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Testing Impact Measures in Spatial Autoregressive Models

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Abstract

Researchers often make use of linear regression models in order to assess the impact of policies on target outcomes. In a correctly specified linear regression model, the marginal impact is simply measured by the linear regression coefficient. However, when dealing with both synchronic and diachronic spatial data, the interpretation of the parameters is more complex because the effects of policies extend to the neighboring locations. Summary measures have been suggested in the literature for the cross-sectional spatial linear regression models and spatial panel data models. In this article, we compare three procedures for testing the significance of impact measures in the spatial linear regression models. These procedures include (i) the estimating equation approach, (ii) the classical delta method, and (iii) the simulation method. In a Monte Carlo study, we compare the finite sample properties of these procedures.

Keywords

spatial econometric models, spatial autoregressive models, impact measures, asymptotic approximation, standard errors, inference, MLE, direct effects, indirect effects, total effects

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In evaluating the effectiveness of economic policies, researchers often make use of linear regression models in order to assess their impact on a target outcome. In a standard nonspatial linear regression model, the regression parameters represent the partial derivative of the dependent variable Y with respect to an independent variable X and, as a consequence, they can be straightforwardly interpreted as the impact on variable Y of a unitary increase or of a one percent increase (when in log) of each independent variable X. In contrast, in the spatial econometric models containing spatial lag terms of dependent variable, the interpretation of parameters is less immediate and requires some clarification. In fact, due to the spatial transmission mechanism inherent to spatial modeling, a variation of variable X observed in location i not only has an effect on the value of variable Y in the same location but also on the same variable observed in other neighboring locations (see Anselin 1988; Kelejian, Tavlas, and Hondroyiannis 2006; LeSage and Pace 2009; Debarsy, Ertur, and LeSage 2012; Lee and Yu 2012; Kelejian, Murrell, and Shepotylo 2013; Elhorst 2010, 2014b; Arbia 2014; LeSage and Chih 2016).

In a spatial regression model that has a spatial lag of the dependent variable, the marginal effects accounting will require the analysis of k different $n \times n$ matrices, where k is the number of explanatory variables and n is the number of spatial units. To ease the interpretation and presentation of marginal effects, summary measures, that is, impact measures, have been suggested in the literature. Since the diagonal elements of these $n \times n$ matrices contain the own-partial derivatives, while the off-diagonal elements represent the cross-partial derivatives, LeSage and Pace (2009) define the average of the main diagonal elements as a scalar summary measure of direct effects and the average of the off-diagonal elements as a scalar summary measure of indirect effects. The sum of direct and indirect effects is labeled as the total effect. Other impact measures can also be defined by using the relevant row or column sums of these $n \times n$ matrices for a plethora of purposes. Although the impact measures are functions of estimated parameters, we cannot use *directly* the estimated parameters and the corresponding standard errors to decide whether the impact measures are statistically and economically significant. In order to draw inference on impact measures, we need to estimate their dispersions as well.

The purpose of this article is to develop general methods for the estimation of dispersions of impact measure and investigate their finite sample properties. We first consider three general procedures: (i) the estimating equation approach, (ii) the classical delta method, and (iii) the simulation method. We show how these methods can be used to derive the asymptotic standard errors of the impact measures in cross-sectional spatial autoregressive (SAR) models containing a spatial lag of the dependent variable. Second, we derive the standard error of some well-known impact measures in some particular cases. Third, we investigate the finite sample properties of the proposed methods through an extensive simulation study. Our results on the impact measures are applicable only for exogenous variables introduced linearly in the regression equations.

The estimating equation approach adopted in this article is based on Pierce (1982). In this approach, the statistic of interest, that is, the impact measure, is embedded into the maximum likelihood (ML) estimation framework for the purpose of determining its asymptotic distribution and covariance. Thus, the asymptotic variance formula suggested by Pierce (1982) is a natural by-product of the ML estimation. We show how this approach can be extended to the impact measures suggested for SAR models. In the classical delta method, the first-order Taylor approximations of impact measures along with the asymptotic distribution of estimator are used to determine the asymptotic variances of impact measures. For the details on the delta method, see Oehlert (1992) and van der Vaart (1998). For the applicability of the classical delta method, Elhorst (2010, 23) writes, "However, owing to the complexity of the matrix of partial derivatives [see (6)] and because every empirical application will have its own unique number of observations (N) and spatial weights matrix (W), it is almost impossible to derive one general approach that can be applied under all circumstances." Though the delta method does not provide a single formula that can be used for all spatial models, we show that this method can be easily used to determine the asymptotic standard errors of some well-known impact measures with simple adjustments in the general expressions derived from the first-order Taylor approximations.

For cross-sectional models, LeSage and Pace (2009) suggested that the empirical distribution of the impact measures can be constructed using a large number of simulated parameters drawn from the asymptotic distribution of parameters. We call this method the simulation method. Alternatively, LeSage and Pace (2009) also suggested to derive estimates of the dispersions for the impact measures by Bayesian Markov chain Monte Carlo (MCMC). Since MCMC estimation yields samples drawn from the posterior distribution of the model parameters, these can be used to produce a posterior distribution for the impact measures. This approach is widely accepted in the literature and found application in the existing software (e.g., in the package *spdep* of R), although it presents a series of drawbacks. First of all, the achievement of the convergence of the sampler in nontrivial cases is computationally time-consuming. Second, while available for scalar summary measures, no result is yet available for the standard errors of the vector measures referring to the impacts in the various locations that constitute the study area. Finally, the accuracy of the MCMC method depends crucially on the (multivariate normal) distributional assumptions.

The article is organized as follows. In the second section, we specify the SAR model and provide assumptions that are required for the consistency and asymptotic normality of the ML estimator (MLE). In the third section, we describe various impact measures for the SAR models. In the fourth section, we provide general expressions for the asymptotic standard error of various impact measures described in the third section. In the fifth section, we describe our Monte Carlo setting and report the simulation results for (i) the Pierce method, (ii) the delta method, and (iii) the simulation method. The sixth section concludes and suggests possible extensions of the approach presented here. The simulation results and some technical derivations are relegated to appendices.

The Model Specification

We consider the following SAR model:

$$Y = \lambda_0 WY + X\beta_0 + \xi, \tag{2.1}$$

where $Y = (y_1, y_1, \ldots, y_n)'$ is the $n \times 1$ vector of dependent variable, $X = (\mathbf{x}_1, \ldots, \mathbf{x}_k)$ is the $n \times k$ matrix of nonstochastic regressors with the matching parameter vector β_0, W is the $n \times n$ exogenously given spatial weight matrix that has zero diagonal elements and $\xi = (\xi_1, \xi_2, \ldots, \xi_n)'$ is the $n \times 1$ vector of regression disturbance terms. X includes an intercept term. We assume that ξ_i s are i.i.d. normal random variables with mean 0 and variance σ_0^2. The spatial lag term is denoted by WY, and the associated scalar parameter λ_0 is called the SAR parameter. The parameter vector $\theta_0 = (\lambda_0, \beta_0', \sigma_0^2)'$ represents true values, while $\theta = (\lambda, \beta', \sigma^2)'$ any arbitrary value in the relevant parameter space. The quantities Y, W, X, and ξ in equation (2.1) are allowed to depend on the sample size n in order to form triangular arrays (see Lee 2004; Kelejian and Prucha 2010). However, for the notational simplicity, we suppressed the subscript n in equation (2.1). Let $S(\lambda) = (I_n - \lambda W)$, $G(\lambda) = WS^{-1}(\lambda)$, $S(\lambda_0) = S$, and $G(\lambda_0) = G$, where I_n is the $n \times n$ identity matrix. We consider equation (2.1) under the following assumptions.

> **Assumption 1:** The disturbance terms ξ_i s are i.i.d. normal random variables with mean 0 and variance σ_0^2.
>
> **Assumption 2:** (i) The sequences of matrices $\{W\}$ and $\{S\}$ are uniformly bounded in both row and column sums. (ii) $\{S^{-1}(\lambda)\}$ are uniformly bounded in either row or column sums, uniformly in λ in a compact parameter space Λ. (iii) The true λ_0 is in the interior of Λ.
>
> **Assumption 3:** (i) The elements of X are uniformly bounded constants for all n and $\lim_{n\to\infty} \frac{1}{n} X'X$ exists and is nonsingular. (ii) $\lim_{n\to\infty} \frac{1}{n} (X, GX\beta_0)'(X, GX\beta_0)$ exists and is nonsingular.

Assumptions 1 and 2 provide the main features of disturbance terms and weights matrix. The uniform boundedness property of $\{W\}$ and $\{S\}$ in Assumption 2 is considered by Kelejian and Prucha (1998, 2010) in order to limit spatial dependence among units to a tractable degree. The additional uniform boundedness of $\{S^{-1}(\lambda)\}$ is required to justify the ML estimation (Lee, 2004). In the literature, (i) Assumption 3 is usually adopted for analytical simplicity; (ii) Assumption 3 requires that $GX\beta_0$ and X are not asymptotically multicollinear, which ensures the global identification of θ_0 in the ML framework (Lee, 2004). In certain interaction scenarios, elements of weights matrices can be a function of sample size n. For equation (2.1), Lee (2004) assumes a large group interaction setting and specifies elements of the weights matrix by $w_{ij} = O(1/h_n)$, where w_{ij} is the (i,j)th element of W and $\{h_n\}$ is a sequence of real numbers that can be bounded or divergent with the property that $\lim_{n\to\infty} h_n/n = 0$. For simplicity, we assume interaction scenarios in which $\{h_n\}$ is bounded.

Under Assumption 1, the log-likelihood function of the model can be expressed as

$$\log L(\theta) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\sigma^2 + \ln|S(\lambda)| - \frac{1}{2\sigma^2}\xi'(\theta)\xi(\theta),$$

where $\xi(\theta) = S(\lambda)Y - X\beta$. Then, the MLE $\hat{\theta}$ is defined by $\hat{\theta} = \text{argmax}_\theta \log L(\theta)$. Under our stated assumptions, it can be shown that $\hat{\theta}$ is a consistent estimator of θ_0 with the following limiting distribution (Lee, 2004):

$$\sqrt{n}(\hat{\theta} - \theta_0) \underset{\longrightarrow}{d} N(0, \Sigma^{-1}), \tag{2.2}$$

where $\Sigma = \lim_{n\to\infty} E\left(-\frac{1}{n}\frac{\partial^2 \log L(\theta_0)}{\partial\theta\partial\theta''}\right)$ and

$$E\left(-\frac{1}{n}\frac{\partial^2 \log L(\theta_0)}{\partial\theta\partial\theta'}\right) = \begin{bmatrix} \frac{1}{\sigma_0^2 n}X'X & \frac{1}{\sigma_0^2 n}X'GX\beta_0 & 0 \\ \frac{1}{\sigma_0^2 n}(GX\beta_0)'X & \frac{1}{\sigma_0^2 n}(GX\beta_0)'(GX\beta_0) + \frac{1}{n}\text{tr}\left((G+G')G\right) & \frac{1}{\sigma_0^2 n}\text{tr}(G) \\ 0 & \frac{1}{\sigma_0^2 n}\text{tr}(G) & \frac{1}{2\sigma_0^4} \end{bmatrix}.$$

For statistical inference, we can use the MLE $\hat{\theta}$ to construct a plug-in estimator of Σ (Lee, 2004). As we will show in the fourth section, the limiting distribution in equation (2.2) is essential for our results on the impact measures.

Impact Measures in SAR Models

In spatial models, the interpretation of the coefficients is different from nonspatial models due to the possible presence of spatial transmission mechanisms, externalities, and spillovers. In this section, we show how several impact measures are formulated for the SAR models. Under the assumption that S is nonsingular, the model can be written in the reduced form as[1]

$$Y = S^{-1}X\beta_0 + S^{-1}\xi. \tag{3.1}$$

The impact of a unitary change in the variable $\mathbf{x}_k$ in one location, say j, on the variable y observed in location i can then be described through the partial derivatives $\partial\mathbb{E}(y_i)/\partial x_{jk}$ which can be arranged in the following matrix:

$$\text{IMP} = \frac{\partial \mathbb{E}(Y)}{\partial \mathbf{x}'_k} = \begin{bmatrix} \frac{\partial \mathbb{E}(y_1)}{\partial x_{1k}} & \frac{\partial \mathbb{E}(y_1)}{\partial x_{2k}} & \cdots & \frac{\partial \mathbb{E}(y_1)}{\partial x_{nk}} \\ \frac{\partial \mathbb{E}(y_2)}{\partial x_{1k}} & \frac{\partial \mathbb{E}(y_2)}{\partial x_{2k}} & \cdots & \frac{\partial \mathbb{E}(y_2)}{\partial x_{nk}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathbb{E}(y_n)}{\partial x_{1k}} & \frac{\partial \mathbb{E}(y_n)}{\partial x_{2k}} & \cdots & \frac{\partial \mathbb{E}(y_n)}{\partial x_{nk}} \end{bmatrix} = S^{-1}\beta_k, \tag{3.2}$$

where β_k is the kth element of β_0. On this basis, we can derive a series of impact measures for each of the independent variables x_{ik} included in the model (Arbia 2014; Elhorst 2010, 2014b; LeSage and Pace 2009). In particular, three scalar measures can be derived. The first, called the Average Direct Impact (ADI), refers to the average total impact of a change in x_{ki} on y_i for $i = 1, \ldots, n$, which can be calculated by taking the average of all diagonal entries in the matrix $S^{-1}\beta_k$:

$$\text{ADI} = \frac{1}{n}\text{tr}(S^{-1}\beta_k) = \frac{1}{n}\sum_{i=1}^{n}\text{IMP}_{ii}, \tag{3.3}$$

where $\text{IMP}_{ii} = \partial\mathbb{E}(y_i)/\partial x_{ik}$. The second impact measure, called Average Total Impact (ATI), is a global measure defined simply as the average of all entries in the matrix $S^{-1}\beta_k$:

$$\text{ATI} = \frac{1}{n} l'_n S^{-1} l_n \beta_k = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\text{IMP}_{ij}, \tag{3.4}$$

where $\text{IMP}_{ij} = \partial\mathbb{E}(y_i)/\partial x_{jk}$ and l_n is the $n \times 1$ vector of ones. The third impact measure is the Average Indirect Impact (AII) and is defined as the difference between ATI and ADI:

$$\text{AII} = \text{ATI} - \text{ADI}, \tag{3.5}$$

and is thus simply the average of all off-diagonal entries of matrix $S^{-1}\beta_k$.

