LIPSCHITZ REGULARITY OF THE EIGENFUNCTIONS ON OPTIMAL DOMAINS

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ABSTRACT. We study the optimal sets $\Omega^* \subseteq \mathbb{R}^d$ for spectral functionals of the form $F\big(\lambda_1(\Omega), \ldots, \lambda_p(\Omega)\big)$, which are bi-Lipschitz with respect to each of the eigenvalues $\lambda_1(\Omega)$, $\lambda_2(\Omega)$, ..., $\lambda_p(\Omega)$ of the Dirichlet Laplacian on Ω, a prototype being the problem

$$\min\big\{ \lambda_1(\Omega) + \cdots + \lambda_p(\Omega) \, : \, \Omega \subseteq \mathbb{R}^d, \ |\Omega| = 1 \big\}.$$

We prove the Lipschitz regularity of the eigenfunctions $u_1, \ldots, u_p$ of the Dirichlet Laplacian on the optimal set Ω^* and, as a corollary, we deduce that Ω^* is open. For functionals depending only on a generic subset of the spectrum, as for example $\lambda_k(\Omega)$, our result proves only the existence of a Lipschitz continuous eigenfunction in correspondence to each of the eigenvalues involved.

1. INTRODUCTION

In this paper we study the domains of prescribed volume, which are optimal for functionals depending on the spectrum of the Dirichlet Laplacian. Precisely, we consider shape optimization problems of the form

$$\min\Big\{ F\big(\lambda_1(\Omega), \ldots, \lambda_p(\Omega)\big) \, : \, \Omega \subseteq \mathbb{R}^d, \ |\Omega| = 1 \Big\}, \tag{1.1}$$

where $F : \mathbb{R}^p \to \mathbb{R}$ is a given continuous function, increasing in each variable, and $\lambda_k(\Omega)$, for $k = 1, \ldots, p$, denotes the k-th eigenvalue of the Dirichlet Laplacian on Ω, i.e. the k-th element of the spectrum of the Dirichlet Laplacian (due to the volume constraint $|\Omega| = 1$, the Dirichlet Laplacian on Ω has compact resolvent and its spectrum is discrete).

The optimization problems of the form (1.1) naturally arise in the study of physical phenomena as, for example, heat diffusion or wave propagation inside a domain $\Omega \subseteq \mathbb{R}^d$. Despite of their simple formulation, these problems turn out to be quite challenging and their analysis usually depends on sophisticated variational techniques. Even the question of the existence of a minimizer for the simplest spectral optimization problem

$$\min\Big\{ \lambda_k(\Omega) \, : \, \Omega \subseteq \mathbb{R}^d, \ |\Omega| = 1 \Big\}, \tag{1.2}$$

was answered only recently for general $k \in \mathbb{N}$ (see [7] and [23]). This question was first formulated in the 19-th century by Lord Rayleigh in his treatise *The Theory of Sound* [26] and it was related to the specific case $k = 1$. It was proved only in the 1920's by Faber and Krahn that the minimizer in this case is the ball. From this result one can easily deduce the Krahn-Szegö inequality, which states that a union of two equal and disjoint balls is optimal for (1.2) with $k = 2$, i.e. it has the smallest second eigenvalue λ_2 among all sets of prescribed measure. An explicit construction of an optimal set for higher eigenvalues is an extremely difficult task. Balls are not always optimal, in fact it was proved by Keller and Wolf in 1994 (see [27]) that a union of disjoint balls is not optimal for λ_{13} in two dimensions. It was recently proved by Berger and Oudet (private communication) that the later result holds for all $k \in \mathbb{N}$ large enough, which confirmed the previous numerical results obtained in [24] and [3].

The classical variational approach to prove existence and regularity of minimizers failed to provide a solution to the spectral problems (1.1) until the 1990's, the main reason being the lack of an appropriate topology on the space of domains $\Omega \subseteq \mathbb{R}^d$. A suitable convergence,

called γ-*convergence*, was introduced by Dal Maso and Mosco (see [15, 16]) in the 1980's and was used by Buttazzo and Dal Maso (see [13]) for proving in 1993 a very general existence result for (1.1), under the additional constraint that $\Omega \subseteq D$, being $D \subseteq \mathbb{R}^d$ a given open and bounded set (the *box*). The presence of this geometric obstacle provided the necessary compactness, needed to obtain the existence of an optimal domain in the class of *quasi-open* sets, which are the superlevel sets $\{u > 0\}$ of the Sobolev functions $u \in \mathcal{H}^1(\mathbb{R}^d)$ (exactly as open sets are the superlevel sets of continuous functions). The proof of existence of a quasi-open minimizer for (1.2) and, more generally, for (1.1) in the entire space $\mathbb{R}^d$ was concluded in 2011 with the independent results of Bucur (see [7]) and Mazzoleni and Pratelli (see [23]). Moreover, it was proved that the optimal sets are bounded (see [7] and [21]) and of finite perimeter (see [7]).

The regularity of the optimal sets or of the corresponding eigenfunctions turned out to be quite a difficult question, due to the min-max nature of the spectral cost functionals, and it remained as an open problem since the general Buttazzo–Dal Maso existence theorem. The only result that provides the complete regularity of the free boundary $\partial\Omega$ of the optimal set Ω considers only the minimizers of (1.2) in the special case $k = 1$ (and under the additional constraint that Ω is contained in a box): more precisely, in this case Briançon and Lamboley [5] proved that the free boundary of an optimal set is smooth. The implementation of this result for higher eigenvalues presents some major difficulties since the techniques, developed by Alt and Caffarelli in [1], used in the proof are exclusive for functionals defined through a minimization, and not a min-max procedure, on the Sobolev space $\mathcal{H}_0^1(\Omega)$.

In this paper we study the regularity of the eigenfunctions (or state functions) on the optimal set Ω^* for the problem (1.1). Our main tool is a result proved by Briançon, Hayouni and Pierre [6], inspired by the pioneering work of Alt and Caffarelli (see [1]) on the regularity for a free boundary problem. It states that a function $u \in \mathcal{H}^1(\mathbb{R}^d)$, satisfying an elliptic PDE on the set $\Omega = \{|u| > 0\}$, is Lipschitz continuous on the whole $\mathbb{R}^d$ if it satisfies the following *quasi-minimality* property:

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, dx \leq \int_{\mathbb{R}^d} |\nabla v|^2 \, dx + cr^d, \qquad \forall v \in \mathcal{H}^1(\mathbb{R}^d) \ \text{s.t.} \ u = v \text{ on } \mathbb{R}^d \setminus B_r(x), \qquad (1.3)$$

for every ball $B_r(x) \subseteq \mathbb{R}^d$ (see Theorem 3.3).

