

## On Newton-Cartan trace anomalies

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Roberto Auzzi,<sup>a,b</sup> Stefano Baiguera<sup>a</sup> and Giuseppe Nardelli<sup>a,c</sup>

<sup>a</sup>*Dipartimento di Matematica e Fisica, Università Cattolica del Sacro Cuore,  
Via Musei 41, 25121 Brescia, Italy*

<sup>b</sup>*INFN Sezione di Perugia,  
Via A. Pascoli, 06123 Perugia, Italy*

<sup>c</sup>*TIFPA — INFN, c/o Dipartimento di Fisica, Università di Trento,  
38123 Povo (TN), Italy*

*E-mail:* [roberto.auzzi@unicatt.it](mailto:roberto.auzzi@unicatt.it), [giuseppe.nardelli@unicatt.it](mailto:giuseppe.nardelli@unicatt.it),  
[stefano.baiguera92@gmail.com](mailto:stefano.baiguera92@gmail.com)

**ABSTRACT:** We classify the trace anomaly for parity-invariant non-relativistic Schrödinger theories in  $2 + 1$  dimensions coupled to background Newton-Cartan gravity. The general anomaly structure looks very different from the one in the  $z = 2$  Lifshitz theories. The type A content of the anomaly is remarkably identical to that of the relativistic  $3 + 1$  dimensional case, suggesting the conjecture that an  $a$ -theorem should exist also in the Newton-Cartan context.

**KEYWORDS:** Anomalies in Field and String Theories, Space-Time Symmetries, Renormalization Group

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## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Newton-Cartan geometry</b>	<b>3</b>
2.1	Newton-Cartan theory and DLCQ	4
<b>3</b>	<b>Conformal anomaly (<math>z = 2</math>)</b>	<b>5</b>
3.1	Consequences of the causality constraint	5
3.2	Classification of tensors	6
3.3	Weyl transformations	7
3.4	The anomaly	8
3.5	Consistency conditions	9
<b>4</b>	<b>Conclusions</b>	<b>10</b>
<b>A</b>	<b>Considerations on the case with general dynamical exponent <math>z \neq 2</math></b>	<b>11</b>
<b>B</b>	<b>Useful formulas</b>	<b>12</b>

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## 1 Introduction

Renormalization group (RG) can be understood as a set of trajectories in the space of theories from the UV to IR; the implicit coarse graining procedure suggest that the number of effective degrees of freedom decreases from UV to IR. This intuition is correct just if a correct counting of effective degrees of freedom is used. In the case of relativistic  $1 + 1$  unitary dimensional theories, it exists indeed a function, introduced by Zamolodchikov [1], which is monotonically decreasing along a RG trajectory. This function coincides with the central charge  $c$  of the conformal field theory (CFT) in the UV and in the IR, which is related to the trace anomaly of the energy-momentum tensor:

$$T_{\mu}^{\mu} = \mathcal{A} = cR. \tag{1.1}$$

Some of these results can be extended to four dimensional theories. In particular it was conjectured in [2] that such monotonically decreasing function exists and that it coincides with the conformal anomaly coefficient  $a$  at the conformal fixed point:<sup>1</sup>

$$T_{\mu}^{\mu} = \mathcal{A} = aE_4 - cW_{\mu\nu\alpha\beta}^2 + a'D^2R. \tag{1.2}$$

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<sup>1</sup>In this equation the anomaly coefficient  $a'$  is a scheme-dependent quantity [3], while  $a$  and  $c$  are genuine scheme-independent anomalies. Conformal anomalies have a long history, see e.g. [4, 5].

A perturbative proof of this conjecture was given in [6–8]. A non-perturbative proof of a weak version of this theorem (which states that the central charge  $a$  of the IR conformal fixed point is less than the one in the UV, if both these fixed points exist) was given in [9, 10].

In odd dimensions there is no relativistic trace anomaly. Still, there is a candidate for a monotonically decreasing function of the coupling along the RG flow, which is related to a scheme-independent contribution to the free-energy of the theory on a sphere [11, 12]. This suggests that for relativistic theories the RG flow is an irreversible process as higher momentum modes are integrated out. In even dimension, the central charges  $c$  and  $a$  provide a definition of the effective number of degrees of freedom that is monotonically decreasing from UV to IR.

It is interesting to ask if similar quantities may exist in the case of non-relativistic RG flows. Non-relativistic scale invariance is realized in Nature in interesting condensed matter systems, such as unitary Fermi gases. Both in the Lifshitz and in the Galilean case, scale invariance is in general characterized by a different scaling of the time and space coordinates, which can be parameterized by the dynamical exponent  $z$ :

$$x^i \rightarrow e^\sigma x^i, \quad t \rightarrow e^{z\sigma} t. \quad (1.3)$$

Conformal invariance is achieved just in the Galilean case for  $z = 2$  (Schrödinger invariance) and in the relativistic case ( $z = 1$ ).

In the case of Lifshitz theories, a detailed study of trace anomalies for various dimensions and values of  $z$  was carried on in [13–16]. The result does not give any reasonable candidate for a decreasing  $a$ -function; several anomalies are indeed possible at the scale-invariant fixed points, but their Weyl variation vanishes identically (type B anomalies [17]). An analysis like the one developed in [6–8] for relativistic theories would suggest that no monotonically-decreasing anomaly coefficient is present in the Lifshitz case.