Two vector measures are also available defined as the Average Total Impact To (ATIT) an observation and the Average Total Impact From (ATIF) an observation. ATIT is a measure related to the impact produced on one single observation by all other observations. For each observation i, this is calculated as the sum of the i th row of matrix $S^{-1}\beta_k$:

$$\text{ATIT}_i = \frac{1}{n}\mathbf{e}'_i S^{-1} l_n \beta_k = \frac{1}{n}\sum_{j=1}^{n}\text{IMP}_{ij}, \quad i = 1, \ldots, n, \tag{3.6}$$

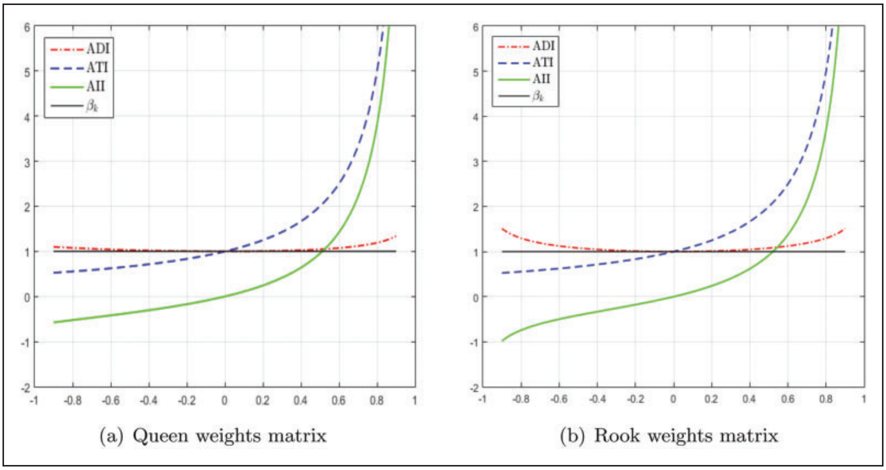

Figure 1. The effect of λ_0 on impact measures.

where $\mathbf{e}_i$ is the ith unitary vector. In contrast, ATIF is related to the impact produced by one single observation on all other observations. For each observation, this is calculated as the sum of the jth column of matrix $S^{-1}\beta_k$:

$$\text{ATIF}_i = \frac{1}{n} l_n' S^{-1} \mathbf{e}_i \beta_k = \frac{1}{n} \sum_{i=1}^{n} \text{IMP}_{ij}, \quad i = 1, \ldots, n. \tag{3.7}$$

Our results on ADI, ATI, and AII indicate that the magnitude of these impact measures depends on (i) the specification adopted for W, (ii) the strength of spatial dependence measured by λ_0, and (iii) the magnitude of coefficient estimate for β_k. In the case of ATIT and ATIF measures, besides these factors, the position of the region in the space also affects the magnitudes of ATIT and ATIF measures. From the series expansion $S^{-1}\beta_k = (I - \lambda_0 W)^{-1}\beta_k = (I + \lambda_0 W + \lambda_0^2 W^2 + \lambda_0^3 W^3 + \ldots)\beta_k$, it is also obvious that the sign of λ_0 will affect the magnitude of all impact measures. In particular, when $\lambda_0 < 0$, we have alternating signs in the series expansion due to the alternation between odd and even powers. As a consequence, the negative effect will be moderated by the presence of positive effects produced by the even powers. To illustrate the effect of λ_0 on the magnitudes of ADI, ATI, and AII, we set $\beta_k = 1$ and consider row-normalized rook and queen contiguity–based weight matrices over 10×10 regular square lattice grid. We calculate the magnitude of each impact measure as λ_0 varies from -0.9 to 0.9. The results are illustrated in Figure 1. The figure shows that the sign of λ_0 not only affects the sign of ATI and AII measures, but it also affects their magnitudes. As expected, the magnitudes of impact measures in absolute value are relatively larger when λ_0 gets positive large values. In the case of ADI measure, we have $\text{ADI} = (1 + \lambda_0^2 \text{tr}(W^2)/n + \lambda_0^3 \text{tr}(W^3)/n + \ldots)\beta_k$. In this expansion, the magnitudes of odd powers are less than that of even powers, and the trace terms are nonnegative since all elements of W are nonnegative. Thus, in this case, the sign of ADI measure is completely determined by the sign of β_k. The figure also shows that the magnitude of ADI measure in absolute value is relatively slightly larger when λ_0 is positive and large, especially in the case of queen weights matrix.

The Asymptotic Standard Errors of Impact Measures

In this section, we consider three general methods to derive the asymptotic standard errors of the impact measures described in the previous section. The first method is based on the estimating equation approach suggested by Pierce (1982; the Pierce method hereafter). The second approach is the classical approach based on the delta method. The final approach is the simulation method suggested by LeSage and Pace (2009).

We start with the Pierce method and provide a general argument by following Pierce (1982). Let $y_1, \ldots, y_n$ be the sequence of (not necessarily identical nor independent) random variables whose joint density function depends on a vector of parameters ψ. Let $\hat{\psi} = \hat{\psi}(Y)$ be the MLE of ψ, where $Y = (y_1, y_2, \ldots, y_n)'$. Let $U(Y, \hat{\psi})$ be a vector-valued statistic. Under some regularity conditions, Pierce (1982) suggests a method that can be used to determine the asymptotic variance of certain type of statistics. The first condition is about the joint limiting distribution of $\sqrt{n}(\hat{\psi} - \psi)$ and $\sqrt{n}U(Y, \psi)$. Pierce (1982) assumes that these two random variables have a limiting joint multivariate normal distribution, namely,

$$\begin{bmatrix} \sqrt{n}U(Y, \psi) \\ \sqrt{n}(\hat{\psi} - \psi) \end{bmatrix} \xrightarrow{d} N\left(0, \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}\right), \tag{4.1}$$

where the variance–covariance matrix may depend continuously on ψ. Note that this assumption is stated for the unfeasible statistic $U(Y, \psi)$. For the second regularity condition, Pierce (1982) assumes that there exists a matrix B, possibly depending continuously on ψ, such that

$$\sqrt{n}U(Y, \hat{\psi}) = \sqrt{n}U(Y, \psi) + B\sqrt{n}(\hat{\psi} - \psi) + o_p(1). \tag{4.2}$$

When U is differentiable with respect to ψ, this result follows from a first-order expansion and B is simply given by

$$B = \lim_{n\to\infty} \mathbb{E}\left(\frac{\partial U(Y, \psi)}{\partial \psi'}\right). \tag{4.3}$$

Finally, third required condition is that $\mathbb{E}(U(Y, \psi))$ is independent with ψ. Under these conditions, Pierce (1982) show that

$$\sqrt{n}U(Y, \hat{\psi}) \xrightarrow{d} N(0, V_{11} - BV_{22}B'). \tag{4.4}$$

This result is based on the expansion in equation (4.2), which implies that

$$\mathrm{Var}\left(\sqrt{n}U(Y,\hat{\psi})\right) = V_{11} + BV_{22}B' + V_{12}B' + BV_{21}. \tag{4.5}$$

The second assumption, that is $\mathbb{E}(U(Y,\psi))$ is independent with ψ, can be used to simplify equation (4.5). Let $l(Y,\psi)$ be the log-likelihood function of the sample. Then, under the second assumption, we have

$$\frac{\partial \mathbb{E}(U(Y,\psi))}{\partial \psi'} = \frac{\partial}{\partial \psi'}\int U(Y,\psi)\exp(l(Y,\psi))\mathrm{d}Y = 0. \tag{4.6}$$

Changing the order of integration and differentiation above yields

$$\int \frac{\partial U(Y,\psi)}{\partial \psi'}\exp(l(Y,\psi))\mathrm{d}Y + \int \sqrt{n}U(Y,\psi)\left(\frac{1}{\sqrt{n}}\frac{\partial l(Y,\psi)}{\partial \psi}\right)'\exp(l(Y,\psi))\mathrm{d}Y = 0. \tag{4.7}$$

This last result implies that

$$-B' = \mathrm{Cov}\left(\frac{1}{\sqrt{n}}\frac{\partial l(Y,\psi)}{\partial \psi}, \sqrt{n}U(Y,\psi)\right). \tag{4.8}$$

Using the asymptotic normality of score function under certain regularity conditions (see Newey and McFadden 1994), we can show that $V_{22}\frac{1}{\sqrt{n}}\frac{\partial l(Y,\psi)}{\partial \psi}$ is asymptotically equivalent to $\sqrt{n}(\hat{\psi}-\psi)$. Hence,

$$-V_{22}B' = \mathrm{Cov}\left(V_{22}\frac{1}{\sqrt{n}}\frac{\partial l(Y,\psi)}{\partial \psi}, \sqrt{n}U(Y,\psi)\right) \approx \mathrm{Cov}(\sqrt{n}(\hat{\psi}-\psi), \sqrt{n}U(Y,\psi)) = V_{21}. \tag{4.9}$$

This last result can be considered as a generalized information matrix equality (Newey and McFadden 1994). Then, the Pierce result in equation (4.4) is obtained by substituting $V_{21} = -V_{22}B'$ and $V_{12} = -BV_{22}$ into equation (4.5).

Next, we apply the general result in equation (4.4) to our stated impact measures to determine their corresponding asymptotic variances. We set $\hat{\psi} = \hat{\lambda}$ in formulating the statistics of interest for our impact measures ADI, AII, ATI, ATIF, and ATIT. These statistics are listed below.

1. $U^{\mathrm{ADI}}(Y,\hat{\lambda}) = \frac{1}{n}\mathrm{tr}(S^{-1}(\hat{\lambda}))\hat{\beta}_k - \frac{1}{n}\mathrm{tr}(S^{-1})\beta_k$.
2. $U^{\mathrm{ATI}}(Y,\hat{\lambda}) = \frac{1}{n}l_n'S^{-1}(\hat{\lambda})l_n\hat{\beta}_k - \frac{1}{n}l_n'S^{-1}l_n\beta_k$.
3. $U^{\mathrm{AII}}(Y,\hat{\lambda}) = \frac{1}{n}l_n'S^{-1}(\hat{\lambda})l_n\hat{\beta}_k - \frac{1}{n}\mathrm{tr}(S^{-1}(\hat{\lambda})\hat{\beta}_k) - \frac{1}{n}l_n'S^{-1}l_n\beta_k + \frac{1}{n}\mathrm{tr}(S^{-1}\beta_k)$.
4. $U^{\mathrm{ATIT}_i}(Y,\hat{\lambda}) = \frac{1}{n}\mathbf{e}'_iS^{-1}(\hat{\lambda})l_n\hat{\beta}_k - \frac{1}{n}\mathbf{e}'_iS^{-1}l_n\beta_k, \quad i = 1,\ldots,n$.
5. $U^{\mathrm{ATIF}_i}(Y,\hat{\lambda}) = \frac{1}{n}l_n'S^{-1}(\hat{\lambda})\mathbf{e}_i\hat{\beta}_k - \frac{1}{n}l_n'S^{-1}\mathbf{e}_i\beta_k, \quad i = 1,\ldots,n$.

Using the Pierce method, we determine the asymptotic distributions of these statistics in the following proposition.

Proposition 1: Under our stated assumptions, the following results hold.

1. In the case of $U^{\mathrm{ADI}}(Y,\hat{\lambda})$, we have

$$\sqrt{n}U^{\mathrm{ADI}}(Y,\hat{\lambda}) = \frac{1}{\sqrt{n}}\left(\mathrm{tr}\left(S^{-1}(\hat{\lambda})\hat{\beta}_k\right) - \mathrm{tr}(S^{-1})\beta_k\right)$$
$$\xrightarrow{d} N\left(0, \lim_{n\to\infty}\frac{1}{n^2}\mathrm{tr}^2(S^{-1})\mathrm{Var}(\sqrt{n}\hat{\beta}_k) - \frac{1}{n^2}\mathrm{tr}^2(WS^{-2})\beta_k^2\mathrm{Var}\left(\sqrt{n}(\hat{\lambda}-\lambda_0)\right)\right). \tag{4.10}$$

2. In the case of $U^{ATI}(Y,\hat{\lambda})$, we have

$$\sqrt{n}U^{\mathrm{ATI}}(Y,\hat{\lambda}) = \frac{1}{\sqrt{n}}\left(l_n'S^{-1}(\hat{\lambda})l_n\hat{\beta}_k - l_n'S^{-1}l_n\beta_k\right)$$
$$\xrightarrow{d} N\left(0, \lim_{n\to\infty}\left(\frac{1}{n}l_n'S^{-1}l_n\right)^2\mathrm{Var}(\sqrt{n}\hat{\beta}_k) - \left(\frac{1}{n}l_n'S^{-1}WS^{-1}l_n\beta_k\right)^2\mathrm{Var}\left(\sqrt{n}(\hat{\lambda}-\lambda_0)\right)\right). \tag{4.11}$$

3. In the case of $U^{\mathrm{AII}}(Y,\hat{\lambda})$, we have

$$\sqrt{n}U^{\mathrm{AII}}(Y,\hat{\lambda}) = \frac{1}{\sqrt{n}}\left(l_n'S^{-1}(\hat{\lambda})l_n\hat{\beta}_k - \mathrm{tr}\left(S^{-1}(\hat{\lambda})\hat{\beta}_k\right) - l_n'S^{-1}l_n\beta_k + \mathrm{tr}(S^{-1}\beta_k)\right)$$
$$\xrightarrow{d} N(0, \lim_{n\to\infty}\left(\frac{1}{n}l_n'S^{-1}l_n - \frac{1}{n}\mathrm{tr}(S^{-1})\right)^2\mathrm{Var}(\sqrt{n}\hat{\beta}_k)$$
$$-\left(\frac{1}{n}l_n'S^{-1}WS^{-1}l_n\beta_k - \frac{1}{n}\mathrm{tr}(WS^{-2})\beta_k\right)^2\mathrm{Var}\left(\sqrt{n}(\hat{\lambda}-\lambda_0)\right). \tag{4.12}$$

4. In the case of $U^{\mathrm{ATIT}_i}(Y,\hat{\lambda})$, we have

$$\sqrt{n}U^{\mathrm{ATIT}_i}(Y,\hat{\lambda}) = \frac{1}{\sqrt{n}}\left(\mathbf{e}'_iS^{-1}(\hat{\lambda})l_n\hat{\beta}_k - \mathbf{e}'_iS^{-1}l_n\beta_k\right)$$
$$\xrightarrow{d} N\left(0, \lim_{n\to\infty}\left(\frac{1}{n}\mathbf{e}'_iS^{-1}l_n\right)^2\mathrm{Var}(\sqrt{n}\hat{\beta}_k) - \left(\frac{1}{n}\mathbf{e}'_iS^{-1}WS^{-1}l_n\beta_k\right)^2\mathrm{Var}\left(\sqrt{n}(\hat{\lambda}-\lambda_0)\right)\right). \tag{4.13}$$

5. In the case of $U^{\text{ATIF}_i}(Y, \hat{\lambda})$, we have

$$\sqrt{n}U^{\text{ATIF}_i}(Y,\hat{\lambda}) = \frac{1}{\sqrt{n}}\left(l_n' S^{-1}(\hat{\lambda})\mathbf{e}_i\hat{\beta}_k - l_n' S^{-1}\mathbf{e}_i\beta_k\right)$$
$$\xrightarrow{d} N\left(0, \lim_{n\to\infty}\left(\frac{1}{n}l_n' S^{-1}\mathbf{e}_i\right)^2 \text{Var}(\sqrt{n}\hat{\beta}_k) - \left(\frac{1}{n}l_n' S^{-1}WS^{-1}\mathbf{e}_i\beta_k\right)^2 \text{Var}\left(\sqrt{n}(\hat{\lambda}-\lambda_0)\right)\right). \tag{4.14}$$

Proof: See Appendix A.

We can use the plug-in estimators to estimate the asymptotic variances in Proposition 1. For example, the estimated variance of $U^{\text{ADI}}(Y, \hat{\lambda})$ can be formulated as[2]

$$\widehat{\text{Var}}\left(U^{\text{ADI}}(Y,\hat{\lambda})\right) = \left(\frac{1}{n^3}\text{tr}^2\left(S^{-1}(\hat{\lambda})\right)\widehat{\text{Var}}(\sqrt{n}\hat{\beta}_k) - \frac{1}{n^3}\text{tr}^2\left(WS^{-2}(\hat{\lambda})\right)\hat{\beta}_k^2\widehat{\text{Var}}\left(\sqrt{n}(\hat{\lambda}-\lambda_0)\right)\right), \tag{4.15}$$

where $\widehat{\text{Var}}(\sqrt{n}\hat{\beta}_k)$ and $\widehat{\text{Var}}(\sqrt{n}(\hat{\lambda}-\lambda_0))$ can be recovered from the plug-in estimator of $\sqrt{n}(\hat{\theta}-\theta_0)$ in equation (2.2). Similarly, the plug-in estimators for other asymptotic variances in Proposition 1 can be formulated.