Since the variational characterization of the eigenvalue λ_k is given through a min-max procedure, the transfer of the minimality properties of Ω to an eigenfunction u_k is a non-trivial task. In fact, it can be proved that the eigenfunction u_k is a quasi-minimizer in the sense of (1.3), provided that the eigenvalue $\lambda_k(\Omega^*)$ is simple. But since this is not expected to be true in general, we need to use an approximation procedure with sets Ω_ε, which are solutions of suitably modified spectral optimization problems. We will study the Lipschitz continuity of the eigenfunctions u_k^ε on each Ω_ε and then pass to the limit to recover the Lipschitz continuity of u_k on Ω^* (see Theorem 5.3). The uniformity of the Lipschitz constants will be assured, roughly speaking, by the optimality conditions that the state functions satisfy on the free boundary of Ω_ε.

The main result of the paper is Theorem 5.6, which applies to shape supersolutions of functionals of the form

$$\Omega \mapsto F\big(\lambda_{k_1}(\Omega), \ldots, \lambda_{k_p}(\Omega)\big) + |\Omega|,$$

where $F : \mathbb{R}^p \to \mathbb{R}$ is increasing and bi-Lipschitz in each variable. Precisely, if a set Ω^* satisfies

$$F\big(\lambda_{k_1}(\Omega^*), \ldots, \lambda_{k_p}(\Omega^*)\big) + |\Omega^*| \leq F\big(\lambda_{k_1}(\Omega), \ldots, \lambda_{k_p}(\Omega)\big) + |\Omega|,$$

for all measurable sets Ω containing Ω^*, then there exists a family of L^2-orthonormal eigenfunctions $u_{k_1}, \ldots, u_{k_p}$, corresponding respectively to $\lambda_{k_1}(\Omega), \ldots, \lambda_{k_p}(\Omega)$, which are Lipschitz continuous on $\mathbb{R}^d$.

In some particular cases, as for example linear combinations of the form

$$F\big(\lambda_1(\Omega), \ldots, \lambda_p(\Omega)\big) = \sum_{i=1}^{p} \alpha_i \lambda_i(\Omega)\,,$$

with strictly positive α_i for every $i = 1, \ldots, p$, the minimizers are then proved to be open sets (see Corollary 6.3). For this particular case, at least in two dimensions, it is also possible to give a more direct proof which does not rely on the Alt–Caffarelli regularity techniques (see [22]).

In [7], the analysis of shape subsolutions gave some qualitative information on the optimal sets, in particular their boundedness and finiteness of the perimeter. Nevertheless, it is known that a subsolution may not be equivalent to an open set. Continuity of the state functions in free boundary problems relies, in general, on outer perturbations. Consequently the study of supersolutions became a fundamental target, which is partially attained in this paper. In the case of subsolutions, the problem could be reduced to the analysis of a unique state function, precisely the torsion function, by controlling the variation of the k-th eigenvalue for an inner geometric domain perturbation with the variation of the torsional rigidity. As far as we know, an analogous approach for the analysis of shape supersolutions can not be performed since one can not control the variation of the torsional rigidity by the variation of the k-th eigenvalue.

This paper is organized as follows: in Section 2 we recall some tools about Sobolev-like spaces, capacity and γ-convergence; in Section 3 we deal with the Lipschitz regularity for quasi-minimizers of the Dirichlet energy and then, in Section 4, we apply these results to eigenfunctions of the Dirichlet Laplacian corresponding to a simple eigenvalue. Then in Section 5 we introduce the notion of shape supersolution and we prove our main results Theorem 5.3 and Theorem 5.6, concerning the Lipschitz regularity of the eigenfunctions associated to the general problem (1.1). At last, in Section 6, we show that for some functionals the optimal sets are open.

2. Preliminary results

We will use the following notation and conventions throughout the paper:

- C_d denotes a constant depending only on the dimension d, which might increase from line to line;
- ω_d denotes the volume of the unit ball in $\mathbb{R}^d$ and thus $d\omega_d$ is the area of the unit sphere;
- $\mathcal{H}^m$ denotes the m-dimensional Hausdorff measure in $\mathbb{R}^d$;
- if the domain of integration is not specified, then it is assumed to be the whole space $\mathbb{R}^d$;
- the mean value of a function $u : \Omega \to \mathbb{R}$ is denoted by

$$\fint_\Omega u\,dx := \frac{1}{|\Omega|} \int_\Omega u\,dx\,.$$

2.1. **Sobolev spaces and PDEs on measurable domains.** For a measurable set $\Omega \subseteq \mathbb{R}^d$ we define the Sobolev-like space $\widetilde{\mathcal{H}}_0^1(\Omega) \subseteq \mathcal{H}^1(\mathbb{R}^d)$ as

$$\widetilde{\mathcal{H}}_0^1(\Omega) = \left\{ u \in \mathcal{H}^1(\mathbb{R}^d) \ : \ |\{u \neq 0\} \setminus \Omega| = 0 \right\}.$$

If Ω is an open set with a Lipschitz boundary, then $\widetilde{\mathcal{H}}_0^1(\Omega)$ coincides with the usual Sobolev space $\mathcal{H}_0^1(\Omega)$, which is the closure of $C_c^\infty(\Omega)$ with respect to the norm $\|u\|_{\mathcal{H}^1} := \big(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\big)^{1/2}$. For a generic set (even an open one) the equality above is false and only the inclusion $\mathcal{H}_0^1(\Omega) \subseteq \widetilde{\mathcal{H}}_0^1(\Omega)$ in general holds.

Remark 2.1. *Another natural way to extend the notion of a Sobolev space to non-open domains $\Omega \subseteq \mathbb{R}^d$ is the following*

$$\widehat{\mathcal{H}}_0^1(\Omega) := \left\{ u \in \mathcal{H}^1(\mathbb{R}^d) : \ \mathrm{cap}(\{u \neq 0\} \setminus \Omega) = 0 \right\}, \tag{2.1}$$

where for every $E \subseteq \mathbb{R}^d$ the capacity *of E is defined as*

$$\mathrm{cap}(E) = \min \left\{ \|v\|_{\mathcal{H}^1(\mathbb{R}^d)}^2 \ : \ v \in \mathcal{H}^1(\mathbb{R}^d), \ v \geq 1 \ a.e. \ in \ a \ neighborhood \ of \ E \right\}.$$

The advantage of this definition is that it coincides with the usual one for every open set $\Omega \subseteq \mathbb{R}^d$, not just for Lipschitz ones (see [20]). Nevertheless, the spaces $\widetilde{\mathcal{H}}_0^1(\Omega)$ appear more suitable for the study of the regularity of minimizers of spectral functionals, hence we will use these spaces in the present work.

