Topology and its Applications 284 (2020) 107361

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Numerical irreducibility criteria for handlebody links

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A R T I C L E   I N F O

Article history:
Received 5 April 2020
Accepted 11 August 2020
Available online 17 August 2020

MSC:
57M25
57M27
57M05

Keywords:
Reducibility
Handlebody links
Knot sum

A B S T R A C T

In the paper we introduce new criteria for a handlebody link to be irreducible. These criteria are numerical and can be computed by a code. They are able to detect the irreducibility of all handlebody knots in the Ishii-Kishimoto-Moriuchi-Suzuki knot table and most handlebody links in the link table produced by G. Paolini and the authors. We also prove the existence of irreducible handlebody links of any given type.

1. Introduction

A handlebody link HL is a union of finitely many handlebodies of positive genus embedded in the 3-sphere $\mathbb{S}^3$; two handlebody links are equivalent if they are ambient isotopic [13], [3]. Throughout the paper handlebody links are non-split unless otherwise specified.

A handlebody link HL is reducible if there exists a cutting 2-sphere $\mathfrak{S}$ in $\mathbb{S}^3$ such that $\mathfrak{S}$ and HL intersect at an incompressible disk D in HL (Fig. 1.1); otherwise it is irreducible. Note that a cutting sphere $\mathfrak{S}$ of a reducible handlebody link HL factorizes it into two handlebody links $\mathrm{HL}_1, \mathrm{HL}_2$, where $\mathrm{HL}_i := \mathrm{HL} \cap B_i$, and B_i, $i = 1, 2$, are the closures of components of the complement $\mathbb{S}^3 \setminus \mathfrak{S}$ (Fig. 1.1); the factorization is denoted by

$$\mathrm{HL} = (\mathrm{HL}_1, h_1)–(\mathrm{HL}_2, h_2), \tag{1.1}$$

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https://doi.org/10.1016/j.topol.2020.107361
0166-8641/

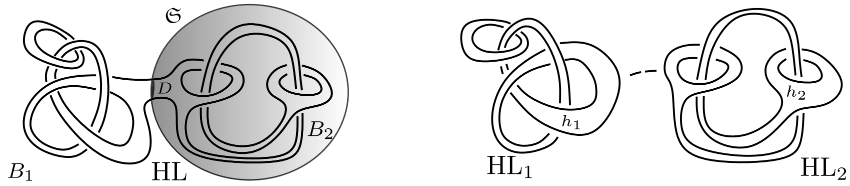


Fig. 1.1. A reducible handlebody link and its factors.

and HL_i, $i = 1, 2$, is called a factor of the factorization, where h_1, h_2 are components of $\mathrm{HL}_1, \mathrm{HL}_2$ containing D, respectively.

Handlebody links are often studied via diagrams of their spines [3], but it is, in general, difficult to detect the irreducibility from their diagrams—the complexity being in the IH-move [3]. As a result, the task is usually left to irreducibility criteria. [4] develops several quandle-invariant-based criteria for handlebody knots, which are used to build the knot table in [5]; some geometric criteria using knotted handle decomposition are discussed in [6]. The present paper concerns numerical criteria for handlebody *links* that employ homomorphisms of the knot group, the fundamental group of a handlebody link complement, on a finite group.

We denote by $\mathrm{ks}_G(\mathrm{HL})$ the number of conjugacy classes of homomorphisms from the knot group G_{HL} to a finite group G, where two homomorphisms are in the same conjugacy class if they are conjugate [7]. And, we say a handlebody link HL is of type $[n_1, n_2, ..., n_m]$ if it consists of n_i handlebodies of genus i, $i = 1, \dots, m$; a handlebody link is r-generator if its knot group is of rank r. Note that r is necessarily larger than or equal to the genus $g(\mathrm{HL})$ of HL, which is the sum $\sum_{i=1}^{m} i \cdot n_i$ of the genera of components of HL.

Let A_4, A_5 be the alternating groups of degree $4, 5$, respectively. The main results of the paper are the following.

Theorem 1.1 ***(Necessary conditions for reducibility**–A_4). Let* HL *be a reducible handlebody link of genus g. If the trivial knot is a factor of some factorization of* HL*, then*

$$12 \mid \mathrm{ks}_{A_4}(\mathrm{HL}) + 6 \cdot 3^{g-1} + 2 \cdot 4^{g-1}; \tag{1.2}$$

if a 2-generator knot is a factor of some factorization of HL*, then*

$$12 + 24k \mid \mathrm{ks}_{A_4}(\mathrm{HL}) + (6 + 16k) \cdot 3^{g-1} + (2 + 6k) \cdot 4^{g-1}, k = 0 \text{ or } 1; \tag{1.3}$$

if a 2-generator link is a factor of some factorization of HL*, then*

$$48 + 24k \mid \mathrm{ks}_{A_4}(\mathrm{HL}) + (26 + 16k) \cdot 3^{g-1} + (8 + 6k) \cdot 4^{g-1}, k = 0, 1, 2, 3 \text{ or } 4. \tag{1.4}$$

Theorem 1.2 ***(Necessary conditions for reducibility**–A_5). Let* HL *be a reducible handlebody link of genus g. If the trivial knot is a factor of some factorization of* HL*, then*

$$60 \mid \mathrm{ks}_{A_5}(\mathrm{HL}) + 14 \cdot 4^{g-1} + 19 \cdot 3^{g-1} + 22 \cdot 5^{g-1}. \tag{1.5}$$

From these necessary conditions we derive an irreducibility test for handlebody knots of genus up to 3 and handlebody links of various types.

Corollary 1.3. *Given an r-generator handlebody knot* HK *of genus g, if $r = g + 1$ and* HK *fails to satisfy either* (1.2) *or* (1.5)*, then* HK *is irreducible; if $r = g+2$ and* HK *fails to satisfy* (1.3)*, then* HK *is irreducible.*

Table 1
Irreducibility test for handlebody links with more than one component.

no. of components	type	$r = g$	$r = g + 1$
		HL is irreducible if it fails criterion/criteria	
2	$[1, 1]$	(1.2) or (1.5)	(1.3)
	$[0, 2]$	(1.2) or (1.5)	(1.3)
	$[1, 0, 1]$	(1.2) or (1.5)	(1.3) & (1.4)
	$[0, 1, 1]$	(1.2) or (1.5)	not applicable
	$[1, 0, 0, 1]$	(1.2) or (1.5)	not applicable
3	$[2, 1]$	(1.2) & (1.4)	(1.3) & (1.4)
	$[1, 2]$	(1.2) & (1.4)	not applicable
	$[2, 0, 1]$	(1.2) & (1.4)	
4	$[3, 1]$	(1.2) & (1.4)	not applicable

The situation with multi-component handlebody links is more complicated as there are more possible combinations, so we summarize it in a tabular format in Table 1, which is also a corollary of Theorems 1.1 and 1.2. The right two columns in Table 1 list criteria which if a handlebody link fails, it is irreducible. Be aware of "& (i.e. and)" and "*or*" in those two columns.