Another asymptotic method that can be used to determine the asymptotic variances of impact measures is the classical delta method (Taşpınar, Doğan, and Vijverberg 2018). In general, the delta method is used to determine (i) the variance of a function of a random variable, (ii) the bias correction for the expectation of a function of a random variable, and (iii) the limiting distribution of a function of a random variable (Oehlert 1992; van der Vaart 1998). In the following proposition, we show how this method can be used to derive the limiting distribution of each impact measure considered in the second section.

Proposition 2: Let J be the asymptotic covariance of $\sqrt{n}(\hat{\lambda}-\lambda_0, \hat{\beta}_k-\beta_k)'$. Then, under our stated assumptions, the following results holds.

1. For the ADI measure, we have

$$\frac{1}{\sqrt{n}}\left(\text{tr}\left(S^{-1}(\hat{\lambda})\hat{\beta}_k\right) - \text{tr}(S^{-1}\beta_k)\right)\xrightarrow{d} N\left(0, \lim_{n\to\infty} A_1JA_1'\right), \tag{4.16}$$

where $A_1 = \left[\frac{1}{n}\text{tr}(S^{-1}G\beta_k), \frac{1}{n}\text{tr}(S^{-1})\right]$.

2. For the ATI measure, we have

$$\frac{1}{\sqrt{n}}\left(\hat{\beta}_k l_n' S^{-1}(\hat{\lambda}) l_n - \beta_k l_n' S^{-1} l_n\right) \xrightarrow{d} N\left(0, \lim_{n\to\infty} A_2 J A_2'\right), \tag{4.17}$$

where $A_2 = [\frac{1}{n}\beta_k l_n' S^{-1} G l_n, \frac{1}{n} l_n' S^{-1} l_n]$.

3. In the case of AII measure, we have

$$\frac{1}{\sqrt{n}}\left(\left(\hat{\beta}_k l_n' S^{-1}(\hat{\lambda}) l_n - \mathrm{tr}\left(S^{-1}(\hat{\lambda})\hat{\beta}_k\right)\right) - \left(\beta_k l_n' S^{-1} l_n - \mathrm{tr}(S^{-1})\beta_k\right)\right) \xrightarrow{d} N\left(0, \lim_{n\to\infty} (A_2 - A_1) J (A_2 - A_1)'\right). \tag{4.18}$$

4. For the ATIT$_i$ measure, we have

$$\frac{1}{\sqrt{n}}\left(\mathbf{e}'_i S^{-1}(\hat{\lambda}) l_n \hat{\beta}_k - \mathbf{e}'_i S^{-1} l_n \beta_k\right) \xrightarrow{d} N(0, \lim_{n\to\infty} A_3 J A_3'), \tag{4.19}$$

where $A_3 = \left[\frac{1}{n}\mathbf{e}'_i S^{-1} G l_n \beta_k, \frac{1}{n}\mathbf{e}'_i S^{-1} l_n\right]$.

5. For the ATIF$_i$ measure, we have

$$\frac{1}{\sqrt{n}}\left(l_n' S^{-1}(\hat{\lambda}) \mathbf{e}_i \hat{\beta}_k - l_n' S^{-1} \mathbf{e}_i \beta_k\right) \xrightarrow{d} N(0, \lim_{n\to\infty} A_4 J A_4'), \tag{4.20}$$

where $A_4 = \left[\frac{1}{n} l_n' S^{-1} G \mathbf{e}_i \beta_k, \frac{1}{n} l_n' S^{-1} \mathbf{e}_i\right]$.

Proof: See Appendix B.

The asymptotic variances stated in Proposition 2 can estimated by the corresponding plug-in estimators. For example, Proposition 2 indicates that the asymptotic variance of ADI measure can be estimated by $\frac{1}{n}\hat{A}_1 \hat{J} \hat{A}'_1$, where $\hat{A}_1 = [\frac{1}{n}\mathrm{tr}(S^{-1}(\hat{\lambda})G(\hat{\lambda})\hat{\beta}_k), \frac{1}{n}\mathrm{tr}(S^{-1}(\hat{\lambda}))]$, and $\hat{J}$ is the estimated asymptotic covariance of $\sqrt{n}(\hat{\lambda} - \lambda_0, \hat{\beta}_k - \beta_k)'$. The estimates of other asymptotic variances in Proposition 2 can be obtained similarly.

Remark 1: Note that our suggested estimators for the asymptotic variance of impact measures in Proposition 2 are specific to the *k*th explanatory variable. The estimators for other explanatory variables can be easily obtained by adjusting only the J term. For example, the estimators for the various impact measures of the *j*th regressor is obtained by defining the $\hat{J}$ term as the estimated asymptotic covariance of $\sqrt{n}(\hat{\lambda} - \lambda_0, \hat{\beta}_j - \beta_j)'$.

The simulation approach suggested by LeSage and Pace (2009) utilizes the parameter estimates and the estimated asymptotic covariance matrix of a consistent estimator. Let L be a lower-triangular matrix recovered from the Cholesky decomposition of $\mathrm{Var}(\hat{\theta})$ and ϑ be a random vector that has a multivariate standard normal distribution. Then, random draws of the parameter vector are generated according to

$$\theta^r = \hat{\theta} + L \times \vartheta^r, \quad \text{for} \quad r = 1, \ldots, R. \tag{4.21}$$

A sequence of impact measures can be calculated by using the sequence $\{\theta^r\}$ for $r = 1, \ldots, R$. The mean and the standard deviation calculated from each sequence of impact measures can be used as the point estimate and the standard error of the corresponding impact measure. LeSage and Pace (2009) also consider the Bayesian estimation method for SAR models. In the Bayesian MCMC approach, a sequence of random draws is generated for each parameter. Similarly, a sequence of random draws can be generated for each scalar summary measure of impact estimates. Hence, the mean and the standard deviation calculated from each sequence of impact measures can be used as the point estimate and the standard error of the corresponding impact measure.

Remark 2: The three methods that we presented in the preceding paragraphs can be extended to the following spatial Durbin model:

$$Y = \lambda_0 WY + X\beta_0 + WX\delta_0 + \xi, \tag{4.22}$$

where WX is the spatial lag of X with the matching parameter vector δ_0. From the reduced form $Y = S^{-1}X\beta_0 + S^{-1}WX\delta_0 + S^{-1}\xi$, we have

$$\mathrm{IMP} = \frac{\partial \mathbb{E}(Y)}{\partial \mathbf{x}'_k} = \begin{bmatrix} \frac{\partial \mathbb{E}(y_1)}{\partial x_{1k}} & \frac{\partial \mathbb{E}(y_1)}{\partial x_{2k}} & \cdots & \frac{\partial \mathbb{E}(y_1)}{\partial x_{nk}} \\ \frac{\partial \mathbb{E}(y_2)}{\partial x_{1k}} & \frac{\partial \mathbb{E}(y_2)}{\partial x_{2k}} & \cdots & \frac{\partial \mathbb{E}(y_2)}{\partial x_{nk}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathbb{E}(y_n)}{\partial x_{1k}} & \frac{\partial \mathbb{E}(y_n)}{\partial x_{2k}} & \cdots & \frac{\partial \mathbb{E}(y_n)}{\partial x_{nk}} \end{bmatrix} = S^{-1}\beta_k + S^{-1}W\delta_k, \tag{4.23}$$

where β_k and δ_k are the kth elements of β_0 and δ_0, respectively. Then, in this case, the impact measures are in the following forms:

1. $\text{ADI} = \frac{1}{n}\,\text{tr}(S^{-1})\beta_k + \frac{1}{n}\text{tr}(S^{-1}W)\delta_k$.
2. $\text{ATI} = \frac{1}{n}\,l_n' S^{-1} l_n \beta_k + \frac{1}{n} l_n' S^{-1} W l_n \delta_k$.
3. $\text{AII} = \text{ATI} - \text{ADI} = \frac{1}{n} l_n' S^{-1} l_n \beta_k + \frac{1}{n} l_n' S^{-1} W l_n \delta_k - \frac{1}{n}\text{tr}(S^{-1})\beta_k - \frac{1}{n}\text{tr}(S^{-1}W)\delta_k$.
4. $\text{ATIT}_i = \frac{1}{n}\,\mathbf{e}'_i S^{-1} l_n \beta_k + \frac{1}{n}\mathbf{e}'_i S^{-1} W l_n \delta_k, \quad i = 1, \ldots, n$.
5. $\text{ATIF}_i = \frac{1}{n} l_n' S^{-1} \mathbf{e}_i \beta_k + \frac{1}{n} l_n' S^{-1} W \mathbf{e}_i \delta_k, \quad i = 1, \ldots, n$.

Following our arguments given for the proofs of Propositions 1 and 2, the Pierce method and the delta method can be used to determine the asymptotic distributions of these statistics. We provide these results in Appendix C.

> **Remark 3:** Note that the calculations of impact measures require the evaluation of $S^{-1}(\hat{\lambda})$. Also our results in Propositions 1 and 2 indicate that the dispersions of impact measures also require the evaluation of $S^{-1}(\hat{\lambda})$. It is clear that the computational cost is relatively high in the case of simulation method as it requires multiple evaluations of $S^{-1}(\hat{\lambda})$. When the sample size is large, the evaluation of $S^{-1}(\hat{\lambda})$ can be time consuming and even may not be feasible due to memory problems. LeSage and Pace (2009) suggest an approximation approach based on the series expansion $S^{-1}(\lambda) = \sum_{j=0}^{\infty}\lambda^j W^j$. In this approach, we can approximate the infinite sum with the truncated sum $\sum_{j=0}^{q}\lambda^j W^j$, where q is a large number and thus avoid the computational problems associated with the inversion of $S(\lambda)$. See LeSage and Pace (2009) and Elhorst (2014a) on the software demonstrations.

A Monte Carlo Study

In this section, we design a Monte Carlo simulation to investigate the finite sample properties of the methods described in the preceding section. We assume the following data generating process:

$$y_i = \lambda_0 \sum_{i \neq j} w_{ij} y_j + \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \xi_i, \tag{5.1}$$

for $i = 1, 2, \ldots, n$, where $n \in \{400, 900\}$. We specify two weights matrices corresponding to rook and queen contiguity cases. Assume that n spatial units are randomly allocated into a lattice of $k \times m$ squares, where $k = m = \sqrt{n}$. In the rook contiguity case, $w_{ij} = 1$ if the spatial spatial unit j is in a square that is adjacent (left/right/above or below) to the square of the spatial unit i. In the queen contiguity case, $w_{ij} = 1$ if the spatial unit j is in a square that is adjacent to or shares a corner with the square of the spatial unit i. In both cases, W is then row normalized.

For the regressors x_1 and x_2, we allow for spatial correlations in both regressors and set $x_1 = 0.7Wx_1 + \epsilon_1$ and $x_2 = 0.3Wx_2 + \epsilon_2$, where the elements of ϵ_1 and ϵ_2 are drawn independently from a uniform distribution on the unit interval (Pace, LeSage, and Zhu 2012). We set $(\beta_0, \beta_1, \beta_2)' = (0.2, 0.5, -0.5)'$. In order to allow for weak, moderate, and strong spatial dependence, we assume that the autoregressive parameter λ_0 takes on values from the set $\{-0.8, -0.5, -0.2, 0, 0.2, 0.5, 0.8\}$. We consider two cases for the distribution of ξ_i. In the first case, ξ_i's are drawn independently from the normal distribution that has mean zero and variance σ_0^2. To analyze the impact of nonnormality in disturbances, in the second case, we set $\xi_i = c \times \vartheta_i$, where c is a constant and ϑ_i is a random variable that has the student's t distribution with 5 degrees of freedom. To measure the degree of signal-to-noise in our setting, we use the following R^2 measure (Pace, LeSage, and Zhu 2012):

$$R^2 = 1 - \frac{\sigma_0^2 \mathrm{tr}(S'^{-1} S^{-1})}{\beta_0' X' S'^{-1} S^{-1} X \beta_0 + \sigma_0^2 \mathrm{tr}(S'^{-1} S^{-1})}. \tag{5.2}$$

We fix the signal-to-noise ratio to $R^2 = 0.5$ as λ_0 varies over $\{-0.8, -0.5, -0.2, 0, 0.2, 0.5, 0.8\}$. To do so, we solve $R^2 = 0.5$ for σ_0^2 in equation (5.2) and obtain

$$\sigma_0^2(\lambda_0) = \frac{\beta_0' X' S'^{-1} S^{-1} X \beta_0}{\mathrm{tr}(S'^{-1} S^{-1})}. \tag{5.3}$$

We then determine $\sigma_0^2(\lambda_0)$ values as λ_0 varies over $\{-0.8, -0.5, -0.2, 0, 0.2, 0.5, 0.8\}$ and use these values in our simulation for the normal distribution case. In the nonnormal case, we set $c = \sqrt{3/5} \times \sigma_0(\lambda_0)$, so that $R^2 = 0.5$ in all cases. As a result, the signal-to-noise ratio is fixed to 0.5 in all cases. For each specification, the resampling is carried out 5,000 times.

We will focus on the relative performance of the following methods: (i) the Pierce method, (ii) the delta method, and (iii) the simulation method.[3] The performance of each method will be analyzed in the context of the ADI, AII, and ATI measures. For each impact measure, we report (i) the empirical standard deviation (referred to as Emp.), (ii) the estimated standard error based on the Pierce method (say Pier.), (iii) the estimated standard error based on the delta method (say Del.), and (iv) the estimated standard error based on the simulation method (say Sim.). For the estimated standard error, we also calculated their percentage deviation from the empirical standard deviation.[4] A low percentage deviation for a method indicates that the method provides a good approximation to the finite sample distribution of the impact measure, while a large percentage deviation shows that the method provides a poor approximation. Furthermore, we will analyze the finite sample properties of the relevant Wald test for each impact measure in terms of size and power.

The simulation results are presented in Tables D1–D7. In order to give an overall assessment for the performance of each method, in the following tables, we highlight the estimated standard errors that have percentage deviations in the $(-5\%, +5\%)$

interval in Tables **D1** and **D2**. These estimated standard errors are presented in blue color and bold font. We summarize our main findings as follows.