The analysis of the Newton-Cartan (NC) conformal anomaly was initiated in [18], where an infinite number of possible terms entering the anomaly was found. In this situation, it is difficult to figure out the existence of an  $a$ -theorem, due to the infinitely many coefficients that are in principle present, and the infinite number of Wess-Zumino consistency conditions to solve. With these premises, the natural conclusion would be that non-relativistic theories can not admit an  $a$ -theorem: either there are not type A anomalies (Lifshitz theories) or there are too many (Schrödinger theories). We shall show that this is not the case.

We shall perform an analysis of the conformal anomalies for Schrödinger theories in  $2 + 1$  dimensions. The discussion of these systems is not moot, because there are several physical models which realize these symmetries, the simplest example being free fields. Conformal invariance of  $2 + 1$ -dimensional systems is relevant, for example, in anyons [19]. In addition, investigation of  $2 + 1$  dimensional non-relativistic CFTs gives useful insights for fermions at unitarity in  $3 + 1$  dimensions [20]. Other interesting examples are provided by holography, e.g. [21, 22].

The analysis of the anomaly is important in order to identify possible quantities which may decrease along the non-relativistic RG flow. For this purpose, the non-relativistic

theory is coupled to a NC background with torsion. As a tool to write terms which respect all the symmetries of the problem, we shall use discrete light-cone quantization (DLCQ): the NC background is obtained via dimensional reduction of a relativistic theory on a null circle [23, 24]. We shall show that the requirement that the NC background respects causality, drastically reduces the possible terms in the anomaly to a finite number. We classify them and we solve the Wess-Zumino consistency conditions in the case of parity-invariant theories. We find that

$$2T_0^0 + T_i^i = \mathcal{A} = a\sigma E_4 - c\sigma W^2 + b\sigma J^2 + \mathcal{A}_{\text{ct}}, \quad (1.4)$$

where the  $\mathcal{A}_{\text{ct}}$  (see eq. (4.3)) corresponds to terms which can be eliminated by scheme-dependent counterterms. The terms  $E_4$  and  $W^2$  are the DLCQ reduction of the four-dimensional Euler density and squared Weyl tensor; the term  $J^2$  is defined in eq. (4.2). Both  $W^2$  and  $J^2$  have vanishing Weyl variation (type B anomalies). Instead, there is only one term which has a non vanishing Weyl variation and, remarkably, it is precisely the same term of the corresponding 3 + 1 dimensional theory. It is the  $E_4$  term, and its coefficient is the natural candidate for a monotonically decreasing function from the UV to the IR.

In  $d$  spatial dimensions,<sup>2</sup> the NC anomaly can appear only for discrete values of  $z$ . More precisely (see appendix A), for

$$z = 2n - d, \quad (1.5)$$

where  $n$  is a positive integer. It is possible to extend the analysis to other values of  $d$  and  $z$ , satisfying eq. (1.5). The Weyl weight of  $E_4$  is  $-4$  for every  $z$ : the  $E_4$  term can be present in the anomaly in  $d = 2$  just for  $z = 2$ . The identification of the coefficient of the  $E_4$  anomaly as a monotonically decreasing quantity is possible only for  $z = 2$ , which forces  $d$  to be even. This justifies  $d = 2$  as the minimal choice with non-trivial conformal anomaly.

## 2 Newton-Cartan geometry

Newton-Cartan (NC) geometry allows to reformulate Newtonian gravity in a way which is coordinate independent; for a review see [25]. Recently, works by Son and collaborators [26–29] showed that it can be used as a powerful tool to study condensed matter systems with galilean invariance; the main idea is to use it as source for energy-momentum tensor for quantum field theory description of several condensed matter systems. NC geometry is an interesting topic by itself; some useful references include [23, 24, 30–37]. Strongly-coupled system with Galilean invariance can be studied holographically [21, 22]; also in this approach the NC geometry is a natural formalism [38–42].

The basic data of Newton-Cartan geometry are a positive definite symmetric tensor  $h^{\mu\nu}$  with rank  $d$  (which corresponds to the spatial inverse metric) and a nowhere-vanishing vector  $n_\mu$  (which corresponds to the local time direction), with the condition

$$n_\mu h^{\mu\alpha} = 0. \quad (2.1)$$

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<sup>2</sup>In  $d = 0$  the anomaly vanishes because no non-zero scalar can be built from the NC geometrical data.

The 1-form  $n = n_\mu dx^\mu$  is not necessarily closed (indeed it should not be if we want to use it as source for the energy current). Another necessary ingredient is a non-dynamical gauge field  $A_\mu$ , which acts as a source for the particle number symmetry.

In order to be able to define a metric with lower indices and a connection, one should also choose a velocity field  $v^\mu$ , with the condition  $n_\mu v^\mu = 1$ . Given  $(h^{\alpha\beta}, n_\mu, v^\nu)$  one can then uniquely define  $h_{\mu\nu}$ , with:

$$h^{\mu\alpha} h_{\alpha\nu} = \delta_\nu^\mu - v^\mu n_\nu = P_\nu^\mu, \quad h_{\mu\alpha} v^\alpha = 0, \quad (2.2)$$

where  $P_\nu^\mu$  is the projector onto spatial directions.