The set of irreducibility criteria is put to test in Section 4, and its effectiveness is evidenced by results in Tables 2 and 3, which show that it detects the irreducibility of all handlebody knots, which are of type $[0, 1]$, in the Ishii-Kishimoto-Moruichi-Suzuki knot table [5], and the irreducibility of all handlebody links, which are of type $[1, 1], [2, 1]$ or $[3, 1]$, in the handlebody link table in [1] except for 6_9 and 6_{12}.

The major constraint of the irreducibility test is that the rank of the knot group G_{HL} cannot be too large and the difference between the rank and the genus $g(\mathrm{HL})$ needs to be small. Nevertheless, they are applicable to a large number of interesting cases, and can be computed easily with a code.

The paper is organized as follows: Section 2 recalls basic properties of handlebody links and knot groups. The necessary conditions for reducibility, Theorems 1.1 and 1.2, are proved in Section 3. Section 4 records results of the irreducibility test applying to various families of handlebody links. Lastly, the existence of irreducible handlebody links of any given type is proved by a concrete construction making use of a generalized knot sum for handlebody links.

2. Preliminaries

Throughout the paper we work in the piecewise linear category. We use HL to refer to a handlebody link (including a handlebody knot), and use HK, K or L when referring specifically to a handlebody knot, knot or link, respectively. $G_\bullet$ denotes the knot group of $\bullet = \mathrm{HL}, \mathrm{HK}, K$ or L, and $\simeq$ stands for an isomorphism of groups. We start with some basic properties of reducible handlebody links and of free product of groups.

Definition 2.1. The rank $\mathrm{rk}(G)$ of a finitely generated group G is the smallest cardinality of a generating set of G.

Definition 2.2. A handlebody link is r-generator if its knot group is of rank r.

The rank respects the free product of groups [2]:

Lemma 2.1 ***(Grushko theorem)***. *If $G = G_1 * G_2$, then*

$$\mathrm{rk}(G) = \mathrm{rk}(G_1) + \mathrm{rk}(G_2).$$

Lemma 2.2. *A g-generator handlebody knot* HK *of genus g is trivial.*

Proof. By the exact sequence of group homology [11], the deficiency d of the knot group of HK is at most g; on the other hand, the Wirtinger presentation induces a presentation with deficiency g, so we have $d = g$. By [8, Satz 1], [12], the knot group is free, and therefore HK is trivial. □

The following corollaries are a consequence of Lemmas 2.1 and 2.2 and the fact that $\mathrm{HL} = (\mathrm{HL}_1, h_1)$–$(\mathrm{HL}_2, h_2)$ implies then $g(\mathrm{HL}) = g(\mathrm{HL}_1) + g(\mathrm{HL}_2)$. These corollaries, together with Theorems 1.1 and 1.2, give Corollary 1.3 and Table 1.

Corollary 2.3. *A $(g+1)$-generator handlebody knot* HK *of genus $g = 2, 3$ is reducible if and only if the trivial knot is a factor of some factorization of* HK*.*

Corollary 2.4. *A 2-component, g-generator handlebody link* HL *of genus $g \leq 5$ is reducible if and only if the trivial knot is a factor of some factorization of* HL*.*

Corollary 2.5. *A genus g, $(g+1)$-generator handlebody link* HL *of type $[1, 1]$ or $[0, 2]$ is reducible if and only if the trivial knot or a 2-generator knot is a factor of some factorization of* HL*.*

Corollary 2.6. *A 3- or 4-component, g-generator handlebody link* HL *of genus $g \leq 5$ is reducible if and only if the trivial knot or a 2-generator link is a factor of some factorization of* HL*.*

Corollary 2.7. *A 5-generator handlebody link* HL *of type $[1, 0, 1]$ or $[2, 1]$ is reducible if and only if the trivial knot, 2-generator knot, or 2-generator link is a factor of some factorization of* HL*.*

3. Irreducibility test

3.1. Homomorphisms to a finite group

Definition 3.1. Given a handlebody link HL and a finite group G, $\mathrm{ks}_G(\mathrm{HL})$ is the number of conjugacy classes of homomorphism from G_{HL} to G, $\mathrm{ks}^G_H(\mathrm{HL})$ is the number of conjugacy classes of homomorphisms from G_{HL} to a subgroup of G isomorphic to H, and $\mathrm{ks}^w_G(\mathrm{HL})$ is the number of homomorphisms from G_{HL} to G.

Lemma 3.1. *Suppose any subgroup of G either has trivial centralizer or is abelian, and any two maximal abelian subgroups of G have trivial intersection. Let H_i, $i = 1, \dots, n$, be isomorphism types of maximal abelian subgroups of G, and l_i be the number of maximum abelian subgroups isomorphic to H_i. Then for any handlebody link* HL*, $\mathrm{ks}_G(\mathrm{HL})$ can be expressed in terms of $\mathrm{ks}^w_G(\mathrm{HL})$ and $\mathrm{ks}^G_{H_i}(\mathrm{HL})$:*

$$\mathrm{ks}_G(\mathrm{HL}) = \mathrm{ks}^G_{H_1}(\mathrm{HL}) + \cdots + \mathrm{ks}^G_{H_n}(\mathrm{HL}) - n + 1 + \frac{\mathrm{ks}^w_G(\mathrm{HL}) - l_1(\mathrm{ks}^w_{H_1}(\mathrm{HL}) - 1) - \cdots - l_n(\mathrm{ks}^w_{H_n}(\mathrm{HL}) - 1) - 1}{|G|}. \quad (3.1)$$

