= \tfrac{1}{2}\left(b_\eta^M(\tau)\sigma_\lambda^M\right)^2 + b_\eta^M(\tau)\left(b^M\left(\sigma_\lambda^M\right)^2 - k^M\right) + \mathbb{E}\left[e^{(\eta-\gamma)Z^M} - e^{(1-\gamma)Z^M}\right] \\
\left(b_\eta^E\right)'(\tau) &= \tfrac{1}{2}\left(b_\eta^E(\tau)\sigma_\lambda^E\right)^2 + b_\eta^E(\tau)\left(b^E\left(\sigma_\lambda^E\right)^2 - k^E\right) + \mathbb{E}\left[e^{(\eta-\gamma)Z^E}\right]\mathbb{E}\left[e^{\left(b^E+b_\eta^E(\tau)\right)S}\right] - \mathbb{E}\left[e^{(1-\gamma)Z^E}\right]\mathbb{E}\left[e^{b^E S}\right]
\end{aligned} \tag{46}$$

where $d_\eta(0) = b_\eta^M(0) = b_\eta^E(0) = 0$ and μ_D is defined in (44).

Boundary conditions $d_\eta(0) = b_\eta^M(0) = b_\eta^E(0) = 0$ are justified by the fact that the asset at $\tau = 0$ pays the current dividend. We observe that (46) is a master–slave type system, that is the first ordinary differential equation of the system depends on the other two, which are two decoupled ordinary differential equations. The second ordinary differential equation for b_η^M is as in [14]. Therefore, we recall the solution provided in [14]

$$b_\eta^M(\tau) = \frac{2\mathbb{E}\left[e^{(1-\gamma)Z^M} - e^{(\eta-\gamma)Z^M}\right]\left(1-e^{-\zeta_\eta^M\tau}\right)}{\left(\zeta_\eta^M + b^M\left(\sigma_\lambda^M\right)^2 - k^M\right)\left(1-e^{-\zeta_\eta^M\tau}\right) - 2\zeta_\eta^M} \tag{47}$$

where

$$\zeta_\eta^M = \sqrt{\left(b^M\left(\sigma_\lambda^M\right)^2 - k^M\right)^2 + 2\mathbb{E}\left[e^{(1-\gamma)Z^M} - e^{(\eta-\gamma)Z^M}\right]\left(\sigma_\lambda^M\right)^2}. \tag{48}$$

Assuming that S is identically zero, we have that the solution of the third ordinary differential equation is

$$b_\eta^E(\tau) = \frac{2\mathbb{E}\left[e^{(1-\gamma)Z^E} - e^{(\eta-\gamma)Z^E}\right]\left(1-e^{-\zeta_\eta^E\tau}\right)}{\left(\zeta_\eta^E + b^E\left(\sigma_\lambda^E\right)^2 - k^E\right)\left(1-e^{-\zeta_\eta^E\tau}\right) - 2\zeta_\eta^E}, \tag{49}$$

where

$$\zeta_\eta^E = \sqrt{\left(b^E\left(\sigma_\lambda^E\right)^2 - k^E\right)^2 + 2\mathbb{E}\left[e^{(1-\gamma)Z^E} - e^{(\eta-\gamma)Z^E}\right]\left(\sigma_\lambda^E\right)^2}. \tag{50}$$

Note that externally-excited jumps impact only on the first differential equation in (46), while self-excited jumps have an impact on the third differential equation in (46) and, via $b_\eta^E(\tau)$ and b^E, on the first one as well. Hence, in general, the presence of self-excited jumps impends a closed-form solution for d_η. Whereas, for S identically equal to zero (no self-excited jumps), we have

$$\begin{aligned}
d_\eta(\tau) &= \left(\mu_D - g(n_G,n_B) - \rho + (1-\eta)\gamma\left(\sigma^A(n_G,n_B)\right)^2 - \sum_{j=M,E}\frac{k^j\bar{\lambda}^j}{\left(\sigma_\lambda^j\right)^2}\left(\zeta_\eta^j + b^j\left(\sigma_\lambda^j\right)^2 - k^j\right)\right)\tau \\
&\quad + \rho_X\left(\left(1-n_B\left(1-e^X\right)\right)^{\eta-\gamma} - \left(1-n_B\left(1-e^X\right)\right)^{1-\gamma}\right)\tau \\
&\quad + \int_0^\tau \rho_\lambda\mathbb{E}\left[e^{b^E R}\left(e^{b_\eta^E(s)R}-1\right)\right]ds \\
&\quad - \sum_{j=M,E} 2k^j\bar{\lambda}^j\mathbb{E}\left[e^{(1-\gamma)Z^M} - e^{(\eta-\gamma)Z^M}\right]\log\left(\frac{\left(\zeta_\eta^j + b^j\left(\sigma_\lambda^j\right)^2 - k^j\right)\left(e^{-\zeta_\eta^j\tau}-1\right) + 2\zeta_\eta^j}{2\zeta_\eta^j}\right).
\end{aligned} \tag{51}$$

Under our assumptions $Z^j < 0$, $\sigma_\lambda^j > 0$ and $\eta > 1$, we have that $b_\eta^M(\tau)$ and, in case $S = 0$, $b_\eta^E(\tau)$, $d_\eta^E(\tau)$ are well defined for all values of τ. Moreover, $b_\eta^j(\tau)$ is negative. Indeed, the term inside the square root in the definition of ζ_η^j is positive. Moreover, $\zeta_\eta^j > |b^j\left(\sigma_\lambda^j\right)^2 - k^j| \geq b^j\left(\sigma_\lambda^j\right)^2 - k^j$ which implies that the denominator of $b_\eta^j(\tau)$ is negative while the argument of the logarithm of $d_\eta(\tau)$ is positive.

In presence of self-excited jumps, solving the last differential equation in (46) to determine $b_\eta^E(\tau)$ is more challenging and, in general, an explicit solution cannot be found. However, we can state the following proposition.

Proposition 6. *If* $x \mapsto \mathbb{E}\left[e^{(b^E+x)S}\right]$ *is* $C^1\left((-\infty,\epsilon)\right)$*, for some* $\epsilon > 0$*, then the Cauchy problem*

$$\begin{cases} \left(b_\eta^E\right)'(\tau) = \frac{1}{2}\left(b_\eta^E(\tau)\sigma_\lambda^E\right)^2 + b_\eta^E(\tau)\left(b^E\left(\sigma_\lambda^E\right)^2 - k^E\right) + \mathbb{E}\left[e^{(\eta-\gamma)Z^E}\right]\mathbb{E}\left[e^{\left(b^E+b_\eta^E(\tau)\right)S}\right] - \mathbb{E}\left[e^{(1-\gamma)Z^E}\right]\mathbb{E}\left[e^{b^E S}\right] \\ b_\eta^E(0) = 0 \end{cases} \tag{52}$$

has a unique strictly decreasing solution $b_\eta^E(\tau) < 0$ *for all* $\tau > 0$*. Moreover, if* $\mathbb{E}\left[S^2 e^{b^E S}\right]$ *is finite, this solution approaches, as* $\tau \to +\infty$*, a unique equilibrium point* $\bar{b}_\eta^E$ *in* $(-\infty, 0)$ *which solves*

$$\frac{1}{2}\left(\bar{b}_\eta^E\sigma_\lambda^E\right)^2 + \bar{b}_\eta^E\left(b^E\left(\sigma_\lambda^E\right)^2 - k^E\right) + \mathbb{E}\left[e^{(\eta-\gamma)Z^E}\right]\mathbb{E}\left[e^{\left(b^E+\bar{b}_\eta^E\right)S}\right] - \mathbb{E}\left[e^{(1-\gamma)Z^E}\right]\mathbb{E}\left[e^{b^E S}\right] = 0. \tag{53}$$

Furthermore, denoting by $b_{\eta,0}^E(\tau)$ *the solution of* (52) *corresponding to* $S = 0$*, we get*

$$b_\eta^E(\tau) < b_{\eta,0}^E(\tau) \; for \; all \; \tau > 0. \tag{54}$$

From (54) of the proposition above, we deduce the following meaningful result.

Corollary 3. *Under the assumptions of Proposition 6, the presence of self-excited jumps further reduces the price–dividend ratio (45).*

Remark 4. Proposition 6 applies when S is exponentially distributed, $S \sim Exp(\alpha)$ with $\alpha > 0$, because $\mathbb{E}\left[e^{(b^E+x)S}\right] = \frac{\alpha}{\alpha - b^E - x} \in C^1\left((-\infty,\epsilon)\right)$, with $\epsilon = \alpha - b^E > 0$.

4.4. The market premium of environmental risks

Now, we want to formally define the expected instantaneous return on equities r^e, understood as the sum of the percentage drift in prices, the instantaneous dividend yield and the expected change in prices in the event of a disaster. In order to do this, we apply Ito's formula to $P = DG(\lambda^M, \lambda^E)$ and, using (43), (45), we obtain

$$\begin{aligned} \frac{dP}{P_-} \quad = \quad & \mu_{P_-}\, dt + \eta \textstyle\sum_{i=G,B} n_i\sigma_i dW_i + \frac{1}{G_-}\textstyle\sum_{j=M,E} G_{\lambda^j}\sigma_\lambda^j\sqrt{\lambda_-^j}\,dW_\lambda^j + \left(e^{\eta Z^M - 1}\right)\Delta N^M + \left[\left(1 - n_B\left(1 - e^X\right)\right)^\eta - 1\right]\Delta N^X \\ & + \left(\frac{\displaystyle\int_0^\infty e^{d_\eta(\tau) + b_\eta^M(\tau)\lambda_-^M + b_\eta^E(\tau)(\lambda_-^E + S)}d\tau}{\displaystyle\int_0^\infty e^{d_\eta(\tau) + b_\eta^M(\tau)\lambda_-^M + b_\eta^E(\tau)\lambda_-^E}d\tau} - 1\right)\Delta N^E + \left(\frac{\displaystyle\int_0^\infty e^{d_\eta(\tau) + b_\eta^M(\tau)\lambda_-^M + b_\eta^E(\tau)(\lambda_-^E + R)}d\tau}{\displaystyle\int_0^\infty e^{d_\eta(\tau) + b_\eta^M(\tau)\lambda_-^M + b_\eta^E(\tau)\lambda_-^E}d\tau} - 1\right)\Delta N^{ext}. \end{aligned} \tag{55}$$

where

$$\mu_P = \mu_D + \frac{1}{G}\sum_{j=M,E} G_{\lambda^j} k^j(\bar{\lambda}^j - \lambda^j) + \frac{1}{2}\sum_{j=M,E} G_{\lambda^j,\lambda^j}(\sigma_\lambda^j)^2\lambda^j$$

with μ_D defined in (44), $G_{\lambda^j} = \frac{\partial G}{\partial \lambda^j}$ and $G_{\lambda^j,\lambda^j} = \frac{\partial^2 G}{\partial {\lambda^j}^2}$.

