On the geometry of some unitary Riemann surface braid group representations and Laughlin-type wave functions

Mauro Spera
Dipartimento di Matematica e Fisica "Niccolò Tartaglia"
Università Cattolica del Sacro Cuore
Via dei Musei 41, 25121 Brescia, Italia
E-mail: mauro.spera@unicatt.it

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Abstract

In this note we construct the simplest unitary Riemann surface braid group representations geometrically by means of stable holomorphic vector bundles over complex tori and the prime form on Riemann surfaces. Generalised Laughlin wave functions are then introduced. The genus one case is discussed in some detail also with the help of noncommutative geometric tools, and an application of Fourier-Mukai-Nahm techniques is also given, explaining the emergence of an intriguing Riemann surface braid group duality.

Keywords: Riemann surface braid groups, stable holomorphic vector bundles, prime form, Laughlin wave functions, noncommutative geometry.

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1 Introduction

In this paper we study the simplest unitary representations of the braid group associated to a general Riemann surface from a geometrical standpoint. Besides being interesting in itself, such an investigation could prove useful in topological quantum computing ([49, 67]), where unitary braid group representations are employed for constructing quantum gates (topology would then automatically enforce robustness and fault tolerance), with the Fractional Quantum Hall Effect (FQHE) possibly yielding the physical clue to its practical implementation ([49]).

Recall that the FQHE arises for a (Coulomb) interacting spin-polarised 2d-electron gas, at low temperature and in the presence of a strong magnetic field.

It is usually observed in semiconductor structures, such as electrons trapped in a thin layer of GaAs surrounded by AlGaAs, Si-MOSFETs (see e.g. [17]) and it has been recently detected in graphene ([16]) as well. The ground state of such a system can be approximately (but most effectively) described by a *Laughlin wave function* of the form (in a plane geometry, [41, 17]):

$$\prod_{i<j}(z_i - z_j)^m e^{-\sum_{i=1}^{N} |z_i|^2} \qquad (L)$$

Here N is the number of electrons in the sample, m is an odd integer (this ensuring Fermi statistics). One notes the appearance of the ground state of a quantum harmonic oscillator. The quantity $\nu = \frac{1}{m}$ is the *filling factor* intervening in the fractional quantization of the *Hall conductance*:

$$\sigma_H = \nu \frac{e^2}{h}$$

and, in the limit $N \to \infty$, equals the electron density per state: $\nu = \frac{N}{N_S}$ with N_S the number of magnetic flux quanta: $N_S = B \cdot \mathcal{A}/\Phi_0$ (B is the modulus of the constant magnetic field (acting perpendicularly to the layer), $\mathcal{A}$ is the area of the given sample, whereas $\Phi_0 := hc/e$ is the flux quantum). The number N_S also gives the degeneracy of the lowest Landau level (for the free system), which appears as a degenerate ground state of a quantum harmonic oscillator.

On the mathematical side, Landau levels admit elegant algebro-geometric descriptions along the lines of geometric quantization (see e.g. [68, 63, 39]): for instance, if the layer is a (closed) Riemann surface of genus g, the lowest Landau level is the space of holomorphic sections of a suitable holomorphic line bundle ([35, 61]); on a torus ($g = 1$) it can be realised as a space of theta functions, see e.g. [26, 50], and also [57] and below).

Now, *on the one hand*, it turns out that the elementary excitations around the Laughlin ground state are *quasiparticles/holes* having *fractional charge* $\pm\nu e$ ([41, 17]) and *anyon statistics* $(-1)^\nu$ ([30, 17]), and this leads to considering the *braid group* associated to the N-point configuration space of the given layer (N now being the number of quasiparticles/holes). Wave functions for quasiparticles/holes can be cast in the form (L), with ν replacing m (see [30, 17]).

On the other hand, the filling factor $\nu = 1/m$ (together with others) for a *torus* sample has been interpreted as the *slope* (that is, degree over rank) of a *stable* holomorphic vector bundle over the corresponding "spectral", or "Brillouin manifold" (which is again a torus, parametrising all admissible boundary conditions, see [29, 65, 26] and below); therefore, the filling factor has a *topological* meaning. (For ν integral one recovers the interpretation of the integral Quantum Hall Effect via the first Chern class of a line bundle over the Brillouin manifold, see e.g. [62, 45, 17].)

One of the aims of the present note – which is a greatly expanded and substantially improved version of [60] – is to show that the above coincidence has an abstract braid group theoretical origin: we consider a general closed Riemann surface – so that the role of the Brillouin manifold is played by the *Jacobian* of the surface (cf. [61]) – and its associated braid group, with the Bellingeri presentation ([7]); then the equalities, in the genus one case,

$$\nu := \text{filling factor} = \text{statistical parameter} = \text{slope of a stable vector bundle}$$

can be derived from a group theoretical perspective, and can be suitably generalised.(See e.g. [37] for a recent comprehensive coverage of braid groups.)

Our first observation is that, in our context, braiding can approached via representations of the Weyl-Heisenberg group corresponding to the (rational) statistical parameter ν, both infinite dimensional and finite dimensional. Then, generalising [57], we observe that the infinite dimensional representations can be constructed geometrically on L^2-sections of holomorphic Hermitian stable bundles over the Jacobian of the Riemann surface under consideration. Stable bundles are irreducible holomorphic vector bundles over Kähler manifolds admitting a Hermitian-Einstein structure (HE) – namely a (unique) Hermitian connection with central constant curvature – in view of the Donaldson-Uhlenbeck-Yau theorem ([47, 22, 64, 40]). Specifically, the representation of the Weyl-Heisenberg group we look for stems from suitable parallel transport operators associated to the HE-connection (which will have constant curvature, essentially given by the statistical parameter ν). The solution is actually reduced to finding suitable *projectively flat HE-bundles over Jacobians*, which can be obtained via the classical Matsushima construction ([44, 31, 40]). In particular, we get a "slope-statistics" formula $\mu = \nu g!$ (with μ denoting the slope of a holomorphic vector bundle). We also show that, at least for genera g = 2,3, which allow for totally split Jacobians (cf. [23]), one can take box products of bundles on elliptic curves.

The other important geometrical ingredient needed to describe the statistical behaviour governing "particle" exchange is the Klein prime form on a Riemann surface, manufactured via theta function theoretic tools. The problem of extracting general roots of a line bundle then arises and it is circumvented by exploiting a universal property of the prime form. Then we define, following Halperin [41, 30, 17], (Laughlin type) vector valued wave functions obeying, in general, *fractional* statistics and having their "centre of mass" part represented by holomorphic sections of the above bundles (see also [29, 35, 65, 11]). Theorem 3.2 summarizes the whole construction of the RS-braid group representations we aimed at.

The successive developments portray an interesting "braid duality", and run as follows. Focussing in particular on the $g = 1$ case, we show that everything can be made even more explicit by resorting to A. Connes' noncommutative geometric setting ([20, 21]) for noncommutative tori and to the notion of noncommutative theta vector introduced by A. Schwarz ([55]), encompassing the classical notions. The upshot is that the "centre of mass" parts of Laughlin wave functions are precisely the Schwarz theta vectors. A notable feature is now the following: the space of theta vectors naturally determines a finite dimensional braid group representation corresponding to the reciprocal parameter $\nu' = 1/\nu$, which, via Matsushima, gives rise to a projectively flat HE-bundle with the corresponding slope. Therefore a (Matsushima-Connes (MC)) "duality" emerges. We shall then prove that this duality is essentially the one given by the Fourier-Mukai-Nahm (FMN) transform. In particular, the noncommutative theta vector approach will be used to calculate the Nahm-transformed connection explicitly. For fully NCG treatments to both integral and fractional QHE see e.g. [8, 42, 43].

The paper is organised as follows: in Section 2 we discuss Bellingeri's presentation of the Riemann surface braid group ([7]) and show that its simplest unitary representations can be constructed via unitary representations of the Weyl-Heisenberg group, both infinite dimensional and finite dimensional. Then we set for a geometric construction of these representations, discussing (Section 3) stable bundles on Jacobians and Klein's prime form on a Riemann surface. This makes a definition of generalised Laughlin wave functions possible. In Section 4 we discuss specific constructions of stable bundles via noncommutative geometry, used in an ancillary way, and realise the centre of mass part of Laughlin wave functions via Schwarz noncommutative theta vectors. We discuss the previously mentioned MC-duality and elucidate its relationship with FMN. Also, a comment on the noncommutative version of the FM-transform due to Polishchuk and Schwarz ([52]) is included. In Section 5 we revert to physics, instantiating the previous developments; in particular, we propose a physical interpretation of the MC-FMN duality generalising the quasihole/electron one $((-1)^{\frac{1}{m}} \leftrightarrow (-1)^m)$ one. The final section recapitulates our conclusions. We inserted short technical digressions throughout the paper in an effort of enhancing its readability.

2 The Riemann surface braid group and its simplest unitary representations

The braid group on n strands $B(X, n)$ pertaining to a topological space X is by definition the fundamental group of the associated configuration space $C_n(X)$ consisting of all n-ples of distinct points, up to order or, equivalently, of all n-point subsets of X. In our case $X = \Sigma_g$, a closed orientable surface (actually, a Riemann surface) of genus $g \geq 1$. Its associated braid group $B(\Sigma_g, n)$ admits, among others, the following presentation due to P. Bellingeri ([7]). The generators are $\sigma_1, \ldots, \sigma_{n-1}; a_1, \ldots, a_g, b_1, \ldots, b_g$ (the former are the standard braid group generators, the latter have a simple geometric interpretation, in terms of the natural dissection of the surface by means of a $4g$-gon, see [7]). The presence of the extra generators is natural: if, say, two points loop around each other, their trajectories can at the same time wind around the handles of the surface.

In the presentation, in addition to the ordinary braid relations

$$(B1) : \sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1} \qquad j = 1, 2, \dots n-2$$

$$(B2) : \sigma_i \sigma_j = \sigma_j \sigma_i, \qquad |i-j| > 1$$

one has "mixed" relations

$$\begin{array}{ll}
(R1): & a_r \sigma_i = \sigma_i a_r, \qquad 1 \le r \le g, \quad i \ne 1 \\
& b_r \sigma_i = \sigma_i b_r, \qquad 1 \le r \le g, \quad i \ne 1 \\
(R2): & \sigma_1^{-1} a_r \sigma_1^{-1} a_r = a_r \sigma_1^{-1} a_r \sigma_1^{-1}, \qquad 1 \le r \le g, \\
& \sigma_1^{-1} b_r \sigma_1^{-1} b_r = b_r \sigma_1^{-1} b_r \sigma_1^{-1}, \qquad 1 \le r \le g, \\
(R3): & \sigma_1^{-1} a_s \sigma_1 a_r = a_r \sigma_1^{-1} a_s \sigma_1, \qquad s < r \\
& \sigma_1^{-1} b_s \sigma_1 b_r = b_r \sigma_1^{-1} b_s \sigma_1, \qquad s < r \\
& \sigma_1^{-1} a_s \sigma_1 b_r = b_r \sigma_1^{-1} a_s \sigma_1, \qquad s < r \\
& \sigma_1^{-1} b_s \sigma_1 a_r = a_r \sigma_1^{-1} b_s \sigma_1, \qquad s < r \\
(R4): & \sigma_1^{-1} a_r \sigma_1^{-1} b_r = b_r \sigma_1^{-1} a_r \sigma_1, \qquad 1 \le r \le g \\
(TR): & [a_1, b_1^{-1}] \cdots [a_g, b_g^{-1}] = \sigma_1 \sigma_2 \cdots \sigma_{n-1}^2 \cdots \sigma_2 \sigma_1
\end{array}$$

with the usual group theoretical convention $[a,b] = aba^{-1}b^{-1}$, used in this Subsection only. We shall restrict ourselves to *unitary* representations

$$\rho : B(\Sigma_g, n) \to U(\mathcal{H})$$

($U(\mathcal{H})$ being the unitary group on a complex separable Hilbert space $\mathcal{H}$) with $\rho(\sigma_j) = \sigma\, I$, $j = 1, \dots, n-1$ (I being the identity operator on $\mathcal{H}$). One writes

$$\sigma = e^{2\pi\sqrt{-1}\theta} \equiv e^{\pi\sqrt{-1}\nu} = (-1)^\nu$$

and calls θ (a priori defined up to integers) the *statistics parameter* (same for $\nu = 2\theta$, with abuse of language). The relations B1, B2, R1 and R2 are automatically fulfilled, the relations R3 become:

$$[\rho(a_s), \rho(a_r)] = [\rho(b_s), \rho(b_r)] = I, \qquad r, s = 1, \dots, g \qquad (\diamond)$$

whereas R4 yields:

$$[\rho(a_r), \rho(b_r)] = \sigma^2 I, \qquad r = 1, \dots, g$$

Condition TR gives, in turn, after checking that

$$[\rho(a_r), \rho(b_r^{-1})] = \sigma^{-2} I \qquad (\diamond\diamond)$$

the constraint

$$\sigma^{2(n-1+g)} = 1$$

furnishing (for $n-1+g \neq 0$)

$$\theta = \frac{q}{2(n-1+g)}, \quad q \in \mathbf{Z} \quad \text{or, equivalently} \quad \nu = \frac{q}{n-1+g},$$

that is, *fractional statistics*, in general. Notice that if $\sigma^2 = 1$, that is $\theta \in 1/2 \cdot \mathbf{Z}$ (slight abuse of notation) we recover ordinary Fermi-Bose statistics (see also below).