Proof. The difference

$$\mathrm{ks}_G(\mathrm{HL}) - \left(\mathrm{ks}^G_{H_1}(\mathrm{HL}) + \cdots + \mathrm{ks}^G_{H_n}(\mathrm{HL}) - n + 1\right) \quad (3.2)$$

is the number of conjugacy classes of homomorphisms $G_{\mathrm{HL}} \to G$ whose images have trivial centralizers. For such a homomorphism ϕ, we have

$$\phi \neq g \cdot \phi \cdot g^{-1}, \quad \text{for any non-trivial element } g \in G;$$

thus the conjugacy class of ϕ contains $|G|$ members. Now, since the intersection of any two maximal abelian subgroups is trivial, the difference

$$\mathrm{ks}^w_G(\mathrm{HL}) - l_1(\mathrm{ks}^w_{H_1}(\mathrm{HL}) - 1) - \cdots - l_n(\mathrm{ks}^w_{H_n}(\mathrm{HL}) - 1) - 1 \tag{3.3}$$

is the number of homomorphisms $G_{\mathrm{HL}} \to G$ whose images have trivial centralizers. Therefore dividing (3.3) by $|G|$ gives us (3.2), that is,

$$\frac{\mathrm{ks}^w_G(\mathrm{HL}) - l_1(\mathrm{ks}^w_{H_1}(\mathrm{HL}) - 1) - \cdots - l_n(\mathrm{ks}^w_{H_n}(\mathrm{HL}) - 1) - 1}{|G|} = \mathrm{ks}_G(\mathrm{HL}) - \left(\mathrm{ks}^G_{H_1}(\mathrm{HL}) + \cdots + \mathrm{ks}^G_{H_n}(\mathrm{HL}) - n + 1\right).$$

This proves the formula (3.1). □

It is not difficult to check that A_4, A_5 satisfy conditions in Lemma 3.1, whence we derive the following formulas.

Corollary 3.2. *Let $\mathbb{Z}_n$ be the cyclic group of order n, and $V_4 \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then*

$$\mathrm{ks}_{A_4}(\mathrm{HL}) = \mathrm{ks}^{A_4}_{V_4}(\mathrm{HL}) + \mathrm{ks}^{A_4}_{\mathbb{Z}_3}(\mathrm{HL}) - 1 + \frac{\mathrm{ks}^w_{A_4}(\mathrm{HL}) - 4(\mathrm{ks}^w_{\mathbb{Z}_3}(\mathrm{HL}) - 1) - \mathrm{ks}^w_{V_4}(\mathrm{HL})}{12} \tag{3.4}$$

$$\begin{aligned}\mathrm{ks}_{A_5}(\mathrm{HL}) = {}& \mathrm{ks}^{A_5}_{V_4}(\mathrm{HL}) + \mathrm{ks}^{A_5}_{\mathbb{Z}_3}(\mathrm{HL}) + \mathrm{ks}^{A_5}_{\mathbb{Z}_5}(\mathrm{HL}) - 2 \\ &+ \frac{\mathrm{ks}^w_{A_5}(\mathrm{HL}) - 10(\mathrm{ks}^w_{\mathbb{Z}_3}(\mathrm{HL}) - 1) - 5(\mathrm{ks}^w_{V_4}(\mathrm{HL}) - 1) - 6(\mathrm{ks}^w_{\mathbb{Z}_5}(\mathrm{HL}) - 1) - 1}{60}.\end{aligned} \tag{3.5}$$

If any two subgroups of G isomorphic to H are conjugate, then given an injective homomorphism $H \xrightarrow{\iota} G$, the number n_H of conjugacy classes of elements in G representable by elements in $\iota(H)$ is independent of ι. If furthermore $\iota(H)$ is the centralizer of every non-trivial element $h \in \iota(H)$, then $\mathrm{ks}^w_H(\mathrm{HL}), \mathrm{ks}^G_H(\mathrm{HL})$ can be computed explicitly as follows.

Lemma 3.3. *Given an abelian group H, and an injective homomorphism $\iota : H \to G$, suppose subgroups of G isomorphic to H are all conjugate, and the centralizer of every non-trivial element $h \in \iota(H)$ is $\iota(H)$. If $g(\mathrm{HL}) = g$, then*

$$\mathrm{ks}^w_H(\mathrm{HL}) = |H|^g \quad \textit{and} \quad \mathrm{ks}^G_H(\mathrm{HL}) = (n_H - 1) \cdot \frac{|H|^g - |H|}{|H| - 1} + n_H.$$

Proof. Firstly, since H is abelian, any homomorphism from G_{HL} to H factors through the abelianization of G_{HL}, which is the free abelian group $\mathbb{Z}^g$ of rank g. Particularly, $\mathrm{ks}^w_H(\mathrm{HL})$ (resp. $\mathrm{ks}^G_H(\mathrm{HL})$) is equal to the numbers (resp. of conjugacy classes) of homomorphisms from $\mathbb{Z}^g$ to H (resp. to $\iota(H)$ in G). This implies the first identity.

For the second identity, we let

$$\mathrm{ks}^G_H(\mathrm{HL}) = l_g$$

and $\mathrm{id}, h_2, \ldots, h_{n_H} \in \iota(H) < G$ be selected representatives of the n_H conjugacy classes of elements in G. Note that if $g = 1$, we have $l_1 = n_H$.

For $g > 1$, up to conjugation, we may assume the g-th copy of $\mathbb{Z}^g$ is sent to $h \in \{\mathrm{id}, h_2 \ldots, h_{n_H}\}$. There are l_{g-1} homomorphisms when $h = \mathrm{id}$, and $|H|^{g-1}$ homomorphisms when $h = h_i, i = 2, \ldots, n_H$, because the centralizer of h_i is $\iota(H)$. As a result, we obtain the recursive formula

$$l_g = l_{g-1} + (n_H - 1) \cdot |H|^{g-1},$$

and hence

$$l_g - l_1 = \sum_{k=2}^{g}(l_k - l_{k-1}) = \sum_{k=2}^{g}(n_H - 1) \cdot |H|^{k-1} = (n_H - 1) \cdot \frac{|H|^g - |H|}{|H| - 1}. \tag{3.6}$$

This implies the second equality after we substitute $l_1 = n_H$ into (3.6). □

Maximal abelian subgroups of A_4, A_5 satisfy conditions assumed in Lemma 3.3, and hence we have the formulas:

$$\mathrm{ks}^w_{\mathbb{Z}_3}(\mathrm{HL}) = 3^g; \quad \mathrm{ks}^w_{V_4}(\mathrm{HL}) = 4^g; \quad \mathrm{ks}^w_{\mathbb{Z}_5}(\mathrm{HL}) = 5^g, \tag{3.7}$$

$$\mathrm{ks}^{A_4}_{\mathbb{Z}_3}(\mathrm{HL}) = 3^g; \quad \mathrm{ks}^{A_4}_{V_4}(\mathrm{HL}) = \frac{4^g - 4}{3} + 2, \tag{3.8}$$

$$\mathrm{ks}^{A_5}_{\mathbb{Z}_3}(\mathrm{HL}) = \frac{3^g - 3}{2} + 2; \quad \mathrm{ks}^{A_5}_{V_4}(\mathrm{HL}) = \frac{4^g - 4}{3} + 2, \quad \mathrm{ks}^{A_5}_{\mathbb{Z}_5}(\mathrm{HL}) = \frac{5^g - 5}{2} + 3. \tag{3.9}$$