The instantaneous expected return on equity r^e is then defined as

$$\begin{aligned} r^e = \mu_P + \frac{D}{P} + & \sum_{j=M,E}\lambda^j\mathbb{E}\left(e^{\eta Z^j} - 1\right) + \rho_X\left[\left(1 - n_B\left(1 - e^X\right)\right)^\eta - 1\right] \\ & + \lambda^E\mathbb{E}_{\lambda^M,\lambda^E}\left(\frac{\displaystyle\int_0^\infty e^{d_\eta(\tau) + b_\eta^M(\tau)\lambda^M + b_\eta^E(\tau)\lambda^E + b_\eta^E(\tau)S}d\tau}{\displaystyle\int_0^\infty e^{d_\eta(\tau) + b_\eta^M(\tau)\lambda^M + b_\eta^E(\tau)\lambda^E}d\tau} - 1\right) \\ & + \rho_\lambda\mathbb{E}_{\lambda^M,\lambda^E}\left(\frac{\displaystyle\int_0^\infty e^{d_\eta(\tau) + b_\eta^M(\tau)\lambda^M + b_\eta^E(\tau)\lambda^E + b_\eta^E(\tau)R}d\tau}{\displaystyle\int_0^\infty e^{d_\eta(\tau) + b_\eta^M(\tau)\lambda^M + b_\eta^E(\tau)\lambda^E}d\tau} - 1\right) \\ & + \lambda^E\,\mathbb{E}_{\lambda^M,\lambda^E}\left[\left(e^{\eta Z^E} - 1\right)\left(\frac{\displaystyle\int_0^\infty e^{d_\eta(\tau) + b_\eta^M(\tau)\lambda^M + b_\eta^E(\tau)\lambda^E + b_\eta^E(\tau)S}d\tau}{\displaystyle\int_0^\infty e^{d_\eta(\tau) + b_\eta^M(\tau)\lambda^M + b_\eta^E(\tau)\lambda^E}d\tau} - 1\right)\right], \end{aligned} \tag{56}$$

where $\mathbb{E}_{\lambda^M,\lambda^E}$ denotes the conditional expectation given λ^M, λ^E.

Comparing to previous works without jumps in the intensity processes of macroeconomic and environmental risks, the dynamics of prices is more involved and this is due to the presence of jumps in the price–dividend ratio G. Consequently, the expression of the risk premium below accounts for extra terms.

Proposition 7 (*Risk Premium*)**.** *The instantaneous aggregate risk premium can be written as*

$$\begin{aligned}
r^e - r \quad = \quad & \underbrace{\gamma\eta\left(\sigma^A\left(n_G, n_B\right)\right)^2}_{\text{Risk-Premium in Standard Model}} \quad + \quad \underbrace{\sum_{j=M,E} \lambda^j \left(-\frac{G_{\lambda j}}{G}\right) b^j \left(\sigma_\lambda^j\right)^2}_{\text{Risk-Premium for time-variation in disaster risk}} \\
& + \underbrace{\rho_X \left[1 - \left(1 - n_B\left(1 - e^X\right)\right)^{-\gamma}\right]\left[\left(1 - n_B\left(1 - e^X\right)\right)^{\eta} - 1\right]}_{\text{Policy Premium}} \\
& + \quad \underbrace{\lambda^M \mathbb{E}\left[\left(e^{-\gamma Z^M} - 1\right)\left(1 - e^{\eta Z^M}\right)\right]}_{\substack{\text{Correlated movements between pricing-kernel} \\ \text{and market price in the event of macroeconomic disaster}}} \\
& + \underbrace{\lambda^E \mathbb{E}_{\lambda^M,\lambda^E}\left[\left(e^{-\gamma Z^E + b^E S} - 1\right)\left(1 - e^{\eta Z^E}\frac{\int_0^\infty e^{d_\eta(\tau) + b_\eta^M(\tau)\lambda^M + b_\eta^E(\tau)\lambda^E + b_\eta^E(\tau) S}\, d\tau}{\int_0^\infty e^{d_\eta(\tau) + b_\eta^M(\tau)\lambda^M + b_\eta^E(\tau)\lambda^E}\, d\tau}\right)\right]}_{\substack{\text{Correlated movements between pricing-kernel} \\ \text{and market price in the event of environmental disaster adjusted for self-excited jumps}}} \\
& + \underbrace{\rho_\lambda \mathbb{E}_{\lambda^M,\lambda^E}\left[\left(e^{b^E R} - 1\right)\left(1 - \frac{\int_0^\infty e^{d_\eta(\tau) + b_\eta^M(\tau)\lambda^M + b_\eta^E(\tau)\lambda^E + b_\eta^E(\tau) R}\, d\tau}{\int_0^\infty e^{d_\eta(\tau) + b_\eta^M(\tau)\lambda^M + b_\eta^E(\tau)\lambda^E}\, d\tau}\right)\right]}_{\text{Extra premium due to externally-excited jumps}},
\end{aligned} \tag{57}$$

where $\mathbb{E}_{\lambda^M,\lambda^E}$ denotes the conditional expectation given λ^M, λ^E.

All terms in (57) are positive, indicating a positive premium for diffusion risk (first term), for time-varying risks of both environmental and macroeconomic disasters (second term), for policy risk (third term), for static macroeconomic disaster risk (fourth term) and, finally, for static environmental disaster risk (last two terms). Regarding this last type of risk, in case of no self-excited jumps (i.e., $S = 0$) the second last term in (57) reduces to

$$\lambda^E \mathbb{E}_{\lambda^M,\lambda^E}\left[\left(e^{-\gamma Z^E} - 1\right)\left(1 - e^{\eta Z^E}\right)\right] \tag{58}$$

which is still positive and lower than the corresponding term with $S > 0$, while in the case of no externally excited jumps (i.e., $R = 0$) the last term in (57) disappears. Hence, the presence of self-excited and externally excited jumps contributes to exacerbate the market's risk premium required for environmental disasters and policy shocks.

In the absence of disasters, from (55) we see that the instantaneous volatility of the stock is given by

$$\left(\sigma_P^T \sigma_P\right)^{\frac{1}{2}} = \left(\eta^2\left(\sigma^A(n_G, n_B)\right)^2 + \sum_{j=M,E} \lambda^j \left(\frac{G_{\lambda j}}{G}\right)^2 \left(\sigma_\lambda^j\right)^2\right)^{\frac{1}{2}} \tag{59}$$

The formula above underline that, as also shown in [14], market's volatility is higher than that generated by a simple diffusion model without jumps or by a jump model with constant intensity, which would be given by $\eta\sigma^A(n_G, n_B)$. This confirms that time-varying intensity is fundamental to explaining the volatility puzzle. In addition to [14], we emphasize that the diffusion component driving the time-varying intensities is essential for explaining the volatility puzzle. Indeed, an intensity process that includes only self-excited and externally excited components – but not the diffusion component – would not affect asset volatility.

4.5. Default-premium on government bonds

Disasters often coincide with at least a partial default on government securities, hence we can use the formula for the risk-free rate in (36) and the state-price density to also deduce their effect on sovereign credit risk. This point is of empirical relevance if one tries to match the behavior of the risk-free asset to the rate of return on government securities in the data. We therefore allow for partial default on government debt, and consider the rate of return on this defaultable security. Specifically, following [12,42], we assume that conditionally on the event of a disaster of type $j \in \{M, E\}$, there will be a default of government liabilities with probability q^j and, in the case of a policy shock, the default probability will be q^X.[9] Furthermore, as in [12], in the event of a

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[9] Let us note that a possible generalization is to assume that the default probability on government bonds is dependent on the amount of losses in case of jumps.

default, the percentage loss is taken to be equal to the percentage decline in consumption. We denote by L the price process that results from rolling over instantaneous government debt. Since consumption follows (A.20) in Appendix A, the dynamics of L is described by

$$\frac{dL}{L_-} = r^L_- dt + \sum_{j\in\{M,E\}} dQ^{j^L} + n_B dQ^{X^L}, \tag{60}$$

where r^L is the reward in case of no default,[10] while $Q^{X^L} = \left(e^{X^L}-1\right)N^X$ and $Q^{j^L} = \sum_{n=1}^{N_t^j}\left(e^{Z_n^{j^L}}-1\right)$, for $j \in \{M,E\}$, with

$$Z_n^{j^L} = \begin{cases} Z_n^j & \text{with probability } q^j \\ \\ 0 & \text{otherwise} \end{cases} \quad ; \quad X^L = \begin{cases} X & \text{with probability } q^X \\ \\ 0 & \text{otherwise} \end{cases} \tag{61}$$

are the jumps components.

The instantaneous expected return[11] on government debt is defined by

$$r^b := r^L - \underbrace{\mathbb{E}\left[1-e^{Z^M}\right]q^M\lambda^M}_{\substack{\text{Expected loss in instantaneous}\\ \text{rewards in case of Macroeconomic Event}}} - \underbrace{\mathbb{E}\left[1-e^{Z^E}\right]q^E\lambda^E}_{\substack{\text{Expected loss in instantaneous rewards}\\ \text{in case of Macroeconomic Event}}} - \underbrace{n_B\left(1-e^X\right)q^X\rho_X}_{\substack{\text{Expected loss in instantaneous rewards}\\ \text{in case of Policy Shocks}}} . \tag{62}$$

In the next proposition, whose proof is omitted as it goes along the way of those of Propositions 5 and 7, we find the equilibrium relation between r^b and r.

Proposition 8. *Let q^j, $j = M,E,X$, be the default probabilities in (61) and let r^b the instantaneous expected return defined in (62). Then, the spread on government bonds reads as*

$$\begin{aligned} r^b - r \quad = \quad & \underbrace{\mathbb{E}\left[\left(e^{-\gamma Z^M}-1\right)\left(1-e^{Z^M}\right)\right]q^M\lambda^M}_{\textit{Macroeconomic Disasters}} + \underbrace{\mathbb{E}\left[\left(e^{-\gamma Z^E+b^E S}-1\right)\left(1-e^{Z^E}\right)\right]q^E\lambda^E}_{\textit{Environmental Disasters}} \\ & \underbrace{+n_B\left(1-e^X\right)\left(\left(1-n_B\left(1-e^X\right)\right)^{-\gamma}-1\right)q^X\rho_X}_{\textit{Transition Risk}} . \end{aligned} \tag{63}$$

The spreads (against the risk free asset, that is $r^L - r$ and $r^b - r$) are represented by three components, which capture, respectively the effects of macroeconomic and environmental disasters and, lastly, the transition risk due to, e.g., the abrupt repricing of large climate-sensitive assets in public ownership such as coal mines or energy utilities, see, e.g., [43]. All these three terms are positive since they have the interpretation of a disaster risk premium for sovereign debt: the percentage change in marginal utility is multiplied by the percentage loss on the government debt claim. Eq. (63) shows that sovereign bond yield spreads should increase, especially for governments with greater exposure to brown assets, i.e., higher n_B, because of climate policy that responds to deviations from low temperatures [44]. This effect is amplified by the presence of self-excited jumps.

To draw further insights, we recall the expression of the three rates r^b, r^L and r

$$\begin{aligned} r^b \quad = \quad & \underbrace{\rho + g\left(n_G,n_B\right) - \gamma\left(\sigma^A(n_G,n_B)\right)^2}_{\text{Standard Model}} - \underbrace{\mathbb{E}\left[e^{-\gamma Z^M}\left(1-e^{Z^M}\right)\right]\left(1-q^M\right)\lambda^M}_{\text{Macroeconomic Risk}} - \underbrace{\mathbb{E}\left[\left(1-e^{Z^M}\right)\right]q^M\lambda^M}_{\text{Macroeconomic Disasters}} \\ & - \underbrace{\mathbb{E}\left[e^{-\gamma Z^E}\left(1-e^{Z^E}\right)\right]}_{\text{Environmental Risk}}\lambda^E\,\mathbb{E}\left[e^{b^E S}\right]\left(1-q^E\right) - \underbrace{n_B\left(1-e^X\right)\left(1-n_B\left(1-e^X\right)\right)^{-\gamma}\rho_X\left(1-q^X\right)}_{\text{Policy Risk}}, \\ & - \underbrace{\mathbb{E}\left[\left(1-e^{Z^E}\right)\right]q^E\lambda^E}_{\text{Environmental Disasters}} - \underbrace{n_B\left(1-e^X\right)q^X\rho_X}_{\text{Transition Risk}} . \end{aligned} \tag{64}$$

$$\begin{aligned} r^L \quad = \quad & \underbrace{\rho + g\left(n_G,n_B\right) - \gamma\left(\sigma^A(n_G,n_B)\right)^2}_{\text{Standard Model}} - \underbrace{\mathbb{E}\left[e^{-\gamma Z^M}\left(1-e^{Z^M}\right)\right]\left(1-q^M\right)\lambda^M}_{\text{Macroeconomic Risk}} \\ & - \underbrace{\mathbb{E}\left[e^{-\gamma Z^E}\left(1-e^{Z^E}\right)\right]\mathbb{E}\left[e^{b^E S}\right]\left(1-q^E\right)\lambda^E}_{\text{Environmental Risk}} - \underbrace{n_B\left(1-e^X\right)\left(1-n_B\left(1-e^X\right)\right)^{-\gamma}\left(1-q^X\right)\rho_X}_{\text{Policy Risk}}, \end{aligned} \tag{65}$$

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[10] Note that r^L is the expected value of the instantaneous reward $\frac{dL}{L_-}$ under risk-neutral measure, while r^b is the expected value of the instantaneous reward $\frac{dL}{L_-}$ under the physical measure.