Next we introduce the following tensor product Hilbert space:

$$\mathcal{H} := H_1 \otimes H_2$$

with H_1 carrying a representation of the *Weyl-Heisenberg Canonical Commutation Relations (CCR)* ([66], see also e.g. [51]) up to obvious inessential notational changes:

$$V(\vec{\beta})\, U(\vec{\alpha}) = e^{2\pi\sqrt{-1}\cdot\nu\,\vec{\alpha}\cdot\vec{\beta}} U(\vec{\alpha})\, V(\vec{\beta})$$

with $\vec{\alpha}$, $\vec{\beta} \in \mathbf{R}^g$, and where H_2 is one-dimensional. Clearly $H_1 \otimes H_2 \cong H_1$, but we keep the distinction in view of our subsequent physical applications. Now take, after denoting by $(e_1, e_2, \ldots e_g)$ the canonical basis of $\mathbf{R}^g$:

$$\rho_1(a_r) = U(e_r), \qquad \rho_1(b_r^{-1}) = V(e_r), \qquad r = 1, 2, \ldots g$$

Upon setting

$$\begin{array}{ll} \rho(a_r) = \rho_1(a_r) \otimes I_{H_2}, \quad \rho(b_r^{-1}) = \rho_1(b_r^{-1}) \otimes I_{H_2}, & r = 1, \ldots, g \\ \rho(\sigma_j) = I_{H_1} \otimes \sigma\, I_{H_2} \equiv I_{H_2} \otimes \rho_2(\sigma_j), & j = 1, \ldots, n-1 \end{array}$$

and in view of ($\diamond$) and ($\diamond\diamond$), we immediately get our first result:

Theorem 2.1 *(i) Any representation of the Weyl-Heisenberg Commutation relations yields, via the map*

$$\rho : B(\Sigma_g, n) \to U(\mathcal{H})$$

defined above, an infinite dimensional unitary representation of the Riemann surface braid group $B(\Sigma_g, n)$ on the Hilbert space $\mathcal{H}$.

(ii) Irreducible finite dimensional unitary RS-braid group representations $\hat{\rho}$ also exist, stemming from the finite version of Weyl-Heisenberg commutation relations.

The representations in (ii) correspond in fact to particular rational *noncommutative tori* (see also Section 4).

Proof. It is enough to prove (ii) and this is quite standard: set $\nu := q/r$, with q and r relatively prime positive integers, and consider, for $g = 1$, unitaries U_i, $i = 1, 2$ on $\mathbf{C}^r$ satisfying

$$U_1 U_2 = e^{-2\pi\sqrt{-1}\nu} U_2 U_1,$$

and set

$$\hat{\rho}(a_1) = U_1, \qquad \hat{\rho}(b_1^{-1}) = U_2$$

Irreducibility easily entails $U_j^r = 1$, $j = 1, 2$ (so the cyclic group $\mathbf{Z}/r\mathbf{Z}$ is involved). Concretely, one may take $U_1 = \mathrm{diag}(1, e(\nu), e(2 \cdot \nu), \ldots e((r-1) \cdot \nu))$ and U_2 = matrix of the shift map $e_i \to e_{i-1}$, $i = 1, 2, \ldots r$, $e_0 = e_r$, with $(e_1, \ldots, e_r)$ being the canonical basis of $\mathbf{C}^r$ and where we defined, for real x, $e(x) := e^{2\pi\sqrt{-1}x}$. If $g > 1$ one can take a tensor product of g copies of the latter representation, which has then rank r^g. The number of strands is then given by

$$n = r + 1 - g$$

and one completes the definition of $\hat{\rho}$ accordingly. If, conversely, n is given, then one takes $r = n + g - 1 \geq 1$ and takes an integer q relatively prime with r. In the $g = 1$ case one has of course $n = r$. □

3 Geometric realisation of ρ

We shall now outline a geometric construction of the Hilbert spaces H_j, $j = 1, 2$ and of the representation ρ.

3.1 Jacobians, theta functions, and the CCR

Let us first briefly recall some basic concepts in Riemann surface theory needed in the sequel. For a thorough treatment we refer e.g. to [25, 38, 46, 48]. Consider a closed Riemann surface Σ_g – a complex structure and a (Kähler) metric thereon are thus understood – and a canonical dissection thereof in terms of a $4g$-gon, leading to a basis of 1-cycles α_j, β_j, $j = 1, 2, \ldots g$. One then finds a basis of Abelian differentials (holomorphic 1-forms) ω_j, $j = 1, 2, \ldots g$ such that

$$\int_{\alpha_i} \omega_j = \delta_{ij}, \qquad \int_{\beta_i} \omega_j = Z_{ij}$$

with the $g \times g$ *symmetric matrix* $Z = (Z_{ij})$ having *positive-definite* imaginary part $\Im Z$.

Let Λ be the lattice generated by the columns of the *period matrix*

$$\Pi = (I, Z)$$

(I being the $g \times g$ identity matrix). The Jacobian $J(\Sigma_g)$ attached to the Riemann surface Σ_g is the g-dimensional complex torus

$$J(\Sigma_g) := H^1(\Sigma_g, \mathbf{R})/H^1(\Sigma_g, \mathbf{Z}) \cong \mathbf{C}^g/\Lambda$$

Actually, $J(\Sigma_g)$ is a Kähler manifold, and its Kähler form ω can be cast in the form

$$\omega = \sum_j dq_j \wedge dp_j = \frac{\sqrt{-1}}{2} \sum_{\alpha,\beta} W_{\alpha\beta}\, dz_\alpha \wedge d\bar{z}_\beta$$

with respect to suitable (Darboux) symplectic coordinates $(q_1, p_1, q_2, p_2, \dots q_g, p_g)$ (and their complex counterparts) of $J(\Sigma_g)$, and where $W := (\Im Z)^{-1}$.

The above conditions for a complex torus (Riemann conditions), without any reference to a Riemann surface, define a general *principally polarized Abelian variety.* It is well known that Abelian varieties, that is, complex tori admitting a polarization, namely, a positive Hermitian form on V such that its imaginary part, restricted to the lattice Λ is *integral*, are exactly the complex tori embeddable in projective space. See forthcoming Subsections for further discussion.

Subsequently, let us briefly recall the *Abel map* $\mathcal{A}$ ([25, 46, 48]):

$$\mathcal{A} : \Sigma_g \longrightarrow J(\Sigma_g)$$

$$x \mapsto \mathcal{A}(x) := (\textstyle\int_{x_0}^{x} \omega_i)_{i=1,2\dots,g} \text{ mod periods}$$

(with a choice of a base point x_0). More generally, one has a map (denoted by the same symbol)

$$\mathcal{A} : C_n(\Sigma_g) \longrightarrow J(\Sigma_g)$$

$$(x_1, x_2, \dots, x_n) \mapsto \mathcal{A}(\textstyle\sum_{j=1}^{n} x_j) := (\sum_{j=1}^{n} \int_{x_0}^{x_j} \omega_i)_{i=1,2\dots,g} \text{ mod periods}$$

The notation $\sum_{j=1}^{n} x_j$ stands for the *(positive) divisor* on Σ_g consisting of the points x_j, $j = 1, 2, \dots n$. The configuration space $C_n(\Sigma_g)$ is a complex n-dimensional manifold and $\mathcal{A}$ is a holomorphic map (cf. [48]).

We also record the *theta functions* (with rational characteristics, for $\vec{a}$, $\vec{b} \in \mathbf{Q}^g$):

$$\vartheta(\begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix})(\vec{z}, Z) = \sum_{\vec{n} \in \mathbf{Z}^g} \exp\left[\pi\sqrt{-1}(\vec{n}+\vec{a})^T Z(\vec{n}+\vec{a}) + 2\pi\sqrt{-1}(\vec{n}+\vec{a})^T(\vec{z}+\vec{b})\right]$$

For $\vec{a} = \vec{b} = \vec{0}$ one gets the standard theta function (which is indeed defined, together with the other ones, for a general Abelian variety, for Z fulfilling the same conditions as above).

A theta function corresponds to the unique holomorphic section (up to a scalar) of a holomorphic line bundle $\Theta \to J(\Sigma_g)$ actually coming from (holomorphic) geometric quantization of $(J(\Sigma_g), \omega)$, as a consequence of the Riemann-Roch theorem (see e.g. [25, 46, 48]). The first Chern class $c_1(\Theta)$ – see next Section as well – equals the de Rham class of the Kähler form ω. (For geometric quantization see e.g. [68, 63, 27, 34].)

We recall at this point, for further repeated use, that, given a smooth Hermitian vector bundle (E, h) over a complex manifold M, equipped with a connection $\nabla : \Lambda^0 \to \Lambda^1$ whose curvature Ω_∇ is of type $(1,1)$, then E possesses a natural holomorphic structure – i.e. the $(0,1)$-component of the connection (it fulfils the integrability condition $\overline{\nabla}^2 = 0$) – such that ∇ coincides with the canonical (Chern-Bott) connection, the latter being characterized by compatibility with both the metric h and the holomorphic structure (i.e. in a local holomorphic frame one has $\overline{\nabla} = \bar{\partial}$), see e.g. [4], Theorem 5.1. We record for clarity and future use the explicit local formula

$$\nabla = d + h^{-1}\partial h, \qquad \Omega_\nabla = h^{-1}\bar{\partial}\partial h - (h^{-1}\bar{\partial}h) \wedge (h^{-1}\partial h)$$

(stemming from the general local formula for a connection $\nabla = d + \omega$, $\Omega_\nabla = d\omega + \omega \wedge \omega$).

Now, the important point for us is that the line bundle Θ, equipped with its canonical connection carries an irreducible representation of the CCR, by Riemann-Roch combined with the von Neumann Uniqueness Theorem ([66]), since theta appears essentially as the ground state of a quantum harmonic oscillator: the annihilation operators read, in fact

$$A_j \sim \nabla_{\bar{z}_j} = \frac{\partial}{\partial \bar{z}_j}$$

(see [57] for details and see also below, Sections 4 and 5).

Actually one has a family of such representations, labelled by the Picard group $Pic^0(J(\Sigma_g)) \cong J(\Sigma_g)$, namely, the receptacle of the isomorphism classes of holomorphic line bundles having vanishing first Chern class. The same observation applies to the tensor powers Θ^k of the theta line bundle, whose holomorphic sections are the k-level theta functions, and yield a k^g-dimensional space. Variation on $J(\Sigma_g)$ yields a rank k-holomorphic vector bundle (the *spectral bundle*) having a crucial physical role (see below, Sections 4 and 5 for further discussion).

We shall extend the above picture to holomorphic Hermitian stable vector bundles in due course.