Plugging (3.7), (3.8) into (3.4) and (3.7), (3.9) into (3.5) gives us the following

Corollary 3.4. *For a genus g handlebody link* HL, *we have*

$$\mathrm{ks}^w_{A_4}(\mathrm{HL}) = 12 \cdot \mathrm{ks}_{A_4}(\mathrm{HL}) - 8 \cdot 3^g - 3 \cdot 4^g$$
$$\mathrm{ks}^w_{A_5}(\mathrm{HL}) = 60 \cdot \mathrm{ks}_{A_5}(\mathrm{HL}) - 20 \cdot 3^g - 15 \cdot 4^g - 24 \cdot 5^g.$$

For the sake of convenience, we let $\mathbf{ks}_G(G')$ denote the set of conjugacy classes of homomorphisms from G' to G; especially, $\mathrm{ks}_G(\mathrm{HL}) = |\mathbf{ks}_G(G_{\mathrm{HL}})|$.

Lemma 3.5. *For a 2-generator knot K, $\mathrm{ks}_{A_4}(K) = 4$ or 6. In each case, $\mathbf{ks}_{A_4}(G_K)$ contains four conjugacy classes represented by homomorphisms whose images are abelian. If $\mathrm{ks}_{A_4}(K) = 6$, the two additional conjugacy classes are represented by surjective homomorphisms.*

Proof. Since any non-surjective homomorphism $\phi : G_K \to A_4$ factors through the abelianization of G_K, $\mathrm{Im}(\phi)$ is either trivial or isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$. By (3.8), the number of conjugacy classes of non-surjective homomorphisms are

$$\mathrm{ks}^{A_4}_{V_4}(K) + \mathrm{ks}^{A_4}_{\mathbb{Z}_3}(K) - 1 = 3 + 2 - 1 = 4,$$

and hence $\mathrm{ks}_{A_4}(K) \geq 4$.

Now, consider a two-generator presentation of G_K:

$$< a, b \mid w(a, b) = 1 > \tag{3.10}$$

and its abelianization:

$$G_K \xrightarrow{\pi} G_K/[G_K, G_K] \simeq \mathbb{Z} =< g >; \tag{3.11}$$

let $g^{3n+l}, g^{3n'+l'}$ be the image of a, b under (3.11), respectively. Suppose both l and l' are non-zero, then either $3 \mid l' - l$ or $3 \mid l' - 2l$. If $3 \mid l' - l$, we replace b with b' by $b' = a^{-1}b$; this implies a new presentation of G_K:

$$G_K =< a, b' \mid w'(a, b') = 1 >,$$

where $w'(a, b') = w(a, ab')$, and b' vanishes under the composition

$$G_K \xrightarrow{\pi} G_K/[G_K, G_K] \simeq \mathbb{Z} \xrightarrow{\pm} \mathbb{Z}_3 \simeq A_4/[A_4, A_4].$$

Similarly, if $3 \mid 2l - l'$, we replace b with b'' by $b'' = a^{-2}b$ to get a new presentation

$$G_K =< a, b'' \mid w''(a, b'') = 1 >,$$

where $w''(a, b'') = w(a, a^2 b'')$, and b'' vanishes under the composition

$$G_K \xrightarrow{\pi} G_K/[G_K, G_K] \simeq \mathbb{Z} \xrightarrow{\pm} \mathbb{Z}_3 \simeq A_4/[A_4, A_4].$$

Therefore, given a surjective homomorphism ϕ, we may assume $\phi(b)$ in (3.10) is in the commutator of A_4 and hence of order 2, and $\phi(a)$ is of order 3. Up to conjugation, there are only two such homomorphisms: one corresponds to $\phi(a) = (123)$, the other $\phi(a) = (132)$; note that every two elements of order 2 in A_4 are conjugate with respect to (123) or (132). This shows there are at most two surjective homomorphisms from G_K to A_4, and they always appear in pairs because there exists an automorphism of A_4 sending (123) to (132), namely

$$\begin{aligned} \Phi_{(23)} : A_4 &\to A_4 \\ x &\mapsto (23)x(23). \quad \square \end{aligned} \tag{3.12}$$

Lemma 3.6. *If L is a 2-generator link, then $\mathrm{ks}_{A_4}(L)$ is 14, 16, 18, 20 or 22. In each case, $\mathbf{ks}_{A_4}(G_L)$ contains 14 elements represented by homomorphisms whose images are abelian. If $\mathrm{ks}_{A_4}(L) > 14$, then any additional conjugacy class is represented by surjective homomorphisms.*

Proof. Suppose $\phi : G_L \to A_4$ is non-surjective, then it factors through the abelianization of G_L, so by (3.8), the number of conjugacy classes of non-surjective homomorphisms can be computed by

$$\mathrm{ks}_{V_4}^{A_4}(K) + \mathrm{ks}_{\mathbb{Z}_3}^{A_4}(K) - 1 = 9 + 6 - 1 = 14$$

and particularly $\mathrm{ks}_{A_4}(L) \geq 14$.

Suppose $\phi : G_L \to A_4$ is onto, and

$$< a, b \mid w(a, b) = 1 >$$

is a presentation of G_L. Then either both $\phi(a)$ and $\phi(b)$ are of order 3 or one of them is of order 3 and the other of order 2. In the former case, up to conjugation, there are four possibilities:

$$\begin{aligned} \mathrm{I} : \phi(a) &= (123), & \phi(b) &= (124); \\ \mathrm{II} : \phi(a) &= (123), & \phi(b) &= (142); \\ \mathrm{III} : \phi(a) &= (132), & \phi(b) &= (124); \\ \mathrm{IV} : \phi(a) &= (132), & \phi(b) &= (142). \end{aligned}$$

By (3.12) $w(\phi(a), 124) = 1$ if and only if $w(\Phi_{(23)}\big(\phi(a)\big), (142)) = 1$ since

$$w(\Phi_{(23)}(\phi(a)), (142)) = \Phi_{(23)}\big(w(\phi(a), (143))\big) = \Phi_{(23)}\Big((123)w\big(\phi(a), (124)\big)(132)\Big).$$

Therefore, I and IV appear in pair; so do II and IV, for a similar reason. Now, if one of $\phi(a)$ and $\phi(b)$ is of order 2, we also have four possibilities:

$$\begin{aligned}
\mathrm{I}' &: \phi(a) = (123), && \phi(b) = (12)(34);\\
\mathrm{II}' &: \phi(a) = (132), && \phi(b) = (12)(34);\\
\mathrm{III}' &: \phi(a) = (12)(34), && \phi(b) = (123);\\
\mathrm{IV}' &: \phi(a) = (12)(34), && \phi(b) = (132).
\end{aligned}$$

They appear in pairs as in the previous case. Thus, $\mathrm{ks}_{A_4}(L)$ is an even integer between 14 and 22. □

3.2. Necessary conditions for reducibility

We divide the proof of Theorems 1.1 and 1.2 into three lemmas.