1. In all tables, the empirical standard deviations become larger when the SAR parameter is positive and large. The same pattern is also true for the estimated standard errors reported by each method. That is, all methods report relatively larger estimated standard errors as λ_0 increases from -0.8 to 0.8. Consider the ADI of X_1 in Table D1. When $\lambda_0 = 0.2$ in the Rook contiguity case for $n = 400$, the reported values for the empirical standard deviation, the Pierce method, the delta method, and the simulation method are, respectively, 0.106, 0.107, 0.106, and 0.107, while the corresponding values are 0.346, 0.340, 0.339, and 0.340 when $\lambda_0 = 0.8$. The extensive simulation results in Arraiz et al. (2010) also show that the MLE of λ_0 reports relatively large empirical standard deviations and root mean square errors in the context of a SARAR(1,1) specification when λ_0 increases from -0.8 to 0.8.
2. In all tables, the empirical standard deviations and the estimated standard errors become relatively smaller when the sample size increases to $n = 900$. In terms of empirical standard deviations and estimated standard errors, the simulation results based on the rook contiguity case are similar to those based on the queen contiguity case. Also, the comparison of results in Table D1 and D2 indicates that the nonnormality of disturbance term has negligible effects on the performance of each method.
3. Looking at the results in Tables D1 and D2 for the ADI measure, all methods produce estimates that are mostly in the interval of $(-5\%, +5\%)$ for both X_1 and X_2. There are only some exceptions when λ_0 is negative and large in the case of Pierce and simulation methods. For example, when $\lambda_0 = -0.8$ in the Rook contiguity case for the Pierce method, and when $\lambda_0 = -0.8$ in the Queen contiguity case for the simulation method, the percentage deviations do not lie in the interval $(-5\%, +5\%)$. Overall, these results clearly suggest that all methods have very similar finite sample properties for the ADI measure.
4. Next, we compare the performance of each method for the AII measure. The delta and simulation methods produce estimates that are mostly in the interval of $(-5\%, +5\%)$ for both X_1 and X_2. However, the Pierce method seems to produce standard error estimates that are much smaller than the empirical standard deviations, and increasing the sample size does not yield an improvement. These results clearly show that Pierce method performs worse than the delta and simulation methods for the AII measure.
5. Turning to the ATI measure, the Pierce method again reports estimates that are smaller than the corresponding empirical standard deviations in Tables D1 and D2. The only occasions when the percentage deviations are in the $(-5\%, +5\%)$ interval for the Pierce method is the Rook contiguity case

when true λ is negative and large. Similarly, increasing sample size does not yield an improvement in the Pierce method. On the other hand, the delta and simulation methods produce estimates that are mostly in the interval of $(-5\%, +5\%)$ for both X_1 and X_2.

Next, we use the same Monte Carlo setting to investigate the finite sample size and power properties of the standard Wald statistics for testing linear simple hypotheses on the impact measures. Using a nominal size of 0.05 and different values of λ_0, we investigate the size properties for the null hypotheses $H_0^1 : \text{ADI} = \frac{1}{n}\text{tr}(S^{-1}\beta_k)$, $H_0^2 : \text{ATI} = \frac{1}{n}l_n' S^{-1} l_n \beta_k$, and $H_0^3 : \text{AII} = \text{ATI} - \text{ADI}$, while the power properties for the null hypotheses $H_0^4 : \text{ADI} = 0$, $H_0^5 : \text{ATI} = 0$, and $H_0^6 : \text{AII} = 0$. Note that we set the hypothesized values to the corresponding true values in the case of H_0^1, H_0^2, and H_0^3. The simulation results for the empirical size properties are reported in Tables D3 and D4 and for the empirical power properties in Tables D6 and D7. In these tables, T_p, T_d, and T_s denote, respectively, the Wald statistic using the estimated standard errors calculated from the corresponding Pierce, delta, and simulation methods. Our main findings are listed in the following:

1. We start with interpreting the results on the empirical size properties of test statistics. Considering the Wald statistics for testing H_0^1 in Tables D3 and D4, we see that all statistics generally report empirical size values that are very close to the nominal size value of 0.05. In particular, all statistics perform similarly under both the rook and queen contiguity cases in general, but T_s is moderately undersized in Queen contiguity case when λ_0 takes large negative values. These results are consistent with our results pertaining to the ADI measure reported in Tables D1 and D2, where all methods generally produce estimated standard errors that are very close to the corresponding empirical standard deviations.
2. We now consider the empirical size properties of statistics for testing H_0^2. In Tables D3 and D4, we see that T_p is oversized highlighting the fact that the estimated standard errors based on the Pierce method are smaller than the corresponding empirical standard deviations, which we have documented in Tables D1 and D2. The results also indicate that T_d and T_s have small size distortions in all cases, and they outperform T_p in all cases. However, again we see that T_s is severely undersized in Queen contiguity case when λ_0 takes large negative values. Overall, these findings are consistent with our results on the ATI measure reported in Tables D1 and D2.
3. Turning to the empirical size properties of statistics for testing H_0^3, we find that T_p is severely oversized confirming our results in Tables D1 and D2 on the estimated standard errors based on the Pierce method for the AII measure. The results also indicate that T_d and T_s have small size distortions in all cases.
4. Next, we consider the empirical power properties of test statistics in Tables D6 and D7. The true values of impact measures in the alternative model are

given in Table D5. We start with the empirical powers of statistics for testing H_0^4. In general, all statistics have similar powers under both the rook and queen contiguity cases, and power increases as the sample size increases. All test statistics for testing H_0^4 report relatively lower power for the cases where $\lambda_0 = 0.5$ and $\lambda_0 = 0.8$, though the true ADI values corresponding to $\lambda_0 = 0.5$ and $\lambda_0 = 0.8$ in Table D5 are further away from the null value of zero. This result is not surprising, since all methods produce relatively large estimated standard errors yielding relatively lower t-statistics for these cases as shown in Tables D3 and D4.

5. Looking at the power properties of all statistics for testing H_0^5, the results are similar to those for H_0^4. As expected though, both T_p and T_d report more power than T_s when λ_0 is large, especially when $\lambda_0 = -0.8$. This confirms our findings from previous tables for the simulation method. However, this gap in power declines as the sample size increases to 900. Again, all test statistics report relatively lower power for the cases where $\lambda_0 = 0.5$ and $\lambda_0 = 0.8$, since all methods produce relatively large estimated standard errors for these cases as documented in Tables D3 and D4.
6. Finally, turning to the power properties of all statistics for testing H_0^6, all statistics have similar powers under both the rook and queen contiguity cases, and the power increases as the sample size increases to $n = 900$. All test statistics report relatively lower power when λ_0 is near to zero. This is not surprising because as seen from Table D5, the true AII values approach to the null value when λ_0 tends to zero. The relatively large estimated standard errors reported in Tables D3 and D4 for the AII measure for the cases where $\lambda_0 = 0.5$ and $\lambda_0 = 0.8$ also cause lower powers for these cases.

Conclusion

In this article, we consider three methods that can be used to estimate the variance of impact measures suggested for spatial models that have spatial dependence in the dependent variable and, thus, allowing for reliable statistical inference on the models' parameters. These methods include (i) the estimating equation approach (the Pierce method), (ii) the classical delta method, and (iii) the simulation method suggested by LeSage and Pace (2009). We provide simple expressions for the variance of various impact measures under each method. In a Monte Carlo simulation, we investigate the finite sample properties of these three methods. Our results show that all three methods have very similar finite sample properties for the ADI measure and they perform satisfactorily. Therefore, the Pierce and delta methods are valid alternatives to reduce the computational burden and to overcome some of the drawbacks of the simulation method. In the case of AII and ATI measures, our simulation results indicate that the delta and simulation methods outperform the Pierce method in all cases.[5]

Finally, we state the possible extensions for future research. Although we derived the variance formulas for various impact measures in the context of a cross-sectional SAR model, our results can easily be extended, among the others, to (i) the static and dynamic spatial panel data models; (ii) the discrete choice models such as spatial logit, probit, or Tobit; (iii) the matrix exponential specification suggested by LeSage and Pace (2007); and (iv) the SAR models with endogenous weights matrices considered in Qu and Lee (2015) and Qu, Lee, and Yu (2017). We leave these extensions for future research.

Appendix A

Proof of Proposition 1

In order to apply the Pierce approach, we need to check for the three assumptions described in the fourth section. All of our test statistics are continuously differentiable with respect to parameter vector. Thus, we only need to check (i) the joint normality assumption in equation (4.1) and (ii) the assumption that $\mathbb{E}\left(\sqrt{n}U(Y,\lambda_0)\right)$ being independent of λ_0. The joint normality assumption holds for all statistics by our result in equation (2.2). For example, consider $U^{\mathrm{ADI}}(Y,\lambda_0)$. Then, under our stated assumptions, the joint normality assumption is satisfied since $\sqrt{n}U^{\mathrm{ADI}}(Y,\lambda_0)=\frac{1}{n}\mathrm{tr}(S^{-1})\sqrt{n}(\hat{\beta}_k-\beta_k)$ has a limiting normal distribution by equation (2.2). Here, note that $\frac{1}{n}\mathrm{tr}(S^{-1})=O(1)$ by Assumption 2. Similarly, it easy to see that the remaining unfeasible statistics $\sqrt{n}U^{\mathrm{ATI}}(Y,\lambda_0)$, $\sqrt{n}U^{\mathrm{AII}}(Y,\lambda_0)$, $\sqrt{n}U^{\mathrm{ATIT}}(Y,\lambda_0)$, and $\sqrt{n}U^{\mathrm{ATIF}}(Y,\lambda_0)$ have limiting normal distributions. Finally, by constructions, all statistics satisfy the assumption that $\mathbb{E}\left(\sqrt{n}U(Y,\lambda_0)\right)$ being independent of λ_0. Thus, in the following, we directly apply equation (4.4) to derive the limiting distribution of impact measures.

We start with $U^{\mathrm{ADI}}(Y,\hat{\lambda})$. The variance term $V_{11}=\mathrm{Var}\left(\sqrt{n}U^{\mathrm{ADI}}(Y,\lambda_0)\right)$ is $V_{11}=\frac{1}{n^2}\mathrm{tr}^2(S^{-1})\mathrm{Var}(\sqrt{n}\hat{\beta}_k)$. Simple calculation shows that the gradient of the statistic is $B=\mathbb{E}\left(\frac{U^{\mathrm{ADI}}(Y,\lambda)}{\partial\lambda}|_{\lambda_0}\right)=\frac{1}{n}\mathrm{tr}(WS^{-2})\beta_k$. Note that $V_{22}=\mathrm{Var}\left(\sqrt{n}(\hat{\lambda}-\lambda_0)\right)$. Then, using Pierce (1982) formula (4.4), we have

$$\begin{aligned}\mathrm{Var}\left(\sqrt{n}U^{\mathrm{ADI}}(Y,\hat{\lambda})\right)=V_{11}-BV_{22}B'=\frac{1}{n^2}\mathrm{tr}^2(S^{-1})\mathrm{Var}(\sqrt{n}\hat{\beta}_k)\\-\frac{1}{n^2}\mathrm{tr}^2(WS^{-2})\beta_k^2\mathrm{Var}\left(\sqrt{n}(\hat{\lambda}-\lambda_0)\right).\end{aligned}\tag{A1}$$

Next, we consider the average total impact measure $\mathrm{ATI}=n^{-1}l_n'S^{-1}l_n\beta_k$. In this case, the variance term $V_{11}=\mathrm{Var}\left(\sqrt{n}U^{\mathrm{ATI}}(Y,\lambda_0)\right)$ is $V_{11}=\left(\frac{1}{n}l_n'S^{-1}l_n\right)^2$

$\mathrm{Var}(\sqrt{n}\hat{\beta}_k)$, and the gradient term is given by $B = \mathbb{E}\left(\frac{U^{\mathrm{ATI}}(Y,\lambda)}{\partial\lambda}|_{\lambda_0}\right) = \frac{1}{n}l_n' S^{-1}WS^{-1}l_n\beta_k$. Then, using equation (4.4), we obtain

$$\mathrm{Var}\left(\sqrt{n}U^{\mathrm{ATI}}(Y,\hat{\lambda})\right) = \left(\frac{1}{n}l_n' S^{-1}l_n\right)^2 \mathrm{Var}\left(\sqrt{n}\hat{\beta}_k\right) - \left(\frac{1}{n}l_n' S^{-1}WS^{-1}l_n\beta_k\right)^2 \mathrm{Var}\left(\sqrt{n}(\hat{\lambda}-\lambda_0)\right).$$

Next, we turn to the AII $= \frac{1}{n}\, l_n' S^{-1}l_n\beta_k - \frac{1}{n}\mathrm{tr}(S^{-1}\beta_k)$. Then, we have $V_{11} = \left(\frac{1}{n}l_n' S^{-1}l_n - \frac{1}{n}\mathrm{tr}\left((S^{-1})\right)^2 \mathrm{Var}(\sqrt{n}\hat{\beta}_k)$. The preceding calculations show that

$$B = \mathbb{E}\left(\frac{U^{\mathrm{AII}}(Y,\lambda)}{\partial\lambda}|_{\lambda_0}\right) = \frac{1}{n}l_n' S^{-1}WS^{-1}l_n\beta_k - \frac{1}{n}\mathrm{tr}(WS^{-2})\beta_k. \tag{A2}$$

Then, using the Pierce (1982) formula in equation (4.4), we obtain

$$\mathrm{Var}\left(\sqrt{n}U^{\mathrm{AII}}(Y,\hat{\lambda})\right) = \left(\frac{1}{n}l_n' S^{-1}l_n - \frac{1}{n}\mathrm{tr}(S^{-1})\right)^2 \mathrm{Var}(\sqrt{n}\hat{\beta}_k) - \left(\frac{1}{n}l_n' S^{-1}WS^{-1}l_n\beta_k - \frac{1}{n}\mathrm{tr}(WS^{-2})\beta_k\right)^2 \mathrm{Var}\left(\sqrt{n}\left(\hat{\lambda}-\lambda_0\right)\right). \tag{A3}$$

In the case of ATIT_i, the variance of the unfeasible version is $V_{11} = \left(\frac{1}{n}\mathbf{e}'_i S^{-1}l_n\right)^2 \mathrm{Var}(\sqrt{n}\hat{\beta}_k)$. Simple calculations show that $B = \frac{1}{n}\mathbf{e}'_i S^{-1}WS^{-1}l_n\beta_k$. Then, applying the Pierce formula yields

$$\mathrm{Var}\left(\sqrt{n}U^{\mathrm{ATIT}_i}\left(Y,\hat{\lambda}\right)\right) = \left(\frac{1}{n}\mathbf{e}'_i S^{-1}l_n\right)^2 \mathrm{Var}\left(\sqrt{n}\hat{\beta}_k\right) - \left(\frac{1}{n}\mathbf{e}'_i S^{-1}WS^{-1}l_n\beta_k\right)^2 \mathrm{Var}\left(\sqrt{n}\left(\hat{\lambda}-\lambda_0\right)\right). \tag{A4}$$

Finally, in the the case of ATIF_i, the required terms are $V_{11} = \left(\frac{1}{n}l_n' S^{-1}\mathbf{e}_i\right)^2 \mathrm{Var}(\sqrt{n}\hat{\beta}_k)$ and $B = \frac{1}{n}l_n' S^{-1}WS^{-1}\mathbf{e}_i\beta_k$. Then, the Pierce formula yields

$$\mathrm{Var}\left(\sqrt{n}U^{\mathrm{ATIF}_i}(Y,\hat{\lambda})\right) = \left(\frac{1}{n}l_n' S^{-1}\mathbf{e}_i\right)^2 \mathrm{Var}\left(\sqrt{n}\hat{\beta}_k\right) - \left(\frac{1}{n}l_n' S^{-1}WS^{-1}\mathbf{e}_i\beta_k\right)^2 \mathrm{Var}\left(\sqrt{n}\left(\hat{\lambda}-\lambda_0\right)\right). \tag{A5}$$