Causality induces the following Frobenius integrability condition on the 1-form  $n$ :

$$n \wedge dn = 0, \quad (2.3)$$

see e.g. in [16, 29]. This constraint is equivalent to the fact that  $n$  can be locally expressed as  $n = g df$ , where  $f, g$  are functions. The condition in eq. (2.3) admits a non-zero torsion<sup>3</sup> in the NC connection [29]; this is chosen in a way which is compatible with causality.

The symmetries of the Newton-Cartan theory include, besides diffeomorphisms and local U(1) gauge symmetry, the Milne boosts, which is a local version of the Galilean boost. If we denote by  $\psi_\mu$  the local parameter of the Milne boost, the Newton-Cartan geometry fields transform in the following way:

$$\begin{aligned} v'^\mu &= v^\mu + h^{\mu\nu} \psi_\nu \\ h'_{\mu\nu} &= h_{\mu\nu} - (n_\mu P_\nu^\rho + n_\nu P_\mu^\rho) \psi_\rho + n_\mu n_\nu h^{\rho\sigma} \psi_\rho \psi_\sigma, \\ A'_\mu &= A_\mu + P_\mu^\rho \psi_\rho - \frac{1}{2} n_\mu h^{\alpha\beta} \psi_\alpha \psi_\beta, \end{aligned} \quad (2.4)$$

while  $n_\mu$  and  $h^{\mu\nu}$  are invariant. The following quantities are Milne invariants:

$$v_A^\alpha = v^\alpha - h^{\alpha\xi} A_\xi, \quad (h_A)_{\alpha\beta} = h_{\alpha\beta} + A_\alpha n_\beta + A_\beta n_\alpha, \quad \phi_A = A^2 - 2v \cdot A. \quad (2.5)$$

where  $A^2 = h^{\mu\nu} A_\mu A_\nu$  and  $A \cdot v = v^\mu A_\mu$ .

In the case in which a 2 + 1 dimensional theory with non-zero magnetic momentum coupling is coupled to gravity, the Milne boost transformations of  $A^\mu$  must be appropriately modified, see e.g. [28, 31]. This situation is physically important for the quantum Hall effect. In this case there is no parity invariance; we leave this as a topic for future investigations.

## 2.1 Newton-Cartan theory and DLCQ

It is possible to get non-relativistic Newton-Cartan theory with zero magnetic momentum coupling as a dimensional reduction along the null direction  $x^-$  of the following relativistic non-degenerate metric [23, 24]:

$$\begin{aligned} G_{MN} &= \begin{pmatrix} 0 & n_\mu \\ n_\nu & n_\mu A_\nu + n_\nu A_\mu + h_{\mu\nu} \end{pmatrix} = \begin{pmatrix} 0 & n_\mu \\ n_\nu & (h_A)_{\mu\nu} \end{pmatrix}, \\ G^{MN} &= \begin{pmatrix} A^2 - 2v \cdot A & v^\mu - h^{\mu\sigma} A_\sigma \\ v^\nu - h^{\nu\sigma} A_\sigma & h^{\mu\nu} \end{pmatrix} = \begin{pmatrix} \phi_A & v_A^\mu \\ v_A^\mu & h^{\mu\nu} \end{pmatrix}. \end{aligned} \quad (2.6)$$

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<sup>3</sup>Torsionless condition is equivalent to  $dn = 0$ .

The first row and column of this metric correspond to the null direction  $x^-$ , the others to  $x^\mu$ . The metric components are independent from  $x^-$ , and the following null vector generates an isometry:

$$n^M = (1, 0, \dots), \quad n_M = (0, n_\mu). \quad (2.7)$$

We will refer to eq. (2.6) as the discrete light-cone quantization (DLCQ) metric. The 1-form  $n$  is naturally embedded in  $d + 2$  dimensions as  $n = n_A dx^A$ . Different ways to decompose  $G_{\mu\nu}$  as a linear combination of  $h_{\mu\nu}$  and  $A_{(\mu} n_{\nu)}$  are related by a Milne boost transformation.

The Levi-Civita connection from the metric in eq. (2.6) is as follows:

$$\begin{aligned} \Gamma_{--}^- = \Gamma_{--}^\mu = 0, \quad \Gamma_{\alpha-}^- = \frac{1}{2} v_A^\sigma \tilde{F}_{\alpha\sigma}, \quad \Gamma_{\alpha-}^\mu = \frac{1}{2} h^{\mu\sigma} \tilde{F}_{\alpha\sigma}, \\ \Gamma_{\alpha\beta}^- = \frac{1}{2} (\phi_A \tilde{S}_{\alpha\beta} + v_A^\sigma Q_{\alpha\beta\sigma}), \quad \Gamma_{\alpha\beta}^\mu = \frac{1}{2} (v_A^\mu \tilde{S}_{\alpha\beta} + h^{\mu\sigma} Q_{\alpha\beta\sigma}), \end{aligned} \quad (2.8)$$

where we introduced the convenient quantities:

$$\begin{aligned} \tilde{F}_{\alpha\mu} &= \partial_\alpha n_\mu - \partial_\mu n_\alpha, \\ \tilde{S}_{\alpha\beta} &= \partial_\alpha n_\beta + \partial_\beta n_\alpha, \\ Q_{\alpha\beta\sigma} &= (\partial_\alpha (h_A)_{\beta\sigma} + \partial_\beta (h_A)_{\alpha\sigma} - \partial_\sigma (h_A)_{\alpha\beta}). \end{aligned} \quad (2.9)$$

We should not confuse the DLCQ Levi-Civita connection in eq. (2.8) with the NC connection, e.g. [29]. Indeed, the former is torsionless and the other admits a non-vanishing torsion if  $dn \neq 0$ .