[11] The expected value computed with respect to the physical measure.

$$\begin{aligned} r \quad = \quad & \underbrace{\rho + g\left(n_G, n_B\right) - \gamma\left(\sigma^A(n_G, n_B)\right)^2}_{\text{Standard Model}} - \underbrace{\mathbb{E}\left[e^{-\gamma Z^M}\left(1 - e^{Z^M}\right)\right]\lambda^M}_{\text{Macroeconomic Risk}} \\ & - \underbrace{\mathbb{E}\left[e^{-\gamma Z^E}\left(1 - e^{Z^E}\right)\right]\lambda^E}_{\text{Environmental Risk}} \quad \underbrace{\mathbb{E}\left[e^{b^E S}\right]}_{\text{Extra Risk from self-excitement}} \quad - \underbrace{n_B\left(1 - e^X\right)\left(1 - n_B\left(1 - e^X\right)\right)^{-\gamma}\rho_X}_{\text{Policy Risk}} . \end{aligned} \tag{66}$$

In summary, all three rates r, r^L and r^b decrease in relation to the intensities λ^M, λ^E and ρ^X. This happens because the representative agent is risk-averse ($\gamma > 1$) and responds to an increase in the risk of future losses, caused by the rise in the intensity values, by increasing precautionary savings. However, r^L and r^b are less sensitive to changes in λ^M, λ^E and ρ^X than r because of an opposing effect. For each of these intensities, the greater its value, the greater is the risk of a default, and therefore the greater the return investors demand for holding the government bill (bond). In particular, for r^L this second effect offsets the precautionary saving effect when $q^M = q^E = q^X = 0$. Because of expected loss in instantaneous rewards in case of macroeconomic events, environmental events and policy shocks (see the last three terms in (62)), r^b decreases more than r^L, but still less than r.

5. Baseline parameterization and numerical illustration

In this section, we present a baseline parameterization to quantify the contribution of the proposed innovative modeling components. For this purpose, we rely on the rare-disaster literature, especially [12,14], to set the core macroeconomic and preference parameters. For the environmental and transition risk components, we draw on [15] and standard climate stress-test scenarios. To capture the asymmetric, fat-tailed nature of real-world disaster damages, we model the disaster sizes and self-exciting jumps as stochastic variables. Specifically, we assume that macroeconomic and environmental disaster sizes are modeled as negative exponential variables, $Z^j = -Y^j$ where $Y^j \sim Exp(\nu_j)$ for $j \in \{M, E\}$, following standard double-exponential jump–diffusion frameworks (e.g., [45]). Similarly, self-exciting jumps have amplitudes drawn from an exponential distribution $S \sim \mathrm{Exp}(\alpha)$, while external jumps follow $R \sim \mathrm{Exp}(\alpha_R)$.

This choice is motivated by both empirical realism and analytical tractability. Empirically, the economic damages and the subsequent cascading effects of environmental disasters exhibit significant right-skewness and heavy tails. The exponential distribution elegantly captures this asymmetry, assigning a higher probability to moderate aggregation effects whilst rigorously accounting for rare chains of catastrophic events. Analytically, e.g., the simple form of the moment-generating function of the exponential, $\mathbb{E}[e^{b^E S}] = \frac{\alpha}{\alpha - b^E}$, reduces the highly complex equation to a manageable cubic polynomial, yielding a semi-explicit form for the coefficient of the function of the value b^E, as demonstrated in Proposition 2.

Table 1 summarizes the parameterization we have adopted. We carefully set $\nu_M = 4.5$ (implying an expected macro drop of 22.2%) and $\nu_E = 10.0$ (expected physical damage of 10%) to ensure that all conditions imposed by Theorem 1 and Proposition 2 on the moment generating functions $\mathbb{E}[e^{(1-\gamma)Z^j}] = \frac{\nu_j}{\nu_j - (\gamma - 1)}$ are strictly met. Furthermore, the model imposes (see Proposition 2) the bound $\mathbb{E}[S] < \frac{\rho + k^E}{\mathbb{E}[e^{(1-\gamma)Z^E}]}$ on the magnitude of self-excited jumps. Due to the choice $\gamma = 3$, $Z^E \sim -Exp(10)$, this implies that $\mathbb{E}[S]$ must be strictly less than 0.050. Economically, this inequality dictates that the economy can only sustain a finite degree of clustering, otherwise the feedback loop of the Hawkes process becomes too severe and the agent's demand for the risk-free asset accelerates asymptotically, pushing out risky investments and causing the equilibrium interest rate to collapse.

With regard to the exogenous intensity jump amplitude which follows an exponential distribution $R \sim Exp(\alpha_R)$, noting that the arrival rate of these exogenous shocks (ρ_λ) is independent of the state variable λ^E, the moment generating function of R does not affect the equation governing b^E. Therefore, we only need to impose the mild regularity condition $\alpha_R > b^E$ to ensure that $\mathbb{E}[e^{b^E R}]$ (appearing in the coefficient d of the value function, see Theorem 1) is finite. Economically, this represents a structural upper bound on the expected magnitude of the exogenous shocks, $\mathbb{E}[R] < 1/b^E$. If the baseline exogenous jumps are excessively large, the precautionary savings motive of a highly risk-averse agent ($\gamma > 1$) explodes, preventing the existence of a stationary general equilibrium. Setting the expected exogenous jump to $\mathbb{E}[R] = 0.015$ yields $\alpha_R \approx 66.67$, which safely strictly satisfies this integrability bound.

Finally, baseline macroeconomic volatilities ($\sigma_G, \sigma_B = 4\%$) coupled with a slight brown premium strictly ensure an internal portfolio allocation $n_B \in (0, 1)$, completely satisfying the structural policy shock constraint (21) imposed by Theorem 1.

5.1. Sensitivity analysis: Precautionary savings and the indirect utility coefficient

To fully grasp the economic mechanism driving the collapse of the risk-free rate, it is necessary to examine the behavior of the indirect utility coefficient b^E. In our affine framework, b^E represents the representative agent's sensitivity to physical climate risk and strictly governs the magnitude of the precautionary savings channel.

Fig. 2 illustrates the numerical solution for b^E (derived from the relevant root of the cubic Eq. (19)) as a function of the expected magnitude of self-exciting jump, $\mathbb{E}[S]$.

As the expected severity of the disaster clustering increases, the agent is more likely to experience prolonged cascading environmental shocks. Because the agent exhibits recursive preferences with a high aversion to extreme wealth variations ($\gamma = 3$), the demand for precautionary savings accelerates. Mathematically, this results in a strictly convex upward-sloping relationship between b^E and $\mathbb{E}[S]$.

Table 1
Baseline parameter values (in annual terms) for fully stochastic specification.

Category	Parameter	Value	Source/Implication
Preferences	Time preference (ρ)	0.012	[14]
	Risk aversion (γ)	3.00	[12]
Asset Dynamics	Green asset drift (μ_G)	0.020	Baseline return (2.0%)
	Brown asset drift (μ_B)	0.025	Brown Premium to satisfy Theorem 1
	Asset volatilities (σ_G, σ_B)	0.040	Macroeconomic baseline
	Covariance (σ_{GB})	0.000	[15]
Macro Risk	Long-run mean intensity ($\bar{\lambda}^M$)	0.017	[12]
	Mean reversion (k^M)	0.080	[14]
	Volatility (σ_λ^M)	0.050	Feller condition satisfied
	Disaster parameter (ν_M)	4.500	$\mathbb{E}[Z^M] \approx -0.222$, bound $\nu_M > 2$ met
Physical Risk	Base long-run intensity ($\tilde{\lambda}^E$)	0.050	[15]
	Mean reversion (k^E)	0.050	[15]
	Volatility (σ_λ^E)	0.040	[15]
	Disaster parameter (ν_E)	10.000	$\mathbb{E}[Z^E] = -0.100$, bound $\nu_E > 2$ met
Clustering	Self-exciting parameter (α)	100.00	$\mathbb{E}[S] = 0.010$, existence bound met
	Exogenous frequency (ρ_λ)	0.020	Baseline heuristic (1 in 50 years)
	Exogenous jump parameter (α_R)	66.67	$\mathbb{E}[R] = 0.015$, bound $\alpha_R > b^E$ met
Policy Risk	Policy shock frequency (ρ_X)	0.050	5% annual probability
	Brown asset drop (X)	−0.162	≈ 15% drop upon regulation

Fig. 2. Sensitivity of the indirect utility coefficient (b^E) to the expected self-exciting jump magnitude ($\mathbb{E}[S]$). The convex solid curve shows that as the expected clustering severity rises, the agent's sensitivity to climate risk (and thus the precautionary savings motive) accelerates. The horizontal dashed line isolates the pure clustering premium relative to a standard disaster model without self-excitement.

The horizontal dashed line in Fig. 2 represents the baseline model without clustering ($\mathbb{E}[S] = 0$). The widening wedge between the convex curve and the baseline horizontal line isolates the pure impact of the Hawkes mechanism. It visually demonstrates that neglecting the self-exciting nature of environmental disasters leads to a systematic and severe underestimation of the agent's climate risk aversion, ultimately failing to capture the true downward pressure exerted on the equilibrium risk-free rate.

5.2. Welfare implications: How clustering reduces the value function

Beyond the pricing implications for the risk-free rate, the presence of self-excited jumps exerts a profoundly negative impact on the general welfare of the representative agent, represented by the value function $V(A, \lambda^M, \lambda^E)$.

In our affine jump–diffusion framework, the value function inherits the functional form of the Epstein–Zin–Weil recursive utility. For an investor with a relative risk aversion strictly greater than unity ($\gamma > 1$), the value function is strictly negative. Specifically, indirect utility depends on the state variable λ^E through the exponential multiplier $\exp(b^E \lambda^E)$.

As established in our previous analysis, the presence of self-exciting jumps ($\mathbb{E}[S] > 0$) strictly increases the indirect utility coefficient b^E. Consequently, for any given strictly positive level of environmental risk intensity ($\lambda^E > 0$), the clustering effect amplifies the exponential multiplier, leading to a further decline in the value function.

Economically, this dynamic captures the severe damage to welfare induced by compounding risks. Without self-excitement ($S = 0$), a physical disaster represents an isolated negative shock to the economy. However, with self-excitement ($S > 0$), the agent

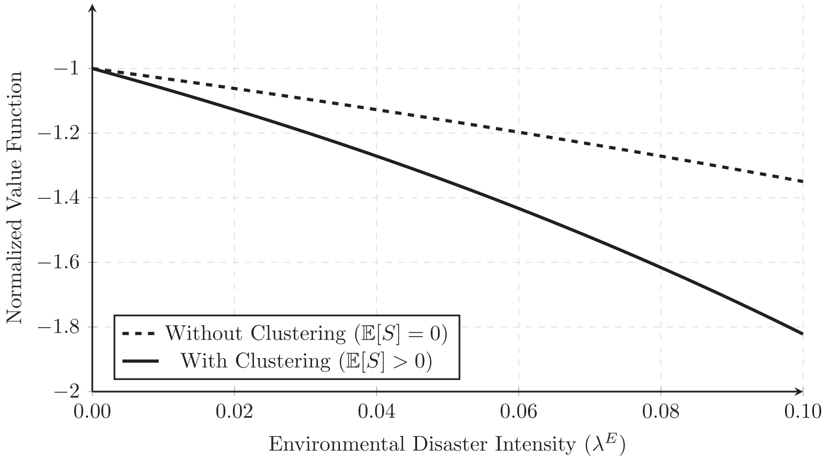

Fig. 3. The impact of self-exciting jumps on the normalized value function. For a highly risk-averse agent ($\gamma > 1$), utility is strictly negative. The presence of clustering ($\mathbb{E}[S] > 0$) increases the indirect utility coefficient b^E. This amplifies the penalty of the environmental intensity state λ^E, causing the value function (solid line) to deteriorate much more rapidly compared to a standard jump–diffusion model without clustering (dashed line).