3.2 Stable bundles

For further background material related to this Subsection we mostly refer to [40]. Let M be a n-dimensional compact Kähler manifold, with Kähler metric $\tilde{g}$ and associated Kähler form ω, $\mathcal{E} \to M$ a vector bundle over it, with rank $\mathrm{rk}(\mathcal{E})$. Recall that the *degree* $\deg(\mathcal{E})$ of $\mathcal{E}$ is given by

$$\deg(\mathcal{E}) = \int_M c_1(\mathcal{E}) \wedge \omega^{n-1}$$

where the first Chern class $c_1(\mathcal{E}) \in H^2(M, \mathbf{Z})$ can be computed via the curvature form Ω of any connection ∇ on $\mathcal{E}$; with a slight notational abuse:

$$c_1(\mathcal{E}) = \mathrm{Tr}_{\mathrm{End}(\mathcal{E})}(-\frac{\Omega}{2\pi\sqrt{-1}})$$

where $\mathrm{End}(\mathcal{E})$ is the endomorphism bundle associated with $\mathcal{E}$, and Tr denotes trace.

The *Chern Character* (with values in $H^{\mathrm{even}}(M, \mathbf{Q})$) reads

$$\mathrm{Ch}(\mathcal{E}) = \mathrm{Tr}_{\mathrm{End}(\mathcal{E})}(\exp[-\frac{\Omega}{2\pi\sqrt{-1}}])$$

and can be organised through the *Chern character vector*

$$(\mathrm{Ch}_0(\mathcal{E}) = \mathrm{rk}(\mathcal{E}), \mathrm{Ch}_1(\mathcal{E}) = c_1(\mathcal{E}), \ldots, \mathrm{Ch}_n\mathcal{E})$$

The Chern character satisfies the ring homomorphism properties

$$\mathrm{Ch}(\mathcal{E}_1 \oplus \mathcal{E}_2) = \mathrm{Ch}(\mathcal{E}_1) + \mathrm{Ch}(\mathcal{E}_2), \qquad \mathrm{Ch}(\mathcal{E}_1 \otimes \mathcal{E}_2) = \mathrm{Ch}(\mathcal{E}_1) \cdot \mathrm{Ch}(\mathcal{E}_2)$$

The *slope* $\mu(\mathcal{E})$ of $\mathcal{E}$ is by definition

$$\mu(\mathcal{E}) := \frac{\deg(\mathcal{E})}{\mathrm{rk}(\mathcal{E})}$$

A holomorphic vector bundle $\mathcal{E} \to M$ is said to be *semistable* if for every holomorphic subbundle $\mathcal{F}$ one has $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$, *stable* if strict equality holds (for a proper subbundle) ([47, 40]). The above condition can be phrased in differential geometric terms in view of the Donaldson-Uhlenbeck-Yau theorem (see [47, 22, 64, 40]): briefly, this goes as follows. Recall that a Hermitian metric h on $\mathcal{E}$ is called a *Hermitian-Einstein* (HE) metric if, given its Chern-Bott connection ∇, one has

$$\sqrt{-1}\Lambda\Omega = \lambda I_{\mathrm{End}(\mathcal{E})}$$

with Λ denoting here contraction with the Kähler form ω and λ a real constant.

Then the *Kobayashi-Hitchin correspondence* states that an irreducible holomorphic vector bundle (i.e. without proper holomorphic direct summands) admits a HE-metric if and only if it is stable; it has been eventually proved for any compact Kähler manifold in [64]. In particular, a projectively flat holomorphic vector bundle admitting a projectively flat Hermitian structure, i.e. equipped with a Hermitian metric whose corresponding canonical connection has constant curvature, is stable.

Concerning the space H_1, and taking into account the discussion of the preceding Subsection, we are thus naturally led to look for a Hermitian holomorphic vector bundle $\mathcal{E} \to J(\Sigma_g)$ over the Jacobian $J(\Sigma_g)$ of the Riemann surface in question, equipped with a HE-connection ∇ having constant curvature equal

(up to a $2\pi\sqrt{-1}$ factor) to $2\theta = \nu$, and this will give rise to a holomorphic stable bundle with slope $\mu(\mathcal{E}) \propto \nu$ (we shall soon verify that the precise factor will be $g!$).
The construction of such bundles over a generic Abelian variety is classical (cf. in particular [44, 31, 40]) and we briefly review it below, tailoring the exposition to our specific needs.

3.3 The Matsushima construction

In this subsection we briefly describe the Matsushima construction ([44] see also [31, 40]), albeit not in full generality.

Take an n-dimensional complex vector space V and a (maximal) lattice $L \subset V$. The quotient group V/L is an n-dimensional complex torus. A holomorphic vector bundle $F \to V/L$ with rank m is determined by what is called a $GL(m, \mathbf{C})$-valued *theta (or automorphy) factor* J, that is, a holomorphic map

$$J : L \times V \to GL(m, \mathbf{C})$$

satisfying

$$J(\alpha + \beta, u) = J(\alpha, \beta + u) J(\beta, u) \qquad \forall\, \alpha, \beta \in L,\, u \in V$$

Call F_J the holomorphic bundle obtained from J. Now, given a Hermitian form H on V whose imaginary part A takes *rational* values on L, one considers the nilpotent group $V \times \mathbf{C}^*$ (Heisenberg group) with multiplication

$$(u, a) \cdot (v, b) = (u + v, \exp[\pi H(u, v) ab]), \qquad \forall\ (u, a), (v, b) \in V \times \mathbf{C}^*$$

The subgroup $G_H(L) = L \times \mathbf{C}^*$ acts holomorphically on the right on $V \times \mathbf{C}^*$, giving rise to a principal bundle over V/L. Given a holomorphic representation $\rho : G_H(L) \to GL(m, \mathbf{C})$, one gets a rank m holomorphic vector bundle $E \to V/L$ as an associated bundle, with theta factor J_ρ given by

$$J_\rho(\alpha, u) = \rho\big(-\alpha, \exp\{H(u, \alpha) + \frac{1}{2} H(\alpha, \alpha)\}\big) \qquad \forall\, \alpha \in L,\, u \in V$$

The vector bundle is then

$$E = (V/L \times \mathbf{C}^m)/L$$

with the action of L specified via

$$\gamma \cdot (z, \zeta) := (\gamma + z, J_\rho(\gamma, z)\zeta) \qquad \gamma \in L,\ (z, \zeta) \in V/L \times \mathbf{C}^m$$

Now the crucial result of the Matsushima-Hano theory ([44, 31]) is that a holomorphic bundle $F \to V/L$ is projectively flat (i.e. its associated projective bundle $P(F) \to V/L$ admits a flat connection or, equivalently, its transition functions can be chosen to be constant) if and only if there exists a Hermitian form H together with a representation ρ as above such that $F \cong F_{J_\rho}$. Let us spell out further details.

Also introduce the Heisenberg group $G_A(L) = L \times \mathbf{C}^*$, with operation

$$(\alpha, a) \cdot (\beta, b) = (\alpha + \beta, \exp[\sqrt{-1}\pi A(\alpha, \beta) ab])$$

which turns out to be isomorphic to $G_H(L)$. Given then a representation ρ_A of $G_A(L)$, its associated *theta factor* J_H reads

$$J_H(\alpha, u) = \exp[\pi H(u, \alpha) + \frac{1}{2} H(\alpha, \alpha)] \rho_A(-\alpha, 1)$$

Now, specifically, take $n = g$ and a real basis $(e_1, \dots, e_g, e'_1, \dots, e'_g)$ for V generating the lattice L, and let A be defined by

$$A(e_i, e_j) = A(e'_i, e'_j) = 0, \quad A(e_i, e'_j) = \nu \, \delta_{ij}$$

with $\nu = q/r$, $r > 0$, $q > 0$ and $g.c.d(r, q) = 1$. Then r is the smallest integer such that rA is integral valued on L.

Then let us set (obvious notation)

$$L = < e_1, \dots, e_g, e'_1, \dots, e'_g > \cong < e_1, \dots, e_g, > \oplus < e'_1, \dots, e'_g > \equiv L_1 \oplus L'_1$$

$$N = < re_1, \dots, re_g, re'_1, \dots, re'_g > \equiv N_1 \oplus N'_1$$

and

$$K := L/N, \qquad K_1 = L_1/N_1 \cong (\mathbf{Z}/r\mathbf{Z})^g$$

The group K_1 has order r^g. Consider then its *group algebra* $C(K_1)$, that is, the r^g-dimensional vector space of all complex valued functions on K_1. A basis f_j, $j = 1, \dots r^g$ is provided by the characteristic functions $\chi_{\{\kappa\}}$, $\kappa \in K_1$.

The Matsushima *Schrödinger representation* D_A of $G_A(L)$ is defined on the above basis via the position:

$$D_A(\alpha, \alpha', a) f_\kappa = a \cdot \exp\{2\pi\sqrt{-1}[-\frac{1}{2} A(\alpha, \alpha') + A(\beta, \alpha')]\} f_{\kappa - \alpha}$$

with the class of β in K_1 equalling κ. The corresponding theta factor then reads:

$$J_H(\alpha + \alpha', u) = \exp\{\pi H(u, \alpha + \alpha') + \frac{1}{2} H(\alpha + \alpha', \alpha + \alpha')\} D_A(-\alpha, -\alpha', 1)$$

A *(vector) theta function* (generalising indeed the ordinary theta function) is an entire function θ on V defined by the requirement, for $u \in V$ and α, $\alpha' \in L$,

$$\theta(\alpha + \alpha' + u) = J_H(\alpha + \alpha', u) \, \theta(u)$$

and can be written as a vector in $\mathbf{C}^{r^g}$ made up of its components with respect to the basis $\{f_\kappa\}$:

$$\theta(u) = \sum \theta_\kappa(u) f_\kappa$$

Moreover, one has, for $\nu \in N_1$,

$$\theta_\kappa(\nu + \alpha' + u) = j_\kappa(\nu + \alpha', u)\, \theta_\kappa(u)$$

and

$$\theta_\kappa(u) = \exp\{H(u,\beta) - \frac{1}{2}H(\beta,\beta)\}\theta_0(u)$$

(0 is here the unit element in K_1), showing that everything depends on θ_0 and ultimately on the automorphic factor j_0 associated to a holomorphic line bundle $L \to V/M$, with $M = N_1 \oplus L_1'$ (note that the natural map $V/M \to V/L$ is a covering map of tori with kernel $L/M \cong K_1$). The upshot is that the space $H^0(\mathcal{E}_{D_A})$ of the holomorphic sections of the bundle $\mathcal{E}_{D_A} \to V/L$ attached to the Schrödinger representation - naturally identified with the space of vector theta functions - is isomorphic to the space $H^0(\mathcal{L})$ of holomorphic sections of the bundle $\mathcal{L} \to V/M$, which are essentially given by the q-level theta functions, so, by Appel-Humbert theory (see e.g. [38]) or equivalently by Riemann-Roch, has complex dimension q^g.

Let us briefly recall, for completeness, the explicit construction by Matsushima of a connection on $F \to V/L$, in order to compute its Chern classes. Actually, *a posteriori*, it will be the HE-connection we are interested in.