Let us denote by  $D_A$  the covariant derivative from the connection (2.8). Also we denote with  $R$ ,  $R_{ABCD}$  and  $R_{AB}$  the scalar curvature, the Riemann and the Ricci tensors associated with the Levi-Civita connection of the metric eq. (2.6).

From the condition that  $n_M$  is a Killing vector for the metric, it follows:

$$0 = \mathcal{L}_n(G_{MN}) = D_M n_N + D_N n_M. \quad (2.10)$$

Using the fact that  $n_A$  is null, one gets also  $D_M n^M = 0$  and  $n^S D_S n_M = 0$ . We can define the 2-form

$$\tilde{F} = \tilde{F}_{AB} dx^A \wedge dx^B = 2dn. \quad (2.11)$$

In components:

$$\tilde{F}_{MN} = \partial_M n_N - \partial_N n_M = 2D_M n_N, \quad \tilde{F}_{-\alpha} = 0. \quad (2.12)$$

### 3 Conformal anomaly ( $z = 2$ )

#### 3.1 Consequences of the causality constraint

The causality condition eq. (2.3) implies that  $dn = n \wedge w$  for some one form  $w$ :

$$\tilde{F}_{AB} = n_{[A} w_{B]}, \quad w_A = (0, w_\alpha), \quad n^A w_A = 0. \quad (3.1)$$

In fact, if  $n \wedge dn = 0$ , it is locally possible to write

$$n = n_A dx^A = e^g \partial_A f dx^A, \quad (3.2)$$

where  $g, f$  are two functions. Then  $\tilde{F} = 2e^g dg \wedge df$ , and so a possible choice is

$$w_A = -2\partial_A g. \quad (3.3)$$

Note that  $w_A$  in eq. (3.1) is not uniquely determined; for example it could be shifted by

$$w_A \rightarrow w_A + p n_A, \quad (3.4)$$

where  $p$  is an arbitrary function (independent from  $x^-$ ) without affecting  $\tilde{F}_{AB}$ . The vector  $w_B$  determines also the covariant derivative of  $n_A$  (see eq. (2.12)).

### 3.2 Classification of tensors

In this section we will show that scalars built with direct contractions of curvature tensors with  $n_A$  vanish; the only terms that we will need to keep track of are the contractions with  $w_A$ , which is the object that parameterize the derivative of  $n_A$ .

We define the following tensors:

$$\begin{aligned} K^A_M &= R^A_{BMN} n^B n^N, & K &= K^A_A = R_{AB} n^A n^B, \\ T^A_{BC} &= R^A_{BCD} n^D, & T_B &= T^A_{BA}. \end{aligned} \quad (3.5)$$

Using eq. (3.1), we get that

$$K_{AB} = \chi n_A n_B, \quad \chi = \frac{1}{16} G^{MN} w_M w_N. \quad (3.6)$$

Then just the  $K^{--}$  component of  $K^{AB}$  is non-zero; moreover  $K^A_B$  is nihilpotent i.e.  $K^A_B K^B_C = 0$  and traceless  $K = K^A_A = 0$ . Also note that  $\chi$  is invariant under the shift in eq. (3.4).

Let us define

$$\Omega_{AB} = \frac{1}{16} (w_A w_B - 4D_A w_B), \quad \Omega = \Omega_{AB} G^{AB}. \quad (3.7)$$

We can use the ambiguity in eq. (3.4) to render  $\Omega_{AB}$  symmetric, using the choice in eq. (3.3). The following property is useful:

$$\Omega_{AB} n^B = \Omega_{BA} n^B = \chi n_A, \quad (3.8)$$

We can write the tensors (3.5) in terms of  $\Omega_{AB}$ :

$$T_{ABC} = \Omega_{CB} n_A - \Omega_{CA} n_B, \quad T_A = R_{AB} n^B = (\chi - \Omega) n_A. \quad (3.9)$$

Due to the fact that  $n^A$  generates an isometry, the following Lie derivatives vanish:

$$\mathcal{L}_n(X) = 0, \quad X = R_{MNPQ}, R_{MN}, R, K_{AB}, T_A, T_{ABC}. \quad (3.10)$$

Let us consider a generic term contained in the anomaly. One can set up a systematic procedure to get rid of the terms which contain an explicit  $n_A$  dependence:

- We can trade any derivative of  $n_A$  with products of  $n_A$ ,  $w_A$  and derivatives of  $w_A$ , by repeated use of eq. (2.12) and (3.1).