Since $\widetilde{\mathcal{H}}_0^1(\Omega)$ is a closed subspace of $\mathcal{H}^1(\mathbb{R}^d)$, one can define the Dirichlet Laplacian on Ω through the weak solutions of elliptic problems on Ω. More precisely, given a set of finite measure $\Omega \subseteq \mathbb{R}^d$ and a function $f \in L^2(\Omega)$, we say that the function $u \in \mathcal{H}^1(\mathbb{R}^d)$ satisfies the equation

$$-\Delta u = f \qquad \text{in } \Omega \tag{2.2}$$

if $u \in \widetilde{\mathcal{H}}_0^1(\Omega)$ and for every $v \in \widetilde{\mathcal{H}}_0^1(\Omega)$ it is $\langle \Delta u + f, v \rangle = 0$, where for every $v \in \mathcal{H}^1(\mathbb{R}^d)$ we set

$$\langle \Delta u + f, v \rangle := - \int_{\mathbb{R}^d} \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^d} f v \, dx. \tag{2.3}$$

Equivalently, u is a solution of (2.2) if it is a minimizer in $\widetilde{\mathcal{H}}_0^1(\Omega)$ of the functional $J_f : \mathcal{H}^1(\mathbb{R}^d) \to \mathbb{R}$ defined as

$$J_f(v) := \frac{1}{2} \int_{\mathbb{R}^d} |\nabla v|^2 \, dx - \int_{\mathbb{R}^d} v f \, dx, \qquad v \in \mathcal{H}^1(\mathbb{R}^d).$$

Remark 2.2. *It is straightforward to check that, if u is a solution of (2.2) in Ω, then u also belongs to $\widetilde{\mathcal{H}}_0^1(\{u \neq 0\})$, and it is a solution of the equation*

$$-\Delta u = f \qquad in \ \{u \neq 0\}.$$

If Ω is an open set with smooth boundary and u is a solution of (2.2), then the operator $\Delta u + f : \mathcal{H}^1(\mathbb{R}^d) \to \mathbb{R}$ defined in (2.3) has the simple expression

$$\langle \Delta u + f, v \rangle = \int_{\partial \Omega} \frac{\partial u}{\partial n} v \, d\mathcal{H}^{d-1}, \qquad \forall \, v \in \mathcal{H}^1(\mathbb{R}^d).$$

We can prove now that, in the general case of a measurable set Ω, the operator $\Delta u + f$ is a measure concentrated on the boundary of Ω.

Proposition 2.3. *Let $\Omega \subseteq \mathbb{R}^d$ be a set of finite measure, $f \in L^2(\Omega)$, and $u \in \widetilde{\mathcal{H}}_0^1(\Omega)$ be a solution of (2.2). Then there is a capacitary Radon measure μ (this means, $\mu(E) = 0$ for every set E with zero capacity) such that, for every $v \in \mathcal{H}^1(\mathbb{R}^d)$, one has*

$$\langle \Delta u + f, v \rangle = \int_{\mathbb{R}^d} v \, d\mu. \tag{2.4}$$

Moreover, μ satisfies the following properties:

  (i) If $u \geq 0$, then the measure μ is positive.
  (ii) The support of μ is contained in the topological bondary $\partial \Omega$ of Ω.

Proof. Suppose first that $u \geq 0$, and define the functions $p_n : \mathbb{R}^+ \to [0,1]$ as

$$p_n(t) = nt \quad \text{if } t \in [0, 1/n], \qquad\qquad p_n(t) = 1 \quad \text{if } t \geq 1/n.$$

Then, for every non-negative $\varphi \in C_c^\infty(\mathbb{R}^d)$, since $p_n(u)\varphi \in \widetilde{\mathcal{H}}_0^1(\Omega)$ we can evaluate

$$
\begin{aligned}
0 = \langle \Delta u + f, p_n(u)\varphi \rangle &= \int_{\mathbb{R}^d} -\nabla u \cdot \nabla(p_n(u)\varphi) + f p_n(u)\varphi \, dx \\
&= \int_{\mathbb{R}^d} -p_n(u)\nabla u \cdot \nabla \varphi - p_n'(u)|\nabla u|^2 \varphi + f p_n(u)\varphi \, dx \\
&\leq \int_{\mathbb{R}^d} p_n(u)\left(-\nabla u \cdot \nabla \varphi + f\varphi \right) dx \xrightarrow[n \to \infty]{} \int_{\mathbb{R}^d} -\nabla u \cdot \nabla \varphi + f\varphi \, dx = \langle \Delta u + f, \varphi \rangle.
\end{aligned}
$$

The functional $\Delta u + f$ on $C_c^\infty(\mathbb{R}^d)$ is then a positive distribution; keeping in mind that a positive distribution is always a measure, we get a positive Radon measure μ such that the equality (2.4) is true for every smooth function v. Thanks to the definition (2.3), an immediate approximation argument shows that μ is a capacitary measure, then we get at once that the right term in (2.4) makes sense also for any $v \in \mathcal{H}^1(\mathbb{R}^d)$, and that the equation (2.4) is true in $\mathcal{H}^1(\mathbb{R}^d)$.

Consider now the case of a generic function $u \in \widetilde{\mathcal{H}}_0^1(\Omega)$, and call $\Omega^\pm = \{u \gtrless 0\}$. It is immediate to observe that u^+ solves the equation $-\Delta u = f_{|\Omega^+}$ in Ω^+, thus the argument above implies that $\Delta u^+ + f_{|\Omega^+}$ corresponds to a positive capacitary measure μ^+, and the very same argument shows also that $\Delta u^- + f_{|\Omega^-}$ is a negative capacitary measure μ^-. Since it is straightforward to check that $\left(\Delta u^+ + f_{|\Omega^+}\right) + \left(\Delta u^- + f_{|\Omega^-}\right) = \Delta u + f$, the claim is then proved with the (signed) measure $\mu = \mu^+ + \mu^-$.