Appendix B

Proof of Proposition 2

Using a first-order Taylor approximation and equation (2.2), it follows that

$$\frac{1}{\sqrt{n}}\left(\mathrm{tr}\left(S^{-1}\left(\hat{\lambda}\right)\hat{\beta}_k\right) - \mathrm{tr}\left(S^{-1}\beta_k\right)\right) = A_1 \times \sqrt{n}\left(\hat{\lambda} - \lambda_0, \hat{\beta}_k - \beta_k\right)' + o_p(1) \\ \xrightarrow{d} N(0, \lim_{n\to\infty} A_1 J A_1'), \tag{B1}$$

where $A_1 = [\frac{1}{n}\,\mathrm{tr}(S^{-1}G\beta_k), \frac{1}{n}\mathrm{tr}(S^{-1})]$, and J is the asymptotic covariance of $\sqrt{n}(\hat{\lambda} - \lambda_0, \hat{\beta}_k - \beta_k)'$. Similarly, for the ATI measure, the first-order Taylor approximation along with equation (2.2) gives

$$\frac{1}{\sqrt{n}}\left(\hat{\beta}_k l_n' S^{-1}\left(\hat{\lambda}\right) l_n - \beta_k l_n' S^{-1} l_n\right) = A_2 \times \sqrt{n}(\hat{\lambda} - \lambda_0, \hat{\beta}_k - \beta_k)' + o_p(1) \\ \xrightarrow{d} N(0, \lim_{n\to\infty} A_2 J A_2'), \tag{B2}$$

where $A_2 = \left[\frac{1}{n}\,\beta_k l_n' S^{-1} G l_n, \frac{1}{n} l_n' S^{-1} l_n\right]$. In the case of AII measure, using a first-order Taylor expansion and equation (2.2), we obtain

$$\frac{1}{\sqrt{n}}\left(\left(\hat{\beta}_k l_n' S^{-1}\left(\hat{\lambda}\right) l_n - \mathrm{tr}\left(S^{-1}\left(\hat{\lambda}\right)\hat{\beta}_k\right)\right) - \left(\beta_k l_n' S^{-1} l_n - \mathrm{tr}(S^{-1})\beta_k\right)\right) \\ = (A_2 - A_1) \times \sqrt{n}\left(\hat{\lambda}_n - \lambda_0, \hat{\beta}_k - \beta_k\right)' + o_p(1) \xrightarrow{d} N\left(0, \lim_{n\to\infty}(A_2 - A_1) J (A_2 - A_1)'\right). \tag{B3}$$

Next, we derive the asymptotic distributions of vector measures. Using a first-order Taylor expansion and equation (2.2) for the ATIT_i measure, we derive

$$\frac{1}{\sqrt{n}}\left(\mathbf{e}'_i S^{-1}(\hat{\lambda}) l_n \hat{\beta}_k - \mathbf{e}'_i S^{-1} l_n \beta_k\right) = A_3 \times \sqrt{n}\left(\hat{\lambda} - \lambda_0, \hat{\beta}_k - \beta_k\right)' + o_p(1) \\ \xrightarrow{d} N(0, \lim_{n\to\infty} A_3 J A_3'), \tag{B4}$$

where $A_3 = \left[\frac{1}{n}\,\mathbf{e}'_i S^{-1} G l_n \beta_k, \frac{1}{n}\mathbf{e}'_i S^{-1} l_n\right]$. Finally, a similar argument for the ATIF_i measure gives

$$\frac{1}{\sqrt{n}}\left(l_n' S^{-1}(\hat{\lambda})\mathbf{e}_i \hat{\beta}_k - l_n' S^{-1}\mathbf{e}_i \beta_k\right) = A_4 \times \sqrt{n}(\hat{\lambda} - \lambda_0, \hat{\beta}_k - \beta_k)' + o_p(1) \\ \xrightarrow{d} N(0, \lim_{n\to\infty} A_4 J A_4'), \tag{B5}$$

where $A_4 = \left[\frac{1}{n}\, l_n' S^{-1} G \mathbf{e}_i \beta_k, \frac{1}{n} l_n' S^{-1} \mathbf{e}_i\right]$.

Appendix C

The Limiting Distribution of the Impact Measures in Spatial Durbin Models

To apply the Pierce method, we consider the following statistics of interest:

1. $U^{\text{ADI}}(Y,\hat{\lambda}) = \frac{1}{n}\,\text{tr}(S^{-1}(\hat{\lambda}))\hat{\beta}_k + \frac{1}{n}\,\text{tr}(S^{-1}(\hat{\lambda})W)\hat{\delta}_k - \frac{1}{n}\,\text{tr}(S^{-1})\beta_k - \frac{1}{n}\text{tr}(S^{-1}W)\delta_k.$
2. $U^{\text{ATI}}(Y,\hat{\lambda}) = \frac{1}{n}\,l_n'S^{-1}(\hat{\lambda})l_n\hat{\beta}_k + \frac{1}{n}\,l_n'S^{-1}(\hat{\lambda})Wl_n\hat{\delta}_k - \frac{1}{n}\,l_n'S^{-1}l_n\beta_k - \frac{1}{n}l_n'S^{-1}Wl_n\delta_k.$
3. $U^{\text{AII}}(Y,\hat{\lambda}) = [\frac{1}{n}\,l_n'S^{-1}(\hat{\lambda})l_n - \frac{1}{n}\,\text{tr}(S^{-1}(\hat{\lambda}))]\hat{\beta}_k + [\frac{1}{n}\,l_n'S^{-1}(\hat{\lambda})Wl_n - \frac{1}{n}\text{tr}(S^{-1}(\hat{\lambda})W)]\hat{\delta}_k - \left(\frac{1}{n}\,l_n'S^{-1}l_n\beta_k + \frac{1}{n}\,l_n'S^{-1}Wl_n\delta_k - \frac{1}{n}\,\text{tr}(S^{-1})\beta_k - \frac{1}{n}\text{tr}(S^{-1}W)\delta_k\right).$
4. $U^{\text{ATIT}_i}(Y,\hat{\lambda}) = \frac{1}{n}\,\mathbf{e}'_iS^{-1}(\hat{\lambda})l_n\hat{\beta}_k + \frac{1}{n}\mathbf{e}'_iS^{-1}(\hat{\lambda})Wl_n\hat{\delta}_k - \left(\frac{1}{n}\,\mathbf{e}'_iS^{-1}l_n\beta_k + \frac{1}{n}\mathbf{e}'_iS^{-1}Wl_n\delta_k\right).$
5. $U^{\text{ATIF}_i}(Y,\hat{\lambda}) = \frac{1}{n}\,l_n'S^{-1}(\hat{\lambda})\mathbf{e}_i\hat{\beta}_k + \frac{1}{n}l_n'S^{-1}(\hat{\lambda})W\mathbf{e}_i\hat{\delta}_k - \left(\frac{1}{n}\,l_n'S^{-1}\mathbf{e}_i\beta_k + \frac{1}{n}l_n'S^{-1}W\mathbf{e}_i\delta_k\right).$

We start with $U^{\text{ADI}}(Y,\hat{\lambda})$. The variance term V_{11} is $V_{11} = \frac{1}{n^2}\,\text{tr}^2(S^{-1})\text{Var}(\sqrt{n}\hat{\beta}_k) + \frac{1}{n^2}\,\text{tr}^2(S^{-1}W)\text{Var}(\sqrt{n}\hat{\delta}_k) + \frac{2}{n^2}\text{tr}(S^{-1})\text{tr}(S^{-1}W)\text{Cov}(\sqrt{n}\hat{\beta}_k,\sqrt{n}\hat{\delta}_k)$. Simple calculation shows that the gradient of the statistic is $B = \mathbb{E}\left(\frac{U^{\text{ADI}}(Y,\lambda)}{\partial\lambda}|_{\lambda_0}\right) = \frac{1}{n}\,\text{tr}(WS^{-2})\beta_k + \frac{1}{n}\text{tr}(G^2)\delta_k$. Note that $V_{22} = \text{Var}(\sqrt{n}(\hat{\lambda}-\lambda_0))$. Then, using Pierce (1982) formula (4.4), we have

$$\begin{aligned}\text{Var}\left(\sqrt{n}U^{\text{ADI}}(Y,\hat{\lambda})\right) &= \frac{1}{n^2}\text{tr}^2(S^{-1})\text{Var}(\sqrt{n}\hat{\beta}_k) + \frac{1}{n^2}\text{tr}^2(S^{-1}W)\text{Var}(\sqrt{n}\hat{\delta}_k)\\ &+ \frac{2}{n^2}\text{tr}(S^{-1})\text{tr}(S^{-1}W)\text{Cov}(\sqrt{n}\hat{\beta}_k,\sqrt{n}\hat{\delta}_k)\\ &- \left[\frac{1}{n}\text{tr}(WS^{-2})\,\beta_k + \frac{1}{n}\text{tr}(G^2)\delta_k\right]^2\text{Var}\left(\sqrt{n}(\hat{\lambda}-\lambda_0)\right).\end{aligned} \tag{C1}$$

In the case of ATI, we have $V_{11} = \left[\frac{1}{n}l_n'S^{-1}l_n\right]^2\text{Var}(\sqrt{n}\hat{\beta}_k) + \left[\frac{1}{n}l_n'S^{-1}Wl_n\right]^2\text{Var}(\sqrt{n}\hat{\delta}_k) + 2\left[\frac{1}{n}l_n'S^{-1}l_n\right]\left[\frac{1}{n}l_n'S^{-1}Wl_n\right]\text{Cov}(\sqrt{n}\hat{\beta}_k,\sqrt{n}\hat{\delta}_k)$, and the gradient term is given by $B = \mathbb{E}\left(\frac{U^{\text{ATI}}(Y,\lambda)}{\partial\lambda}|_{\lambda_0}\right) = \frac{1}{n}l_n'S^{-1}WS^{-1}l_n\beta_k + \frac{1}{n}l_n'G^2l_n\delta_k$. Then, using equation (4.4), we obtain

$$\mathrm{Var}\Big(\sqrt{n}U^{\mathrm{ATI}}(Y,\hat{\lambda})\Big) = \left[\frac{1}{n}l'_n S^{-1} l_n\right]^2 \mathrm{Var}(\sqrt{n}\hat{\beta}_k) + \left[\frac{1}{n}l'_n S^{-1} W l_n\right]^2 \mathrm{Var}(\sqrt{n}\hat{\delta}_k) \\ + 2\left[\frac{1}{n}l'_n S^{-1} l_n\right]\left[\frac{1}{n}l'_n S^{-1} W l_n\right]\mathrm{Cov}(\sqrt{n}\hat{\beta}_k, \sqrt{n}\hat{\delta}_k) \\ - \left[\frac{1}{n}l'_n S^{-1} W S^{-1} l_n \beta_k + \frac{1}{n}l'_n G^2 l_n \delta_k\right]^2 \mathrm{Var}\Big(\sqrt{n}(\hat{\lambda}-\lambda_0)\Big). \tag{C2}$$

In the case of AII, we have

$$V_{11} = \left[\frac{1}{n}l'_n S^{-1}(\hat{\lambda}) l_n - \frac{1}{n}\mathrm{tr}(S^{-1}(\hat{\lambda}))\right]^2 \mathrm{Var}(\sqrt{n}\hat{\beta}_k) \\ + \left[\frac{1}{n}l'_\beta S^{-1}(\hat{\lambda}) W l_n - \frac{1}{n}\mathrm{tr}(S^{-1}(\hat{\lambda})W)\right]^2 \mathrm{Var}(\sqrt{n}\hat{\delta}_k) \\ +2\left[\frac{1}{n}l'_n S^{-1}(\hat{\lambda}) l_n - \frac{1}{n}\mathrm{tr}(S^{-1}(\hat{\lambda}))\right]\left[\frac{1}{n}l'_\lambda S^{-1}(\hat{\lambda}) W l_n - \frac{1}{n}\mathrm{tr}(S^{-1}(\hat{\lambda})W)\right]\mathrm{Cov}(\sqrt{n}\hat{\delta}_k, \sqrt{n}\hat{\beta}_k), \tag{C3}$$

and

$$B = \mathbb{E}\left(\frac{U^{\mathrm{AII}}(Y,\lambda)}{\partial\lambda}|_{\lambda_0}\right) = \left[\frac{1}{n}l'_n S^{-1} W S^{-1} l_n - \frac{1}{n}\mathrm{tr}(WS^{-2})\right]\beta_k + \left[\frac{1}{n}l'_n G^2 l_n - \frac{1}{n}\mathrm{tr}(G^2)\right]\delta_k. \tag{C4}$$

Then, substituting equations (C3) and (C4) into equation (4.4), we will obtain the asymptotic variance:

$$\mathrm{Var}\Big(\sqrt{n}U^{\mathrm{AII}}(Y,\hat{\lambda})\Big) = \left[\frac{1}{n}l'_n S^{-1}(\hat{\lambda}) l_n - \frac{1}{n}\mathrm{tr}(S^{-1}(\hat{\lambda}))\right]^2 \mathrm{Var}(\sqrt{n}\hat{\beta}_k) \\ + \left[\frac{1}{n}l'_n S^{-1}(\hat{\lambda}) W l_n - \frac{1}{n}\mathrm{tr}(S^{-1}(\hat{\lambda})W)\right]^2 \mathrm{Var}(\sqrt{n}\hat{\delta}_k) \\ +2\left[\frac{1}{n}l'_n S^{-1}(\hat{\lambda}) l_n - \frac{1}{n}\mathrm{tr}(S^{-1}(\hat{\lambda}))\right]\left[\frac{1}{n}l'_n S^{-1}(\hat{\lambda}) W l_n - \frac{1}{n}\mathrm{tr}(S^{-1}(\hat{\lambda})W)\right]\mathrm{Cov}(\sqrt{n}\hat{\delta}_k, \sqrt{n}\hat{\beta}_k) \\ - \left(\left[\frac{1}{n}l'_n S^{-1} W S^{-1} l_n - \frac{1}{n}\mathrm{tr}(WS^{-2})\right]\beta_k + \left[\frac{1}{n}l'_n G^2 l_n - \frac{1}{n}\mathrm{tr}(G^2)\right]\delta_k\right)^2 \mathrm{Var}\Big(\sqrt{n}(\hat{\lambda}-\lambda_0)\Big). \tag{C5}$$

In the case of ATIT_i, the variance of the unfeasible version is $V_{11} = [\frac{1}{n}\mathbf{e}'_i S^{-1}(\hat{\lambda})l_n]^2 \text{Var}(\sqrt{n}\hat{\beta}_k) + [\frac{1}{n}\mathbf{e}'_i S^{-1}(\hat{\lambda})Wl_n]^2 \text{Var}(\sqrt{n}\hat{\delta}_k) + 2[\frac{1}{n}\mathbf{e}'_i S^{-1}(\hat{\lambda})l_n]$ $[\frac{1}{n}\mathbf{e}'_i S^{-1}(\hat{\lambda})Wl_n]\text{Cov}(\sqrt{n}\hat{\beta}_k, \sqrt{n}\hat{\delta}_k)$. Simple calculations show that $B = \frac{1}{n}$ $\mathbf{e}'_i S^{-1}WS^{-1}l_n\beta_k + \frac{1}{n}\mathbf{e}'_i S^{-1}WS^{-1}Wl_n\delta_k$. Then, applying the Pierce formula in equation (4.4) yields

$$\begin{aligned}\text{Var}\left(\sqrt{n}U^{\text{ATIT}_i}(Y,\hat{\lambda})\right) = &\left[\frac{1}{n}\mathbf{e}'_i S^{-1}(\hat{\lambda})l_n\right]^2 \text{Var}(\sqrt{n}\hat{\beta}_k) + \left[\frac{1}{n}\mathbf{e}'_i S^{-1}(\hat{\lambda})Wl_n\right]^2 \text{Var}(\sqrt{n}\hat{\delta}_k)\\ &+ 2\left[\frac{1}{n}\mathbf{e}'_i S^{-1}(\hat{\lambda})l_n\right]\left[\frac{1}{n}\mathbf{e}'_i S^{-1}(\hat{\lambda})Wl_n\right]\text{Cov}(\sqrt{n}\hat{\beta}_k, \sqrt{n}\hat{\delta}_k)\\ &- \left[\frac{1}{n}\mathbf{e}'_i S^{-1}WS^{-1}l_n\beta_k + \frac{1}{n}\mathbf{e}'_i S^{-1}WS^{-1}Wl_n\delta_k\right]^2 \text{Var}\left(\sqrt{n}(\hat{\lambda}-\lambda_0)\right).\end{aligned} \tag{C6}$$