Lemma 3.7. *Given a reducible handlebody link* HL *of genus g, if the trivial knot is a factor of some factorization of* HL*, then*

$$12 \mid \mathrm{ks}_{A_4}(\mathrm{HL}) + 6\cdot 3^{g-1} + 2\cdot 4^{g-1} \quad \textit{and} \quad 60 \mid \mathrm{ks}_{A_5}(\mathrm{HL}) + 14\cdot 4^{g-1} + 19\cdot 3^{g-1} + 22\cdot 5^{g-1}.$$

Proof. By the assumption, the knot group G_{HL} is isomorphic to the free product $\mathbb{Z} * G_{\mathrm{HL}'}$, where HL' is a handlebody link of genus $g-1$.

Recall that $\mathbf{ks}_{A_4}(\mathbb{Z})$ contains four elements by (3.8). Let $\phi_1, \phi_2, \phi_3^1, \phi_3^2$ be homomorphisms representing these four conjugacy classes with $\mathrm{Im}(\phi_1)$ trivial, $\mathrm{Im}(\phi_2)$ isomorphic to $\mathbb{Z}_2$, and $\mathrm{Im}(\phi_3^i), i = 1, 2$, isomorphic to $\mathbb{Z}_3$. Then observe that, given a homomorphism $\phi : G_{\mathrm{HL}} \to A_4$, by conjugating with some element in A_4, we may assume its restriction $\phi|_{\mathbb{Z}}$ is one of $\{\phi_1, \phi_2, \phi_3^1, \phi_3^2\}$.

Case 1: $\phi|_{\mathbb{Z}} = \phi_1$**.** Let $\phi, \psi : G_{\mathrm{HL}} \to A_4$ be two homomorphisms with

$$\phi|_{\mathbb{Z}} = \psi|_{\mathbb{Z}} = \phi_1.$$

Then they are in the same conjugacy class if and only if their restrictions $\phi|_{G_{\mathrm{HL}'}}, \psi|_{G_{\mathrm{HL}'}}$ are conjugate, so there are $\mathrm{ks}_{A_4}(\mathrm{HL}')$ conjugacy classes in this case.

Case 2: $\phi|_{\mathbb{Z}_2} = \phi_2$**.** Let $\phi, \psi : G_{\mathrm{HL}} \to A_4$ be two homomorphisms with

$$\phi|_{\mathbb{Z}} = \psi|_{\mathbb{Z}} = \phi_2.$$

Then they are in the same conjugacy class if and only if

$$\phi|_{G_{\mathrm{HL}'}} = g\cdot\psi|_{G_{\mathrm{HL}'}}\cdot g^{-1}, \text{ for some } g \in V_4.$$

Hence in case 2, the number of conjugacy classes is

$$\frac{\mathrm{ks}^w_{A_4}(\mathrm{HL}') - \mathrm{ks}^w_{V_4}(\mathrm{HL}')}{4} + \mathrm{ks}^w_{V_4}(\mathrm{HL}').$$

Case 3: $\phi|_{\mathbb{Z}} = \phi_3^i$, $i = 1$ or 2. Let $\phi, \psi : G_{\mathrm{HL}} \to A_4$ be two homomorphisms with

$$\phi|_{\mathbb{Z}} = \psi|_{\mathbb{Z}} = \phi_3^i, i = 1(\text{resp. } 2).$$

Then they are in the same conjugacy class if and only if

$$\phi|_{G_{\mathrm{HL}'}} = g \cdot \psi|_{G_{\mathrm{HL}'}} \cdot g^{-1}, \text{ for some } g \in \mathrm{Im}(\phi_3^i), i = 1(\text{resp. } 2),$$

and therefore for each i, there are

$$\frac{\mathrm{ks}^w_{A_4}(\mathrm{HL}') - \mathrm{ks}^w_{\mathbb{Z}_3}(\mathrm{HL}')}{3} + \mathrm{ks}^w_{\mathbb{Z}_3}(\mathrm{HL}')$$

conjugacy classes.

Summing the three cases up gives us the formula of $\mathrm{ks}_{A_4}(\mathrm{HL})$ in terms of the ks-invariants of HL':

$$\mathrm{ks}_{A_4}(\mathrm{HL}) = \mathrm{ks}_{A_4}(\mathrm{HL}') + \frac{\mathrm{ks}^w_{A_4}(\mathrm{HL}') - \mathrm{ks}^w_{V_4}(\mathrm{HL}')}{4} + \mathrm{ks}^w_{V_4}(\mathrm{HL}') + 2 \cdot \left(\frac{\mathrm{ks}^w_{A_4}(\mathrm{HL}') - \mathrm{ks}^w_{\mathbb{Z}_3}(\mathrm{HL}')}{3} + \mathrm{ks}^w_{\mathbb{Z}_3}(\mathrm{HL}') \right). \quad (3.13)$$

Combining (3.13) with (3.7) and Corollary 3.4, we get the equation

$$\mathrm{ks}_{A_4}(\mathrm{HL}) = 12 \cdot \mathrm{ks}_{A_4}(\mathrm{HL}') - 6 \cdot 3^{g-1} - 2 \cdot 4^{g-1},$$

which implies the first assertion.

$\mathrm{ks}_{A_5}(\mathrm{HL})$ can be computed in a similar manner. First note that $\mathbf{ks}_G(\mathbb{Z})$ contains five elements by (3.9), and they are represented by homomorphisms

$$\phi_1, \phi_2, \phi_3, \phi_5^1, \phi_5^2 \quad (3.14)$$

with $\mathrm{Im}(\phi_1)$ trivial, $\mathrm{Im}(\phi_2)$ isomorphic to $\mathbb{Z}_2$, $\mathrm{Im}(\phi_3)$ isomorphic to $\mathbb{Z}_3$, and $\mathrm{Im}(\phi_5^i)$, $i = 1, 2$, isomorphic to $\mathbb{Z}_5$. As with the case of A_4, given a homomorphism $\phi : G_{\mathrm{HL}} \to A_5$, by conjugating with some element in A_5, its restriction on $\mathbb{Z}$ is one of the representing homomorphisms in (3.14). The number of conjugacy classes of homomorphisms that restrict to ϕ_1 is $\mathrm{ks}_{A_5}(\mathrm{HL}')$, and the number of conjugacy classes of homomorphisms that restrict to ϕ_2, ϕ_3 or $\phi_5^i, i = 1, 2$, is