Let $\pi : V \to V/L$ the covering map; choose an open covering $\{U_i\}_{i\in\mathcal{I}}$ of V/L – with $\mathcal{I}$ an index set – made of connected sets, together with a connected component $\widetilde{U}_i \subset \pi^{-1}(U_i)$. Let $\rho_i = \pi^{-1} : U_i \to \widetilde{U}_i$. If $U_i \cap U_j \neq \emptyset$, for all $x \in U_i \cap U_j$, there exists a unique $\sigma_{ji} \in L$ such that

$$\rho_i(x) = \rho_j(x) + \sigma_{ji}$$

let J be a $GL(m, \mathbf{C})$-valued theta factor and set

$$g_{ij}(x) = J(\sigma_{ji}, \rho_j(x))$$

The $\{g_{ij}\}$ yield holomorphic transition functions for F. Let us define a connection on F via local connection forms $\{\omega_i\}$ on U_i, $i \in \mathcal{I}$, fulfilling, for $x \in U_i \cap U_j$

$$\omega_j = g_{ij}^{-1} dg_{ij} + g_{ij}^{-1}\omega_i g_{ij}$$

Take J of the form

$$J(\alpha, u) = \exp\{\pi H(u,\alpha)\}\, C(\alpha)$$

(the theta factor J_H associated with the Schrödinger representation D_A is indeed of this form). Working in coordinates for V, one has $H(u,\alpha) = \sum h_{ab} u_a \bar{\alpha}_b$, and consequently

$$g_{ij}(x) = \exp\{\pi \sum_{a,b} h_{ab} u_a(\rho_i(x))(\overline{\sigma_{ji}})_b\}\, C(\sigma_{ji})$$

Introduce local complex coordinates $z_a^{(i)} := u_a \circ \rho_i$. On $U_i \cap U_j$ one has $dz_a^{(i)} = dz_a^{(j)}$. If ζ_a is the holomorphic 1-form on V/L such that $\pi^* \zeta_a = du_a$, one has, on each U_i,

$$\zeta_a = dz_a^{(i)}$$

Now let

$$\omega_i = -\big(\pi \sum_{a,b} h_{ab} \bar{z}_b^{(i)} \zeta_a\big) \cdot I_m$$

(I_m being the $m \times m$-identity matrix). It is easily verified that the collection $\omega = \{\omega_i\}$ gives rise to a connection form. The curvature form $\Omega = \{\Omega_i\}$ reads then

$$\Omega_i = d\omega_i + \omega_i \wedge \omega_i = d\omega_i = \big(\pi \sum_{a,b} h_{ab} \zeta_a \wedge \bar{\zeta}_b\big) \cdot I_m$$

and this expression is global.

Let us specialise the above construction to the case $g = 1$, with slight notational changes, for future purposes and in order to make contact with [40] as well.

Start from a r-dimensional representation of the finite Weyl-Heisenberg group corresponding to a 2-lattice $\Gamma \subset \mathbf{C}$, giving rise to the complex -torus $\mathbf{C}/\Gamma$:

$$U(\gamma + \gamma') = U(\gamma) \cdot U(\gamma')\, e^{\frac{\sqrt{-1}}{2\,r} A(\gamma', \gamma)}$$

with A being the imaginary part of a Hermitian form R on $\mathbf{C}$ such that

$$\frac{1}{2\pi} A(\gamma, \gamma') \in \mathbf{Z}$$

In our setting we take (abuse of notation, and in standard coordinates)

$$A = -2\pi q \cdot \omega = -2\pi q \cdot dx \wedge dy = -\sqrt{-1}\pi q \cdot dz \wedge d\bar{z}$$

Equivalently

$$U_1 U_2 = U_2 U_1\, e^{-2\pi\sqrt{-1}\frac{q}{r}}$$

Therefore

$$R(z, w) = \pi q \cdot z\, \bar{w}$$

We get a factor of automorphy (theta factor)

$$j(\gamma, z) = U(\gamma) \cdot \exp\big[\frac{1}{2r} R(z, \gamma) + \frac{1}{4r} R(\gamma, \gamma)\big], \qquad (\gamma, z) \in \Gamma \times \mathbf{C}$$

yielding in turn - via the above procedure - a stable bundle $\mathcal{E}_\nu \to \mathbf{C}/\Gamma$ having rank r, equipped with the Hermitian metric

$$h(z) = \exp\big[-\frac{1}{2r} R(z, z)\big]\, I_r$$

leading to a Chern-Bott connection ∇ having the required constant curvature

$$\Omega_\nabla = \sqrt{-1}\,\frac{A}{r}\,I_r = -2\pi\sqrt{-1}\,\nu\cdot e_1\wedge e_2\,I_r$$

The space of holomorphic sections $H^0(\mathcal{E}_\nu)$ is then q-dimensional (see also next Subsection) and, in the case $r=1$, we retrieve the q-level theta functions. Of course the above treatment is consistent with the early findings of Atiyah ([2]).

Therefore, the conclusion is that *a projectively flat HE-bundle on the Jacobian* $J(\Sigma_g)$ *with HE-connection with curvature* $-2\pi\sqrt{-1}\nu\,\omega$ *can be manufactured via the Matsushima construction*, and, by a result of Hano ([31]), this is essentially the only way to achieve this. In the following Subsection we detect the admissible ranks and h^0's via an application of the Riemann-Roch-Hirzebruch (RRH) theorem and discuss specific examples of the construction, touching upon the notable algebro-geometric problem of finding the totally split Jacobians.

3.4 RRH considerations

The problem of identifying the projectively flat HE-vector bundles one can employ for representing the RS-braid group can be addressed via the RRH-theorem, together with a cohomology vanishing theorem. We refer to [9, 33] for background and full details, and also below, Subsection 4.3 for further applications.

One considers, on the Clifford module $\Lambda(T^{0,1}M)^*\otimes W$, the Dolbeault-Dirac (or spin^c) operator

$$D_W := \sqrt{2}(\overline{\partial}_W + \overline{\partial}^*_W)$$

associated to the Dolbeault complex attached to a Hermitian holomorphic vector bundle W over a Kähler manifold M. The Hodge theorem implies then that

$$\mathrm{Ker}(D_W) \cong H^\bullet(M, \mathcal{O}(W))$$

(the cohomology of W). As usual, we set $h^i(M,W)) := \dim\, H^i(M,\mathcal{O}(W))$. The index "ind" of the spin^c operator is the dimension of the above kernel, viewed as a *superspace*:

$$\mathrm{ind}(D_W) = \dim(\mathrm{Ker}\ \overline{\partial}_W) - \dim(\mathrm{Ker}\ \overline{\partial}^*_W)$$

The RRH-theorem (viewed à la Atiyah-Singer) yields then the formula

$$\mathrm{ind}(D_W) = \chi(W) = \sum_{i=0}^{n} h^i(M,W) = \int_M \mathrm{Ch}(W)\mathrm{Td}(M)$$

($\chi(W)$ is the holomorphic Euler characteristic of W, $\mathrm{Ch}(W)$ its Chern Character, and $\mathrm{Td}(M)$ the Todd class of M – we shall not need the explicit expression for the latter). We are now prepared to state and prove the following general result.

Theorem 3.1 *(i) Let $\mathcal{E}$ be a projectively flat holomorphic vector bundle over $J(\Sigma_g)$ (or, more generally, over an Abelian variety) carrying a HE-connection ∇ with constant curvature $\Omega_\nabla = -2\pi\sqrt{-1}\,\nu\cdot\omega := -2\pi\sqrt{-1}\,q/r\cdot\omega$, (with $r>0$, $q>0$ and g.c.d$(r,q)=1$). Then one has,*

$$R := \mathrm{rk}(\mathcal{E}) = k\,r^g, \qquad h^0(\mathcal{E}) = k\,q^g$$

with k a positive integer.
(ii) The following slope-statistics formula *holds:*

$$\mu(\mathcal{E}) = \nu\, g!$$

Proof. Given the above preparations, let us assume that M is a torus, and again denote the Chern-Bott connection by ∇ (one has $\overline{\nabla} = \overline{\partial}_W$ in a holomorphic frame). The Weitzenböck formula (removing suffixes), yields

$$\Delta = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial} = \overline{\nabla}^*\overline{\nabla} + \widetilde{\Omega}$$

with $\widetilde{\Omega} \in \mathrm{End}(W)$ the (Clifford-contracted) curvature endomorphism, and, in turn, leads to the Bochner-Kodaira formula, if $\psi \in \Lambda^{0,\bullet}(M,W)$:

$$\langle \psi, \Delta\psi\rangle = ||\overline{\nabla}\psi||^2 + \langle\psi, \widetilde{\Omega}\,\psi\rangle$$

(we used the fact that the canonical bundle over a torus is trivial). But in our situation, if $i>0$, the second term in the r.h.s. is *positive* for $\psi \neq 0$, and this, together with $\overline{\partial}\psi = \overline{\partial}^*\psi = 0$, implies that $h^i(M,W) = 0$.

Let us give a few extra details. The Clifford-contracted curvature endomorphism $\widetilde{\Omega}$ can be written in the form

$$\widetilde{\Omega} = \sum_{i,j} \Omega_\nabla(\partial_{z_i}, \partial_{\bar{z}_j}) c_i^* c_j$$

where we set, to fix ideas (working locally) $\psi = \tilde{\psi}\, dz_{i_1} \wedge \ldots \wedge dz_{i_\ell} \in \Lambda^{0,\ell}(W)$

$$c_i^*(\psi) = d\bar{z}_i \wedge \psi, \qquad c_i = \iota(d\bar{z}_i)(\psi)$$

The above operators (wedge and contraction) can be viewed as *fermionic creation and annihilation operators* (see also e.g. [24], Ch.15), with $\Lambda^{0,\bullet}$ the corresponding Fock space and in our case

$$\widetilde{\Omega} = \pi\nu \sum_i c_i^* c_i = \pi\nu\cdot N$$

(N is the fermionic *number operator*), eventually furnishing

$$\langle\psi, \widetilde{\Omega}\,\psi\rangle = \ell\,\pi\nu\,||\psi||^2$$

and with this our claim, for $\ell > 0$.

Moreover, since M is a torus, its Todd class $\mathrm{Td}(M)$ is trivial as well, therefore

$$h^0(\mathcal{E}) = \int_{J(\Sigma_g)} \mathrm{Ch}(\mathcal{E}) = \int_{J(\Sigma_g)} \mathrm{Tr}\exp\left[-\frac{\Omega_\nabla}{2\pi\sqrt{-1}}\right] = R\cdot\frac{q^g}{g!\cdot r^g}\int_{J(\Sigma_g)} \omega^g = R\cdot\frac{q^g}{r^g}$$

Now $h^0(\mathcal{E})$ must be a non negative integer so r^g divides R (since $g.c.d(q^g, r^g) = 1$) and this yields the conclusion. Notice that, in particular, $c_1(\mathcal{E}) = [k\, r^{g-1}\, q\cdot\omega]$, so we get (ii). □

Therefore, we see that Matsushima construction produces examples for $k = 1$. Upon taking the direct sum of $k > 1$ copies of a Matsushima bundle one produces the more general examples (as an immediate consequence of the properties of the Chern character).

We shall now show how to construct explicit examples for higher genera via products of elliptic curves $\Sigma_1 \cong J(\Sigma_1)$. We need to remember that we are working on Jacobians and not on general Abelian varieties.

Now take again $k = 1$ in the previous result and consider the holomorphic vector bundle on $J(\Sigma_1)^g := J(\Sigma_1) \times \ldots J(\Sigma_1)$ (g copies):

$$\mathcal{E} := \prod_{i=1}^{g}{}^{\otimes}\, \pi_i^*(\mathcal{E}_\nu) \to J(\Sigma_1)^g$$

(i.e. the *box product* of g copies $\mathcal{E}_{r,q}$), where π_i, $i = 1, 2, \ldots g$ denotes projection onto the ith-factor of $J(\Sigma_1) \times \ldots J(\Sigma_1)$. It is clear that that $\mathcal{E}$ has rank r^g and carries an obvious connection ∇ with constant curvature

$$\Omega_\nabla = -2\pi\sqrt{-1}\,\nu \sum_{i=1}^{g} \omega^{(i)}$$

(with $\omega^{(i)}$ the Kähler form of the i-th factor), whence it is stable.

The RRH-theorem again yields

$$h^0(\mathcal{E}) = \int_{J(\Sigma_1)\times\ldots J(\Sigma_1)} \mathrm{Ch}(\mathcal{E}) = q^g$$

Now, we have to pull-back the above bundle to the Jacobian $J(\Sigma_g)$ via an isogeny ϕ (i.e. a surjective morphism with finite kernel)

$$\phi:\ J(\Sigma_g) \to J(\Sigma_1)^g$$

with degree $\deg\phi$ and this can be done for a restricted class of curves see [23]; moreover, upon observing that

$$\mathrm{Ch}_g(\phi^*\mathcal{E}) = \deg\phi \cdot \mathrm{Ch}_g(\mathcal{E})$$

one must have $\deg \phi = 1$, that is, ϕ must be an isomorphism, and examples occur at least for $g = 2, 3$ (again cf. [23]).