Finally, in the the case of ATIF $_i$, the required terms are $V_{11} = \left[\frac{1}{n}l'_n S^{-1}(\hat{\lambda})\mathbf{e}_i\right]^2$ $\text{Var}(\sqrt{n}\hat{\beta}_k) + \left[\frac{1}{n}l'_n S^{-1}(\hat{\lambda})W\mathbf{e}_i\right]^2 \text{Var}(\sqrt{n}\hat{\delta}_k) + 2\left[\frac{1}{n}l'_n S^{-1}(\hat{\lambda})\mathbf{e}_i\right]\left[\frac{1}{n}l'_n S^{-1}(\hat{\lambda})W\mathbf{e}_i\right]\text{Cov}$ $(\sqrt{n}\hat{\beta}_k, \sqrt{n}\hat{\delta}_k)$ and $B = \frac{1}{n}l'_n S^{-1}WS^{-1}\mathbf{e}_i\beta_k + \frac{1}{n}l'_n S^{-1}WS^{-1}W\mathbf{e}_i\delta_k$. Then, the Pierce formula in equation (4.4) yields

$$\begin{aligned}\text{Var}\left(\sqrt{n}U^{\text{ATIF}_i}(Y,\hat{\lambda})\right) = &\left[\frac{1}{n}l'_n S^{-1}(\hat{\lambda})\mathbf{e}_i\right]^2 \text{Var}(\sqrt{n}\hat{\beta}_k) + \left[\frac{1}{n}l'_n S^{-1}(\hat{\lambda})W\mathbf{e}_i\right]^2 \text{Var}(\sqrt{n}\hat{\delta}_k)\\ &+ 2\left[\frac{1}{n}l'_n S^{-1}(\hat{\lambda})\mathbf{e}_i\right]\left[\frac{1}{n}l'_n S^{-1}(\hat{\lambda})W\mathbf{e}_i\right]\text{Cov}(\sqrt{n}\hat{\beta}_k, \sqrt{n}\hat{\delta}_k)\\ &- \left[\frac{1}{n}l'_n S^{-1}WS^{-1}\mathbf{e}_i\beta_k + \frac{1}{n}l'_n S^{-1}WS^{-1}W\mathbf{e}_i\delta_k\right]^2 \text{Var}\left(\sqrt{n}(\hat{\lambda}-\lambda_0)\right).\end{aligned} \tag{C7}$$

Next, we determine the asymptotic distributions of statistics by using the delta method. For the ADI measure, using a first-order Taylor approximation and equation (2.2), it can be shown that

$$\begin{aligned}&\frac{1}{\sqrt{n}}\left(\text{tr}(S^{-1}(\hat{\lambda}))\hat{\beta}_k + \text{tr}(S^{-1}(\hat{\lambda})W)\hat{\delta}_k - \text{tr}(S^{-1})\beta_k - \text{tr}(S^{-1}W)\delta_k\right)\\ &= A_1 \times \sqrt{n}(\hat{\lambda}-\lambda_0, \hat{\beta}_k - \beta_k, \hat{\delta}_k - \delta_k)' + o_p(1) \xrightarrow{d} N(0, \lim_{n\to\infty} A_1 J A'_1),\end{aligned} \tag{C8}$$

where $A_1 = [\frac{1}{n}\,\mathrm{tr}(S^{-1}G\beta_k) + \frac{1}{n}\,\mathrm{tr}(G^2), \frac{1}{n}\,\mathrm{tr}(S^{-1}), \frac{1}{n}\mathrm{tr}(S^{-1}W)]$, and J is the asymptotic covariance of $\sqrt{n}(\hat{\lambda} - \lambda_0, \hat{\beta}_k - \beta_k, \hat{\delta}_k - \delta_k)'$. In the case of ATI measure, the first-order Taylor approximation and equation (2.2) gives

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\left(l_n'S^{-1}(\hat{\lambda})l_n\hat{\beta}_k + l_n'S^{-1}(\hat{\lambda})Wl_n\hat{\delta}_k - l_n'S^{-1}l_n\beta_k - \frac{1}{n}l_n'S^{-1}Wl_n\delta_k\right) \\
&= A_2 \times \sqrt{n}(\hat{\lambda} - \lambda_0, \hat{\beta}_k - \beta_k, \hat{\delta}_k - \delta_k)' + o_p(1) \xrightarrow{d} N(0, \lim_{n\to\infty} A_2JA_2'),
\end{aligned} \tag{C9}$$

where $A_2 = [\frac{1}{n}\,\beta_k l_n'S^{-1}Gl_n + \frac{1}{n}\,l_n'G^2l_n\delta_k, \frac{1}{n}\,l_n'S^{-1}l_n, \frac{1}{n}l_n'S^{-1}Wl_n]$. In the case of AII measure, the first-order Taylor expansion along with equation (2.2) yields

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\Big([l_n'S^{-1}(\hat{\lambda})l_n - \mathrm{tr}(S^{-1}(\hat{\lambda}))]\hat{\beta}_k + [l_n'S^{-1}(\hat{\lambda})Wl_n - \mathrm{tr}(S^{-1}(\hat{\lambda})W)]\hat{\delta}_k \\
&-\Big(l_n'S^{-1}l_n\beta_k + l_n'S^{-1}Wl_n\delta_k - \mathrm{tr}(S^{-1})\beta_k - \mathrm{tr}(S^{-1}W)\delta_k\Big)\Big) \\
&= (A_2 - A_1) \times \sqrt{n}(\hat{\lambda}_n - \lambda_0, \hat{\beta}_k - \beta_k, \hat{\delta}_k - \delta_k)' + o_p(1) \xrightarrow{d} N\Big(0, \lim_{n\to\infty}(A_2 - A_1)J(A_2 - A_1)'\Big).
\end{aligned} \tag{C10}$$

The first-order Taylor expansion for the ATIT_i measure gives

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\Big(\mathbf{e}'_iS^{-1}(\hat{\lambda})l_n\hat{\beta}_k + \mathbf{e}'_iS^{-1}(\hat{\lambda})Wl_n\hat{\delta}_k - (\mathbf{e}'_iS^{-1}l_n\beta_k + \mathbf{e}'_iS^{-1}Wl_n\delta_k)\Big) \\
&= A_3 \times \sqrt{n}(\hat{\lambda} - \lambda_0, \hat{\beta}_k - \beta_k, \hat{\delta}_k - \delta_k)' + o_p(1) \rightarrow dN(0, \lim_{n\to\infty} A_3JA_3'),
\end{aligned} \tag{C11}$$

where $A_3 = \left[\frac{1}{n}\,\mathbf{e}'_iS^{-1}Gl_n\beta_k + \frac{1}{n}\,\mathbf{e}'_iG^2l_n, \frac{1}{n}\,\mathbf{e}'_iS^{-1}l_n, \frac{1}{n}\mathbf{e}'_iS^{-1}Wl_n\right]$. Finally, in the case of ATIF_i, we have

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\Big(l_n'S^{-1}(\hat{\lambda})\mathbf{e}_i\hat{\beta}_k + l_n'S^{-1}(\hat{\lambda})W\mathbf{e}_i\hat{\delta}_k - (l_n'S^{-1}\mathbf{e}_i\beta_k + l_n'S^{-1}W\mathbf{e}_i\delta_k)\Big) \\
&= A_4 \times \sqrt{n}(\hat{\lambda} - \lambda_0, \hat{\beta}_k - \beta_k, \hat{\delta}_k - \delta_k)' + o_p(1)\, d\longrightarrow N(0, \lim_{n\to\infty} A_4JA_4'),
\end{aligned} \tag{C12}$$

where $A_4 = \left[\frac{1}{n}\,l_n'S^{-1}G\mathbf{e}_i\beta_k + \frac{1}{n}\,l_n'G^2\mathbf{e}_i\delta_k, \frac{1}{n}\,l_n'S^{-1}\mathbf{e}_i, \frac{1}{n}l_n'S^{-1}W\mathbf{e}_i\right]$.

Appendix D

Simulation Results

Table D1. Empirical and Estimated Standard Errors: Normal Case.

			Direct								Indirect								Total							
			X_1				X_2				X_1				X_2				X_1				X_2			
		λ_0	Emp.	Pier.	Del.	Sim.	Emp.	Pier.	Del.	Sim.	Emp.	Pier.	Del.	Sim.	Emp.	Pier.	Del.	Sim.	Emp.	Pier.	Del.	Sim.	Emp.	Pier.	Del.	Sim.
Rook	$n = 400$	−.8	.054	.042	.054	.054	.064	.058	.065	.065	0.040	0.004	0.039	0.040	0.042	0.023	0.043	0.044	0.021	0.020	0.020	0.020	0.028	0.026	0.027	0.027
		−.5	.063	.059	.062	.062	.079	.077	.078	.078	0.034	0.005	0.034	0.034	0.036	0.020	0.037	0.037	0.037	0.035	0.037	0.037	0.050	0.046	0.050	0.050
		−.2	.076	.074	.075	.075	.099	.096	.097	.097	0.031	0.000	0.032	0.032	0.032	0.000	0.032	0.033	0.062	0.056	0.061	0.061	0.084	0.076	0.082	0.082
		.0	.088	.088	.088	.088	.115	.114	.114	.114	0.034	0.000	0.034	0.035	0.034	0.000	0.034	0.036	0.088	0.080	0.087	0.088	0.118	0.108	0.117	0.118
		.2	.106	.107	.106	.107	.140	.139	.139	.139	0.050	0.000	0.050	0.052	0.059	0.000	0.056	0.059	0.132	0.120	0.132	0.133	0.180	0.162	0.177	0.178
		.5	.164	.161	.160	.160	.212	.210	.210	.210	0.160	0.088	0.155	0.162	0.203	0.138	0.201	0.211	0.308	0.272	0.298	0.304	0.402	0.366	0.398	0.406
		.8	.346	.340	.339	.340	.445	.447	.448	.449	1.021	0.829	0.983	1.053	1.326	1.140	1.297	1.397	1.349	1.192	1.306	1.373	1.754	1.609	1.734	1.824
	$n = 900$	−.8	.038	.031	.038	.038	.046	.042	.045	.046	0.027	0.009	0.027	0.027	0.030	0.018	0.030	0.030	0.015	0.014	0.015	0.015	0.020	0.019	0.019	0.019
		−.5	.043	.042	.044	.044	.056	.054	.055	.055	0.022	0.007	0.023	0.023	0.025	0.015	0.025	0.025	0.026	0.025	0.026	0.026	0.035	0.033	0.035	0.035
		−.2	.054	.053	.053	.053	.070	.068	.068	.068	0.021	0.000	0.021	0.021	0.022	0.000	0.022	0.022	0.045	0.041	0.044	0.044	0.059	0.054	0.058	0.058
		.0	.063	.063	.063	.063	.079	.080	.080	.080	0.023	0.000	0.023	0.023	0.023	0.000	0.023	0.023	0.063	0.058	0.063	0.063	0.082	0.077	0.083	0.083
		.2	.077	.077	.076	.076	.100	.099	.098	.098	0.035	0.000	0.034	0.035	0.039	0.000	0.039	0.039	0.097	0.087	0.095	0.095	0.127	0.116	0.124	0.125
		.5	.114	.115	.115	.115	.148	.149	.149	.149	0.111	0.070	0.112	0.114	0.142	0.106	0.142	0.145	0.215	0.200	0.217	0.219	0.283	0.266	0.284	0.286
		.8	.251	.247	.246	.246	.322	.320	.321	.321	0.751	0.644	0.736	0.758	0.965	0.865	0.954	0.984	0.991	0.904	0.971	0.993	1.278	1.194	1.266	1.296
Queen	$n = 400$	−.8	.063	.061	.063	.068	.074	.074	.075	.068	0.040	0.017	0.041	0.090	0.043	0.027	0.043	0.092	0.033	0.028	0.033	0.091	0.041	0.035	0.041	0.092
		−.5	.075	.072	.073	.073	.088	.087	.088	.088	0.041	0.000	0.041	0.041	0.043	0.000	0.043	0.043	0.049	0.041	0.048	0.049	0.060	0.051	0.060	0.061
		−.2	.089	.088	.089	.089	.107	.107	.107	.107	0.042	0.000	0.042	0.043	0.042	0.000	0.042	0.044	0.076	0.064	0.075	0.076	0.094	0.081	0.093	0.094
		.0	.104	.104	.104	.104	.129	.126	.127	.127	0.048	0.000	0.047	0.049	0.049	0.000	0.047	0.050	0.106	0.091	0.107	0.108	0.136	0.115	0.131	0.133
		.2	.127	.128	.128	.128	.155	.155	.155	.155	0.069	0.000	0.067	0.071	0.075	0.000	0.072	0.078	0.163	0.140	0.161	0.165	0.201	0.176	0.199	0.203
		.5	.197	.195	.194	.195	.240	.238	.238	.238	0.208	0.113	0.203	0.218	0.252	0.157	0.241	0.260	0.384	0.335	0.376	0.388	0.473	0.420	0.462	0.476
		.8	.437	.421	.421	.422	.522	.515	.516	.518	1.425	1.121	1.317	1.459	1.732	1.397	1.615	1.793	1.837	1.570	1.720	1.847	2.231	1.939	2.116	2.281
	$n = 900$	−.8	.041	.039	.041	.042	.048	.046	.047	.047	0.027	0.011	0.027	0.036	0.028	0.016	0.028	0.037	0.021	0.018	0.021	0.034	0.026	0.022	0.026	0.036
		−.5	.048	.047	.048	.048	.055	.054	.055	.055	0.027	0.000	0.027	0.027	0.027	0.005	0.027	0.027	0.032	0.026	0.031	0.031	0.038	0.032	0.038	0.038
		−.2	.057	.057	.057	.057	.068	.067	.067	.067	0.027	0.000	0.028	0.028	0.027	0.000	0.028	0.028	0.048	0.041	0.049	0.049	0.058	0.050	0.058	0.059
		.0	.069	.068	.068	.068	.079	.079	.079	.079	0.032	0.000	0.032	0.032	0.032	0.000	0.032	0.032	0.070	0.059	0.069	0.069	0.083	0.071	0.083	0.083
		.2	.082	.083	.083	.083	.098	.097	.097	.097	0.044	0.000	0.045	0.046	0.048	0.000	0.047	0.049	0.104	0.091	0.105	0.106	0.128	0.110	0.125	0.126
		.5	.127	.126	.126	.126	.149	.148	.148	.148	0.132	0.036	0.134	0.138	0.156	0.098	0.155	0.160	0.245	0.220	0.245	0.248	0.293	0.264	0.291	0.296
		.8	.278	.274	.273	.274	.328	.322	.323	.323	0.926	0.780	0.893	0.930	1.089	0.939	1.055	1.100	1.189	1.068	1.154	1.187	1.405	1.277	1.365	1.410

Note: Emp. = empirical standard deviation; Pier. = estimated standard error based on the Pierce method; Del. = estimated standard error based on the delta method; Sim. = estimated standard error based on the simulation method.