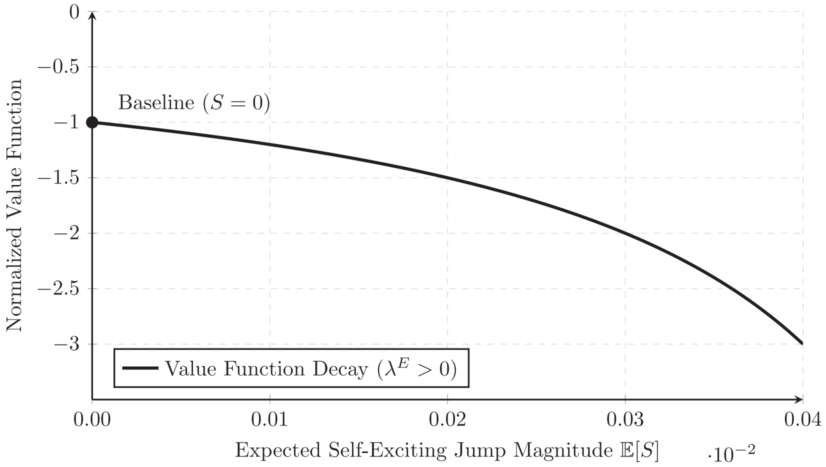

Fig. 4. The isolated impact of the clustering severity ($\mathbb{E}[S]$) on the normalized value function, evaluated at a positive environmental risk intensity ($\lambda^E > 0$). The compounding fear of cascading disasters drives a severe, convex deterioration in the agent's expected utility.

internalizes that a single physical shock will systematically increase the probability of subsequent disasters, creating a cascading sequence of capital destruction. A highly risk-averse agent severely penalizes this compounding uncertainty, resulting in a steeper decline of the expected lifetime utility.

Fig. 3 visually demonstrates this mechanism by plotting the normalized value function ($V_{norm} = -\exp(b^E\lambda^E)$) against the intensity of environmental disasters λ^E. The solid curve, representing the economy with clustering, consistently lies below the dashed baseline curve, illustrating that the presence of Hawkes-type mutually and self-exciting jumps strictly reduces the value function.

To fully isolate the welfare reduction induced by the clustering mechanism, we can fix the state of environmental risk at a strictly positive level ($\lambda^E > 0$) and analyze the sensitivity of the value function directly with respect to the expected magnitude of the self-exciting jumps, $\mathbb{E}[S]$. Fig. 4 plots the relationship between the expected jump size $\mathbb{E}[S]$ and the value function. At $\mathbb{E}[S] = 0$, the economy faces standard isolated disasters, establishing a baseline negative welfare level. However, as the clustering severity increases, the value function gets worse at an increasing rate. In fact, the value function is proportional to $-\exp(b^E\lambda^E)$ and, as $\mathbb{E}[S]$ increases, b^E increases convexly, reflecting the agent's absolute intolerance for cascading risks.

5.3. Sensitivity analysis: Risk-free rate and climate risk intensity

To further understand the model's dynamics, it is highly instructive to analyze the sensitivity of the equilibrium risk-free rate (r) to changes in the environmental disasters intensity (λ^E).

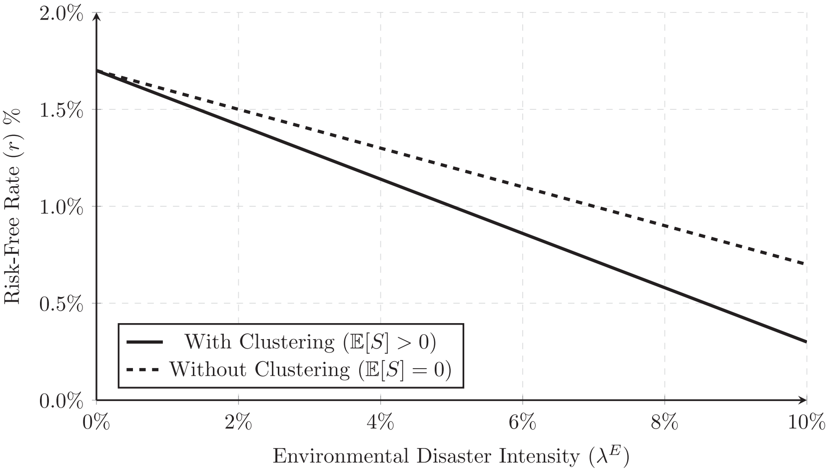

Fig. 5. Sensitivity of the equilibrium risk-free rate (r) to the intensity of environmental disasters (λ^E). The solid line illustrates the baseline model with self-exciting jumps (clustering), while the dashed line represents the standard model without clustering. Even within realistic ranges of climate risk probability (< 10%), the clustering mechanism acts as a severe structural amplifier, driving the interest rate toward the zero lower bound much more aggressively than isolated shocks.

From an analytical standpoint, it is worth noting that the intensity of physical risk λ^E does not enter the determination of the optimal portfolio allocation n_B (Eq. (22)), nor does it affect the indirect utility coefficient b^E (Eq. (19)). Consequently, these variables remain invariant to shifts in λ^E. Inspecting the equilibrium risk-free rate in Eq. (36), it becomes evident that the relationship between r and λ^E is linear. We decompose the rate into a baseline component (net of macroeconomic diffusion and policy risks) minus a climate risk penalty, where the slope is determined by the expectations over the stochastic disaster severity Z^E and the self-exciting jump S:

$$r = \bar{r}_{base} - \left(\mathbb{E}\left[e^{-\gamma Z^E}\left(1 - e^{Z^E}\right)\right]\mathbb{E}\left[e^{b^E S}\right]\right)\lambda^E \tag{67}$$

This linear sensitivity yields two crucial economic insights, visually summarized in Fig. 5. First, as the intensity of climate risk increases, the representative agent's precautionary savings motive intensifies linearly. If λ^E reaches sufficiently high levels, the equilibrium risk-free rate is driven in a forceful direction toward the zero lower limit, which potentially constrains the effectiveness of standard monetary policy interventions.

Second, the presence of clustering acts as a severe structural amplifier. A higher expected value of the self-exciting jump amplitude ($\mathbb{E}[S]$) steepens the negative slope of the risk-free rate. This visually confirms the findings of Corollary 2: neglecting the clustering nature of environmental shocks (i.e., assuming $S = 0$) leads to severe underestimate the downward effect of high climate stress on interest rates.

Fig. 6 shows that the risk-free rate r in Eq. (36) is penalized much more aggressively when S is stochastic (specifically, S follows an exponential distribution $S \sim Exp(\alpha)$) than in the deterministic case. From a mathematical point of view, as demonstrated in Proposition 2, replacing a deterministic jump with an exponentially distributed jump replaces the standard exponential moment generating function with a rational function, $\mathbb{E}[e^{b^E S}] = \frac{\alpha}{\alpha - b^E}$, transforming the b^E root-finding problem into a cubic equation (Eq. (19)). Economically, the exponential distribution has a fat right tail, meaning that the probability of an extreme clustering event is strictly positive. The representative agent, endowed with recursive preferences and a high aversion to extreme wealth variations, demands a disproportionately higher precautionary savings premium to hold this risk.

5.4. Asset pricing implications: Jumps and the equity risk premium

Proposition 7 formalizes the mechanism through which the complex jump–diffusion dynamics strictly increase the equilibrium equity risk premium demanded by investors to hold the risky assets. In our framework, the total risk premium is a composite compensation for continuous macroeconomic diffusion risks, transition policy shocks, and the catastrophic physical climate risks governed by the Hawkes process.

The physical climate risk premium is intrinsically tied to the investor's indirect utility coefficient b^E and the state variable λ^E. Because the representative agent exhibits recursive preferences ($\gamma > 1$), any mechanism that exacerbates the perceived tail risk will command a higher expected return.

Fig. 7 illustrates the distinct, compounding effects of the two jump mechanisms on the total equity risk premium With regard to the self-exciting Jumps (S), the expected magnitude of self-exciting jumps ($\mathbb{E}[S]$) dictates the severity of disaster clustering. As demonstrated earlier, an increase in $\mathbb{E}[S]$ drives a strictly convex increase in b^E. Consequently, as shown on the horizontal axis of Fig. 7, the risk premium is a strictly convex, upward-sloping function of $\mathbb{E}[S]$. The market strictly penalizes the compounding nature

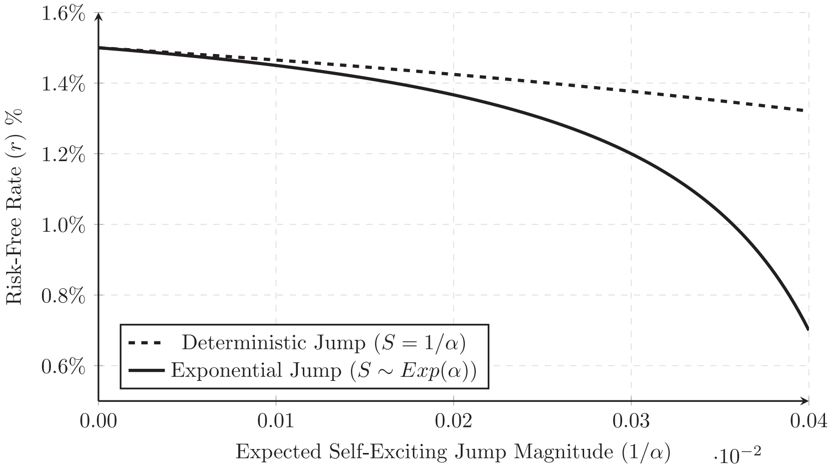

Fig. 6. Sensitivity of the risk-free rate to the expected magnitude of self-exciting jumps ($1/\alpha$). The dashed line represents a constant jump size, while the solid line represents an exponentially distributed jump. The fat tail of the exponential distribution accelerates precautionary savings, driving the interest rate down significantly faster than the deterministic baseline even for moderate expected clustering levels.

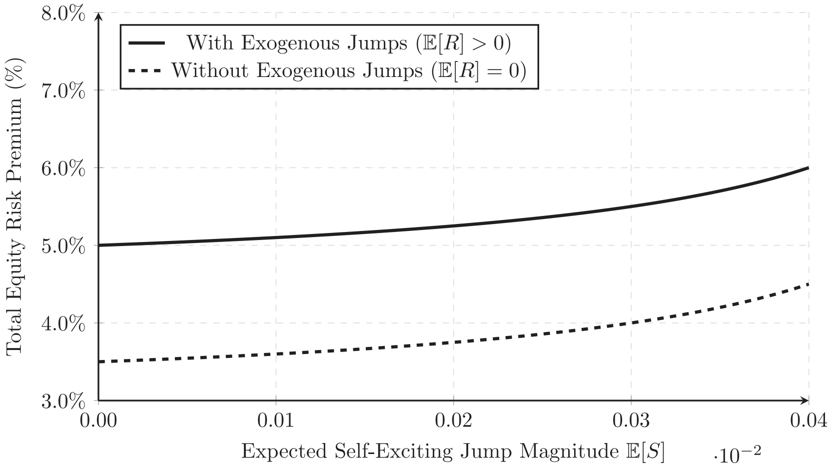

Fig. 7. The impact of self-exciting (S) and externally-excited (R) jumps on the Equity Risk Premium, confirming Proposition 7. The clustering parameter $\mathbb{E}[S]$ generates a non-linear, convex surge in the required risk premium due to the compounding fear of cascading disasters. Concurrently, the presence of exogenous shocks ($\mathbb{E}[R] > 0$) acts as a structural background penalty, shifting the entire risk premium curve upward by a constant factor.

of cascading disasters. With regard to the externally-excited jumps (R), the exogenous jumps introduce a baseline background risk. Because their arrival rate (ρ_λ) is constant and independent of the environmental state λ^E, the expected exogenous jump size ($\mathbb{E}[R]$) does not alter the convexity of the pricing kernel (i.e., it does not enter the Riccati equation for b^E). Instead, an increase in $\mathbb{E}[R]$ imposes a parallel upward shift on the entire risk premium curve.