The fact that μ is concentrated on the topological boundary of Ω comes trivially by approximation, since for every smooth φ concentrated either in the interior of Ω or in the interior of $\mathbb{R}^d \setminus \Omega$ one has $\langle \Delta u + f, \varphi \rangle = 0$ by definition. $\qquad \square$

2.2. Solutions of PDEs with bounded data.

In this subsection we quickly recall some properties of the solutions $u \in \widetilde{\mathcal{H}}_0^1(\Omega)$ of (2.2), in the case when Ω is a measurable set with finite measure, and the data $f \in L^\infty(\Omega)$. First of all, an L^∞ estimate for u holds, namely

$$
\|u\|_{L^\infty} \leq \frac{|\Omega|^{2/d}\|f\|_{L^\infty}}{2d\omega_d^{2/d}}, \tag{2.5}
$$

and the equality achieved when Ω is a ball and $f \equiv \text{const}$ on Ω (see for instance [25]). Moreover, since the function

$$
v(x) = u(x) + \frac{\|f\|_{L^\infty}}{2d}|x|^2,
$$

is clearly subharmonic on $\mathbb{R}^d$, because $\Delta v = \Delta u + \|f\|_{L^\infty} = -f + \|f\|_{L^\infty} \geq 0$, then it is simple to notice that every point of $\mathbb{R}^d$ is a Lebesgue point for u. More in detail, whenever $v \in \mathcal{H}^1(\mathbb{R}^d)$ is a function such that Δv is a measure on $\mathbb{R}^d$, then the following estimate holds for any $x \in \mathbb{R}^d$ and $r > 0$ (for a proof, see for instance [6, Lemma 3.6]):

$$
\fint_{\partial B_r(x)} v \, d\mathcal{H}^{d-1} - v(x) = \frac{1}{d\omega_d} \int_{\rho=0}^r \rho^{1-d} \Delta v(B_\rho(x)) \, d\rho, \tag{2.6}
$$

where $\Delta v(B_r(x))$ is the measure of $B_r(x)$ with respect to the measure Δv.

Most of the perturbation techniques that we will use to get the Lipschitz continuity of the state functions u on the optimal sets Ω will provide us information about the mean values $\fint_{B_r} u \, dx$ or $\fint_{\partial B_r} u \, d\mathcal{H}^{d-1}$. In order to transfer this information to the gradient $|\nabla u|$, we will make use of the following classical result (see for example [19] for a proof).

Remark 2.4 (Gradient estimate). *If $u \in \mathcal{H}^1(B_r)$ is such that $-\Delta u = f$ in B_r and $f \in L^\infty(B_r)$, then the following estimates hold*

$$\|\nabla u\|_{L^\infty(B_{r/2})} \le C_d \|f\|_{L^\infty(B_r)} + \frac{2d}{r} \|u\|_{L^\infty(B_r)} \,,$$

$$\|u\|_{L^\infty(B_{2r/3})} \le \frac{r^2}{2d} \|f\|_{L^\infty(B_r)} + C_d \fint_{\partial B_r} |u| \, d\mathcal{H}^{d-1} \,. \tag{2.7}$$

Actually, while for the first estimate it is really important that the equation $-\Delta u = f$ is valid in the whole B_r, the second estimate holds true also for balls B_r centered in a point $x \in \partial\Omega$, where Ω is an open set such that $u \in \mathcal{H}_0^1(\Omega)$ and $-\Delta u = f$ is valid in Ω. Even if this fact is known, we will add a simple proof of it during the proof of Theorem 3.3 (which is the only point of the paper where we need it).

2.3. **Eigenfunctions and eigenvalues of measurable sets.** We will consider the eigenvalues of the Dirichlet Laplacian on the linear subspace $\widetilde{\mathcal{H}}_0^1(\Omega) \subseteq \mathcal{H}^1(\mathbb{R}^d)$. In general, given a closed linear subspace H of $\mathcal{H}^1(\mathbb{R}^d)$ such that the embedding $H \subseteq L^2(\mathbb{R}^d)$ is compact, one defines the spectrum σ_H of the Laplace operator $-\Delta$ on H as $\sigma_H = (\lambda_1(H), \dots, \lambda_k(H), \dots)$, where the k-th eigenvalue is defined as

$$\lambda_k(H) := \min_{S_k} \max_{u \in S_k \setminus \{0\}} \frac{\int |\nabla u|^2 \, dx}{\int u^2 \, dx} \,, \tag{2.8}$$

and the minimum ranges over all k-dimensional subspaces S_k of H.

Given a measurable set Ω with finite measure and $k \in \mathbb{N}$, we define then the k-th eigenvalue of the Dirichlet Laplacian on Ω as $\lambda_k(\Omega) := \lambda_k(\widetilde{\mathcal{H}}_0^1(\Omega))$. The sequence

$$\frac{1}{\lambda_1(\Omega)} \ge \frac{1}{\lambda_2(\Omega)} \ge \dots \ge \frac{1}{\lambda_k(\Omega)} \ge \dots$$

constitutes precisely the spectrum of the compact operator $R_\Omega : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, which associates to each $f \in L^2(\mathbb{R}^d)$ the solution u of the equation (2.2). Thus, there is a sequence of eigenfunctions $u_k \in \widetilde{\mathcal{H}}_0^1(\Omega)$ orthonormal in L^2 and satisfying the equation

$$-\Delta u_k = \lambda_k(\Omega) u_k \qquad \text{in } \Omega \,.$$

Remark 2.5 (L^∞ bound for the eigenfunctions). *The eigenfunctions u_k admit the following L^∞ estimate (for a proof we refer to [17, Example 2.1.8]):*

$$\|u_k\|_{L^\infty} \le e^{1/8\pi} \lambda_k(\Omega)^{d/4} \,. \tag{2.9}$$

As a consequence, the arguments of Section 2.2 imply that every point of $\mathbb{R}^d$ is a Lebesgue point for u_k and that the function

$$x \mapsto |u_k(x)| + \frac{e^{1/8\pi} \lambda_k(\Omega)^{\frac{d+4}{4}}}{2d} |x|^2$$

is subharmonic in $\mathbb{R}^d$. Applying then (2.7), we get that for any ball B_r essentially contained in Ω (this means, $\Omega \setminus B_r$ is negligible) one has

$$\|\nabla u_k\|_{L^\infty(B_{r/3})} \le C_d \left(\lambda_k(\Omega)^{\frac{d+4}{4}} + \frac{1}{r} \fint_{\partial B_r} |u_k| \, d\mathcal{H}^{d-1} \right) \,.$$

2.4. **The γ- and the weak γ-convergence of measurable sets.** Through the paper, we will make extensively use of two variational notions of convergence defined on the measurable sets of finite Lebesgue measure, namely the γ-convergence and the weak γ-convergence. To introduce these notions, we start by calling, for every measurable set $\Omega \subseteq \mathbb{R}^d$ with finite measure, $w_\Omega \in \widetilde{\mathcal{H}}_0^1(\Omega)$ the solution of the problem

$$-\Delta w_\Omega = 1 \qquad \text{in } \Omega \,.$$

The measurable set $\Omega \subseteq \mathbb{R}^d$ is determined, as a domain of the Sobolev space $\widetilde{\mathcal{H}}_0^1(\Omega)$, by w_Ω, which is usually called *energy function*, or *torsion function*. In fact, we have the equality (see for example [10])

$$\widetilde{\mathcal{H}}_0^1(\Omega) = \widetilde{\mathcal{H}}_0^1(\{w_\Omega > 0\}) = \widehat{\mathcal{H}}_0^1(\{w_\Omega > 0\}).$$

If the measurable set Ω is such that $\left| \Omega \Delta \{w_\Omega > 0\} \right| = 0$, then we can choose its representative in the family of measurable sets to be precisely the set $\{w_\Omega > 0\}$.