Table D2. Empirical and Estimated Standard Errors: Nonnormal Case.

			Direct								Indirect								Total							
			X_1				X_2				X_1				X_2				X_1				X_2			
		λ_0	Emp.	Pier.	Del.	Sim.	Emp.	Pier.	Del.	Sim.	Emp.	Pier.	Del.	Sim.	Emp.	Pier.	Del.	Sim.	Emp.	Pier.	Del.	Sim.	Emp.	Pier.	Del.	Sim.
Rook	$n = 400$	−.8	.053	.042	**.053**	**.054**	.064	.058	**.064**	**.065**	0.039	0.004	**0.039**	**0.040**	0.042	0.023	**0.043**	**0.043**	0.020	**0.020**	**0.020**	**0.020**	0.028	**0.026**	**0.027**	**0.027**
		−.5	.062	.058	**.061**	**.062**	.078	**.076**	**.078**	**.078**	0.034	0.000	**0.034**	**0.034**	0.036	0.020	**0.037**	**0.037**	0.037	0.035	**0.037**	**0.037**	0.050	0.046	**0.049**	**0.049**
		−.2	.076	**.074**	**.075**	**.075**	.096	**.096**	**.096**	**.096**	0.032	0.000	**0.032**	**0.032**	0.032	0.000	**0.032**	**0.033**	0.063	0.056	**0.061**	**0.061**	0.082	0.076	**0.082**	**0.082**
		.0	.087	**.087**	**.087**	**.087**	.113	**.113**	**.113**	**.113**	0.033	0.000	**0.034**	0.035	0.034	0.000	**0.034**	0.036	0.087	0.079	**0.087**	**0.087**	0.117	0.107	**0.116**	**0.117**
		.2	.105	**.106**	**.106**	**.106**	.141	**.138**	**.138**	**.138**	0.049	0.000	**0.050**	**0.052**	0.057	0.018	**0.057**	**0.060**	0.131	0.120	**0.131**	**0.133**	0.180	0.161	**0.176**	**0.178**
		.5	.161	**.160**	**.159**	**.159**	.211	**.209**	**.209**	**.209**	0.156	0.087	**0.155**	**0.162**	0.201	0.138	**0.200**	**0.210**	0.301	0.271	**0.297**	**0.304**	0.399	0.365	**0.397**	**0.406**
		.8	.344	**.338**	**.337**	**.338**	.449	**.444**	**.445**	**.446**	1.015	0.825	**0.983**	**1.052**	1.336	1.130	**1.302**	**1.397**	1.340	1.190	**1.306**	**1.370**	1.768	1.599	**1.736**	**1.829**
	$n = 900$	−.8	.037	.031	**.037**	**.038**	.045	.041	**.045**	**.045**	0.027	0.008	**0.027**	**0.027**	0.030	0.018	**0.030**	**0.030**	0.015	**0.014**	**0.015**	**0.015**	0.019	**0.019**	**0.019**	**0.019**
		−.5	.044	.042	**.043**	**.044**	.055	**.054**	**.055**	**.055**	0.023	0.007	**0.023**	**0.023**	0.025	0.015	**0.025**	**0.025**	0.026	0.025	**0.026**	**0.026**	0.035	0.033	**0.035**	**0.035**
		−.2	.053	**.053**	**.053**	**.053**	.067	**.068**	**.068**	**.068**	0.021	0.000	**0.021**	**0.021**	0.021	0.000	**0.022**	**0.022**	0.044	0.041	**0.044**	**0.044**	0.058	0.054	**0.057**	**0.058**
		.0	.065	**.062**	**.062**	**.062**	.082	**.080**	**.080**	**.080**	0.022	0.000	**0.023**	**0.023**	0.023	0.000	**0.023**	**0.023**	0.065	0.058	**0.063**	**0.063**	0.084	0.077	**0.082**	**0.083**
		.2	.076	**.076**	**.076**	**.076**	.099	**.098**	**.098**	**.098**	0.034	0.000	**0.034**	**0.035**	0.039	0.011	**0.039**	**0.039**	0.095	0.087	**0.095**	**0.095**	0.126	0.116	**0.124**	**0.125**
		.5	.116	**.115**	**.115**	**.115**	.148	**.149**	**.149**	**.149**	0.113	0.070	**0.112**	**0.114**	0.140	0.106	**0.143**	**0.146**	0.219	0.200	**0.217**	**0.219**	0.280	0.265	**0.284**	**0.285**
		.8	.248	**.246**	**.245**	**.246**	.318	**.319**	**.319**	**.320**	0.741	0.649	**0.738**	**0.759**	0.960	0.868	**0.959**	**0.988**	0.978	0.906	**0.973**	**0.993**	1.268	1.199	**1.270**	**1.297**
Queen	$n = 400$	−.8	.064	.060	**.063**	.069	.076	**.074**	**.075**	.070	0.041	0.017	**0.041**	0.071	0.043	0.027	**0.043**	0.074	0.033	0.028	**0.033**	0.071	0.042	0.035	**0.041**	0.073
		−.5	.074	**.071**	**.073**	**.073**	.089	**.087**	**.087**	**.087**	0.041	0.000	**0.041**	**0.041**	0.042	0.009	**0.042**	**0.043**	0.049	0.040	**0.048**	**0.049**	0.061	0.051	**0.060**	**0.061**
		−.2	.088	**.088**	**.088**	**.088**	.107	**.106**	**.106**	**.106**	0.042	0.000	**0.042**	**0.043**	0.042	0.000	**0.042**	**0.044**	0.075	0.063	**0.075**	**0.076**	0.094	0.080	**0.093**	**0.094**
		.0	.105	**.104**	**.104**	**.104**	.127	**.126**	**.126**	**.126**	0.048	0.000	**0.047**	**0.049**	0.048	0.000	**0.047**	**0.050**	0.106	0.091	**0.106**	**0.107**	0.134	0.114	**0.131**	**0.133**
		.2	.129	**.127**	**.126**	**.127**	.157	**.154**	**.154**	**.154**	0.067	0.000	**0.067**	0.071	0.075	0.026	0.071	**0.077**	0.161	0.139	**0.160**	**0.163**	0.203	0.175	**0.197**	**0.201**
		.5	.196	**.194**	**.193**	**.194**	.238	**.236**	**.236**	**.236**	0.209	0.000	**0.199**	**0.214**	0.245	0.155	**0.239**	0.258	0.383	0.333	**0.373**	**0.385**	0.464	0.416	**0.459**	**0.473**
		.8	.429	**.419**	**.418**	**.420**	.524	**.512**	**.513**	**.515**	1.391	1.117	1.321	1.469	1.709	1.414	1.619	**1.793**	1.797	1.564	**1.724**	**1.858**	2.213	1.954	**2.114**	**2.275**
	$n = 900$	−.8	.042	.039	**.041**	**.042**	.048	**.046**	**.047**	**.048**	0.027	0.011	**0.027**	0.033	0.028	0.016	**0.028**	0.034	0.022	0.018	**0.021**	0.030	0.026	0.022	**0.026**	0.033
		−.5	.048	**.046**	**.047**	**.047**	.055	**.054**	**.055**	**.055**	0.027	0.000	**0.027**	**0.027**	0.027	0.005	**0.027**	**0.027**	0.032	0.026	**0.031**	**0.031**	0.037	0.032	**0.038**	**0.038**
		−.2	.059	**.057**	**.057**	**.057**	.066	**.066**	**.066**	**.066**	0.028	0.000	**0.028**	**0.028**	0.027	0.000	**0.028**	**0.028**	0.050	0.041	**0.049**	**0.049**	0.058	0.050	**0.058**	**0.059**
		.0	.068	**.067**	**.067**	**.067**	.078	**.078**	**.078**	**.079**	0.032	0.000	**0.032**	**0.032**	0.032	0.000	**0.031**	**0.032**	0.070	0.059	**0.069**	**0.069**	0.082	0.071	**0.083**	**0.083**
		.2	.082	**.082**	**.082**	**.082**	.098	**.096**	**.096**	**.096**	0.045	0.000	**0.045**	**0.046**	0.048	0.000	**0.047**	**0.049**	0.106	0.091	**0.105**	**0.106**	0.128	0.110	**0.125**	**0.126**
		.5	.127	**.126**	**.125**	**.125**	.147	**.147**	**.147**	**.147**	0.134	0.072	**0.133**	**0.138**	0.154	0.098	**0.155**	**0.161**	0.247	0.219	**0.245**	**0.248**	0.289	0.263	**0.291**	**0.295**
		.8	.280	**.272**	**.272**	**.273**	.325	**.321**	**.321**	**.322**	0.926	0.772	**0.882**	**0.922**	1.071	0.928	**1.042**	**1.087**	1.192	1.059	**1.144**	**1.179**	1.384	1.265	**1.353**	**1.394**

Note: Emp. = empirical standard deviation; Pier. = estimated standard error based on the Pierce method; Del. = estimated standard error based on the delta method; Sim. = estimated standard error based on the simulation method.

Table D3. Empirical Size of Wald Statistics: Normal Case.

			Direct						Indirect						Total					
			X_1			X_2			X_1			X_2			X_1			X_2		
		λ_0	T_p	T_d	T_s	T_p	T_d	T_s	T_p	T_d	T_s	T_p	T_d	T_s	T_p	T_d	T_s	T_p	T_d	T_s
Rook	$n = 400$	−.8	.132	.049	.046	.072	.048	.046	.700	.054	.049	.288	.049	.047	.060	.050	.051	.055	.046	.046
		−.5	.069	.055	.055	.057	.054	.053	.580	.056	.054	.315	.055	.053	.068	.052	.053	.074	.054	.055
		−.2	.052	.050	.051	.058	.057	.057	.189	.046	.043	.281	.054	.050	.073	.055	.055	.076	.054	.052
		.0	.053	.052	.050	.054	.054	.051	.071	.042	.038	.084	.035	.029	.077	.052	.049	.072	.053	.049
		.2	.048	.049	.049	.052	.052	.051	.166	.082	.072	.297	.092	.079	.079	.053	.050	.073	.050	.048
		.5	.054	.056	.056	.050	.050	.050	.331	.074	.065	.220	.064	.053	.086	.061	.056	.077	.048	.044
		.8	.055	.056	.055	.051	.051	.049	.128	.060	.044	.101	.055	.039	.094	.057	.047	.073	.047	.037
	$n = 900$	−.8	.113	.051	.048	.076	.050	.049	.542	.056	.053	.242	.058	.056	.058	.049	.049	.058	.051	.050
		−.5	.052	.043	.044	.059	.056	.056	.564	.044	.043	.273	.055	.054	.062	.046	.045	.072	.058	.057
		−.2	.052	.050	.050	.052	.052	.052	.181	.048	.048	.281	.058	.054	.076	.055	.054	.073	.054	.053
		.0	.052	.052	.052	.047	.047	.048	.061	.043	.041	.067	.039	.036	.071	.052	.051	.068	.046	.046
		.2	.054	.054	.055	.052	.052	.053	.165	.067	.063	.301	.070	.065	.078	.056	.053	.072	.051	.050
		.5	.045	.046	.048	.047	.047	.048	.245	.062	.058	.165	.054	.050	.070	.047	.044	.068	.046	.043
		.8	.051	.052	.052	.053	.052	.053	.103	.053	.046	.083	.049	.045	.077	.051	.047	.070	.049	.046
Queen	$n = 400$	−.8	.055	.047	.013	.052	.046	.016	.435	.048	.020	.251	.048	.017	.098	.047	.007	.090	.046	.009
		−.5	.057	.052	.050	.052	.050	.048	.487	.053	.038	.409	.055	.044	.111	.052	.043	.095	.048	.041
		−.2	.050	.049	.048	.054	.054	.053	.165	.045	.038	.215	.042	.035	.099	.051	.047	.092	.052	.050
		.0	.052	.052	.052	.058	.058	.058	.086	.038	.033	.103	.034	.028	.096	.051	.049	.102	.057	.054
		.2	.050	.051	.050	.049	.049	.051	.158	.096	.080	.240	.115	.094	.103	.056	.051	.093	.055	.049
		.5	.050	.052	.053	.055	.055	.054	.337	.085	.067	.266	.084	.064	.091	.055	.047	.085	.052	.046
		.8	.064	.064	.063	.053	.052	.053	.133	.059	.039	.105	.043	.027	.103	.057	.040	.080	.042	.028

(continued)

Table D3. (continued)

		Direct						Indirect						Total					
		X_1			X_2			X_1			X_2			X_1			X_2		
	λ_0	T_p	T_d	T_s	T_p	T_d	T_s	T_p	T_d	T_s	T_p	T_d	T_s	T_p	T_d	T_s	T_p	T_d	T_s
$n = 900$	−.8	.062	.050	.038	.062	.057	.046	.455	.048	.025	.276	.054	.032	.102	.052	.019	.103	.058	.025
	−.5	.057	.051	.051	.054	.053	.052	.519	.048	.048	.525	.052	.051	.109	.055	.054	.100	.054	.052
	−.2	.049	.049	.048	.055	.055	.056	.121	.044	.042	.146	.048	.045	.097	.050	.050	.097	.052	.050
	.0	.054	.054	.055	.048	.048	.046	.068	.047	.043	.073	.045	.038	.099	.053	.054	.092	.047	.046
	.2	.047	.047	.047	.055	.055	.056	.114	.065	.059	.184	.079	.069	.090	.050	.048	.092	.055	.053
	.5	.049	.050	.051	.048	.048	.051	.333	.057	.051	.250	.067	.058	.082	.049	.046	.075	.049	.047
	.8	.054	.054	.055	.057	.056	.055	.106	.052	.044	.096	.048	.040	.079	.049	.043	.078	.048	.040

Table D4. Empirical Size of Wald Statistics: Nonnormal Case.