The visual wedge between the two curves in Fig. 7 explicitly isolates the contribution of exogenous shocks, while the accelerating slopes isolate the penalty for self-excitation. Together, they demonstrate that failing to account for both mutually-exciting and externally-excited jump amplitudes leads to a profound underpricing of asset risk in environments exposed to severe climate change.

6. Conclusions

Building on the frameworks of [14,15], we develop an asset pricing model that incorporates both brown and green assets, introducing a novel feature of time-varying risks – both self-excited and externally triggered – that capture the clustering behavior of rare environmental disasters. We conduct a dynamic portfolio-consumption general equilibrium optimization over an infinite time horizon with recursive preferences, deriving closed-form expressions for the risk-free rate, market's risk premium, and government bond default risk. Our results, supported by numerical simulations, show that the self- and externally-excited nature of rare environmental events leads to an increase in the market's risk premium and government bond default risk, while simultaneously reducing the risk-free rate and expected government bond returns. These findings highlight the potential impact of environmental factors, including climate-related financial risks, on key financial variables and suggest that they constrain the effectiveness of monetary policy interventions aimed at supporting the green transition.

In an ongoing research, we aim to calibrate the parameters of the model to real market data and to conduct scenarios analysis. Further extensions of our setup can be explored, such as allowing for correlations among macroeconomic, environmental, and policy rare events, considering alternative jump distributions of the shocks, and studying the problem under different specifications of the representative agent's utility function, potentially enriched with a pro-green behavioral component. In particular, modeling the correlation between macroeconomic and environmental shocks, on the one hand, and transition risk, on the other, would allow us to capture additional dimensions of climate–finance risks that could further reinforce our findings. However, this extension comes at the expense of analytical tractability and would require relying on numerical methods.

CRediT authorship contribution statement

D. Radi: Conceptualization. **M. Santacroce:** Formal analysis. **B. Trivellato:** Methodology.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Technical proofs

Proof of Theorem 1. Substituting the guessed value function (16) in the first stationary condition (13) we obtain (20) or, equivalently, (22). Under condition (21), the existence and uniqueness of a solution of (22) in (0,1) is guaranteed. In fact, from (22) let us define the function

$$\phi(n_B) = n_B - n_{B,min} - \frac{\mu_B - \mu_G + \left(1 - n_B\left(1 - e^X\right)\right)^{-\gamma} \rho_X \left(e^X - 1\right)}{\hat{\sigma}^2_{GB}\,\gamma}. \tag{A.1}$$

It can be easily checked that $\phi'(n_B) = 1 + \frac{(1-n_B(1-e^X))^{-\gamma-1}\rho_X (e^X-1)^2}{\hat{\sigma}^2_{GB}} > 0$ in $[0,1]$, so $\phi(n_B)$ is strictly increasing. Furthermore, inequalities (21) ensure that $\phi(n_B = 0) < 0$ and $\phi(n_B = 1) > 0$. Finally, the optimality of the solution n_B to (20) comes from the concavity of $L^{\upsilon}\left(V\left(A, \lambda^M, \lambda^E\right)\right)$; specifically, we have

$$\begin{aligned}
\frac{\partial^2}{\partial n_B^2} L^{\upsilon}\left(V\left(A, \lambda^M, \lambda^E\right)\right) &= \frac{\partial^2}{\partial n_B^2} V\left(A\left(1 - n_B\left(1 - e^X\right), \lambda^M, \lambda^E\right)\rho_X + \frac{1}{2} V_{AA}\left(A, \lambda^M, \lambda^E\right)\left(\sigma^A\left(n_G, n_B\right)\right)^2 A^2\right) \\
&= -\gamma\left(A\left(1 - n_B(1 - e^X)\right)\right)^{-\gamma-1}(1 - e^X)^2 e^{d + b^M\lambda^M + b^E\lambda^E}\rho_X - \gamma A^{-\gamma-1} e^{d + b^M\lambda^M + b^E\lambda^E}\hat{\sigma}^2_{GB} \\
&= -\gamma A^{-\gamma-1} e^{d + b^M\lambda^M + b^E\lambda^E}\left(\left(1 - n_B(1 - e^X)\right)^{-\gamma-1}(1 - e^X)^2\rho_X + \hat{\sigma}^2_{GB}\right) < 0.
\end{aligned}$$

Now, substituting the value function (16) and the current utility function

$$f\left(C, V\left(A, \lambda^M, \lambda^E\right)\right) = \rho(1 - \gamma) V\left(A, \lambda^M, \lambda^E\right)\left[\log C - \frac{1}{1 - \gamma}\log\left((1 - \gamma) V\left(A, \lambda^M, \lambda^E\right)\right)\right] \tag{A.2}$$

in the first stationary condition (10), we immediately get $C = \rho A$. The optimality of $C = \rho A$ comes again from a concavity argument. Denoting, with an abuse of notation, $f\left(C, V\left(A, \lambda^M, \lambda^E\right)\right)$ by $f\left(A, \lambda^M, \lambda^E\right)$ we can rewrite (A.2) as

$$f\left(A, \lambda^M, \lambda^E\right) = \rho A^{1-\gamma} e^{d + b^M\lambda^M + b^E\lambda^E}\left(\log\rho - \frac{d + b^M\lambda^M + b^E\lambda^E}{1 - \gamma}\right). \tag{A.3}$$

Consider now the optimized Hamilton–Jacobi–Bellman equation in (7), that is

$$L^{v}\left(V\left(A,\lambda^{M},\lambda^{E}\right)\right)+f\left(A,\lambda^{M},\lambda^{E}\right)=0, \tag{A.4}$$

Replacing (16), $C=\rho A$ and (A.3) in (A.4), and taking into account that $n_G+n_B=1$, we can simplify (A.4) as follows:

$$\begin{aligned}&(1-\gamma)\left(\sum_{i=G,B}n_i\mu_i-\rho\right)-\frac{1}{2}\gamma(1-\gamma)\left(n_G^2\sigma_G^2+n_G^2\sigma_G^2+2n_Bn_G\sigma_{GB}\right)+\sum_{j=M,E}b^jk^j\bar{\lambda}^j+\rho_\lambda\left(\mathbb{E}\left[e^{b^ER}\right]-1\right)\\&+(1-\gamma)\rho\left(\log\rho-\frac{d}{1-\gamma}\right)+\rho_X\left(\left(1-n_B\left(1-e^X\right)\right)^{1-\gamma}-1\right)+\lambda^M\left(\mathbb{E}\left[e^{(1-\gamma)Z^M}\right]-1-b^M\left(\rho+k^M\right)+\frac{1}{2}(b^M)^2\left(\sigma_\lambda^M\right)^2\right)\\&+\lambda^E\left(\mathbb{E}\left[e^{(1-\gamma)Z^E}\right]\mathbb{E}\left[e^{b^ES}\right]-1-b^E\left(\rho+k^E\right)+\frac{1}{2}(b^E)^2\left(\sigma_\lambda^E\right)^2\right)=0,\end{aligned} \tag{A.5}$$

where we have collected the constant terms and the terms in λ^E and λ^M. Using the technique of separation of variables, we get the thesis. Solving the quadratic equation for b^M:

$$\frac{1}{2}\left(\sigma_\lambda^M\right)^2\left(b^M\right)^2-\left(\rho+k^M\right)b^M+\mathbb{E}\left[e^{(1-\gamma)Z^M}-1\right]=0, \tag{A.6}$$

and taking the negative root, we obtain (19). ■

Proof of Proposition 1. Under our assumptions, from

$$\Phi'(x;S)=0\iff\left(\sigma_\lambda^E\right)^2x-\left(\rho+k^E\right)=-\mathbb{E}\left[Se^{xS}\right]\mathbb{E}\left[e^{(1-\gamma)Z^E}\right] \tag{A.7}$$

we deduce that there exists a unique stationary point $\tilde{x}<\frac{\rho+k^E}{\left(\sigma_\lambda^E\right)^2}$ where the function $\Phi(x;S)$ reaches the minimum. Assertion (i) then follows from $\Phi(0;S)=\mathbb{E}\left[e^{(1-\gamma)Z^E}\right]-1>0$ and the assumption $\Phi(\tilde{x};S)\le 0$. Assertions (ii) and (iii) are immediate since (26) reduces to a quadratic equation. Finally, assertion (iv) is due to the fact that $\Phi(x;0)\le\Phi_{ap}(x;S)\le\Phi(x;S)$. ■

Proof of Proposition 2.

Case 1) If $S\sim Exp(\alpha)$ and $0<b^E<\alpha$, we have $\mathbb{E}\left[e^{b^ES}\right]=\frac{1}{1-\frac{b^E}{\alpha}}$. Therefore, Eq. (19) for b^E becomes

$$\frac{1}{2}\left(\sigma_\lambda^E\right)^2\left(b^E\right)^2-\left(\rho+k^E\right)b^E+\frac{\alpha}{\alpha-b^E}\mathbb{E}\left[e^{(1-\gamma)Z^E}\right]-1=0, \tag{A.8}$$

whose solutions in $(0,\alpha)$ are the same as the cubic equation

$$\mathcal{P}\left(b^E\right)=A\left(b^E\right)^3+B\left(b^E\right)^2+Cb^E+D=0, \tag{A.9}$$

where the coefficients A, B, C and D are defined as in (31). Note that $\mathcal{P}\left(b^E\right)\to-\infty$ as $b^E\to+\infty$ and $\mathcal{P}(\alpha)=\alpha\mathbb{E}\left[e^{(1-\gamma)Z^E}\right]>0$. Hence, there is always a real and positive root of (A.9) which is greater than α and therefore not of interest for us as we need $0<b^E<\alpha$. Moreover, note that $A<0$, $B>0$, and $D>0$. Therefore, by Descartes's rule of signs the number of positive roots can be either one or three for $C<0$ and is one for $C>0$. Hence, $C<0$, i.e. condition $\mathbb{E}(S)=\frac{1}{\alpha}<\rho+k^E$, is necessary for Eq. (A.9) to have real solutions in $(0,\alpha)$. Now, let us compute the derivative of $\mathcal{P}\left(b^E\right)$:

$$\mathcal{P}'\left(b^E\right)=3A\left(b^E\right)^2+2Bb^E+C. \tag{A.10}$$

It results that

$$\frac{\Delta}{4}=B^2-3AC=A^2\alpha^2+A(k^E+\rho)\alpha+\left(k^E+\rho\right)^2-A$$

is strictly positive for all α since

$$\tilde{\Delta}=-3A^2\left(k^E+\rho\right)^2+4A^3<0.$$

Therefore, the equation $\mathcal{P}'\left(b^E\right)=0$ has two positive real solutions $0<\tilde{b}_1^E<\tilde{b}_2^E$ with

$$\tilde{b}_{1,2}^E=\frac{-B\pm\sqrt{B^2-3AC}}{3A}. \tag{A.11}$$

Furthermore, we get $\mathcal{P}'\left(b^E\right)<0$ in $(0,\tilde{b}_1^E)$ and $(\tilde{b}_2^E,+\infty)$, while $\mathcal{P}'\left(b^E\right)>0$ in $(\tilde{b}_1^E,\tilde{b}_2^E)$. Since $\mathcal{P}(0)<\mathcal{P}(\alpha)$, taking into account the monotonicity of $\mathcal{P}$ we deduce that $\tilde{b}_1$ is smaller than α. Therefore, assuming $\mathcal{P}\left(\tilde{b}_1\right)<0$ ensures the existence of two real roots of the cubic Eq. (A.9) in $(0,\alpha)$. Finally, recall that when a cubic equation has three real roots, these roots can be computed using Cardanos' formulas:

$$b_j^E=2\sqrt{-\frac{p}{3}}\cos\left(\frac{\theta-2\pi j}{3}\right)-\frac{B}{3A},\text{ with }j\in\{0,1,2\} \tag{A.12}$$

where p and θ are defined in (35). Following the simple thought experiment of [14], we select for b^E the root that is equal to zero when $X = 0$ and Z^E is identically equal to zero, that is $b^E = \min_{j=0,1,2}\{b_j^E\}$.