$$\frac{\mathrm{ks}^w_{A_5}(\mathrm{HL}') - \mathrm{ks}^w_{V_4}(\mathrm{HL}')}{4} + \mathrm{ks}^w_{V_4}(\mathrm{HL}'),$$
$$\frac{\mathrm{ks}^w_{A_5}(\mathrm{HL}') - \mathrm{ks}^w_{\mathbb{Z}_3}(\mathrm{HL}')}{3} + \mathrm{ks}^w_{\mathbb{Z}_3}(\mathrm{HL}'),$$
$$\text{or} \quad \frac{\mathrm{ks}^w_{A_5}(\mathrm{HL}') - \mathrm{ks}^w_{\mathbb{Z}_5}(\mathrm{HL}')}{5} + \mathrm{ks}^w_{\mathbb{Z}_5}(\mathrm{HL}'), \quad \text{respectively.}$$

Summing them up yields the formula of $\mathrm{ks}_{A_5}(\mathrm{HL})$:

$$\mathrm{ks}_{A_5}(\mathrm{HL}) = \mathrm{ks}_{A_5}(\mathrm{HL}') + \frac{\mathrm{ks}^w_{A_5}(\mathrm{HL}') - \mathrm{ks}^w_{V_4}(\mathrm{HL}')}{4} + \mathrm{ks}^w_{V_4}(\mathrm{HL}') + \frac{\mathrm{ks}^w_{A_5}(\mathrm{HL}') - \mathrm{ks}^w_{\mathbb{Z}_3}(\mathrm{HL}')}{3} + \mathrm{ks}^w_{\mathbb{Z}_3}(\mathrm{HL}') + 2 \cdot \left(\frac{\mathrm{ks}^w_{A_5}(\mathrm{HL}') - \mathrm{ks}^w_{\mathbb{Z}_5}(\mathrm{HL}')}{5} + \mathrm{ks}^w_{\mathbb{Z}_5}(\mathrm{HL}') \right). \quad (3.15)$$

The formula (3.15), together with (3.7) and Corollary 3.4, implies the identity:

$$\mathrm{ks}_{A_5}(\mathrm{HL}) = 60 \cdot \mathrm{ks}_{A_5}(\mathrm{HL}') - 19 \cdot 3^{g-1} - 14 \cdot 4^{g-1} - 22 \cdot 5^{g-1},$$

and thus the second assertion. □

Lemma 3.8. *Given a reducible handlebody link* HL *of genus g, if a 2-generator knot K is a factor of some factorization of* HL*, then*

$$12 + 24k \mid \mathrm{ks}_{A_4}(\mathrm{HL}) + (6 + 16k) \cdot 3^{g-1} + (2 + 6k) \cdot 4^{g-1},$$

where $k = 0$ or 1.

Proof. By the assumption the knot group G_{HL} is isomorphic to the free product $G_K * G_{\mathrm{HL}'}$, where HL' is a handlebody link of genus $g-1$. By Lemma 3.5, $\mathbf{ks}_{A_4}(G_K)$ might have two more elements than $\mathbf{ks}_{A_4}(\mathbb{Z})$. Let ϕ_s^1, ϕ_s^2 be representing surjective homomorphisms of these two conjugacy classes. Since two homomorphisms

$$\phi, \psi : G_{\mathrm{HL}} \to A_4 \quad \text{with} \quad \phi|_{G_K} = \psi|_{G_K} = \phi_s^i, \quad i = 1 \text{ or } 2 \tag{3.16}$$

are conjugate if and only if

$$\phi|_{G_{\mathrm{HL}'}} = \psi|_{G_{\mathrm{HL}'}},$$

there are $\mathrm{ks}^w_{A_4}(\mathrm{HL}')$ conjugacy classes of homomorphisms with the property (3.16). Adding it to (3.13), we obtain

$$\mathrm{ks}_{A_4}(\mathrm{HL}) = ks_{A_4}(\mathrm{HL}') + \frac{\mathrm{ks}^w_{A_4}(L) - ks^w_{V_4}(\mathrm{HL}')}{4} + \mathrm{ks}^w_{V_4}(\mathrm{HL}') + 2 \cdot \left(\frac{\mathrm{ks}^w_{A_4}(\mathrm{HL}') - \mathrm{ks}^w_{\mathbb{Z}_3}(\mathrm{HL}')}{3} + \mathrm{ks}^w_{\mathbb{Z}_3}(\mathrm{HL}') \right) + 2k \cdot \mathrm{ks}^w_{A_4}(\mathrm{HL}'), \tag{3.17}$$

where $k = 0$ or 1. Plugging (3.7) and Corollary 3.4 into (3.17) implies the identity:

$$\mathrm{ks}_{A_4}(\mathrm{HL}) = (12 + 24k) \cdot \mathrm{ks}_{A_4}(\mathrm{HL}') - (6 + 16k) \cdot 3^{g-1} - (2 + 6k) \cdot 4^{g-1}, k = 0 \text{ or } 1,$$

and therefore the assertion. □

Lemma 3.9. *Given a reducible handlebody link* HL *of genus g, if a 2-generator link L is a factor of some factorization of* HL*, then*

$$48 + 24k \mid \mathrm{ks}_{A_4}(\mathrm{HL}) + (26 + 16k) \cdot 3^{g-2} + (8 + 6k) \cdot 4^{g-2},$$

where $k = 0, 1, 2, 3$ or 4.

Proof. The knot group G_{HL} is isomorphic to the free product $G_L * G_{\mathrm{HL}'}$, where HL' is a handlebody link of genus $g-2$. By Lemma 3.6, $\mathbf{ks}_{A_4}(G_L)$ contains $14 + 2k$ elements, $k = 0, 1, 2, 3$ or 4, where one conjugacy class is for the trivial homomorphism, five for non-trivial homomorphisms whose images are in V_4, eight for homomorphisms whose images are isomorphic to $\mathbb{Z}_3$, and $2k$ for surjective homomorphisms. The same argument as in the proof of Lemmas 3.7 and 3.8 implies

Table 2
Irreducibility of Ishii, Kishimoto, Moriuchi and Suzuki's handlebody knots.