Definition 2.6. *We say that the sequence of sets of finite measure Ω_n*
- *γ-converges to the set Ω, if the sequence w_{Ω_n} converges strongly in $L^2(\mathbb{R}^d)$ to the function w_Ω;*
- *weak γ-converges to the set Ω, if the sequence w_{Ω_n} converges strongly in $L^2(\mathbb{R}^d)$ to a function $w \in \mathcal{H}^1(\mathbb{R}^d)$ such that $\Omega = \{w > 0\}$.*

In the case of a weak γ-converging sequence $\Omega_n \to \Omega$, there is a comparison principle between the limit function w and the energy function w_Ω, namely, the inequality $w \leq w_\Omega$ holds. This follows by the variational characterization of w, through the *capacitary measures*, or it can also be proved directly by comparing the functions w_{Ω_n} to w_Ω (see [10]). Using only this weak maximum principle and the definitions above, one may deduce the following properties of the γ- and the weak γ-convergences (for more details we refer the reader to the papers [11, 13] and the books [8, 20]).

Remark 2.7 (γ- and weak γ-convergences). *If Ω_n γ-converges to Ω, then it also weak γ-converges to Ω. Under the additional assumption that $\Omega \subseteq \Omega_n$ for every $n \in \mathbb{N}$, we have that if Ω_n weak γ-converges to Ω then it also γ-converges to Ω.*

Remark 2.8 (measure and weak γ-convergences). *If Ω_n converges to Ω in $L^1(\mathbb{R}^d)$, i.e. $\left| \Omega_n \Delta \Omega \right| \to 0$, then up to a subsequence Ω_n weak γ-converges to Ω. On the other hand, if Ω_n weak γ-converges to Ω, then we have the following semi-continuity of the Lebesgue measure:*

$$|\Omega| \leq \liminf_{n \to \infty} |\Omega_n|.$$

Remark 2.9 (γ- and Mosco convergences). *(a) Suppose that Ω_n weak γ-converges to Ω. Then, if the sequence $u_n \in \widetilde{\mathcal{H}}_0^1(\Omega_n)$ converges in $L^2(\mathbb{R}^d)$ to $u \in \mathcal{H}^1(\mathbb{R}^d)$, we have that $u \in \widetilde{\mathcal{H}}_0^1(\Omega)$. In particular, we obtain the semi-continuity of λ_k with respect to the weak γ-convergence:*

$$\lambda_k(\Omega) \leq \liminf_{n \to \infty} \lambda_k(\Omega_n).$$

(b) Suppose that Ω_n γ-converges to Ω. Then, for every $u \in \widetilde{\mathcal{H}}_0^1(\Omega)$ there is a sequence $u_n \in \widetilde{\mathcal{H}}_0^1(\Omega_n)$ converging to u strongly in $\mathcal{H}^1(\mathbb{R}^d)$. As a consequence, one has the continuity of λ_k with respect to the γ-convergence:

$$\lambda_k(\Omega) = \liminf_{n \to \infty} \lambda_k(\Omega_n).$$

3. Lipschitz continuity of energy quasi-minimizers

In this section we study the properties of the local quasi-minimizers for the Dirichlet integral. More precisely, let $f \in L^2(\mathbb{R}^d)$ and let $u \in \mathcal{H}^1(\mathbb{R}^d)$ satisfy $-\Delta u = f$ in $\{u \neq 0\}$.

Definition 3.1. *We say that u is a* local quasi-minimizer *for the functional*

$$J_f(u) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx - \int_{\mathbb{R}^d} uf \, dx, \tag{3.1}$$

if there is a positive constant C such that for every $r > 0$ we have

$$J_f(u) \leq J_f(v) + Cr^d, \qquad \forall v \in \mathcal{A}_r(u), \tag{3.2}$$

where the admissible set $\mathcal{A}_r(u)$ is defined as

$$\mathcal{A}_r(u) := \left\{ v \in \mathcal{H}^1(\mathbb{R}^d) \, : \, \exists \, x_0 \in \mathbb{R}^d \text{ such that } \, v - u \in \mathcal{H}_0^1(B_r(x_0)) \right\}.$$

Since J_f is clearly bounded below, it is equivalent to check the validity of (3.2) only for $0 \leq r \leq r_0$ for some given $r_0 > 0$.

Remark 3.2. *We highlight three equivalent characterizations of the local quasi-minimality. First of all, it is equivalent (and this is straightforward from the definition) to the existence of a constant $C > 0$ such that, for every ball $B_r(x_0)$ of radius smaller than r_0 and every $v \in \mathcal{H}_0^1(B_r(x_0))$, it is*

$$|\langle \Delta u + f, v \rangle| \leq \frac{1}{2} \int_{B_r(x_0)} |\nabla v|^2 \, dx + C r^d \,. \tag{3.3}$$

Actually, by the non-linearity of the right term, it is enough that, for some constant $C_1, C_2 > 0$, it is

$$|\langle \Delta u + f, v \rangle| \leq C_1 \int_{B_r(x_0)} |\nabla v|^2 \, dx + C_2 r^d \,. \tag{3.4}$$

Indeed, it is clear that (3.3) implies (3.4); but on the other hand, if (3.4) holds true, then for every $v \in \widetilde{\mathcal{H}}_0^1(B_r(x_0))$ it is enough to apply (3.4) to the function $v/2C_1$ to get the validity of (3.3), with $C = 2C_1 C_2$. The third equivalent formulation is the following: there exists a constant $C_b > 0$ such that, for any ball $B_r(x_0)$ of radius smaller than r_0, and any $v \in \mathcal{H}_0^1(B_r(x_0))$, it is

$$|\langle \Delta u + f, v \rangle| \leq C_b \, r^{d/2} \left(\int_{B_r(x_0)} |\nabla v|^2 \, dx \right)^{1/2} \,. \tag{3.5}$$

Indeed, by the geometric–quadratic mean inequality (3.5) implies (3.4), and on the other hand testing (3.4) with $\tilde{v} := r^{d/2} \|\nabla v\|_{L^2}^{-1} v$ gives (3.5) with $C_b = C_1 + C_2$. A last observation, again coming from the non-linearity of the right term in (3.4), is the following: if we obtain (3.4) only for functions $v \in \mathcal{H}_0^1(B_r(x_0))$ satisfying $\int |\nabla v|^2 \leq 1$, then this is sufficient to obtain (3.5) for every $v \in \mathcal{H}_0^1(B_r(x_0))$, just testing (3.4) with $\tilde{v} = r^{d/2} v / \|\nabla v\|_{L^2}$ (this requires to choose $r_0 \leq 1$, which is admissible as already observed).