To conclude the proof, we show that $\mathcal{P}\left(\tilde{b}_1\right) < 0$ is satisfied for α sufficiently large under the assumption $\mathbb{E}\left[e^{(1-\gamma)Z^E}\right] < 1 + \frac{\left(k^E+\rho\right)^2}{2\left(\sigma_\lambda^E\right)^2}$. In fact, note that

$$\lim_{\alpha\to+\infty}\tilde{b}_1 = \lim_{\alpha\to+\infty}\frac{-B+\sqrt{B^2-3AC}}{3A} = \frac{\rho+k^E}{(\sigma_\lambda^E)^2} \tag{A.13}$$

hence

$$\begin{aligned}
\lim_{\alpha\to+\infty}\mathcal{P}\left(\tilde{b}_1\right) &= \lim_{\alpha\to+\infty}\mathcal{P}\left(\frac{B+\sqrt{B^2-3AC}}{3A}\right)\\
&= A\left(\frac{\rho+k^E}{(\sigma_\lambda^E)^2}\right)^3 + (\rho+k^E)\left(\frac{\rho+k^E}{(\sigma_\lambda^E)^2}\right)^2 + \frac{\rho+k^E}{(\sigma_\lambda^E)^2}\\
&\quad + \lim_{\alpha\to+\infty}\alpha\left(-A\left(\tilde{b}_1\right)^2 - (\rho+k^E)\tilde{b}_1 + \mathbb{E}\left[e^{(1-\gamma)Z^E}-1\right]\right)\\
&= \frac{\rho+k^E}{(\sigma_\lambda^E)^2}\left[1+\frac{(\rho+k^E)^2}{2(\sigma_\lambda^E)^2}\right]\\
&\quad + \left(\lim_{\alpha\to+\infty}\alpha\right)\left(-\frac{(\rho+k^E)^2}{2(\sigma_\lambda^E)^2} + \mathbb{E}\left[e^{(1-\gamma)Z^E}-1\right]\right) = -\infty.
\end{aligned}$$

Case 2) The thesis follows by substituting $\mathbb{E}[S] = \frac{1}{\alpha}$ and $\mathbb{E}\left[S^2\right] = \frac{2}{\alpha^2}$ in (29). ■

Proof of Proposition 3. Multiplying (11) by n_G and (12) by n_B, then summing up and substituting the guessed value function (16) and $n_G + n_B = 1$, we finally obtain (36). ■

Proof of Proposition 4. Applying the integration by parts formula to (37), $\frac{dm}{m_-}$ takes the form[12]

$$\frac{dm}{m} = f_U(C,U)\,dt + \frac{d\,f_C(C,U)}{f_C(C,U)}, \tag{A.14}$$

where

$$f_U(C,U) = \rho(1-\gamma)\log C - \rho\log((1-\gamma)U) - \rho \tag{A.15}$$

and

$$f_C(C,U) = \rho(1-\gamma)\frac{U}{C}. \tag{A.16}$$

We first compute $\frac{d\,f_C(C,U)}{f_C(C,U)}$ in (A.14) to find the stochastic terms of m, and then we will collect all the terms in dt. Employing optimality to (A.16), that is $U = \frac{A^{1-\gamma}}{1-\gamma}e^{d+b^M\lambda^M+b^E\lambda^E}$ and $C = \rho A$, we can write U/C as

$$\frac{U}{C} = \frac{1}{(1-\gamma)\,\rho^{1-\gamma}}C^{-\gamma}e^{d+b^M\lambda^M+b^E\lambda^E}, \tag{A.17}$$

and therefore in (A.14) we get

$$\frac{d\,f_C(C,U)}{f_C(C,U)} = \frac{d(U/C)}{U/C} = \frac{d\left(C^{-\gamma}e^{d+b^M\lambda^M+b^E\lambda^E}\right)}{C^{-\gamma}e^{d+b^M\lambda^M+b^E\lambda^E}} = \frac{d\left(C^{-\gamma}e^{\sum_{j=M,E}b^j\lambda^j}\right)}{C^{-\gamma}e^{\sum_{j=M,E}b^j\lambda^j}}. \tag{A.18}$$

Using the integration by parts formula and separating the jump component we have

$$\frac{d\left(C^{-\gamma}e^{\sum_{j=M,E}b^j\lambda^j}\right)}{C^{-\gamma}e^{\sum_{j=M,E}b^j\lambda^j}} = \frac{d\left(C^{-\gamma}\right)^c}{C^{-\gamma}} + \frac{d\left(e^{\sum_{j=M,E}b^j\lambda^j}\right)^c}{e^{\sum_{j=M,E}b^j\lambda^j}} + \frac{d\langle C^{-\gamma}, e^{\sum_{j=M,E}b^j\lambda^j}\rangle}{C^{-\gamma}e^{\sum_{j=M,E}b^j\lambda^j}} + \frac{\Delta\left(C^{-\gamma}e^{\sum_{j=M,E}b^j\lambda^j}\right)}{C^{-\gamma}e^{\sum_{j=M,E}b^j\lambda^j}}. \tag{A.19}$$

Therefore, in order to compute (A.18), we first find $\frac{d(C^{-\gamma})}{C^{-\gamma}}$ and $\frac{d\left(e^{\sum_{j=M,E}b^j\lambda^j}\right)}{e^{\sum_{j=M,E}b^j\lambda^j}}$. Applying Ito's formula to $C = \rho A$, and using $n_G + n_B = 1$, we obtain

$$\frac{dC}{C} = g\left(n_G,n_B\right)\,dt + \sum_{i=G,B}n_i\sigma_i dW_i + \sum_{j=M,E}dQ^j + n_B dQ^X, \tag{A.20}$$

---

[12] To lighten the notation, hereafter we neglect the subscript $_-$.

where $g\left(n_G, n_B\right) = \sum_{i=G,B} n_i \mu_i - \rho$, whose solution is given by

$$\begin{aligned} C_t &= C_0 \exp\left(\left(g\left(n_G, n_B\right) - \frac{1}{2}\left(\sigma^A(n_G, n_B)\right)^2\right) t + \sum_{i=G,B} n_i \sigma_i W_{i,t}\right) \\ &\quad \times \left(\prod_{n=1}^{N_t^M} e^{Z_n^M}\right)\left(\prod_{n=1}^{N_t^E} e^{Z_n^E}\right)\left(1 - n_B\left(1 - e^X\right)\right)^{N_t^X}. \end{aligned} \tag{A.21}$$

Applying Ito's formula to $C^{-\gamma}$ and using (A.20) and (A.21), we obtain[13]

$$\begin{aligned} \frac{d(C^{-\gamma})}{C^{-\gamma}} &= \frac{-\gamma C^{-\gamma-1} dC^c + \frac{1}{2}\gamma(\gamma+1) C^{-\gamma-2} d\langle C, C\rangle + \Delta C^{-\gamma}}{C^{-\gamma}} \\ &= -\gamma\left(g\left(n_G, n_B\right) - \tfrac{1}{2}(\gamma+1)\left(\sigma^A\left(n_G, n_B\right)\right)^2\right) dt - \gamma \textstyle\sum_{i=G,B} n_i \sigma_i dW_i \\ &\quad + \textstyle\sum_{j=M,E}\left(e^{-\gamma Z^j} - 1\right)\Delta N^j + \left[\left(1 - n_B\left(1 - e^X\right)\right)^{-\gamma} - 1\right]\Delta N^X \end{aligned} \tag{A.22}$$

Similarly, applying Ito's formula to $e^{\sum_{j=M,E} b^j \lambda^j}$, and using (2)–(3), we get

$$\begin{aligned} \frac{d\left(e^{\sum_{j=M,E} b^j \lambda^j}\right)}{e^{\sum_{j=M,E} b^j \lambda^j}} &= \textstyle\sum_{j=M,E} b^j d\left(\lambda^j\right)^c + \tfrac{1}{2} d\langle \sum_{j=M,E} b^j \lambda^j, \sum_{j=M,E} b^j \lambda^j \rangle + e^{b^E \Delta \lambda^E} - 1 \\ &= \left(\sum_{j=M,E} b^j k^j\left(\bar{\lambda}^j - \lambda^j\right) + \frac{1}{2}\sum_{j=M,E}\left(b^j \sigma_\lambda^j\right)^2 \lambda^j\right) dt + \sum_{j=M,E} b^j \sigma_\lambda^j \sqrt{\lambda^j} dW_\lambda^j \\ &\quad + \left(e^{b^E S} - 1\right)\Delta N^E + \left(e^{b^E R} - 1\right)\Delta N^{ext}. \end{aligned} \tag{A.23}$$

From (A.22) and (A.23), we find the jump component of (A.19)

$$\begin{aligned} \frac{\Delta\left(C^{-\gamma} e^{\sum_{j=M,E} b^j \lambda^j}\right)}{C^{-\gamma} e^{\sum_{j=M,E} b^j \lambda^j}} &= \left(e^{-\gamma Z^M} - 1\right)\Delta N^M + \left(e^{-\gamma Z^E + b^E S} - 1\right)\Delta N^E + \left(e^{b^E R} - 1\right)\Delta N^{ext} \\ &\quad + \left[\left(1 - n_B\left(1 - e^X\right)\right)^{-\gamma} - 1\right]\Delta N^X \\ &= d\tilde{Q}^M + d\tilde{Q}^E + d\tilde{Q}^{ext} + d\tilde{Q}^X \end{aligned} \tag{A.24}$$

where $\tilde{Q}^M$, $\tilde{Q}^E$, $\tilde{Q}^{ext}$ and $\tilde{Q}^X$ are given in (41) and (42). Thus, summing the terms of (A.22) and (A.23), we find (A.18). To conclude the proof, we are left with the task of detecting the drift μ_m in (A.14) by collecting all the terms in dt. Employing optimality to (A.15), that is $U = \frac{A^{1-\gamma}}{1-\gamma} e^{d + b^M \lambda^M + b^E \lambda^E}$ and $C = \rho A$, we can write $f_U(C, U)$ as

$$\begin{aligned} f_U(C, U) &= \rho(1-\gamma)\log C - \rho \log((1-\gamma)U) - \rho \\ &= \rho(1-\gamma)\log \rho - \rho d - \rho \sum_{j=M,E} b^j \lambda^j - \rho. \end{aligned} \tag{A.25}$$

Summing (A.25) with the terms in dt of (A.22)–(A.23), and exploiting the optimized Hamilton–Jacobi–Bellman Eq. (A.5), we obtain

$$\begin{aligned} \mu_m &= -\left(\rho + g(n_G, n_B) - \gamma\left(\sigma^A(n_G, n_B)\right)^2\right) - \rho_\lambda\left(\mathbb{E}\left[e^{b^E R}\right] - 1\right) - \mathbb{E}\left[e^{(1-\gamma)Z^M} - 1\right]\lambda^M \\ &\quad - \left(\mathbb{E}\left[e^{(1-\gamma)Z^E}\right]\mathbb{E}\left[e^{b^E S}\right] - 1\right)\lambda^E - \left[\left(1 - n_B\left(1 - e^X\right)\right)^{1-\gamma} - 1\right]\rho_X. \end{aligned} \tag{A.26}$$

Finally, using (36) we can rewrite μ_m as in (40). ■

Proof of Proposition 5. Denote

$$H\left(D_t, \lambda_t^M, \lambda_t^E, s - t\right) = \mathbb{E}_t\left[\frac{m_s}{m_t} D_s\right]. \tag{A.27}$$

Since $H\left(D_s, \lambda_s^M, \lambda_s^E, 0\right) = D_s$ and

$$H\left(D_t, \lambda_t^M, \lambda_t^E, s - t\right) = \mathbb{E}_t\left[\frac{m_s}{m_t} D_s\right] = \mathbb{E}_t\left[\frac{m_s}{m_t} H\left(D_s, \lambda_s^M, \lambda_s^E, 0\right)\right], \tag{A.28}$$

we get that $m_t H\left(D_t, \lambda_t^M, \lambda_t^E, s - t\right)$ is a martingale. Conjecture that

$$H_t \equiv H\left(D_t, \lambda_t^M, \lambda_t^E, s - t\right) = D_t e^{d_\eta(s-t) + b_\eta^M(s-t)\lambda_t^M + b_\eta^E(s-t)\lambda_t^E}. \tag{A.29}$$

---

[13] Let Y^c denote the continuous part of the process Y while $\langle Y, Z\rangle$ is the quadratic covariation of processes Y^c and Z^c.