3.5 Construction of ρ_1

Now consider the projectively flat HE-vector bundles $\mathcal{E}$ of the preceding Subsection and take $H_1 := L^2(\mathcal{E})$, namely the L^2- sections of $\mathcal{E}$ obtained by completing its smooth sections with respect to the inner product

$$\langle \cdot\,, \cdot \rangle := \int_{J(\Sigma_g)} h(\cdot\,, \cdot) \frac{\omega^g}{g!}$$

The braid generators a_i and b_i, $i = 1, 2, \ldots g$, can be realised as *parallel transport* operators pertaining to the Hermitian connection with constant curvature ∇, and actually will yield a representation of the Weyl-Heisenberg Commutation Relations as above. (Again there exists a family of such connections parametrized by the Jacobian itself). Specifically, again with respect to the standard (Darboux) symplectic coordinates $(q_1, p_1, q_2, p_2, \ldots q_g, p_g)$ of $J(\Sigma_g)$, we have

$$[\nabla_{\frac{\partial}{\partial q_j}}, \nabla_{\frac{\partial}{\partial q_k}}] = [\nabla_{\frac{\partial}{\partial p_j}}, \nabla_{\frac{\partial}{\partial p_k}}] = 0; \qquad [\nabla_{\frac{\partial}{\partial q_j}}, \nabla_{\frac{\partial}{\partial p_k}}] = -2\pi\sqrt{-1}\,\nu\,\delta_{jk} \cdot I$$

for $j, k = 1, 2 \ldots g$.

Notice in fact that, by periodicity and the compatibility of ∇ with h, one has

$$0 = \int_{J(\Sigma_g)} X\, h(\cdot\,, \cdot) \frac{\omega^g}{g!} = \int_{J(\Sigma_g)} \big[h(\nabla_X \cdot, \cdot) + h(\cdot, \nabla_X \cdot)\big] \frac{\omega^g}{g!}$$

with $X = \partial/\partial q_j,\ \partial/\partial p_j$, thus the operators $\nabla_{\frac{\partial}{\partial q_j}}$, $\nabla_{\frac{\partial}{\partial p_j}}$ are formally skew-hermitian. By classical functional analytic arguments they are skew-adjoint (cf. [53]).

Thus, under the above assumptions, we get an infinite dimensional representation of the Weyl-Heisenberg Commutation relations (generalising [57]) with multiplicity q^g.

The braid group generators a_i and b_i, $i = 1, \ldots, g$ are then represented as

$$\rho_1(a_r) := \exp(\nabla_{\frac{\partial}{\partial q_r}}), \quad \rho_1(b_r^{-1}) := \exp(\nabla_{\frac{\partial}{\partial p_r}}), \qquad r = 1, 2, \ldots g$$

Remark. The holomorphic hermitian stable bundle $(\mathcal{E}, h, \nabla) \to J(\Sigma_g)$ can be pulled-back (via the Abel map) to

$$(\mathcal{A}^*\mathcal{E}, \mathcal{A}^*h, \mathcal{A}^*\nabla) \to C_n(\Sigma_g)$$

equipped with the pulled-back metric $\mathcal{A}^*h$ and connection $\mathcal{A}^*\nabla$. The corresponding pulled-back representation is well defined on pulled-back sections. The Hilbert space H_1 will be the receptacle, because of the Abel map, of "centre of mass" wave functions, cf. [29, 65, 26], and below.

3.6 The prime form bundle

In this subsection recall the basic ingredient needed for a physically relevant realisation of the one-dimensional space H_2, namely the so-called *prime form* on a Riemann surface. We closely follow [46], to which we refer for full details. The prime form $E(x,y)$ (for $x,\ y\in\Sigma_g$) on $\Sigma_g\times\Sigma_g$ is the unique – up to a scalar – holomorphic section of the holomorphic line bundle $\mathcal{O}(\Delta)\to\Sigma_g\times\Sigma_g$ (Δ being the diagonal in $\Sigma_g\times\Sigma_g$) or, alternatively, a differential form of weight $(-1/2,-1/2)$ on its universal cover $\widetilde{\Sigma}_g\times\widetilde{\Sigma}_g$. Here $\mathcal{O}(D)$ denotes, as usual, the holomorphic line bundle pertaining to the divisor D (roughly speaking, a hypersurface in the base manifold). The prime form E provides a generalization to its genus zero counterpart

$$E_0(x,y)=\frac{x-y}{\sqrt{dx}\,\sqrt{dy}}$$

to which it reduces, up to a third order term, upon working in local coordinates. A bit more explicitly (in terms of theta functions with characteristics)

$$E(x,y)=\frac{\vartheta[\beta](\mathcal{A}(y-x))}{\sqrt{\zeta(x)}\,\sqrt{\zeta(y)}}=-E(y,x)$$

for a non singular, odd theta characteristic β, which, in turn, is interpreted both as a divisor and as an element

$$\beta=\begin{bmatrix}\beta'\\ \beta''\end{bmatrix}\in\frac{\frac{1}{2}\mathbf{Z}^{2g}}{\mathbf{Z}^{2g}} \tag{3.1}$$

The 1-form

$$\zeta=\sum_{i=1}^{g}\frac{\partial\,\vartheta[\beta]}{\partial z_i}(0)\omega_i$$

is the (essentially) unique 1-form vanishing on β: actually its divisor is 2β, whence its square root $\sqrt{\zeta}$ is well-defined. Notice that this arrangement ensures that $E(x,y)$, as a function of x, vanishes only at y, since the theta function $\vartheta[\beta]$ also possesses extra zeros at specific points p_j, $j=1,2,\ldots g-1$ (they are therefore absent in the genus one case).

Set $\Sigma_g^n:=\Sigma_g\times\Sigma_g\times\ldots\times\Sigma_g$ (n copies). Define the following holomorphic line bundle :

$$\mathcal{L}:=\prod_{i<j}^{\otimes}\pi_{ij}^{*}\mathcal{O}(\Delta_{ij})\to\Sigma_g^n$$

where

$$\pi_{ij}:\Sigma_g^n\to(\Sigma_g)_i\times(\Sigma_g)_j$$

(with $(\Sigma_g)_j$ denoting the j-th copy of Σ_g, and Δ_{ij} the diagonal in $(\Sigma_g)_i\times(\Sigma_g)_j$).

3.7 Construction of H_2 and generalised Laughlin wave functions

The bundle $\mathcal{L} \to \Sigma_g^n$ constructed in the previous Subsection naturally descends to a bundle (same notation) $\mathcal{L} \to C_n(\Sigma_g)$ whereupon the braid group $B(\Sigma_g, n)$ naturally acts; however, one ends up with a representation of the symmetric group S_n, yielding ordinary statistics (see Subsection 3.9). If one wishes to implement fractional statistics then one faces the problem of extracting *roots* of line bundles and this cannot be achieved in general for non trivial line bundles. Thus, in order to circumvent this difficulty, we adopt a "minimalistic" approach and resort to a local description, which however retains an intrinsic character with respect to braiding: define the Hilbert space

$$H_2 = < \psi_\nu = \prod_{i<j} (\zeta_i - \zeta_j)^\nu >$$

with ζ being a local coordinate (the behaviour of the prime form near the diagonal is however independent of the choice of the local coordinate). Branching is then produced. Actually, ψ_ν is the "topological" part of the Laughlin wave function discussed in [11], upon regarding the coordinates ζ_i as global coordinates on the configuration space $C_n(\mathbf{C})$.

A scalar product can be introduced in H_2 in an obvious manner. The function ψ_ν then manifestly enjoys the correct transformation law under the exchange of two points $x_i \leftrightarrow x_j$:

$$\psi_\nu \mapsto (-1)^\nu \, \psi_\nu = \sigma \, \psi_\nu$$

The upshot is that we may devise generalised "ground state" *Laughlin wave functions* [41, 30, 29, 35] in $\mathcal{H} = H_1 \otimes H_2$ as follows:

$$\Psi(x_1, \ldots x_n) := \psi_\nu \cdot \xi$$

where $x_1, \ldots x_n$ are distinct points in Σ_g, and ξ is a *holomorphic section* - when existing - of the stable bundle entering the construction (depending on a centre of mass coordinate). These holomorphic sections play the role of the ground states, or fundamental Landau levels, see also Section 5. Notice that they are not invariant under the action of the "full" braid group, since parallel transport does not preserve the holomorphic structure, in general.

In this way we have also generalised the geometric treatment given for the standard braid group by A. Besana and the author [11] as well.

We summarise the developments of this Section in the following theorem/definition

Theorem 3.2 *Let $\mathcal{E} \to J(\Sigma_g)$ be a Matsushima HE-holomorphic vector bundle with slope $\mu(\mathcal{E}) = \nu g! = q/r \cdot g!$. The representation ρ_1 of the CCR on the Hilbert* *space $H_1 = L^2(\mathcal{E})$ – built up as above via parallel transport operators associated with the canonical HE-connection ∇ – together with the position*

$$\rho_2(\sigma_j)\psi := (-1)^\nu \psi, \qquad \psi \in H_2$$

gives rise to a unitary representation

$$\rho : B(\Sigma_g, n) \to U(\mathcal{H})$$

where $n = r + 1 - g$, $\mathcal{H} = H_1 \otimes H_2$. The representation ρ_1 has multiplicity $h^0(\mathcal{E}) = q^g$.

The vectors $\psi = \psi_\nu \, \xi$, $\xi \in H^0(\mathcal{E})$ (ξ is then a Matsushima theta vector) are called Laughlin generalised wave functions.

3.8 Alternative construction of H_2

Another proposal for the Hilbert space H_2 is $H_2 = \langle \widetilde{\Psi} \rangle$, with

$$\widetilde{\Psi}(x_1, \ldots, x_n) = \prod_{i<j} \frac{E(x_i, x_j)^{2\theta}}{E(x_i, x_0)^\theta E(x_j, x_0)^\theta}$$

(again a local expression) which again enjoys the correct transformation law under the exchange of two points $x_i \leftrightarrow x_j$:

$$\widetilde{\Psi} \mapsto (-1)^{2\theta} \, \widetilde{\Psi} = \sigma \, \widetilde{\Psi}$$

This is motivated by the fact that, fixing say, x_j, and taking an arbitrary point $x_0 \neq x_j$, the 1-form

$$\omega_{x_j - x_0}(x_i) := d_{x_i} \log \frac{E(x_i, x_j)}{E(x_i, x_0)}$$

is a differential of the third kind with residues ± 1 at x_j and x_0, respectively. This generalises the complex plane situation, with the form $dz_i/(z_i - z_j)$, which has residues ± 1 at x_j and ∞, respectively.

In the genus one case one we can add the following observation: The tangent bundle $T\Sigma_1$ is trivial, and, since $\Delta \cong \Sigma_1$, one has that the restriction $\mathcal{O}(\Delta) \mid_\Delta \cong T\Delta$ is trivial. Thus the prime form bundle is trivial when restricted to a tubular neighbourhood of the diagonal Δ, which can be taken as an open dense set in $\Sigma_1 \times \Sigma_1$. Restriction to this open dense set renders root extraction possible.

Remarks. 1. Our geometrically constructed representations depend on the complex structure of a Riemann surface, and not just on its topology (also cf. [36]).

2. The root extraction problem deserves further scrutiny: an important step would be the determination of the second cohomology group $H^2(C_n(\Sigma_g), \mathbf{Z})$. The rational cohomology groups of configuration spaces of surfaces have been studied in [15].

3.9 Ordinary statistics recovered

The case $\nu = 2\theta = 1$ gives back Fermi-Dirac statistics, and one can safely employ the prime form bundle (actually $\mathcal{L} \to C_n(\Sigma_g)$) as it stands. Of course one may take tensor powers thereof as well. As for the centre of mass part, also in view of the above considerations, one retrieves the ordinary theta line bundle, having first Chern class (and slope) equal to one, together with the geometric theory of Landau levels discussed in [26, 50], see also [57] and below.