- Every time  $n^A$  is contracted with one of the tensors  $X$ , we can use eqs. (3.6)–(3.9) to simplify it to direct product of  $n^A$ ,  $X$ ,  $w^A$  and its derivatives.
- When  $n^A$  is contracted with a derivative  $D_A X$ , we can use eq. (3.10) to trade it with some other direct product of  $n^A$ ,  $w^A$  and  $X$ , using eq. (2.12) and (3.1).
- When  $n^B$  is contracted with a tensor inside a derivative, we can use Leibniz rule:

$$n^B D_A(X_B) = D_A(n^B X_B) - D_A(n^B) X_B, \quad (3.11)$$

to reduce to a contraction with a tensor which does not have an explicit derivative.

At the end of this procedure, we will end up just with terms in which  $n^A$  is only contracted with  $n_A$  or  $w_A$ , which both vanish. For example:

$$T^{ABC} T_{ABC} = -2\Omega^{CA} n^B \Omega_{CB} n_A = -2\chi^2 n^A n_A = 0. \quad (3.12)$$

This argument shows that in order to classify non-vanishing terms in the anomaly, it is sufficient to consider only terms which can be written using the curvatures,  $w^A$  and their derivatives. It is not necessary to consider terms which depend explicitly on  $n^A$  and its derivatives because they vanish. This will be used to show that only a finite number of terms survives out of the infinite family considered in [18].

### 3.3 Weyl transformations

In our notation, the metric transforms as follows under Weyl transformations:

$$n_\mu \rightarrow e^{2\sigma} n_\mu, \quad h_{\mu\nu} \rightarrow e^{2\sigma} h_{\mu\nu}, \quad v^\mu \rightarrow e^{-2\sigma} v^\mu, \quad h^{\mu\nu} \rightarrow e^{-2\sigma} h^{\mu\nu}, \quad (3.13)$$

while the coordinates do not transform.

A Weyl transformation on the Newton-Cartan background is equivalent to a Weyl transformation in the extra-dimensional metric in eq. (2.6) which is independent from the  $x^-$  coordinate:

$$n^A D_A \sigma = 0. \quad (3.14)$$

eq. (3.13) fixes the transformation properties of the metric in eq. (2.6):

$$G_{MN} \rightarrow e^{2\sigma} G_{MN}, \quad G^{MN} \rightarrow e^{-2\sigma} G^{MN}, \quad n^A \rightarrow n^A, \quad n_A \rightarrow e^{2\sigma} n_A. \quad (3.15)$$

Moreover, starting from eq. (3.15) it is possible to derive the Weyl transformation of all the building blocks entering the anomaly, see table 1.

The NC measure  $\sqrt{\det \gamma}$ , where

$$\gamma_{\mu\nu} = n_\mu n_\nu + h_{\mu\nu}, \quad (3.16)$$

is also equal to  $\sqrt{-\det G_{AB}} = \sqrt{-G}$ . The Weyl weight of this measure is equal to  $d + 2$  in  $d$  spatial dimensions. Consequently, in odd spatial dimensions it is not possible to obtain a trace anomaly. We can write anomalies only for even spatial dimension. In the case of  $d = 0$  all the curvatures and tensor such as  $w^A$  are zero and the anomaly is vanishing. We specialize to the minimal dimension  $d = 2$  in which the anomaly survives.

Term	Weyl variation
$\Gamma_{BC}^A$	$\delta_B^A D_C \sigma + \delta_C^A D_B \sigma - G_{BC} D^A \sigma$
$D^2 \phi$	$D^2(\delta \phi) - 2\sigma D^2 \phi + 2D^A \sigma D_A \phi$
$D_A V_B$	$D_A(\delta V_B) - V_A D_B \sigma - V_B D_A \sigma + G_{AB} V^M D_M \sigma$
$R^A{}_{BMN}$	$-\delta_M^A D_B D_N \sigma + \delta_N^A D_B D_M \sigma + G_{BM} D^A D_N \sigma - G_{BN} D^A D_M \sigma$
$R$	$-2\sigma R - 6D^2 \sigma$
$\chi$	$-2\sigma \chi - \frac{1}{2} w^A \partial_A \sigma$
$\Omega$	$-2\sigma \Omega - w^A \partial_A \sigma + D^2 \sigma$
$w_A$	$-4D_A \sigma$
$R_{MN}$	$-2D_M D_N \sigma - G_{MN} D^2 \sigma$
$\Omega_{CB}$	$D_B D_C \sigma - \frac{1}{4} G_{BC} w^K D_K \sigma$

**Table 1.** Weyl variation of the basic fields entering the anomaly. Here  $V_B$  is a generic vector and  $\phi$  a generic scalar.