Proceeding similarly to (A.19)–(A.24), but with time-dependent coefficients $d_\eta(s-t)$ and $b_\eta^j(s-t)$, we can write H_t as

$$\begin{aligned}\frac{dH}{H_-} &= \left(\mu_D + \textstyle\sum_{j=M,E} b_\eta^j(s-t)k^j\left(\bar{\lambda}^j-\lambda^j\right) + \frac{1}{2}\sum_{j=M,E}\left(b_\eta^j(s-t)\sigma_\lambda^j\right)^2\lambda^j - \left(d_\eta\right)'(s-t) - \sum_{j=M,E}\left(b_\eta^j\right)'(s-t)\lambda^j\right)dt \\ &\quad +\eta\textstyle\sum_{i=G,B} n_i\sigma_i dW_i + \sum_{j=M,E} b_\eta^j(s-t)\sigma_\lambda^j\sqrt{\lambda^j}dW_\lambda^j + d\tilde{Q}_\eta^M + d\tilde{Q}_\eta^E + d\tilde{Q}_\eta^{ext} + d\tilde{Q}_\eta^X\end{aligned} \tag{A.30}$$

where μ_D is defined by (44) and $\tilde{Q}_\eta^M$, $\tilde{Q}_\eta^E$, $\tilde{Q}_\eta^{ext}$ and $\tilde{Q}_\eta^X$, are defined as in (41) and (42) but with η instead of $-\gamma$ and, with $b_\eta^E(s-t)$ instead of b^E. Denote

$$\mu_H = \mu_D + \sum_{j=M,E} b_\eta^j(s-t)k^j\left(\bar{\lambda}^j-\lambda^j\right) + \frac{1}{2}\sum_{j=M,E}\left(b_\eta^j(s-t)\sigma_\lambda^j\right)^2\lambda^j - \left(d_\eta\right)'(s-t) - \sum_{j=M,E}\left(b_\eta^j\right)'(s-t)\lambda^j. \tag{A.31}$$

Applying integration by parts formula to $m_t H_t$ we get

$$\begin{aligned}\frac{d(mH)}{(mH)_-} &= \left(\mu_m + \mu_H - \gamma\eta\left(\sigma^A(n_G,n_B)\right)^2 + \textstyle\sum_{j=M,E} b^j b_\eta^j(s-t)\left(\sigma_\lambda^j\right)^2\lambda^j\right)dt \\ &\quad +(\eta-\gamma)\textstyle\sum_{i=G,B} n_i\sigma_i dW_i + \sum_{j=M,E}\left(b_\eta^j(s-t)+b^j\right)\sigma_\lambda^j\sqrt{\lambda^j}dW_\lambda^j \\ &\quad +d\tilde{\tilde{Q}}^M + d\tilde{\tilde{Q}}^E + d\tilde{\tilde{Q}}^X + d\tilde{\tilde{Q}}^{ext}\end{aligned} \tag{A.32}$$

where, $\tilde{\tilde{Q}}^M$, $\tilde{\tilde{Q}}^E$, $\tilde{\tilde{Q}}^{ext}$ and $\tilde{\tilde{Q}}^X$ are, respectively, defined again as in (41), (42), but this time with $\eta-\gamma$ instead of $-\gamma$ and, with $b^E + b_\eta^E(s-t)$ instead of b^E. We observe that the compensators of the jumps components $\tilde{\tilde{Q}}^M$, $\tilde{\tilde{Q}}^E$, $\tilde{\tilde{Q}}^{ext}$ and $\tilde{\tilde{Q}}^X$ in (A.32) are

$$\begin{aligned}&\left(\lambda^M E\left[e^{(\eta-\gamma)Z^M}-1\right] + \lambda^E\left(E\left[e^{(\eta-\gamma)Z^E}\right]E\left[e^{(b^E+b_\eta^E(s-t))S}\right]-1\right) + \rho_\lambda E\left[e^{(b^E+b_\eta^E(s-t))R}-1\right]\right)dt \\ &\quad +\left(\rho_X\left[\left(1-n_B\left(1-e^X\right)\right)^{\eta-\gamma}-1\right]\right)dt.\end{aligned} \tag{A.33}$$

Since mH is a martingale,[14] taking the conditional expectation in (A.32), we get

$$\begin{aligned}&\mu_m + \mu_H - \gamma\eta\left(\sigma^A\left(n_G,n_B\right)\right)^2 + \textstyle\sum_{j=M,E} b^j b_\eta^j(s-t)\left(\sigma_\lambda^j\right)^2\lambda^j + \rho_\lambda\mathbb{E}\left[e^{\left(b^E+b_\eta^E(s-t)\right)R}-1\right] \\ &\quad +\lambda^M\mathbb{E}\left[e^{(\eta-\gamma)Z^M}-1\right] + \lambda^E\left(\mathbb{E}\left[e^{(\eta-\gamma)Z^E}\right]\mathbb{E}\left[e^{\left(b^E+b_\eta^E(s-t)\right)S}\right]-1\right) \\ &\quad +\rho_X\left[\left(1-n_B\left(1-e^X\right)\right)^{\eta-\gamma}-1\right] = 0.\end{aligned} \tag{A.34}$$

Substituting the expressions of μ_m and μ_H given, respectively in (40) and in (A.31), in Eq. (A.34), after collecting the constant terms and the terms in λ^E and λ^M, we finally obtain

$$\begin{aligned}&-\left(d_\eta\right)'(s-t) + \mu_D - g\left(n_G,n_B\right) - \rho + (1-\eta)\gamma\left(\sigma^A\left(n_G,n_B\right)\right)^2 + \rho_\lambda\mathbb{E}\left[e^{b^E R}\left(e^{b_\eta^E(s-t)R}-1\right)\right] + \textstyle\sum_{j=M,E} b_\eta^j(s-t)k^j\bar{\lambda}^j \\ &\quad +\rho_X\left[\left(1-n_B\left(1-e^X\right)\right)^{\eta-\gamma} - \left(1-n_B\left(1-e^X\right)\right)^{1-\gamma}\right] \\ &\quad +\lambda^M\left\{-\left(b_\eta^M\right)'(s-t) - b_\eta^M(s-t)k^M + \tfrac{1}{2}\left(b_\eta^M(s-t)\sigma_\lambda^M\right)^2 + b^M b_\eta^M(s-t)\left(\sigma_\lambda^M\right)^2 + \mathbb{E}\left[e^{(\eta-\gamma)Z^M} - e^{(1-\gamma)Z^M}\right]\right\} \\ &\quad +\lambda^E\left\{-\left(b_\eta^E\right)'(s-t) - b_\eta^E(s-t)k^M + \tfrac{1}{2}\left(b_\eta^E(s-t)\sigma_\lambda^E\right)^2 + b^E b_\eta^E(s-t)\left(\sigma_\lambda^E\right)^2\right. \\ &\quad \left. +\mathbb{E}\left[e^{(\eta-\gamma)Z^E}\right]\mathbb{E}\left[e^{\left(b^E+b_\eta^E(s-t)\right)S}\right] - \mathbb{E}\left[e^{(1-\gamma)Z^E}\right]\mathbb{E}\left[e^{b^E S}\right]\right\} = 0.\end{aligned} \tag{A.35}$$

The system of differential Eqs. (46) follows immediately. ■

Proof of Proposition 7. To derive an expression for the premium on the aggregate market, we start by multiplying each side of (38) by m_t:

$$m_t P_t = \mathbb{E}_t\left[\int_t^\infty m_u D_u du\right]. \tag{A.36}$$

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[14] We assume mild regularity conditions analogous to those used in [46, Proposition 1] to ensure that the compensated jump processes are martingales.

Since the same equation holds for any $s > t$, we deduce that

$$m_t P_t = \mathbb{E}_t \left[m_s P_s + \int_t^s m_u D_u du \right]. \tag{A.37}$$

Adding $\int_0^t m_u D_u du$ to both sides of (A.37) implies

$$m_t P_t + \int_0^t m_u D_u du = \mathbb{E}_t \left[m_s P_s + \int_0^s m_u D_u du \right], \tag{A.38}$$

which means that $m_t P_t + \int_0^t m_u D_u du$ is a martingale. Applying integration by parts formula to $m_t P_t$ we get

$$\begin{aligned} \frac{d(mP)}{(mP)_-} \quad = \quad & \left(\mu_m + \mu_P - \gamma\eta \left(\sigma^A(n_G, n_B)\right)^2 + \sum_{j=M,E} b^j \frac{G_{\lambda j}}{G_-} \left(\sigma_\lambda^j\right)^2 \lambda^j \right) dt \\ & + (\eta - \gamma) \sum_{i=G,B} n_i \sigma_i dW_i + \sum_{j=M,E} \left(b^j + \frac{G_{\lambda j}}{G_-} \right) \sigma_\lambda^j \sqrt{\lambda^j} dW_\lambda^j \\ & + \text{jump terms}, \end{aligned} \tag{A.39}$$

where the jump terms are as in (A.32), but with $\left(e^{b_\eta^E (s-t) R}\right)$ and $\left(e^{b_\eta^E (s-t) S}\right)$ replaced, respectively, by

$$\frac{\displaystyle\int_0^\infty e^{d_\eta(\tau)+b_\eta^M(\tau)\lambda^M+b_\eta^E(\tau)\lambda^E+b_\eta^E(\tau)R} d\tau}{\displaystyle\int_0^\infty e^{d_\eta(\tau)+b_\eta^M(\tau)\lambda^M+b_\eta^E(\tau)\lambda^E} d\tau} \quad \text{and} \quad \frac{\displaystyle\int_0^\infty e^{d_\eta(\tau)+b_\eta^M(\tau)\lambda^M+b_\eta^E(\tau)\lambda^E+b_\eta^E(\tau)S} d\tau}{\displaystyle\int_0^\infty e^{d_\eta(\tau)+b_\eta^M(\tau)\lambda^M+b_\eta^E(\tau)\lambda^E} d\tau}. \tag{A.40}$$