4 NCG and FMN theoretical aspects of generalised Laughlin wave functions

In this section we resort to noncommutative geometry (NCG) and we shall devise, giving details in the genus one case, a construction of stable bundles, together with their holomorphic sections, via suitable noncommutative theta vectors in the sense of A. Schwarz ([55]). The noncommutative setup will allow the emergence of a natural "statistical" duality which will be related to the Fourier-Mukai-Nahm transform (FMN).

Applications of NCG to the QHE, both integral and fractional, in which NCG is used in its full strength at the very foundational level, are well known (see e.g. [8, 42, 43]). Our approach pursues a different, more classical route in the sense that NCG plays an auxiliary, though relevant, role.

4.1 Noncommutative tori and stable bundles

In this Subsection we concentrate on the $g = 1$ case and we use noncommutative geometry to sketch a construction of "classical" stable bundles thereon. Everything is ultimately based on A. Connes' seminal paper ([20], see also [54, 21]), to which we refer for complete details. We are tacitly identifying, via Swan's theorem, sections of vector bundles with finitely generated projective modules over the algebra of smooth functions on the base manifold: it is precisely this interpretation that renders the transition to a noncommutative environment (i.e. general algebras and finitely generated projective modules thereon) possible, and the classical differential geometric apparatus (connections, curvature and so on) carries through to this new situation.

Recall that the noncommutative torus A_ϑ, $\vartheta \in \mathbf{R}/\mathbf{Z}$, is the universal unital C^*-algebra generated by unitary operators U_j, $j = 1, 2$ satisfying the commutation relation

$$U_1 U_2 = e^{2\pi\sqrt{-1}\vartheta}\, U_2 U_1$$

and it is to be thought of as a deformation of the standard commutative algebra of continuous functions on a torus - via Fourier theory.

Actually we shall use its smooth subalgebra $\mathbf{T}^2_\vartheta$ consisting of all series with rapidly decreasing coefficients:

$$\sum a_{mn} U_1^m U_2^n, \qquad \{a_{mn}\} \in \mathcal{S}(\mathbf{Z}^2)$$

Natural (Hermitian) $\mathbf{T}^2_\theta$-right modules (the analogues of vector bundles) are given by

$$\mathcal{E}_{p,q} = \mathcal{S}(\mathbf{R}, \mathbf{C}^q)$$

(vector valued Schwartz functions), p and q positive integers, g.c.d.$(p,q) = 1$, having "rank" (à la Murray-von Neumann)

$$\mathrm{rk}(\mathcal{E}_{p,q}) := \tau_{\mathrm{End}(\mathcal{E}_{p,q})}(I) = p - \vartheta q$$

(the trace on the endomorphism algebra being involved); also assume $p - \vartheta q > 0$. In detail, the module structure, is given by first setting

$$(V_1\xi)(s) = e(s)\xi(s), \qquad (V_2\xi)(s) = \xi(s - (p/q - \vartheta)), \qquad \xi \in \mathcal{E}_{p,q}$$

Then, one considers the following finite Weyl-Heisenberg commutation relations for $\mathbf{Z}/q\mathbf{Z}$:

$$w_1 w_2 = \bar{e}(p/q) w_2 w_1, \qquad w_1^q = w_2^q = 1.$$

Concretely, one may take, as we already observed: $w_1 = \mathrm{diag}(1, e(p/q), e(2 \cdot p/q), \ldots e((q-1) \cdot /q))$ and $w_2 =$ matrix of the shift map $e_i \to e_{i-1}$, $i = 1, 2, \ldots q$, $e_0 = e_q$, with $(e_1, \ldots, e_q)$ being the canonical basis of $\mathbf{C}^q$.

Then one sets

$$\xi \, U_i = (V_i \otimes w_i) \, \xi, \qquad i = 1, 2$$

The *Connes connection* ([20, 21]) reads:

$$(\nabla_1 \xi)(s) = 2\pi\sqrt{-1} \, \frac{q}{p - \vartheta q} \, s \, \xi(s), \qquad (\nabla_2 \xi)(s) = \xi'(s), \qquad s \in \mathbf{R}$$

(Indices 1 and 2 refer to the canonical basis of $\mathbf{R}^2$ viewed as the Lie algebra of an ordinary torus acting on $\mathcal{A}_\vartheta$). It actually fulfils the appropriate version of the Leibniz rule, it is hermitian with respect to a natural metric, it has constant curvature

$$\Omega := [\nabla_1, \nabla_2] \, e_1 \wedge e_2 = -2\pi\sqrt{-1} \, \frac{q}{p - \vartheta q} \, e_1 \wedge e_2$$

and *first Chern class* $c_1(\mathcal{E}_{p,q})$ given by the usual expression

$$c_1(\mathcal{E}_{p,q}) = \frac{1}{2\pi\sqrt{-1}} \; \tau_{\mathrm{End}(\mathcal{E}_{p,q})}(I)(2\pi\sqrt{-1} \, \frac{q}{p - \vartheta q}) = q$$

We now introduce, mimicking the classical case, a *holomorphic structure* ([58, 59, 52]) on $\mathcal{E}_{p,q}$ via the $\overline{\nabla}$ operator as

$$\overline{\nabla} := \nabla_1 + \sqrt{-1} \nabla_2$$

The kernel of the $\overline{\nabla}$ operator is called the space of *noncommutative theta vectors* ([55], see [57] as well) and it easily seen to be q-dimensional:

$$\xi = \xi(s) = e^{-\pi \frac{q}{p-\vartheta q} s^2} \cdot v \qquad s \in \mathbf{R}, \quad v \in \mathbf{C}^q$$

i.e. a (vector valued) one-dimensional harmonic oscillator ground state. The adjoint $\overline{\nabla}^*$, by contrast, has trivial kernel

$$Ker(\overline{\nabla}^*) = \{0\}$$

This follows easily from the commutation relation

$$[\overline{\nabla}, \overline{\nabla}^*] = 4\pi \frac{q}{p - \vartheta q} I$$

(indeed, the operators $\overline{\nabla}$, resp. $\overline{\nabla}^*$ are (up to constants) annihilation and creation operators). The above result can by cast in the form of an index formula:

$$\mathrm{ind}(\overline{\nabla}) := \dim Ker(\overline{\nabla}) - \dim Ker(\overline{\nabla}^*) = q$$

(cf. Subsection 3.4).

Now observe that the modules $\mathcal{E}_{p,q}$, as vector spaces, are actually *independent of* p. Thus, upon taking $p = r$ a positive integer and $\vartheta = 0$, we obtain a "classical" rank r Hermitian holomorphic vector bundle over a complex torus, which, in view of the above discussion, is indeed stable, with slope = statistical parameter = $-1/(2\pi\sqrt{-1})\cdot$ curvature = q/r. Summing up, we have the following:

Theorem 4.1 *(i) The above modules $\mathcal{E}_{r,q}$, with* g.c.d$(r, q) = 1$*, are the modules of sections of a Hermitian holomorphic stable vector bundle (denoted by the same symbol) over a complex torus Σ_1 equipped with a Hermitian connection with constant curvature, having slope*

$$\mu(\mathcal{E}_{r,q}) = \frac{q}{r} = \nu$$

(ii) The space $H^0(\mathcal{E}_{r,q})$ of holomorphic sections of $\mathcal{E}_{r,q}$ has dimension

$$h^0(\mathcal{E}_{r,q}) = q$$

(iii) In particular, the case $q = r = 1$ yields back the theta function line bundle.
(iv) ("Schwarz = Laughlin"). The "centre of mass" part of generalised Laughlin wave functions can be realised via suitable Schwarz theta vectors.

Actually, a whole *family* of constant curvature connections may be found, and it is labelled by the torus itself (viewed again as $Pic^0(\Sigma_1)$). So, *de facto*, these stable vector bundles essentially coincide with the Matsushima bundles $\mathcal{E}_\nu$ previously discussed.

Remarks. 1. A similar construction of stable bundles over an elliptic curve can be extracted via [52], where a noncommutative Fourier-Mukai transform is discussed (see below for applications of FM).

2. The previous noncommutative geometric construction of "classical" holomorphic stable bundles can be easily generalised upon resorting to the higher dimensional noncommutative tori $\mathbf{T}^{2m}_{\vartheta}$ following e.g. [54, 59, 1, 55]. One deals with a "standard module" $\mathcal{E}$, i.e. a projective $\mathbf{T}^{2m}_{\vartheta}$-module equipped with a constant curvature connection ∇. The bundle is equipped with a holomorphic structure, and the kernel of the antiholomorphic part $\overline{\nabla}$ of the connection ∇ is the $h^0(\mathcal{E})$-dimensional space of the theta vectors in the sense of [55]. The connection naturally gives rise to a representation of the CCR, direct sum of $h^0(\mathcal{E})$ copies of the Schrödinger representation, again in view of the von Neumann uniqueness theorem ([55], [57] and above). The number $h^0(\mathcal{E})$ also equals, according to Schwarz, the K-theory class $\widetilde{\mu}(\mathcal{E})$ of the module $\mathcal{E}$ evaluated on an appropriate symplectic basis of the Lie algebra of the (commutative) torus acting on $\mathbf{T}^{2m}_{\vartheta}$. In a "commutative" environment, one again gets the Matsushima stable holomorphic Hermitian vector bundles previously discussed.

4.2 Statistics of theta vectors

The preceding noncommutative approach sheds light on a notable braid group symmetry.

Let as above $\nu = q/r$, and take one of the above stable bundles $\mathcal{E}_\nu$, constructed à la Matsushima; then set $\nu' = r/q = 1/\nu$ and consider the previous "dual" representation of the finite CCR:

$$w_1 w_2 = e^{-2\pi\sqrt{-1}\cdot\frac{r}{q}}\, w_2 w_1$$

realised on $H^0(\mathcal{E}) \cong \mathbf{C}^q$, corresponding to the *rational* noncommutative torus $\mathbf{T}^2_{\frac{r}{q}}$. We write, functorially:

$$C : \mathcal{E}_\nu \to \mathbf{T}^2_{\nu'}$$

(C stands for Connes). This representation can be associated (cf. Section 2) to a RS-braid group $B(\Sigma_1, q)$ representation with statistics parameter $\sigma' = e^{\pi\sqrt{-1}\cdot\frac{r}{q}} = (-1)^{\nu'}$ and can also be promoted to a stable bundle $\mathcal{E}_{\nu'}$ via the Matsushima (M) construction:

$$M : \mathbf{T}^2_\nu \to \mathcal{E}_\nu$$

So the upshot is the following:

Theorem 4.2 (double interpretation of theta vectors). *The space of theta vectors pertaining to the above geometrical (infinite dimensional) unitary RS-braid group $B(\Sigma_1, r)$ representation with statistics parameter $\sigma = e^{\pi\sqrt{-1}\cdot\frac{q}{r}} = (-1)^\nu$*

also determines a finite dimensional unitary RS-braid group $B(\Sigma_1, q)$ representation with "dual" statistics parameter $\sigma' = e^{\pi\sqrt{-1}\cdot\frac{r}{q}} = (-1)^{\nu'}$. We have an ensuing "Matsushima-Connes (MC) duality"*, written succinctly*

$$M : \mathbf{T}^2_\nu \to \mathcal{E}_\nu, \qquad C : \mathcal{E}_\nu \to \mathbf{T}^2_{\nu'}, \qquad MC : \mathcal{E}_\nu \to \mathcal{E}_{\nu'}, \qquad CM : \mathbf{T}^2_\nu \to \mathbf{T}^2_{\nu'}$$

In the following section we are going to interpret this phenomenon in terms of the Fourier-Mukai-Nahm transform.