We present now a theorem concerning the Lipschitz continuity of the local quasi-minimizers, which is a consequence of the techniques introduced by Briançon, Hayouni and Pierre [6].

Theorem 3.3. *Let $\Omega \subseteq \mathbb{R}^d$ be a measurable set of finite measure, $f \in L^\infty(\Omega)$ and let the function $u \in \mathcal{H}^1(\mathbb{R}^d)$ be a solution of the equation $-\Delta u = f$ in $\widetilde{\mathcal{H}}_0^1(\Omega)$, as well as a local quasi-minimizer for the functional J_f. Then:*

(1) u is Lipschitz continuous on $\mathbb{R}^d$, and its Lipschitz constant depends on d, $\|f\|_{L^\infty}$, $|\Omega|$, r_0, and the constant C_b in (3.5).

(2) the distribution $\Delta|u|$ is a Borel measure on the whole $\mathbb{R}^d$, and in particular

$$\big|\Delta|u|\big|\big(B_r(x)\big) \leq C \, r^{d-1} \tag{3.6}$$

for every $x \in \mathbb{R}^d$ such that $u(x) = 0$ and every $0 < r < r_0/4$, where the constant C depends on d, $\|f\|_{L^\infty}$, $|\Omega|$ and C_b (but not on r_0).

We notice that the local quasi-minimality condition is also necessary for the Lipschitz continuity of u. In fact, it expresses in a weak form the boundedness of the gradient $|\nabla u|$ near the boundary $\partial\Omega$.

The proof of this theorem is implicitly contained in [6, Theorem 3.1], but for the sake of completeness we also reproduce it in the Appendix. We prove now a consequence of the above theorem, which will be very important later. Recall from Section 2.3 that an

eigenfunction $u \in \widetilde{\mathcal{H}}_0^1(\Omega)$, corresponding to the eigenvalue λ, is a solution of $-\Delta u = f$ with $f = \lambda u$; then, in particular, we can ask ourselves whether u is a local quasi-minimizer of the functional J_f.

Theorem 3.4. *Let $\Omega \subseteq \mathbb{R}^d$ be a measurable set of finite measure, and let u be a normalized eigenfunction on Ω with eigenvalue λ, as well as a local quasi-minimizer of J_f, being $f = \lambda u$; in particular, u satisfies (3.5) with some constant C_u and for r smaller than some $r_0 = r_0(u) \leq 1$. Then u is Lipschitz continuous in $\mathbb{R}^d$ and the Lipschitz constant depends only on d, $|\Omega|$, λ, and C_u, but not on r_0.*

Proof. By Theorem 3.3, applied to u and $f := \lambda u$, we already know that u is Lipschitz continuous, then we must only show that its Lipschitz constant is independent on r_0.

Let us then set $\widetilde{\Omega} := \{u \neq 0\}$ (note that $\widetilde{\Omega}$ is open); let also $\bar{x}$ be any point with $R := d(\bar{x}, \widetilde{\Omega}^c) < r_0/8$ and let $y \in \partial\widetilde{\Omega}$ be such that $|y - \bar{x}| = R$. Using the first estimate (2.7) on the ball $B_{R/2}(\bar{x}) \subseteq \widetilde{\Omega}$ we know that

$$|\nabla u(\bar{x})| \leq C_d \lambda \|u\|_{L^\infty} + \frac{4d}{R}\|u\|_{L^\infty(B_{R/2}(\bar{x}))}. \tag{3.7}$$

Let now $z \in B_{R/2}(\bar{x})$ be a point such that $\|u\|_{L^\infty(B_{R/2}(\bar{x}))} \leq |u(z)|$. For any $0 < r < R/2$, the ball $B_r(z)$ is contained in $\widetilde{\Omega}$, so we can apply the second estimate (2.7) on it, to get

$$\|u\|_{L^\infty(B_{R/2}(\bar{x}))} \leq |u(z)| \leq \|u\|_{L^\infty(B_{2r/3}(z))} \leq \frac{R^2}{8d}\lambda\|u\|_{L^\infty} + C_d \fint_{\partial B_r(z)} |u|\,d\mathcal{H}^{d-1}.$$

Being this estimate valid for every $0 < r < R/2$, then of course it is also

$$\|u\|_{L^\infty(B_{R/2}(\bar{x}))} \leq \frac{R^2}{8d}\lambda\|u\|_{L^\infty} + C_d \fint_{B_{R/2}(z)} |u|\,d\mathcal{H}^d. \tag{3.8}$$

Since $B_{R/2}(z) \subseteq B_{2R}(y)$ by construction, we deduce

$$\fint_{B_{R/2}(z)} |u|\,d\mathcal{H}^d = \frac{2^d}{\omega_d R^d}\int_{B_{R/2}(z)} |u| \leq \frac{2^d}{\omega_d R^d}\int_{B_{2R}(y)} |u| = 4^d \fint_{B_{2R}(y)} |u|\,d\mathcal{H}^d. \tag{3.9}$$

Finally, there exists some $r \in (0, 2R)$ such that

$$\fint_{B_{2R}(y)} |u|\,d\mathcal{H}^d \leq \fint_{\partial B_r(y)} |u|\,d\mathcal{H}^{d-1}. \tag{3.10}$$

Putting together (3.7), (3.8), (3.9) and (3.10), we then get

$$|\nabla u(\bar{x})| \leq C_d \lambda\|u\|_{L^\infty} + \frac{C_d}{R}\fint_{\partial B_r(y)} |u|\,d\mathcal{H}^{d-1}. \tag{3.11}$$

Observe now that the ball $B_r(y)$ is not contained in $\widetilde{\Omega}$, hence in this ball we could not apply the gradient estimates (2.7). Nevertheless, $|u|$ belongs to $\mathcal{H}^1(\mathbb{R}^d)$, because u does, and Theorem 3.3 ensures that $\Delta|u|$ is a measure on $\mathbb{R}^d$; thus, we are in position to apply the estimate (2.6) with $v = |u|$ and $x = y$: keeping in mind that $u(y) = 0$ because $y \in \partial\widetilde{\Omega}$ and u is Lipschitz continuous, and using also (3.6) from Theorem 3.3, we get

$$\fint_{\partial B_r(y)} |u|\,d\mathcal{H}^{d-1} = \frac{1}{d\omega_d}\int_{\rho=0}^r \rho^{1-d}\Delta|u|(B_\rho(y))\,d\rho \leq \frac{Cr}{d\omega_d} \leq \frac{2CR}{d\omega_d},$$

which inserted in (3.11), and recalling again Theorem 3.3, finally gives

$$|\nabla u(\bar{x})| \leq C', \tag{3.12}$$

for some constant C' depending on d, λ, $\|u\|_{L^\infty}$, $|\Omega|$ and C_u, but not on r_0. In turn, since Remark 2.5 says that $\|u\|_{L^\infty}$ can be bounded only in terms of λ and d, we have that C' actually depends only on d, λ, $|\Omega|$ and C_u, and of course still not on r_0. Summarizing,

up to now we have shown that the Lipschitz constant of u is independent of r_0 in a $r_0/8$-neighborhood of the boundary of $\widetilde{\Omega}$; we will conclude the proof by showing that an estimate near the boundary implies a (worse) estimate in the whole set $\widetilde{\Omega}$.