As in the proof of Proposition 5, since $m_t P_t + \int_0^t m_u D_u du$ is a martingale, we sum up the compensators of the jumps with the term in dt of (A.39), and set the conditional expectation equal to zero, obtaining

$$\begin{aligned} & \mu_m + \mu_P + \frac{D}{P} - \gamma\eta \left(\sigma^A(n_G, n_B)\right)^2 + \sum_{j=M,E} b^j \frac{G_{\lambda j}}{G_-} \left(\sigma_\lambda^j\right)^2 \lambda^j \\ & + \rho_\lambda \mathbb{E}_{\lambda^M, \lambda^E} \left[\frac{\displaystyle\int_0^\infty e^{d_\eta(\tau)+b_\eta^M(\tau)\lambda^M+b_\eta^E(\tau)\lambda^E} e^{\left(b^E + b_\eta^E(\tau)\right) R} d\tau}{\displaystyle\int_0^\infty e^{d_\eta(\tau)+b_\eta^M(\tau)\lambda^M+b_\eta^E(\tau)\lambda^E} d\tau} - 1 \right] + \rho_X \left[\left(1 - n_B \left(1 - e^X\right)\right)^{\eta-\gamma} - 1 \right] \\ & + \lambda^M \mathbb{E}\left[e^{(\eta-\gamma) Z^M} - 1 \right] + \lambda^E \left(\mathbb{E}\left[e^{(\eta-\gamma) Z^E} \right] \mathbb{E}_{\lambda^M, \lambda^E} \left[\frac{\displaystyle\int_0^\infty e^{d_\eta(\tau)+b_\eta^M(\tau)\lambda^M+b_\eta^E(\tau)\lambda^E} e^{\left(b^E + b_\eta^E(\tau)\right) S} d\tau}{\displaystyle\int_0^\infty e^{d_\eta(\tau)+b_\eta^M(\tau)\lambda^M+b_\eta^E(\tau)\lambda^E} d\tau} \right] - 1 \right) = 0. \end{aligned} \tag{A.41}$$

Now, substituting (40) in (A.41), we immediately obtain

$$\begin{aligned} \mu_P + \frac{D}{P} - r \quad = \quad & \rho_\lambda \left(\mathbb{E}\left[e^{b^E R} \right] - 1 \right) + \mathbb{E}\left[e^{-\gamma Z^M} - 1 \right] \lambda^M + \left(\mathbb{E}\left[e^{-\gamma Z^E} \right] \mathbb{E}\left[e^{b^E S} \right] - 1 \right) \lambda^E \\ & + \left[\left(1 - n_B \left(1 - e^X\right)\right)^{-\gamma} - 1 \right] \rho_X + \gamma\eta \left(\sigma^A(n_G, n_B)\right)^2 - \sum_{j=M,E} b^j \frac{G_{\lambda j}}{G_-} \left(\sigma_\lambda^j\right)^2 \lambda^j \\ & - \rho_\lambda \mathbb{E}_{\lambda^M, \lambda^E} \left[\frac{\displaystyle\int_0^\infty e^{d_\eta(\tau)+b_\eta^M(\tau)\lambda^M+b_\eta^E(\tau)\lambda^E} \mathbb{E}\left[e^{\left(b^E + b_\eta^E(\tau)\right) R} \right] d\tau}{\displaystyle\int_0^\infty e^{d_\eta(\tau)+b_\eta^M(\tau)\lambda^M+b_\eta^E(\tau)\lambda^E} d\tau} - 1 \right] - \left[\left(1 - n_B \left(1 - e^X\right)\right)^{\eta-\gamma} - 1 \right] \rho_X \\ & - \mathbb{E}\left[e^{(\eta-\gamma) Z^M} - 1 \right] \lambda^M - \left(\mathbb{E}\left[e^{(\eta-\gamma) Z^E} \right] \mathbb{E}_{\lambda^M, \lambda^E} \left[\frac{\displaystyle\int_0^\infty e^{d_\eta(\tau)+b_\eta^M(\tau)\lambda^M+b_\eta^E(\tau)\lambda^E} e^{\left(b^E + b_\eta^E(\tau)\right) S} d\tau}{\displaystyle\int_0^\infty e^{d_\eta(\tau)+b_\eta^M(\tau)\lambda^M+b_\eta^E(\tau)\lambda^E} d\tau} \right] - 1 \right) \lambda^E. \end{aligned} \tag{A.42}$$

We finally get the thesis by combining (56) with (A.42). ■

Proof of Proposition 6. Consider the general Cauchy problem

$$\begin{cases} x'(\tau) = f(x(\tau)) \\ x(0) = 0, \end{cases}$$

where the driver $f \in C^1((-\infty, 0])$ and $f(0) < 0$. The existence and uniqueness of its solution is a classical problem in dynamical systems and it is well known that solutions of autonomous ODEs are monotone in τ, see e.g. [47]. Therefore, since $x'(0) = f(0) < 0$, the solution of the above Cauchy problem, as $\tau \to +\infty$, either diverges to $-\infty$ or converges to an equilibrium point $\bar{x} < 0$ such that $f(\bar{x}) = 0$.

With respect to the Cauchy problem (52), since $b_\eta^E(0) = 0$ and

$$\left(b_\eta^E\right)'(0) = \left(\mathbb{E}\left[e^{(\eta-\gamma)Z^E}\right] - \mathbb{E}\left[e^{(1-\gamma)Z^E}\right]\right)\mathbb{E}\left[e^{b^E S}\right] < 0,$$

we deduce that the solution $b_\eta^E(\tau)$ is strictly decreasing and thus $b_\eta^E(\tau) < 0$ for all $\tau > 0$.

Now, consider the steady-state Eq. (53) on $(-\infty, 0]$ rewritten as:

$$\frac{1}{2}\left(\sigma_\lambda^E\right)^2 x^2 + \left(b^E\left(\sigma_\lambda^E\right)^2 - k^E\right)x = \mathbb{E}\left[e^{(1-\gamma)Z^E}\right]\mathbb{E}\left[e^{b^E S}\right] - \mathbb{E}\left[e^{(\eta-\gamma)Z^E}\right]\mathbb{E}\left[e^{\left(b^E+x\right)S}\right]. \tag{A.43}$$

The right-hand side of (A.43) is positive at $x = 0$, concave and increasing in $x < 0$, and converges to $\mathbb{E}\left[e^{(1-\gamma)Z^E}\right]\mathbb{E}\left[e^{b^E S}\right]$ as $x \to -\infty$. Therefore, it intersects the left-hand side of (A.43) at a single point $\bar{x} < 0$. This proves that $b_\eta^E(\tau) < 0$ converges, as $\tau \to +\infty$, to an equilibrium point $\bar{b}_\eta^E < 0$ which solves (53).

Finally, we have that $b_\eta^E(\tau) < b_{\eta,0}^E(\tau)$ for all $\tau > 0$ because, using the same arguments as before for $\left(b_\eta^E - b_{\eta,0}^E\right)(\tau)$, we get $\left(b_\eta^E - b_{\eta,0}^E\right)(0) = 0$ and

$$\left(b_\eta^E - b_{\eta,0}^E\right)'(0) = \underbrace{\left(\mathbb{E}\left[e^{(\eta-\gamma)Z^E}\right] - \mathbb{E}\left[e^{(1-\gamma)Z^E}\right]\right)}_{\langle 0}\underbrace{\left(\mathbb{E}\left[e^{b^E S}\right] - 1\right)}_{\rangle 0} < 0. \quad \blacksquare$$

Proof of Corollary 3. The thesis immediately follows from the inequality $b_\eta^E(\tau) < b_{\eta,0}^E(\tau)$, which also implies $d_\eta(\tau) < d_{\eta,0}(\tau)$ for all $\tau > 0$. Indeed, since $b^E > b_0^E > 0$, where b_0^E is the solution of (19) when $S = 0$, and $k^E, \bar{\lambda}^E > 0$, we have that

$$d_\eta(\tau) - d_{\eta,0}(\tau) \quad = \quad \rho_\lambda \int_0^\tau \underbrace{\left(\mathbb{E}\left[e^{b^E R}\left(e^{b_\eta^E(s)R} - 1\right)\right] - \mathbb{E}\left[e^{b_0^E R}\left(e^{b_{\eta,0}^E(s)R} - 1\right)\right]\right)}_{<0} ds + \int_0^\tau \underbrace{\left(b_\eta^E(s) - b_{\eta,0}^E(s)\right)}_{<0} k^E \bar{\lambda}^E ds < 0. \quad \blacksquare$$

Proof of Proposition 8. We write by parts formula for mL and we use the martingality of the price process L under the measure induced by the state price density,

$$\begin{aligned}\frac{d(mL)}{(mL)_-} \quad &= \quad \mu_{m_-}\,dt - \gamma\sum_{i=G,B} n_i\sigma_i dW_i + \sum_{j=M,E} b^j\sigma_\lambda^j\sqrt{\lambda_-^j}\,dW_\lambda^j + r^L\,dt + \left(e^{-\gamma Z^M + Z^{M^L}} - 1\right)\Delta N^M \\ &+ \quad \left(e^{-\gamma Z^E + b^E S + Z^{E^L}} - 1\right)\Delta N^E + \left(e^{b^E R} - 1\right)\Delta N^{ext} + \left(\left(1 - n_B\left(1 - e^{X^L}\right)\right)\left(1 - n_B\left(1 - e^X\right)\right)^{-\gamma} - 1\right)\Delta N^X.\end{aligned} \tag{A.44}$$

Since mL is a martingale, taking the conditional expectation, we get

$$\begin{aligned}&\mu_m + r^L + \mathbb{E}\left[\left(e^{-\gamma Z^M + Z^{M^L}} - 1\right)\right]\lambda^M + \mathbb{E}\left[\left(e^{-\gamma Z^E + b^E S + Z^{E^L}} - 1\right)\right]\lambda^E + \mathbb{E}\left[e^{b^E R} - 1\right]\rho_\lambda \\ &\qquad \mathbb{E}\left[\left(1 - n_B\left(1 - e^{X^L}\right)\right)\left(1 - n_B\left(1 - e^X\right)\right)^{-\gamma} - 1\right]\rho_X = 0.\end{aligned} \tag{A.45}$$

Now, taking into account the probabilities of government default, we compute the expected values appearing in (A.45):

$$\begin{aligned}&\mathbb{E}\left[\left(e^{-\gamma Z^M + Z^{M^L}} - 1\right)\right] = \mathbb{E}\left[e^{-\gamma Z^M}\mathbb{E}\left[e^{Z^{M^L}} | Z^M\right]\right] - 1 = \mathbb{E}\left[e^{-\gamma Z^M}\right] - q^M\mathbb{E}\left[e^{-\gamma Z^M}\left(1 - e^{Z^M}\right)\right] - 1 \\ &\mathbb{E}\left[\left(e^{-\gamma Z^E + b^E S + Z^{E^L}} - 1\right)\right] = \mathbb{E}\left[e^{-\gamma Z^E + b^E S}\right] - q^E\mathbb{E}\left[e^{-\gamma Z^E + b^E S}\left(1 - e^{Z^E}\right)\right] - 1 \\ &\mathbb{E}\left[\left(1 - n_B\left(1 - e^{X^L}\right)\right)\left(1 - n_B\left(1 - e^X\right)\right)^{-\gamma} - 1\right] = \left(1 - n_B\left(1 - e^X\right)\right)^{-\gamma}\left(1 - q^X n_B\left(1 - e^X\right)\right) - 1.\end{aligned} \tag{A.46}$$

Substituting the expected values of (A.46) and μ_m defined in (40), Eq. (A.45) gives the formula of the reward in case of no default

$$\begin{aligned}r^L \quad = \quad r + &\underbrace{\mathbb{E}\left[e^{-\gamma Z^M}\left(1 - e^{Z^M}\right)\right]q^M\lambda^M}_{\text{Extra reward for taking Macroeconomic Risk}} \\ &+ \underbrace{\mathbb{E}\left[e^{-\gamma Z^E + b^E S}\left(1 - e^{Z^E}\right)\right]q^E\lambda^E}_{\text{Extra reward for taking Physical Rik}} + \underbrace{n_B\left(1 - e^X\right)\left(1 - n_B\left(1 - e^X\right)\right)^{-\gamma}q^X\rho_X}_{\text{Extra reward for taking Policy Shock Rik}}.\end{aligned} \tag{A.47}$$

Afterwards, immediately from the definition of instantaneous expected return (62), we derive the spread on government bonds (63). Note that (A.47) gives also $r^L - r$, the observed premium on government debt against the risk-free rate in samples without disasters. ■

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