			Direct						Indirect						Total					
			X_1			X_2			X_1			X_2			X_1			X_2		
		λ_0	T_p	T_d	T_s	T_p	T_d	T_s	T_p	T_d	T_s	T_p	T_d	T_s	T_p	T_d	T_s	T_p	T_d	T_s
Rook	$n = 400$	−.8	.124	.046	.043	.076	.046	.044	.677	.053	.048	.299	.050	.046	.053	.046	.047	.060	.049	.048
		−.5	.063	.053	.052	.054	.051	.050	.590	.053	.051	.326	.050	.047	.066	.051	.052	.070	.050	.051
		−.2	.055	.052	.053	.049	.049	.050	.189	.046	.043	.275	.053	.048	.076	.056	.055	.071	.048	.049
		.0	.051	.050	.050	.044	.044	.044	.059	.035	.030	.072	.028	.022	.072	.052	.049	.073	.049	.048
		.2	.046	.047	.046	.055	.055	.056	.154	.078	.071	.279	.091	.079	.075	.051	.049	.083	.056	.053
		.5	.053	.053	.052	.050	.050	.050	.319	.067	.058	.213	.061	.051	.080	.052	.046	.077	.051	.047
		.8	.053	.054	.054	.053	.053	.051	.130	.065	.051	.107	.055	.042	.089	.060	.050	.085	.053	.041
	$n = 900$	−.8	.109	.050	.049	.072	.049	.048	.539	.053	.051	.249	.051	.049	.060	.052	.053	.053	.045	.046
		−.5	.066	.053	.052	.052	.048	.050	.565	.056	.055	.279	.050	.050	.068	.052	.052	.066	.049	.051
		−.2	.056	.054	.053	.049	.047	.049	.183	.052	.051	.269	.053	.050	.074	.056	.057	.067	.052	.049
		.0	.055	.055	.055	.053	.053	.054	.050	.035	.034	.055	.032	.030	.079	.055	.056	.068	.049	.048
		.2	.047	.048	.047	.054	.054	.055	.161	.064	.061	.297	.073	.067	.069	.045	.043	.074	.052	.051
		.5	.052	.054	.054	.045	.045	.046	.247	.058	.053	.152	.050	.045	.075	.054	.052	.065	.046	.044
		.8	.055	.055	.054	.050	.050	.048	.097	.053	.046	.084	.052	.045	.073	.052	.048	.071	.051	.045
Queen	$n = 400$	−.8	.063	.050	.016	.059	.055	.017	.443	.050	.023	.249	.053	.017	.100	.049	.007	.097	.054	.009
		−.5	.056	.051	.049	.054	.053	.051	.474	.052	.039	.427	.050	.042	.107	.053	.047	.102	.054	.045
		−.2	.048	.046	.046	.047	.047	.047	.162	.044	.039	.208	.046	.040	.099	.049	.047	.096	.049	.048
		.0	.050	.050	.050	.050	.050	.051	.086	.037	.034	.100	.032	.026	.097	.050	.048	.097	.055	.051
		.2	.050	.051	.050	.048	.048	.048	.160	.095	.078	.227	.107	.086	.092	.051	.047	.093	.053	.049
		.5	.055	.055	.055	.050	.050	.052	.339	.088	.071	.262	.087	.065	.094	.056	.047	.082	.052	.044
		.8	.053	.053	.054	.054	.053	.052	.123	.052	.034	.111	.047	.030	.090	.050	.034	.086	.045	.030

(continued)

Table D4. (continued)

		Direct						Indirect						Total					
		X_1			X_2			X_1			X_2			X_1			X_2		
	λ_0	T_p	T_d	T_s	T_p	T_d	T_s	T_p	T_d	T_s	T_p	T_d	T_s	T_p	T_d	T_s	T_p	T_d	T_s
$n = 900$	−.8	.066	.055	.042	.059	.054	.047	.465	.054	.031	.279	.056	.034	.102	.053	.021	.101	.054	.022
	−.5	.059	.053	.054	.052	.051	.050	.526	.050	.050	.535	.054	.054	.112	.056	.053	.098	.046	.046
	−.2	.057	.056	.054	.049	.049	.049	.133	.047	.046	.149	.042	.039	.104	.050	.049	.094	.049	.048
	.0	.055	.055	.055	.048	.048	.048	.073	.053	.049	.077	.053	.046	.100	.053	.051	.088	.049	.048
	.2	.048	.048	.048	.055	.055	.055	.117	.067	.060	.189	.084	.075	.095	.049	.047	.094	.056	.053
	.5	.050	.051	.052	.048	.048	.050	.337	.063	.055	.250	.065	.057	.080	.049	.048	.080	.052	.049
	.8	.056	.056	.056	.055	.055	.055	.109	.061	.050	.098	.052	.046	.081	.056	.050	.078	.052	.046

Table D5. True Effects Values.

			ADI		AII		ATI	
		λ_0	X_1	X_2	X_1	X_2	X_1	X_2
Rook	$n = 400$	−.8	.642	−.642	−0.364	0.364	0.278	−0.278
		−.5	.538	−.538	−0.205	0.205	0.333	−0.333
		−.2	.505	−.505	−0.089	0.089	0.417	−0.417
		.0	.500	−.500	0.000	0.000	0.500	−0.500
		.2	.505	−.505	0.120	−0.120	0.625	−0.625
		.5	.538	−.538	0.462	−0.462	1.000	−1.000
		.8	.642	−.642	1.858	−1.858	2.500	−2.500
	$n = 900$	−.8	.639	−.639	−0.362	0.362	0.278	−0.278
		−.5	.537	−.537	−0.204	0.204	0.333	−0.333
		−.2	.505	−.505	−0.089	0.089	0.417	−0.417
		.0	.500	−.500	0.000	0.000	0.500	−0.500
		.2	.505	−.505	0.120	−0.120	0.625	−0.625
		.5	.537	−.537	0.463	−0.463	1.000	−1.000
		.8	.639	−.639	1.861	−1.861	2.500	−2.500
Queen	$n = 400$	−.8	.537	−.537	−0.260	0.260	0.278	−0.278
		−.5	.515	−.515	−0.181	0.181	0.333	−0.333
		−.2	.502	−.502	−0.086	0.086	0.417	−0.417
		.0	.500	−.500	0.000	0.000	0.500	−0.500
		.2	.503	−.503	0.122	−0.122	0.625	−0.625
		.5	.522	−.522	0.478	−0.478	1.000	−1.000
		.8	.590	−.590	1.910	−1.910	2.500	−2.500
	$n = 900$	−.8	.537	−.537	−0.259	0.259	0.278	−0.278
		−.5	.514	−.514	−0.181	0.181	0.333	−0.333
		−.2	.502	−.502	−0.086	0.086	0.417	−0.417
		.0	.500	−.500	0.000	0.000	0.500	−0.500
		.2	.503	−.503	0.122	−0.122	0.625	−0.625
		.5	.522	−.522	0.478	−0.478	1.000	−1.000
		.8	.588	−.588	1.912	−1.912	2.500	−2.500

Note: ADI = Average Direct Impact; AII = Average Indirect Impact; ATI = Average Total Impact.

Table D6. Empirical Power of Wald Statistics: Normal Case.

			Direct						Indirect						Total					
			X_1			X_2			X_1			X_2			X_1			X_2		
		λ_0	T_p	T_d	T_s	T_p	T_d	T_s	T_p	T_d	T_s	T_p	T_d	T_s	T_p	T_d	T_s	T_p	T_d	T_s
Rook	$n = 400$	−.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
		−.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
		−.2	1.000	1.000	1.000	0.999	0.999	0.999	0.906	0.860	0.851	0.912	0.858	0.842	1.000	1.000	1.000	0.999	0.999	0.999
		.0	1.000	1.000	1.000	0.993	0.993	0.993	0.071	0.042	0.038	0.084	0.035	0.029	1.000	1.000	1.000	0.993	0.992	0.992
		.2	0.998	0.998	0.998	0.951	0.951	0.951	0.788	0.693	0.646	0.811	0.479	0.389	0.998	0.998	0.998	0.953	0.948	0.945
		.5	0.916	0.917	0.917	0.716	0.716	0.715	0.942	0.897	0.874	0.769	0.644	0.588	0.924	0.914	0.908	0.729	0.702	0.686
		.8	0.471	0.473	0.470	0.287	0.284	0.285	0.531	0.432	0.373	0.348	0.231	0.171	0.511	0.453	0.413	0.323	0.255	0.216
	$n = 900$	−.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
		−.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
		−.2	1.000	1.000	1.000	1.000	1.000	1.000	0.997	0.995	0.994	0.997	0.995	0.994	1.000	1.000	1.000	1.000	1.000	1.000
		.0	1.000	1.000	1.000	1.000	1.000	1.000	0.061	0.043	0.041	0.067	0.039	0.036	1.000	1.000	1.000	1.000	1.000	1.000
		.2	1.000	1.000	1.000	0.999	0.999	0.999	0.989	0.987	0.984	0.990	0.974	0.966	1.000	1.000	1.000	0.999	0.999	0.999
		.5	0.998	0.998	0.999	0.950	0.950	0.950	0.999	0.998	0.996	0.958	0.944	0.937	0.998	0.998	0.998	0.952	0.948	0.947
		.8	0.736	0.736	0.734	0.515	0.514	0.512	0.757	0.721	0.699	0.543	0.487	0.459	0.748	0.729	0.713	0.530	0.498	0.478
Queen	$n = 400$	−.8	1.000	1.000	0.406	1.000	1.000	0.408	1.000	1.000	0.171	1.000	1.000	0.172	1.000	1.000	0.177	1.000	1.000	0.178
		−.5	1.000	1.000	0.991	1.000	1.000	0.990	1.000	1.000	0.959	1.000	1.000	0.958	1.000	1.000	0.962	1.000	1.000	0.963
		−.2	1.000	1.000	1.000	0.997	0.997	0.996	0.710	0.594	0.576	0.724	0.579	0.551	1.000	1.000	1.000	0.997	0.996	0.996
		.0	0.998	0.998	0.998	0.975	0.975	0.976	0.086	0.038	0.033	0.103	0.034	0.028	0.998	0.998	0.998	0.978	0.974	0.972
		.2	0.980	0.980	0.980	0.898	0.898	0.898	0.510	0.275	0.200	0.539	0.151	0.085	0.982	0.979	0.975	0.906	0.893	0.884
		.5	0.766	0.768	0.766	0.590	0.590	0.590	0.828	0.678	0.592	0.677	0.453	0.355	0.783	0.757	0.737	0.616	0.570	0.545
		.8	0.288	0.289	0.287	0.215	0.214	0.212	0.359	0.235	0.156	0.275	0.153	0.078	0.333	0.261	0.205	0.250	0.180	0.127
	$n = 900$	−.8	1.000	1.000	0.908	1.000	1.000	0.909	1.000	1.000	0.596	1.000	1.000	0.599	1.000	1.000	0.599	1.000	1.000	0.601
		−.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
		−.2	1.000	1.000	1.000	1.000	1.000	1.000	0.931	0.902	0.899	0.932	0.909	0.904	1.000	1.000	1.000	1.000	1.000	1.000
		.0	1.000	1.000	1.000	1.000	1.000	1.000	0.068	0.047	0.043	0.073	0.045	0.038	1.000	1.000	1.000	1.000	1.000	1.000
		.2	1.000	1.000	1.000	1.000	1.000	1.000	0.862	0.824	0.804	0.871	0.775	0.736	1.000	1.000	1.000	1.000	1.000	1.000
		.5	0.987	0.987	0.986	0.936	0.936	0.935	0.990	0.983	0.979	0.949	0.920	0.907	0.987	0.986	0.985	0.940	0.932	0.929
		.8	0.574	0.575	0.573	0.451	0.450	0.448	0.611	0.549	0.507	0.488	0.408	0.368	0.597	0.561	0.534	0.476	0.428	0.395

Table D7. Empirical Power of Wald Statistics: Nonnormal Case.

			Direct						Indirect						Total					
			X_1			X_2			X_1			X_2			X_1			X_2		
		λ_0	T_p	T_d	T_s	T_p	T_d	T_s	T_p	T_d	T_s	T_p	T_d	T_s	T_p	T_d	T_s	T_p	T_d	T_s
Rook	$n = 400$	−.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
		−.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
		−.2	1.000	1.000	1.000	1.000	1.000	0.999	0.898	0.842	0.834	0.906	0.842	0.828	1.000	1.000	1.000	1.000	0.999	0.999
		.0	1.000	1.000	1.000	0.993	0.993	0.993	0.059	0.035	0.030	0.072	0.028	0.022	1.000	1.000	1.000	0.994	0.992	0.992
		.2	0.998	0.998	0.998	0.948	0.948	0.948	0.796	0.699	0.648	0.814	0.489	0.395	0.998	0.998	0.998	0.950	0.946	0.943
		.5	0.919	0.919	0.919	0.728	0.727	0.726	0.946	0.901	0.875	0.777	0.658	0.605	0.927	0.917	0.911	0.740	0.717	0.701
		.8	0.470	0.472	0.470	0.301	0.300	0.301	0.531	0.433	0.370	0.354	0.243	0.181	0.510	0.456	0.414	0.335	0.267	0.228
	$n = 900$	−.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
		−.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
		−.2	1.000	1.000	1.000	1.000	1.000	1.000	0.997	0.995	0.995	0.997	0.995	0.995	1.000	1.000	1.000	1.000	1.000	1.000
		.0	1.000	1.000	1.000	1.000	1.000	1.000	0.050	0.035	0.034	0.055	0.032	0.030	1.000	1.000	1.000	1.000	1.000	1.000
		.2	1.000	1.000	1.000	1.000	1.000	0.999	0.990	0.987	0.984	0.990	0.974	0.967	1.000	1.000	1.000	1.000	1.000	0.999
		.5	0.996	0.996	0.996	0.953	0.953	0.951	0.997	0.996	0.995	0.960	0.943	0.937	0.996	0.996	0.996	0.956	0.951	0.948
		.8	0.734	0.735	0.735	0.513	0.513	0.509	0.758	0.721	0.699	0.543	0.482	0.455	0.750	0.727	0.715	0.532	0.496	0.477
Queen	$n = 400$	−.8	1.000	1.000	0.393	1.000	1.000	0.394	1.000	1.000	0.169	1.000	1.000	0.171	1.000	1.000	0.175	1.000	1.000	0.177
		−.5	1.000	1.000	0.992	1.000	1.000	0.993	1.000	0.999	0.963	1.000	0.999	0.963	1.000	1.000	0.966	1.000	1.000	0.966
		−.2	1.000	1.000	1.000	0.997	0.997	0.996	0.713	0.589	0.570	0.729	0.583	0.557	1.000	1.000	1.000	0.997	0.996	0.996
		.0	0.997	0.997	0.997	0.978	0.978	0.978	0.086	0.037	0.034	0.100	0.032	0.026	0.997	0.997	0.997	0.979	0.977	0.975
		.2	0.977	0.977	0.977	0.902	0.902	0.899	0.509	0.265	0.187	0.530	0.153	0.088	0.980	0.977	0.972	0.908	0.895	0.885
		.5	0.767	0.768	0.768	0.601	0.601	0.601	0.832	0.681	0.591	0.689	0.457	0.358	0.786	0.759	0.740	0.625	0.580	0.548
		.8	0.305	0.307	0.305	0.208	0.207	0.204	0.375	0.247	0.154	0.267	0.146	0.078	0.350	0.279	0.215	0.244	0.175	0.121
	$n = 900$	−.8	1.000	1.000	0.911	1.000	1.000	0.912	1.000	1.000	0.604	1.000	1.000	0.607	1.000	1.000	0.607	1.000	1.000	0.611
		−.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
		−.2	1.000	1.000	1.000	1.000	1.000	1.000	0.928	0.894	0.890	0.930	0.900	0.895	1.000	1.000	1.000	1.000	1.000	1.000
		.0	1.000	1.000	1.000	1.000	1.000	1.000	0.073	0.053	0.049	0.077	0.053	0.046	1.000	1.000	1.000	1.000	1.000	1.000
		.2	1.000	1.000	1.000	0.999	0.999	0.999	0.865	0.834	0.814	0.879	0.780	0.749	1.000	1.000	1.000	0.999	0.999	0.999
		.5	0.985	0.985	0.985	0.941	0.941	0.941	0.989	0.982	0.979	0.952	0.925	0.912	0.986	0.985	0.984	0.944	0.938	0.934
		.8	0.573	0.573	0.572	0.437	0.437	0.435	0.612	0.541	0.501	0.478	0.404	0.369	0.599	0.555	0.529	0.461	0.417	0.395

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Notes

1. For parameter spaces suggested for λ_0, see Anselin (1988), LeSage and Pace (2009), Kelejian and Prucha (2010), and Elhorst, Lacombe, and Piras (2012).
2. The consistency of all plug-in estimators in this section can be established by using Liu, Lee, and Bollinger (2010; lemma D.11).
3. We do not consider the Bayesian approach suggested by LeSage and Pace (2009) as our focus is on the classical approach.
4. The percentage deviation is calculated by 100 $\times$ (estimated standard error $-$ empirical standard deviation)/empirical standard deviation.
5. A function written in Matlab is available at https://sites.google.com/site/gcsuleymantaspinar/home/software. The function returns impact measure estimates and their standard errors.

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