To do so, consider the auxiliary function $P \in C^\infty(\widetilde{\Omega})$ defined as

$$P := |\nabla u|^2 + \lambda u^2 - 2\lambda^2 \|u\|_{L^\infty}^2 w_{\widetilde{\Omega}} \,,$$

where $w_{\widetilde{\Omega}} \in \mathcal{H}_0^1(\widetilde{\Omega})$ is the solution of the equation $-\Delta w_{\widetilde{\Omega}} = 1$ in $\widetilde{\Omega}$. A direct computation gives that P is sub-harmonic on the open set $\widetilde{\Omega}$, indeed

$$\Delta P = \left(2[Hess(u)]^2 - 2\lambda |\nabla u|^2\right) + \left(2\lambda |\nabla u|^2 - 2\lambda^2 u^2\right) + 2\lambda^2 \|u\|_{L^\infty}^2 \geq 0 \,.$$

Thus, by the maximum principle we get

$$\sup\left\{P(x) \,:\, x \in \widetilde{\Omega}\right\} \leq \sup\left\{P(x) \,:\, x \in \widetilde{\Omega}, \ \mathrm{dist}(x, \partial\widetilde{\Omega}) < r_0/8\right\} ,$$

and so, recalling the estimate (3.12) near the boundary, we immediately obtain

$$\|\nabla u\|_{L^\infty}^2 \leq \|P\|_{L^\infty} + 2\lambda^2 \|u\|_{L^\infty}^2 \|w_{\widetilde{\Omega}}\|_{L^\infty} \leq C'^2 + \lambda \|u\|_{L^\infty}^2 + 2\lambda^2 \|u\|_{L^\infty}^2 \|w_{\widetilde{\Omega}}\|_{L^\infty} \,.$$

We finally conclude the proof, just recalling again that $\|u\|_{L^\infty}$ can be bounded only in terms of λ and d, and also by the classical bound $\|w_{\widetilde{\Omega}}\|_{L^\infty} \leq C_d |\widetilde{\Omega}|^{2/d}$ (see, for example, [25, Theorem 1]). $\qquad\square$

4. Shape quasi-minimizers for Dirichlet eigenvalues

In this section we discuss the regularity of the eigenfunctions on sets which are minimal with respect to a given (spectral) shape functional; in particular, we will show in Lemma 4.6 that the k-th eigenfunction of a set which is a shape quasi-minimizer for λ_k is Lipschitz as soon as λ_k is a simple eigenvalue for Ω. In what follows we denote by $\mathcal{A}$ the family of subset of $\mathbb{R}^d$ with finite Lebesgue measure endowed with the equivalence relation $\Omega \sim \widetilde{\Omega}$, whenever $|\Omega \Delta \widetilde{\Omega}| = 0$.

Definition 4.1. *We say that the measurable set $\Omega \in \mathcal{A}$ is a* shape quasi-minimizer *for the functional $\mathcal{F} : \mathcal{A} \to \mathbb{R}$ if there exists a constant $C > 0$ such that for every ball $B_r(x)$ and every set $\widetilde{\Omega} \in \mathcal{A}$ with $\Omega \Delta \widetilde{\Omega} \subseteq B_r(x)$ it is*

$$\mathcal{F}(\Omega) \leq \mathcal{F}(\widetilde{\Omega}) + C|B_r| \,.$$

Of course, whenever $\mathcal{F}$ is positive (as is almost always the case in the applications) then we can restrict ourselves in considering balls with radius r smaller than some given $r_0 > 0$.

Remark 4.2. *If the functional $\mathcal{F}$ is non-increasing with respect to inclusions, then Ω is a shape quasi-minimizer if and only if for every ball $B_r(x)$ it is*

$$\mathcal{F}(\Omega) \leq \mathcal{F}\left(\Omega \cup B_r(x)\right) + C|B_r| \,.$$

One may expect that the property of shape quasi-minimality contains some information on the regularity of Ω, or of the eigenfunctions on Ω. Let us quickly see what happens with a very simple example, namely, let us consider the Dirichlet Energy

$$E(\Omega) := \min\left\{J_1(u) : \ u \in \widetilde{\mathcal{H}}_0^1(\Omega)\right\},$$

where the functional J_1 is intended in the sense of (3.1) with $f \equiv 1$, and let Ω be a shape quasi-minimizer for E. Then, calling ω_Ω the energy function, it is clear that for any ball $B_r(x)$ and any $\widetilde{\Omega} \subseteq \mathcal{A}$ such that $\widetilde{\Omega} \Delta \Omega \subseteq B_r(x)$ it is

$$J_1(w_\Omega) = E(\Omega) \leq E(\widetilde{\Omega}) + C|B_r| \leq J_1(w_\Omega + \varphi) + C|B_r| \qquad \forall\, \varphi \in \mathcal{H}_0^1\left(B_r(x)\right).$$

This means that w_Ω is a local quasi-minimizer for the functional J_1, according to Definition 3.1, and then Theorem 3.3 ensures that w_Ω is Lipschitz continuous on $\mathbb{R}^d$.

The case $\mathcal{F} = \lambda_k$ is more involved, since the k-th eigenvalue is not defined through a single state function, but is variationally characterized by a min-max procedure involving

an entire linear subspace of $\widetilde{\mathcal{H}}_0^1(\Omega)$; therefore, we will need to transfer information from the minimality of Ω to the variation of the eigenvalues of Ω, then from this to the variation of the eigenfunctions, and finally from this to the regularity of Ω itself.