handlebody knot	rank	ks_{A_4}	A_4-criterion (1.2)	ks_{A_5}	A_5-criterion (1.5)
HK 4_1	3	30	✓	156	✓
HK 5_1	3	22	?	111	✓
HK 5_2	3	30	✓	156	✓
HK 5_3	3	30	✓	105	✓
HK 5_4	3	22	?	365	✓
HK 6_1	3	30	✓	143	✓
HK 6_2	3	30	✓	105	✓
HK 6_3	3	22	?	83	✓
HK 6_4	3	22	?	111	✓
HK 6_5	3	22	?	97	✓
HK 6_6	3	22	?	97	✓
HK 6_7	3	30	✓	157	✓
HK 6_8	3	22	?	105	✓
HK 6_9	3	30	✓	146	✓
HK 6_{10}	3	22	?	195	✓
HK 6_{11}	3	22	?	73	✓
HK 6_{12}	3	30	✓	135	✓
HK 6_{13}	3	30	✓	156	✓
HK 6_{14}	3	46	?	353	✓
HK 6_{15}	3	46	?	353	✓
HK 6_{16}	3	22	?	267	✓

$$\mathrm{ks}_{A_4}(\mathrm{HL}) = \mathrm{ks}_{A_4}(\mathrm{HL}') + 5 \cdot \left(\frac{\mathrm{ks}^w_{A_4}(\mathrm{HL}') - \mathrm{ks}^w_{V_4}(\mathrm{HL}')}{4} + \mathrm{ks}^w_{V_4}(\mathrm{HL}') \right) + 8 \cdot \left(\frac{\mathrm{ks}^w_{A_4}(\mathrm{HL}') - \mathrm{ks}^w_{\mathbb{Z}_3}(\mathrm{HL}')}{3} + \mathrm{ks}^w_{\mathbb{Z}_3}(\mathrm{HL}') \right) + 2k \cdot \mathrm{ks}^w_{A_4}(\mathrm{HL}'), \quad (3.18)$$

where $k = 0, 1, 2, 3$ or 4. Plugging (3.7) and Corollary 3.4 into (3.18), we obtain

$$\mathrm{ks}_{A_4}(\mathrm{HL}) = (48 + 24k) \cdot \mathrm{ks}_{A_4}(\mathrm{HL}') - (26 + 16k) \cdot 3^{g-2} - (8 + 6k) \cdot 4^{g-2}$$

and hence the lemma. □

4. Examples

4.1. Applications to handlebody knot/link tables

Irreducibility of handlebody knots in [5] and handlebody links in [1] are examined here with the irreducibility criteria (Corollary 1.3 and Table 1). The ks_{A_4}- and ks_{A_5}-invariants of handlebody links are computed by Appcontour [10]; the same software is also used to find an upper bound of the rank of each knot group. In many cases, the upper bound is identical to the rank.

The results of the irreducibility test are recorded in Tables 2 and 3, where the check mark ✓ stands for the corresponding condition(s) not satisfied, and hence the handlebody link is irreducible, and the question mark means the opposite, so its irreducibility is inconclusive. To avoid confusion, HK is added to the name of each handlebody knot in [5]; so is HL to the name of each handlebody link in [1].

Since all handlebody knots in [5] are 3-generator, by Corollary 1.3, if either 12 does not divide $\mathrm{ks}_{A_4}(\mathrm{HK})+26$, or 60 does not divide $\mathrm{ks}_{A_5}(\mathrm{HK}) + 223$, HK is irreducible. On the contrary, in Table 3 different criteria are required to test each case, depending on the rank and the number of component (the column "comp.") based on Table 1. For instance, for a 3-generator handlebody link of type $[1, 1]$, such as HL 4_1, if it fails either of (1.2) and (1.5), it is irreducible. But, for HL 5_1, which is possibly 4-generator, we need to have *both* (1.2) and (1.3) failed in order to draw a conclusion; also, the A_5 criterion is not applicable in this case.

Table 3
Irreducibility of handlebody links in [1].

comp.	handlebody link	rank	ks_{A_4}	A_4-criterion	ks_{A_5}	A_5-criterion
2	$\mathrm{HL}\,4_1$	3	114	✓	600	✓
	$\mathrm{HL}\,5_1$	≤ 4	98	✓		not applicable
	$\mathrm{HL}\,6_1$	3	90	✓	600	✓
	$\mathrm{HL}\,6_2$	3	106	?	689	✓
	$\mathrm{HL}\,6_3$	3	90	✓	469	✓
	$\mathrm{HL}\,6_4$	3	106	?	689	✓
	$\mathrm{HL}\,6_5$	≤ 4	210	✓		not applicable
	$\mathrm{HL}\,6_6$	3	130	?	1380	✓
	$\mathrm{HL}\,6_7$	≤ 4	98	✓		not applicable
	$\mathrm{HL}\,6_8$	3	114	✓	1401	✓
3	$\mathrm{HL}\,6_9$	4	310	?		not applicable
	$\mathrm{HL}\,6_{10}$	4	326	✓		not applicable
	$\mathrm{HL}\,6_{11}$	4	486	✓		not applicable
	$\mathrm{HL}\,6_{12}$	4	502	?		not applicable
	$\mathrm{HL}\,6_{13}$	4	822	✓		not applicable
	$\mathrm{HL}\,6_{14}$	4	486	✓		not applicable
4	$\mathrm{HL}\,6_{15}$	5	1242	✓		not applicable

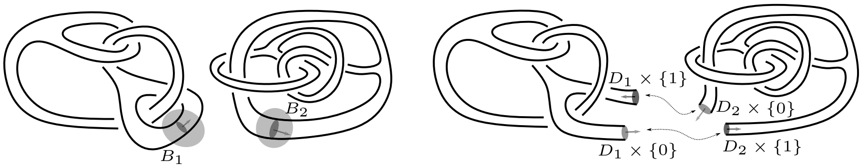


Fig. 4.1. Knot sum of HK 4_1 and HK 5_1 with meridian disks.

4.2. Irreducible handlebody links of a given type

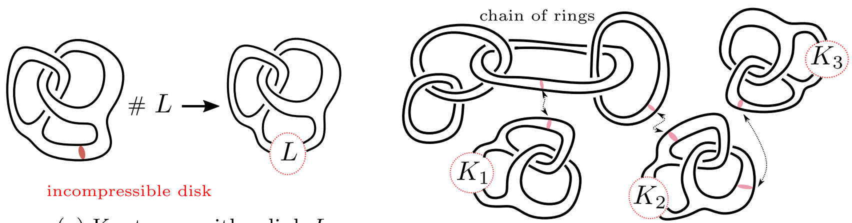
Here we present a construction of irreducible handlebody links of any given type. First we introduce the notion of $\mathcal{D}$-irreducibility for handlebody-link-disk pairs.