### 3.4 The anomaly

The anomaly can be written as a linear combination of 16 linearly-independent terms, built from DLCQ tensors:<sup>4</sup>

$$\mathcal{A}_\sigma = \sum_{k=1}^{16} b_k \mathcal{A}_\sigma^k, \quad \mathcal{A}_\sigma^k = \int \sqrt{-G} d^4x (\sigma A_k), \quad (3.17)$$

where  $b_k$  are anomaly coefficients and

$$\begin{aligned} A_1 &= E_4, & A_2 &= W^2, & A_3 &= D^2 R, & A_4 &= R^2, \\ A_5 &= \chi^2, & A_6 &= \Omega^2, & A_7 &= \chi \Omega, & A_8 &= \chi R, \\ A_9 &= \Omega R, & A_{10} &= \Omega_{AB} \Omega^{AB}, & A_{11} &= \Omega_{AB} R^{AB}, & A_{12} &= \Omega_{AB} w^A w^B, \\ A_{13} &= R_{AB} w^A w^B, & A_{14} &= w^A D_A R, & A_{15} &= D^2 \chi, & A_{16} &= D^2 \Omega. \end{aligned} \quad (3.18)$$

In our conventions the Euler density  $E_4$  and the square of Weyl tensor  $W^2$  are:

$$E_4 = R_{ABMN}^2 - 4R_{AB}^2 + R^2, \quad W_{ABMN}^2 = R_{ABMN}^2 - 2R_{AB}^2 + \frac{1}{3}R^2. \quad (3.19)$$

This basis cover the most generic term with the correct Weyl weight; all the other terms with the same Weyl weight can be reduced to these ones using integration by parts.

One may wonder about the explicit form of  $E_4$  in term of 2 + 1 dimensional fields, at least in some simple example. In the special case of flat space metric  $h_{\mu\nu} = \text{Diag}(0, 1, 1)$  and generic  $n_\alpha$  consistent with the condition (2.3),

$$E_4 = 8(6\chi^2 - 4\chi\Omega + \Omega^2 - \Omega_{AB}\Omega^{AB}). \quad (3.20)$$

<sup>4</sup>NC U(1) gauge transformations are included in the DLCQ diffeomorphisms. Consequently, any DLCQ scalar is automatically U(1) gauge invariant. Milne boosts invariance is also automatic as it is related to equivalent ways to decompose  $G_{\mu\nu}$  as a linear combination of  $h_{\mu\nu}$  and  $A_{(\mu}n_{\nu)}$ .

This expression is not valid if a non-zero spatial curvature is present. A general expression for  $E_4$  is given in eq. (B.10).

### 3.5 Consistency conditions

We solve the Wess-Zumino consistency conditions for the Weyl symmetry [43]:

$$\Delta_{\sigma_1\sigma_2}^{WZ}\mathcal{A} = \delta_{\sigma_2}^W\mathcal{A}_{\sigma_1} - \delta_{\sigma_1}^W\mathcal{A}_{\sigma_2} = 0. \quad (3.21)$$

Using integrations by parts and formulas in appendix B, the commutator of the two Weyl variations can be written as a linear combination of several independent expressions  $C_k$ :

$$\Delta_{\sigma_1\sigma_2}^{WZ}\mathcal{A}^k = \int \sqrt{-G} d^4x \left( \sum_{m=1}^8 M^{km} C_m \right), \quad k = 1 \dots 16, \quad (3.22)$$

where

$$\begin{aligned} C_1 &= (\sigma_{[1}D_A\sigma_{2]})D^A R, & C_2 &= (\sigma_{[1}D_A\sigma_{2]})D^A\chi, & C_3 &= (\sigma_{[1}D_A\sigma_{2]})D^A\Omega, \\ C_4 &= (\sigma_{[1}D_A\sigma_{2]})Rw^A, & C_5 &= (\sigma_{[1}D_A\sigma_{2]})\chi w^A, & C_6 &= (\sigma_{[1}D_A\sigma_{2]})\Omega w^A, \\ C_7 &= (\sigma_{[1}D_AD^2\sigma_{2]})w^A, & C_8 &= (\sigma_{[1}D_A\sigma_{2]})D^2w^A. \end{aligned} \quad (3.23)$$

The transpose of the matrix  $M^{km}$  is:

$$(M^t)^{mk} = \begin{pmatrix} 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & -1 & 0 & -\frac{1}{2} & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 6 & 0 & -1 & 1 & 8 & 64 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 6 & 0 & 1 & 0 & -32 & 0 & -2 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -1 & 0 & -\frac{1}{4} & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & -16 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 4 & -8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & -\frac{1}{2} & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & -8 & 0 & 0 & 0 \end{pmatrix}. \quad (3.24)$$

This matrix has 9 null eigenvectors, which correspond to the consistent combination of the various anomaly terms.

On the other hand, not every consistent combination corresponds to a genuine anomaly. In fact we must eliminate terms which are the Weyl variation of local counterterms:

$$\delta^W \left( \sum_{k=1}^{16} \int \sqrt{-G} d^4x \sigma b_k A_k \right) \quad (3.25)$$

because they correspond to scheme-dependent terms. These local counterterms are chosen in such a way that they are invariant under all the NC symmetries: diffeomorphisms, gauge U(1) and Milne boosts. This eliminates 6 out of 9 consistent terms.

## 4 Conclusions

For a  $d = 2$  Schrödinger invariant theory coupled to a NC background, the three genuinely independent terms in the conformal anomaly can be chosen as:

$$\mathcal{A} = a\sigma E_4 + c\sigma W^2 + b\sigma J^2 + \mathcal{A}_{ct}, \quad (4.1)$$

where

$$J = \Omega - 2\chi + \frac{R}{6}. \quad (4.2)$$

Both  $J^2$  and  $W^2$  have vanishing Weyl variation and they correspond to type B anomalies [17].