Remarks. 1. We add a brief extra comment on the term "dual" employed above. Two noncommutative tori $A := A_\vartheta$ and are called dual, or strongly Morita equivalent $B := A_{\vartheta'}$ if there exists a $A - B$-bimodule E such that they are each other's endomorphism algebra. In the present context, this is tantamount to require that ϑ and ϑ' are on the same $SL(2, \mathbf{Z})$-orbit. In our case, the tori involved have parameters $\vartheta' := -r/q = -(q/r)^{-1} =: -\vartheta^{-1}$. It corresponds, essentially, to the CM-duality for tori discussed above. However, it should be noted that all *rational* noncommutative tori are strongly Morita equivalent (to $C(\mathbf{T}^2)$, the C^*-algebra of continuous functions on $\mathbf{T}^2$).

2. The Matsushima construction matches exactly with the treatment of Landau levels on a torus given by [50] in terms of "symmetry breaking" of the genuine Weyl-Heisenberg group (magnetic translations) to a finite one. The noncommutative geometrical approach improves on this picture, rendering it quite natural.

4.3 FMN-transform via theta vectors

For a comprehensive treatment of the Fourier-Mukai and Nahm transforms see e.g. [6, 32]. Here we confine ourselves to cursory remarks, and avoiding most of the categorial machinery in order to focus on the crucial issues. Let $M = V/\Lambda$ be an Abelian variety, and $\widehat{M}$ its dual one. Let us briefly recall its construction: one takes the dual conjugate space $\bar{V}^*$ and the dual lattice $\Lambda^* = \{\ell \in \bar{V}^* | \, \ell(\Lambda) \subset \mathbf{Z}\}$; then $\widehat{M} = \bar{V}^*/\Lambda^*$. Let $\pi_1 : M \times \widehat{M} \to M$ and $\pi_2 : M \times \widehat{M} \to \widehat{M}$ denote the obvious projections. Let $\mathcal{P} \to M \times \widehat{M}$ be the *Poincaré bundle*, which is uniquely characterised by the fact that its restriction $\mathcal{P}_\xi := \mathcal{P}_{M\times\{\xi\}} \to M$ is the line bundle pertaining to $\xi \in \widehat{M}$, together with the requirement that, restricted to $\widehat{M} \cong \pi_M^{-1}(0)$, it is trivial. It can be also described as in Subsection 3.3, via the Hermitian form H on $V \times \bar{V}^*$ given by:

$$H(v, w, \alpha, \beta) = \overline{\beta(v)} + \alpha(w) \qquad v, w \in V, \; \alpha, \beta \in \bar{V}^*$$

and the semicharacter

$$\chi(\lambda, \mu) = e^{\sqrt{-1}\pi\mu(\lambda)} \qquad \lambda \in \Lambda, \, \mu \in \Lambda^*$$

Let R be the *direct image functor*. The Fourier-Mukai (FM) transform of a holomorphic vector bundle on M (or more generally, a sheaf – as before, one describes a vector bundle via its sections) is the following sheaf on $\widehat{M}$:

$$(FM)(\mathcal{E} \to M) = [R\pi_{2*}(\pi_1^*(\mathcal{E}) \otimes \mathcal{P})] \to \widehat{M}$$

In favourable cases one obtains a bona fide holomorphic vector bundle: we present a concrete instance of this phenomenon in the genus one case, for the (Matsushima) bundles $\mathcal{E} = \mathcal{E}_\nu$ previously treated. The Jacobian $J(\Sigma_1)$ is a self-dual abelian variety and it also parametrizes the flat line bundles over itself, in the sense that its points label holonomies of flat connections thereon, and at the same time the holomorphically inequivalent degree zero (i.e. flat) line bundles $\mathcal{P}_x$, $x \in J(\Sigma_1)$ (by the very definition of the Poincaré bundle). Upon tensoring $\mathcal{E}$ with $\mathcal{P}_\xi$, one obtains a vector bundle $\mathcal{E}_\xi = \mathcal{E} \otimes \mathcal{P}_\xi$ fulfilling the same conditions as $\mathcal{E}$, by the RRH-theorem (cf. the proof of Theorem 3.1 and see also below).

Then by the *index theorem for families*, the cohomology spaces $H^0(\mathcal{E}_x)$, $x \in J(\Sigma_1)$ become the fibres of the sought-for FM-transformed holomorphic vector bundle $FM(\mathcal{E}) \to J(\Sigma_1)$, which is known to be *stable*. (The direct image functor R involves in this case just the 0-cohomology: we are in the so-called IT_0 situation, [6, 32]). Upon subsequent dualization (denoted by *), the first Chern class changes sign and stability persists. Explicitly, one has, for their respective Chern character vectors:

$$\big(\mathrm{Ch}_0(\mathcal{E}), \mathrm{Ch}_1(\mathcal{E})\big) = (r, q), \qquad \big(\mathrm{Ch}_0(FM^*(\mathcal{E})), \mathrm{Ch}_1(FM^*(\mathcal{E}))\big) = (q, r)$$

Notice then that *the Chern Character vectors of the MC-vector bundle and of the FM-transformed bundle agree.*

One has also the *Nahm transform* $\hat{\mathcal{E}}$ of the HE-vector bundle $\mathcal{E}$. It comes equipped with a metric and compatible connection $\hat{\nabla}$ with curvature of type $(1,1)$, which endows it with a holomorphic structure, and one has $\hat{\mathcal{E}} \cong FM(\mathcal{E})$ (see e.g. [61]). Let us briefly review this construction, for a Hermitian vector bundle $E \to \Sigma_1$ equipped with a unitary connection ∇; since the curvature form Ω_∇ is of type $(1,1)$, $E \to \Sigma_1$ becomes a holomorphic vector bundle, with Chern-Bott connection ∇. Let, as in Subsection 3.4, consider the canonical spinor bundle $\mathbf{S} = \Lambda^{0,\bullet}T^*\Sigma_1$ with the natural splitting

$$\mathbf{S} = \mathbf{S}^+ \oplus \mathbf{S}^-, \qquad \mathbf{S}^+ = \Lambda^{0,0}T^*\Sigma_1, \qquad \mathbf{S}^+ = \Lambda^{0,1}T^*\Sigma_1$$

Upon tensoring with the Poincaré line bundle $\mathcal{P} \to \Sigma_1 \times J(\Sigma_1)$ one gets a family of spin^c Dirac operators

$$D_\xi : \Lambda^0(\Sigma_1, \mathbf{S}^+ \otimes E_\xi) \to \Lambda^1(\Sigma_1, \mathbf{S}^- \otimes E_\xi)$$

coinciding with the Dolbeault operators

$$D_\xi = \sqrt{2}\,(\bar{\partial}^*_{E_\xi} + \bar{\partial}_{E_\xi})$$

The Atiyah-Singer theorem for families gives rise to an element $\mathrm{ind}(D) = [\mathrm{Ker} D] - [\mathrm{Coker}\, D]$ in the K-theory of $J(\Sigma_1)$; actually, in view of the Bochner-Kodaira considerations in Subsection 3.4 one abuts at a bona fide vector bundle manufactured from the holomorphic sections

$$\mathrm{Ker} D_\xi = \mathrm{Ker}\, \bar\partial_{E_\xi} = H^0(E_\xi) \Rightarrow h^0(E_\xi) = q \quad \forall \xi \in J(\Sigma_1)$$

Let us recall the construction of the Nahm connection $\hat\nabla$. Upon taking L^2-completions, one has a short exact sequence

$$0 \to \hat E_\xi \to L^2(E_\xi \otimes \mathbf{S}^+) \stackrel{D_\xi}{\to} L^2(E_\xi \otimes \mathbf{S}^-) \to 0$$

The L^2-spaces in the middle are typical fibres of trivial vector bundles over $J(\Sigma_1)$, equipped with a standard flat connection. Subsequent projection defines the Nahm connection $\hat\nabla$ (it is actually a Grassmann connection). We write, concisely, $FMN(\mathcal{E})$ for the FM-bundle, equipped with the Nahm transformed connection. However, by Uhlenbeck-Yau, $\hat\nabla$ coincides with its unique (up to a scalar) HE-connection. *Therefore, up to moduli, (and dualization) the MC-vector bundle agrees with the FMN-transformed bundle.* The Nahm-transformed connection turns out to have constant curvature. We are now going to check all this explicitly via a noncommutative geometric approach.

First, consider the constant curvature connections (parametrized by a torus)

$$\tilde\nabla_1 = \nabla_1 - 2\pi\sqrt{-1}\,\alpha\, I, \qquad \tilde\nabla_2 = \nabla_2 - 2\pi\sqrt{-1}\,\beta\, I, \qquad \alpha, \beta \in [0,1]$$

We remark in passing that they represent Yang-Mills gauge equivalence classes. (Here one has a manifestation of the general principle "symplectic quotient = Mumford quotient", see [27, 47, 3, 5] and [54, 58, 59] as well).

Then consider their associated holomorphic structures

$$\overline{\tilde\nabla} = \tilde\nabla_1 + \sqrt{-1}\tilde\nabla_2 = \overline{\nabla} - 2\pi\sqrt{-1} z I$$

(setting $z = \alpha + \sqrt{-1}\,\beta$). If we compute the corresponding theta vectors, we easily get

$$\xi_z \equiv \xi_{\alpha,\beta} = e^{-\pi \frac{r}{q}(\frac{q}{r}s - z)^2} v, \qquad v \in \mathbf{C}^q$$

One finds, successively (working with scalar functions, since this clearly suffices)

$$|\xi_z|^2 = e^{2\pi\frac{r}{q}\beta^2} e^{-2\pi\frac{r}{q}(\frac{q}{r}s-\alpha)^2}$$

which, after using, for $\gamma > 0$, $\int_{\mathbf{R}} e^{-\gamma x^2} dx = \sqrt{\pi/\gamma}$, leads to L^2-normalized theta vectors

$$\tilde\xi_z = e^{-\pi\frac{r}{q}\beta^2} (2q/r)^{\frac14} \xi_z$$

Now, according to the general recipe, regard the theta vectors as fibres of the Nahm-Fourier-Mukai transformed vector bundle over the torus parametrized by

z and compute the Nahm-transformed connection, or, in physical terminology, adiabatic *Berry-Simon connection* (form) ([10, 56, 18]), (it is enough to work componentwise):

$$z \mapsto A_z = \langle \tilde{\xi}_z, d\, \tilde{\xi}_z \rangle = \langle \tilde{\xi}_z, \partial_\alpha\, \tilde{\xi}_z \rangle d\alpha + \langle \tilde{\xi}_z, \partial_\beta\, \tilde{\xi}_z \rangle d\beta$$

Upon making use of $\int_{\mathbf{R}} e^{-\gamma x^2} x\, dx = 0$, we find:

$$\langle \tilde{\xi}_z, \partial_\alpha\, \tilde{\xi}_z \rangle = -2\pi\sqrt{-1}\, \frac{r}{q}\ \beta, \qquad \langle \tilde{\xi}_z, \partial_\beta\, \tilde{\xi}_z \rangle = 0$$

whence (reintroducing vectors):

$$A = -2\pi\sqrt{-1}\, \frac{r}{q}\ \beta\, d\alpha \cdot I_q$$

having (constant) curvature

$$\Omega = 2\pi\sqrt{-1}\, \frac{r}{q}\ d\alpha \wedge d\beta \cdot I_q$$

as in the Matsushima construction and leading to the correct values $(q, -r)$ for the FMN-transformed bundle.

Remark. In physical literature, what is called the adiabatic curvature is actually the trace of Ω, $\mathrm{Tr}(\Omega)$, and it is the curvature form of the determinant line bundle $\det(\hat{\mathcal{E}})$:

$$\mathrm{Tr}(\Omega) = 2\pi\sqrt{-1}\, r\ d\alpha \wedge d\beta$$

The constancy of $\mathrm{Tr}(\Omega)$ would also follow from the general Bismut-Gillet-Soulé theory (BGS) ([12, 13, 14]) used as in [61], taking into account the fact that for tori, no *analytic torsion* phenomena arise and one can compute via the Quillen metric. Our simple-minded use of noncommutative geometry allowed us to avoid usage of the formidable BGS apparatus.

Alternatively, one could have resorted to Varnhagen's approach via standard theta function theory, but this would have been much more complicated (see [65]).