Definition 4.1 ***($\mathcal{D}$-irreducibility).*** A handlebody link HL is $\mathcal{D}$-irreducible if either its complement $\overline{\mathbb{S}^3 \setminus \mathrm{HL}}$ admits no incompressible disks or it is a trivial knot. A handlebody-link-disk pair (HL, D) is a handlebody link HL together with an oriented incompressible disk $D \subset \mathrm{HL}$. The pair (HL, D) is $\mathcal{D}$-irreducible if there exists no incompressible disk D' in the complement $\overline{\mathbb{S}^3 \setminus \mathrm{HL}}$ with $D' \cap D = \emptyset$. An unknot with a meridian disk is the trivial $\mathcal{D}$-irreducible handlebody-link-disk pair.

$\mathcal{D}$-irreducibility is equivalent to irreducibility for handlebody knots of genus $g \leq 2$ [15] but stronger in general [14, Examples 5.5-6], [1, Remark 3.3]. Any $\mathcal{D}$-irreducible handlebody link with an incompressible disk is a $\mathcal{D}$-irreducible pair. On the other hand, the underlying handlebody link of a $\mathcal{D}$-irreducible handlebody-link-disk pair could be trivial (left handlebody-knot-disk pair in Fig. 4.2a).

Definition 4.2 ***(Knot sum).*** The knot sum of two handlebody-link-disk pairs $(\mathrm{HL}_1, D_1), (\mathrm{HL}_2, D_2)$ is a handlebody link $(\mathrm{HL}_1, D_1)\#(\mathrm{HL}_2, D_2)$ obtained by gluing $\mathrm{HL}_1, \mathrm{HL}_2$ together as follows: Let B_i be a 3-ball with $\mathring{B}_i \cap \mathrm{HL}_i$ a tubular neighborhood $N(D_i)$ of D_i in HL_i, and identify $\overline{N(D_i)}$ with the oriented 3-manifold $D_i \times [0,1]$ using the orientation of D_i, $i = 1, 2$. Then the knot sum is obtained by removing B_i from $\mathbb{S}^3$ and gluing resultant 3-manifolds $\overline{\mathbb{S}^3 \setminus B_1}$, $\overline{\mathbb{S}^3 \setminus B_2}$ via an orientation-reversing homeomorphism $f : \partial(\overline{\mathbb{S}^3 \setminus B_1}) \to \partial(\overline{\mathbb{S}^3 \setminus B_2})$ with $f(D_1 \times \{i\}) = D_2 \times \{j\}$, $i, j \in \{0, 1\}$ and $i - j \equiv 1 \bmod 2$.

The knot sum resembles the order-2 connected sum of spatial graphs [9].



(a) Knot sum with a link L.

(b) Irreducible handlebody link of type $[2, 1, 1]$.

Fig. 4.2. Knot sum of $\mathcal{D}$-irreducible handlebody-link-disk pairs.

Theorem 4.1. *The knot sum of two non-trivial $\mathcal{D}$-irreducible handlebody-link-disk pairs* $(\mathrm{HL}_1, D_1), (\mathrm{HL}_2, D_2)$ *is $\mathcal{D}$-irreducible.*

Proof. We prove by contradiction. Suppose the knot sum

$$\mathrm{HL} \simeq (\mathrm{HL}_1, D_1)\#(\mathrm{HL}_2, D_2)$$

is not $\mathcal{D}$-irreducible, and D is an incompressible disk in $\overline{\mathbb{S}^3 \setminus \mathrm{HL}}$.

Let B be a 3-ball with $B \cap (\overline{\mathbb{S}^3 \setminus \mathrm{HL}})$ the complement of HL_2, and A the intersection annulus $\partial B \cap (\overline{\mathbb{S}^3 \setminus \mathrm{HL}})$. Isotope B and hence A such that the number of components of $A \cap D$ is minimized.
Claim: $A \cap D = \emptyset$. Suppose the intersection is non-empty, then we can choose a component α of $A \cap D$ that is innermost in D. α must be an arc, for otherwise it would contradict either the $\mathcal{D}$-irreducibility of (HL_i, D_i) or the minimality. α cuts D into two disks, one of which, say D', and A intersect at α. Without loss of generality, we may assume D' is in $\overline{\mathbb{S}^3 \setminus B}$.

If α is essential in A, then HL_1 can be identified with the union of a tubular neighborhood of α in B and $\overline{\mathbb{S}^3 \setminus B} \cap \mathrm{HL}$ in $\mathbb{S}^3$. Since $D' \cap \partial D$ is an arc connecting two sides of D_1 in HL_1, D_1 is not separating and therefore a meridian disk of HL_1. In addition, D' and ∂D_1 intersect at only one point, so (HL_1, D_1) is either trivial or not $\mathcal{D}$-irreducible, contradicting the assumption.

If α is inessential in A, let D'' be the disk cut off from A by α. Then $D' \cup D''$ is a compressing disk in HL_1. If $\partial(D' \cup D'')$ is inessential in $\partial\,\mathrm{HL}_1$, the intersection α can be removed—with other intersection arcs intact—by isotoping B. On the other hand, the $\mathcal{D}$-irreducibility of (HL_1, D_1) forces $\partial(D' \cup D'')$ to be inessential in $\partial\,\mathrm{HL}_1$. Thus, we have proved the claim, whence the theorem follows. □

In Fig. 4.2, K_1, K_2, K_3 are knots, and L is a link. If L in Fig. 4.2a is the composition of two Hopf links, the resulting knot sum is $\mathrm{HL}\,6_{12}$. Its irreducibility, which cannot be seen by our irreducibility test, hence follows from Theorem 4.1. The following corollary generalizes Suzuki's example [14, Theorem 5.2].

Corollary 4.2. *Given m non-negative integers $n_1, n_2, \ldots, n_m$ with $n := \sum n_i > 0$, there is an irreducible handlebody link of type* $[n_1, n_2, \ldots, n_m]$.

Proof. Consider a chain of rings with n-component—a knot sum of $n-1$ Hopf links (Fig. 4.2b). Label each ring with a number in $\{1, 2, \ldots, n\}$, and for the ring with label k,

$$\sum_{i=1}^{l-1} n_i < k \leq \sum_{i=1}^{l} n_i,$$

we consider its knot sum with an irreducible handlebody knot of genus l, which can be obtained by performing the knot sum operation iteratively on handlebody knots in [5] with meridian disks (e.g. Fig. 4.1). The resultant handlebody link is necessarily irreducible by Theorem 4.1 and of the prescribed type. □

Acknowledgements

The present paper benefits from the support of the GNAMPA (Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni) of INdAM (Istituto Nazionale di Alta Matematica), and from the support of National Center for Theoretical Sciences.

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