The scheme-dependent part  $\mathcal{A}_{ct}$  is an arbitrary linear combination of the following terms:

$$\begin{aligned} & \sigma D^2 R, \quad \sigma D^2(\Omega - 2\chi), \quad \sigma \left( 12\chi^2 - 4\chi\Omega - \frac{1}{2}\Omega_{AB}w^A w^B \right), \\ & \sigma \left( 2R\chi - 2R\Omega + \frac{w^A D_A R}{2} - 6D^2\chi \right), \\ & \sigma \left( 2\chi R - 2\Omega R + 4\Omega_{AB}R^{AB} - \frac{R_{AB}w^A w^B}{4} \right), \\ & \sigma \left( -\Omega^2 + 2\chi\Omega + \Omega_{AB}^2 - \frac{\Omega_{AB}w^A w^B}{8} + \frac{R_{AB}w^A w^B}{16} \right). \end{aligned} \quad (4.3)$$

This result is rather different from the Lifshitz case [13–16] where just type B anomalies are present. The analog of the  $E_4$  term in this case can be reabsorbed by a local counterterm. This is due to the fact that the number of admissible counterterms is bigger in the Lifshitz case, because the symmetry content is smaller. It is surprising that the structure of the anomaly in eq. (4.1) is similar to the one in the relativistic case with an extra spatial dimension. The only difference is that an extra type B anomaly  $J^2$  is present, and there are several more scheme-dependent terms. In both cases, the  $E_4$  term is the only anomaly with non-trivial Weyl variation. For general  $z$ , the Weyl weights do not allow the presence of the  $E_4$  term in the anomaly (see appendix A).

The coefficient  $a$  is a natural candidate for a decreasing quantity from the UV to IR conformal fixed point. It will be possible to check this statement in concrete examples, both at weak and at strong coupling. It would be interesting to clarify the relation between the anomaly and Anti-de Sitter radius appearing in the holographic calculation in [11] and [44]. Another exciting direction is to explore the relation between scale invariance and conformal invariance in the non relativistic case [45].

We did not consider the case of non-zero magnetic momentum coupling  $g_s$  where a modified Milne boost transformation is required. In this case, the DLCQ description [23, 24] cannot be used. As a consequence, the classification of the possible terms entering the anomaly looks harder. Nonetheless, the  $E_4$  term written as a function of NC quantities is still Milne boost invariant (see eq. (B.10)) because it does not contain  $A_\mu$ . Therefore, even if  $g_s \neq 0$ , the  $a$ -coefficient in eq. (4.1) is still a candidate for a monotonically decreasing quantity.

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## A Considerations on the case with general dynamical exponent $z \neq 2$

In the case of general dynamical exponent  $z$ , the Weyl transformations in eq. (3.13) are modified into:

$$n_{\mu} \rightarrow e^{z\sigma} n_{\mu}, \quad v^{\mu} \rightarrow e^{-z\sigma} v^{\mu}, \quad A_{\mu} \rightarrow e^{(2-z)\sigma} A_{\mu}, \quad \sqrt{\det \gamma} \rightarrow e^{(d+z)\sigma} \sqrt{\det \gamma}, \quad (\text{A.1})$$

while the ones for  $h^{\alpha\beta}$ ,  $h_{\alpha\beta}$  are unchanged. Since the DLCQ metric components in eq. (2.6) do not transform homogeneously, one may wonder if the Weyl weights of the quadratic extra-dimensional curvature invariants are still well-defined.

Given a contravariant vector  $V^A$ , there is the following relation between the Weyl weight of the  $V^{\alpha}$  and of the  $V^{-}$  components:

$$V^{\alpha} \rightarrow e^{x\sigma} V^{\alpha}, \quad V^{-} \rightarrow e^{(x+2-z)\sigma} V^{-}. \quad (\text{A.2})$$

Conversely, for covariant vectors:

$$W_{\alpha} \rightarrow e^{y\sigma} W_{\alpha}, \quad W_{-} \rightarrow e^{(y-2+z)\sigma} W_{-}. \quad (\text{A.3})$$

Note that the weight of the  $V^{-}W_{-}$  contraction is the same as the one of  $V^{\alpha}W_{\alpha}$  and does not depend on  $z$ . The same rules can be iterated for tensors, for example:

$$G^{\alpha\beta} \rightarrow e^{-2\sigma} G^{\alpha\beta}, \quad G^{\alpha-} \rightarrow e^{-z\sigma} G^{\alpha-}, \quad G^{--} \rightarrow e^{(2-2z)\sigma} G^{--}. \quad (\text{A.4})$$

Also, these rules work for the vectors  $w^A$  and  $n^A$  and are consistent with the lowering and raising index procedure. Using the Levi-Civita connection eq. (2.8), it can be shown that if these rule are valid for a tensor, they are valid also for its derivatives. Using the definition of the Riemann tensor in term of commutator of covariant derivatives, it follows that they apply also to the Riemann tensor.