We recapitulate the preceding discussion by means of the following:

Theorem 4.3 *(i) Set $\nu = q/r$, $\nu' = r/q = 1/\nu$. The MC-correspondence and the FMN^*-correspondence (Fourier-Mukai-Nahm, plus dualization):*

$$FMN^* : \mathcal{E}_\nu \to \mathcal{E}_{\nu'}$$

agree up to moduli.

(ii) The FMN^*-correspondence can be explicitly realised via noncommutative theta vectors by means of the direct calculation performed above.

The following theorem shows that no further examples of stable bundles suitable for the construction of RS-braid group representations can be manufactured via the FM-transform of vector bundles on Σ_g.

Theorem 4.4 *Let $E \to \Sigma_g$ be a holomorphic vector bundle. Assume that its FM-transformed bundle $FM(E) \to J(\Sigma_g)$ is projectively flat. Then one has $g = 1$.*

Proof. If $FM(E) \to J(\Sigma_g)$ is projectively flat, then their Chern character classes $\mathrm{Ch}_m(FM(E))$ do not vanish for all $m = 0, 1, \ldots, g$. But the latter vanish, in general, for $m > 1$, e.g. by [61] and this implies $g = 1$. □

4.4 On the noncommutative FM-transform

We would like to add a remark on the noncommutative version of the FM-transform discussed in [52], producing, from a standard module on a noncommutative torus, a "classical" stable vector bundle over an elliptic curve. Our previous calculation can be carried out verbatim for general non commutative theta vectors, depending on θ. In the calculation of the FMN-curvature we find, comparing with the previous calculation, an extra factor

$$-2\pi\sqrt{-1}\theta \, d\alpha \wedge d\beta$$

which can be however be removed via employment of the following *projectively flat* connection

$$\nabla = d - \pi\sqrt{-1}\theta \, (\alpha \, d\beta - \beta \, d\alpha)$$

one the same infinite dimensional bundle. The use of a projectively flat connection allows one to identify the ensuing projective Hilbert spaces, and it is a common procedure in geometric quantization for assuring independence of the physical Hilbert space of the complex polarization, see e.g. [34, 5, 63]). In this way, our construction matches the Polishchuk-Schwarz one.

5 Physical applications

In the present Subsection we revert to physics, in order to further substantiate and instantiate the above developments. The first two Subsections are meant to motivate the physical interpretation of the CM-FMN duality previously introduced, to be discussed in the last Subsection. We do not attempt at a theoretical explanation of appearance of specific fractions in experiments.

5.1 Anyon Laughlin wave functions

The paper [19] contains a derivation of an anyon Laughlin wave function corresponding to a filling factor of the form $\nu = n/(2pn+1)$, where B is the magnetic field, ℓ_0 is the magnetic length (see below) and the suffix LLL stands for lowest Landau level:

$$\Psi_{LLL}(B) = \prod_{i<j}^{N} (z_i - z_j)^{\frac{1}{\nu}} e^{-\frac{1}{4\ell_0^2}\sum_{1=1}^{N} |z_i|^2}$$

A geometric quantization approach to functions of the above type has been given in [11].

The derivation of the latter functions carried out in [19] proceeds via the so-called Chern-Simons (CS) transform, and amounts at finding the ground state of a generalized harmonic oscillator Hamiltonian representing the kinetic energy part of the full Hamiltonian:

$$a_j(\alpha)\Psi_{LLL} = 0$$

where:

$$a_j(\alpha) = \tfrac{1}{\sqrt{2}}\,(2\,\tfrac{\partial}{\partial \bar{z}} + \tfrac{z_j}{2} + \alpha \tfrac{\partial \ln \overline{\chi}}{\partial \bar{z}})$$

$$a_j^{\dagger}(\alpha) = \tfrac{1}{\sqrt{2}}\,(-2\,\tfrac{\partial}{\partial z} + \tfrac{\bar{z}_j}{2} + \alpha \tfrac{\partial \ln \chi}{\partial z})$$

with

$$\chi = \prod_{i<j}^{N} (z_i - z_j)$$

The kinetic part of the Hamiltonian has the oscillator-like expression

$$H_0 = \hbar\omega_c \sum_{j=1}^{N} a_j^{\dagger}(\alpha) a_j(\alpha) + \frac{1}{2} N \hbar\omega_c$$

where m_e is the electron mass, $\omega_c = e|B|/m_e$ is the cyclotron frequency and ℓ_0 is the magnetic length

$$\ell_0 = \sqrt{\hbar/e|B|}$$

Exploitation of PT-invariance and angular momentum considerations yield $\alpha = -2/\nu$ together with the above solution Ψ_{LLL}.

5.2 Lowest Landau level and Laughlin wave functions on a torus

We briefly summarize the exposition of [26] for Landau levels of a free electron, keeping their notation, differing slightly from ours. They start from an elliptic curve

$$\Sigma_1 = \mathbf{C}/(\mathbf{Z} \oplus \tau\mathbf{Z})$$

Set

$$z = x + \sqrt{-1}y, \quad x_i = \tilde{x}_i/L_i, \qquad \tau = (L_2/L_1)e^{\sqrt{-1}\psi}, \Im\tau > 0$$

(L_1 and L_2 being the dimensions of the rectangular sample employed). Holomorphic geometric quantization is then carried out, yielding a holomorphic line bundle $\Theta^m \to \Sigma_1$, m an odd integer, with connection and curvature

$$A = -\frac{2\pi m}{\Im\tau} y dx, \qquad F_A = \frac{2\pi m}{\Im\tau} dx \wedge dy$$

and first Chern class

$$c_1(L^m) = \frac{1}{2\pi}\int_{\Sigma_1} F_A = m \in \mathbf{Z}, \qquad 2\pi m = \frac{eB}{hc} L_1^2$$

The annihilation/creation operators read

$$a = \sqrt{\frac{\Im\tau}{\pi m}}\left(\frac{\partial}{\partial \bar{z}} + \sqrt{-1}\frac{\pi m}{\Im\tau}\Im z\right), \quad a^\dagger = -\sqrt{\frac{\Im\tau}{\pi m}}\left(\frac{\partial}{\partial z} + \sqrt{-1}\frac{\pi m}{\Im\tau}\Im z\right),$$

and the Hamiltonian is

$$H = \hbar\omega_c(a^\dagger a + \frac{1}{2}), \qquad \omega_c = \frac{\hbar}{m_e}\frac{2\pi m}{\Im\tau}, \quad [a, a^\dagger] = 1$$

(m_e denoting the mass of the electron). The ground state space can be identified with the holomorphic sections of the above bundle, i.e. with the m-level theta functions; explicitly

$$f_l(z,\tau) = \vartheta\begin{bmatrix} 0 \\ l/m \end{bmatrix}(z|\tau/m), \quad l = 0, 1, \ldots, m-1$$

The Laughlin wave functions in this situation are the Haldane-Razayi (HR)-wave functions, written for generic boundary conditions, corresponding to phases associated to a double Bohm-Aharonov device (see e.g. [62, 26]) and with Z being the centre of mass coordinate:

$$F_l^{HR} = \vartheta\begin{bmatrix} (l+\phi_1)/m \\ \phi_2 \end{bmatrix}(mZ|m\tau)\prod_{i<j}\vartheta^m\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}(z_i - z_j|\tau), \quad l = 0, 1, \ldots, m-1$$

Now consider a HR-two-quasihole-two electron wave function (see [26], p.317, (34), with slight notational changes and a possible typo corrected); we separately record the centre of mass part

$$(F_l^{2QH})_{cm} = \vartheta\begin{bmatrix} l/m \\ 0 \end{bmatrix}(m(z_1 + z_2) + \eta_1 + \eta_2|m\tau)$$

the purely electronic part

$$(F_l^{2QH})_{el} = \vartheta^m \left[\begin{array}{c} 1/2 \\ 1/2 \end{array} \right] (z_1 - z_2|\tau) \leftrightarrow (-1)^m = (-1)^{\nu'}$$

and the purely quasi-hole part

$$(F_l^{2QH})_{qh} = \vartheta^{\frac{1}{m}} \left[\begin{array}{c} 1/2 \\ 1/2 \end{array} \right] (\eta_1 - \eta_2|\tau) \leftrightarrow (-1)^{\nu}$$

The other ingredients are the "mixed" terms $\vartheta \left[\begin{array}{c} 1/2 \\ 1/2 \end{array} \right] (z_i - \eta_j|\tau)$.

Subsequently, set $\xi = (\phi_1, \phi_2)$ and consider the family $\Theta_\xi^m \to J(\Sigma_1)$ of m-level theta line bundles. One has $h^0(\Theta_\xi^m) = m$. The corresponding FMN^*-transformed vector bundle (spectral bundle) has rank m, degree 1, slope $1/m$. Now, *one can interpret the Bohm-Aharonov angular parameters as quasihole coordinates (varying adiabatically)*. This follows from the general formula (for real a and b)

$$\vartheta \left[\begin{array}{c} a \\ b \end{array} \right] (z + \eta)|\tau) = e^{2\pi\sqrt{-1}a\,(z+b)} \vartheta(z + a\tau + b|\tau)$$

showing that there is a bijective correspondence between theta functions with characteristics and translated ones.

Therefore, the spectral bundle attached to electrons, encoding the centre of mass dynamics, can be interpreted as a quasihole centre of mass bundle, this being signalled by the correct statistical parameter $\nu = 1/m$. Clearly, the above procedure is reversible.

We also record, as a second example, in a plane geometry, the following Laughlin wave function describing two quasiholes ([28], (again $\nu = 1/m$, m odd):

$$\Psi_{\eta_1,\eta_2} = (\eta_1 - \eta_2)^{\frac{1}{m}} \prod_k (z_k - \eta_1)(z_k - \eta_2) \prod_{i>j} (z_i - z_j)^m e^{-\frac{1}{4\ell_0^2}\left[\frac{1}{m}(|\eta_1|^2 + |\eta_2|^2) + \sum_i |z_i|^2\right]}$$

Upon integrating over the electron coordinates one has an effective quasi-hole wavefunction with braiding $(-1)^{\nu}$. By contrast, the electron ground state wave function, which has the familiar expression, exhibits, of course, Fermi statistics.

5.3 ν-anyon/ν'-anyon duality

We are now in a position to explain the underlying physical idea of the duality we discussed in previous Subsections, which can be termed "ν-anyon/ν'-anyon duality", generalising the quasihole/electron one previously discussed. Starting from a ν-anyon representation, we found that the q-dimensional ground state space $H^0(\mathcal{E}_\nu)$ has a dual braid symmetry which gives rise to a ν'-anyon representation, via Matsushima-Connes/ Fourier-Mukai-Nahm. The crucial physical issue is that the change in the holomorphic structure of the bundle $\mathcal{E}_\nu$ and the ensuing variation of the ground state spaces $H^0(\mathcal{E}_\nu)$ involved in the FMN transform can be interpreted as an adiabatic motion of the ν'-anyons, encoded in the centre of mass coordinate. The FMN-transform (plus dualization) ultimately creates an *effective* ν'-anyon wave function (via direct image, i.e. essentially via fibre integration). We stress the fact that we did not analyze the mixed terms of the complete wave function.

6 Conclusions

We proposed a geometrical construction of the simplest unitary Riemann surface braid group representations via the differential geometry of stable holomorphic vector bundles over Jacobians, building on the Matsushima theory and on Weyl-Heisenberg group theoretic techniques. Braiding turned out to be tightly related to curvature, yielding in particular a "slope-statistics" formula. Stable bundles have been also be approached, in turn, via noncommutative geometric techniques; in particular, the notion of noncommutative theta vector, together with the corresponding notion in Matsushima's setup, resulted to be useful for the construction and interpretation of generalised Laughlin wave functions, exhibiting anyon statistics. The problem of extracting roots of line bundles has been overcome by passing to local objects which however still captured braiding phenomena in an intrinsic way. A "ν-anyon/ν'-anyon" duality emerged, closely related to the Fourier-Mukai-Nahm transform, which had been elucidated again via the Schwarz theta vectors.

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