This shows that for every tensor that we can build from the curvatures,  $n^A$ ,  $w^A$  and their derivatives, the rules in eqs. (A.2) and (A.3) can be used to determine the weight of some components in which some minus index is present in term of the components with greek indices. Moreover, every scalar defined contracting these tensors has a definite weight, which can be determined looking at the weight of the part where just greek indices are contracted. This shows that the Weyl weight of terms built from curvature and  $w_A$  and its derivatives are well-defined and independent of  $z$ .

Looking at table 1, all the ingredients that can enter the anomaly have even Weyl weight; then, taking the weight of the measure  $\sqrt{\gamma}$  into account, a non-zero trace anomaly exists just for

$$z = 2n - d, \quad (\text{A.5})$$

where  $n$  is a positive integer.

## B Useful formulas

The following identities are useful in manipulating the anomaly and its Weyl variation:

$$D_A w^A = 4(\chi - \Omega), \quad D_A w_B = \frac{1}{4} w_A w_B - 4\Omega_{AB}, \quad (\text{B.1})$$

$$\Omega^{AB} w_B = w^A \chi - 2D^A \chi, \quad w^A D_A \chi = 8\chi^2 - \frac{1}{2} \Omega_{AB} w^A w^B, \quad (\text{B.2})$$

$$16D^2 \chi = 2 \left( \frac{w^A w_B}{4} - 4\Omega_{AB} \right)^2 + 2(D^2 w_A) w^A, \quad (\text{B.3})$$

$$(D^2 w_A) w^A = 8D^2 \chi - 16\chi^2 - 16\Omega_{AB}^2 + 2\Omega_{AB} w^A w^B, \quad (\text{B.4})$$

$$w^B D^A \Omega_{AB} = 12\chi^2 - 4\chi\Omega - 2D^2 \chi - \frac{3}{4} \Omega_{AB} w^A w^B + 4\Omega_{AB}^2, \quad (\text{B.5})$$

$$D^A D^B \Omega_{AB} = \Omega^2 - 2\chi\Omega - \Omega_{AB}^2 + D^2 \Omega + R^{AB} \Omega_{AB} \\ + \frac{1}{8} (\Omega_{AB} w^A w^B - R_{AB} w^A w^B - w^A D_A R), \quad (\text{B.6})$$

$$w^A D_A \Omega = 12\chi^2 + 4\Omega_{AB}^2 - \Omega_{AB} w^A w^B - 2D^2 \chi + \frac{1}{4} R_{AB} w^A w^B, \quad (\text{B.7})$$

$$D^2 D_A \sigma = D_A D^2 \sigma + R_{BA} D^B \sigma, \quad (\text{B.8})$$

$$D^2 w^A = D_A (D_B w^B) + R_{AB} w^B. \quad (\text{B.9})$$

An explicit expression for  $E_4$  in terms of NC fields is:

$$E_4 = 8(\Omega - 2\chi)^2 - 8h^{\alpha\mu} h^{\beta\nu} \Omega_{\alpha\beta} \Omega_{\mu\nu} + h^{\alpha\mu} h^{\beta\nu} h^{\rho\xi} h^{\sigma\eta} \tilde{R}_{\alpha\rho\beta\sigma} \tilde{R}_{\mu\xi\nu\eta} - 4h^{\alpha\mu} h^{\beta\nu} h^{\rho\sigma} h^{\xi\eta} \tilde{R}_{\alpha\rho\beta\sigma} \tilde{R}_{\mu\xi\nu\eta} \\ + 16h^{\alpha\mu} h^{\beta\nu} h^{\rho\sigma} \tilde{R}_{\alpha\rho\beta\sigma} \Omega_{\mu\nu} + \left( \tilde{R}_{\alpha\rho\beta\sigma} h^{\alpha\beta} h^{\rho\sigma} \right)^2 + 2(6\chi - 4\Omega) \tilde{R}_{\alpha\rho\beta\sigma} h^{\alpha\beta} h^{\rho\sigma} \quad (\text{B.10})$$

where

$$\tilde{R}_{\alpha\rho\beta\sigma} = -\frac{1}{2} \partial_\alpha \partial_\beta h_{\rho\sigma} - \frac{1}{2} \partial_\rho \partial_\sigma h_{\alpha\beta} + \frac{1}{2} \partial_\alpha \partial_\sigma h_{\rho\beta} + \frac{1}{2} \partial_\rho \partial_\beta h_{\alpha\sigma} \\ - \frac{1}{4} h^{\tau_1 \tau_2} (H_{\alpha\beta\tau_1} H_{\rho\sigma\tau_2} - H_{\alpha\sigma\tau_1} H_{\rho\beta\tau_2}) \\ - \frac{1}{4} v^\tau \left( \tilde{S}_{\alpha\beta} H_{\rho\sigma\tau} + \tilde{S}_{\rho\sigma} H_{\alpha\beta\tau} - \tilde{S}_{\alpha\sigma} H_{\rho\beta\tau} - \tilde{S}_{\rho\beta} H_{\alpha\sigma\tau} \right). \quad (\text{B.11})$$

and

$$H_{\alpha\beta\sigma} = (\partial_\alpha (h)_{\beta\sigma} + \partial_\beta (h)_{\alpha\sigma} - \partial_\sigma (h)_{\alpha\beta}). \quad (\text{B.12})$$

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