The main tool to prove Lemma 4.6 is the technical Lemma 4.3 below. There, we consider a generic set $\Omega \in \mathcal{A}$, we take $k \geq l \geq 1$ so that

$$\lambda_k(\Omega) = \cdots = \lambda_{k-l+1}(\Omega) > \lambda_{k-l}(\Omega) \,, \tag{4.1}$$

where by consistence we mean $\lambda_0(\Omega) = 0$, and we fix l normalized orthogonal eigenfunctions corresponding to eigenvalue $\lambda_k(\Omega)$, that we call $u_{k-l+1}, \ldots, u_k$. We will use the following notation: given a vector $\alpha = (\alpha_{k-l+1}, \ldots, \alpha_k) \in \mathbb{R}^l$, we denote by $\boldsymbol{u}_\alpha$ the corresponding linear combination

$$\boldsymbol{u}_\alpha := \alpha_{k-l+1} u_{k-l+1} + \ldots + \alpha_k u_k \,. \tag{4.2}$$

Lemma 4.3. *Let $\Omega \subseteq \mathbb{R}^d$ be a set of finite measure and $k \geq l \geq 1$ be such that (4.1) holds. For every $\eta > 0$ there is a constant $r_0 > 0$ such that, for every $x \in \mathbb{R}^d$, every $0 < r < r_0$, and every l-uple of functions $v_{k-l+1}, \ldots, v_k \in \mathcal{H}_0^1(B_r(x))$, there is a unit vector $\alpha \in \mathbb{R}^l$ such that*

$$\lambda_k(\Omega \cup B_r(x)) \leq \frac{\int |\nabla(\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha)|^2}{\int |\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha|^2} + \eta \int |\nabla \boldsymbol{v}_\alpha|^2 \,. \tag{4.3}$$

The constant r_0 depends on Ω and in particular, if the gap $\lambda_{k-l+1}(\Omega) - \lambda_{k-l}(\Omega)$ vanishes, then so does r_0 as well.

Proof. For the sake of shortness, let us simply write B_r in place of $B_r(x)$, as well as λ_j in place of $\lambda_j(\Omega)$. By the definition of the k-th eigenvalue, we know that

$$\lambda_k(\Omega \cup B_r) \leq \max\left\{ \frac{\int |\nabla u|^2}{\int u^2} : u \in \operatorname{span}\langle u_1, \ldots, u_{k-l}, u_{k-l+1} + v_{k-l+1}, \ldots, u_k + v_k \rangle \right\} \,, \tag{4.4}$$

and the maximum is attained for some linear combination

$$\alpha_1 u_1 + \ldots + \alpha_{k-l} u_{k-l} + \alpha_{k-l+1}(u_{k-l+1} + v_{k-l+1}) + \ldots + \alpha_k(u_k + v_k) \,.$$

One can immediately notice that the vector $\alpha = (\alpha_{k-l+1}, \ldots, \alpha_k) \in \mathbb{R}^l$ must be non-zero if $\lambda_{k-l}(\Omega) < \lambda_k(\Omega \cup B_r)$. And in turn, an immediate argument by contradiction shows that this is always the case if r_0 is small enough; we can then assume that α is a unitary vector. On the other hand, consider the vector $(\alpha_1, \ldots, \alpha_{k-l})$: if it is the null vector, then (4.3) comes directly from (4.4), hence we have nothing to prove. Otherwise, let us call

$$u := \frac{\alpha_1 u_1 + \ldots + \alpha_{k-l} u_{k-l}}{\sqrt{\alpha_1^2 + \ldots + \alpha_{k-l}^2}} \,,$$

so that $\int u^2 = 1$, $\int |\nabla u|^2 \leq \lambda_{k-l}$, and from (4.4) we derive

$$\lambda_k(\Omega \cup B_r) \leq \max_{t \in \mathbb{R}} \left\{ \frac{\int |\nabla(\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha + tu)|^2}{\int |\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha + tu|^2} \right\} \,. \tag{4.5}$$

We can now quickly evaluate, keeping in mind that u and $\boldsymbol{u}_\alpha$ are orthogonal,

$$\frac{\int |\nabla(\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha + tu)|^2}{\int (\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha + tu)^2} \leq \frac{\int |\nabla(\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha)|^2 + 2t \int \nabla(\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha) \cdot \nabla u + t^2 \lambda_{k-l}}{\int |\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha|^2 + 2t \int (\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha)u + t^2}$$

$$= \frac{\int |\nabla(\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha)|^2 + 2t \int_{B_r} \nabla \boldsymbol{v}_\alpha \cdot \nabla u + t^2 \lambda_{k-l}}{\int |\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha|^2 + 2t \int_{B_r} \boldsymbol{v}_\alpha u + t^2} = \frac{A + 2ta + t^2 \lambda_{k-l}}{B + 2tb + t^2} \,,$$

where by shortness we write

$$A = \int_{\mathbb{R}^d} |\nabla(\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha)|^2 \,, \qquad\qquad a = \int_{B_r} \nabla \boldsymbol{v}_\alpha \cdot \nabla u$$

$$B = \int_{\mathbb{R}^d} |\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha|^2 \,, \qquad\qquad b = \int_{B_r} \boldsymbol{v}_\alpha u \,.$$

Calling now for simplicity $D = \sqrt{\int |\nabla \boldsymbol{v}_\alpha|^2}$, and picking a small number $\delta = \delta(\eta) > 0$ to be chosen later, it is clear by the Hölder inequality and the embedding of $\mathcal{H}_0^1(B_r)$ into $L^2(B_r)$ that, if r_0 is small enough, then

$$|a| \leq \|\nabla u\|_{L^2(B_r)} D \leq \delta D\,, \quad |b| \leq \|\boldsymbol{v}_\alpha\|_{L^2(B_r)} \leq \delta D\,, \quad B \geq \int_{\mathbb{R}^d \setminus B_r} \boldsymbol{u}_\alpha^2 \geq 1 - \delta\,. \quad (4.6)$$

On the other hand, we can estimate the quotient A/B as

$$\begin{aligned}
\frac{A}{B} &= \frac{\int |\nabla(\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha)|^2\, dx}{\int |\boldsymbol{u}_\alpha + \boldsymbol{v}_\alpha|^2\, dx} = \frac{\lambda_k + 2\int \nabla \boldsymbol{u}_\alpha \cdot \nabla \boldsymbol{v}_\alpha\, dx + \int |\nabla \boldsymbol{v}_\alpha|^2\, dx}{1 + 2\int \boldsymbol{u}_\alpha \boldsymbol{v}_\alpha\, dx + \int \boldsymbol{v}_\alpha^2\, dx} \\[2mm]
&\geq \frac{\lambda_k - 2D\left(\int_{B_r} |\nabla \boldsymbol{u}_\alpha|^2\, dx\right)^{1/2} + D^2}{1 + 2\left(\int_{B_r} \boldsymbol{u}_\alpha^2\, dx\right)^{1/2}\left(\int_{B_r} \boldsymbol{v}_\alpha^2\, dx\right)^{1/2} + \int \boldsymbol{v}_\alpha^2\, dx} \\[2mm]
&\geq \frac{\lambda_k - 2D\left(\int_{B_{r_0}} |\nabla \boldsymbol{u}_\alpha|^2\, dx\right)^{1/2} + D^2}{1 + 2\left(\int_{B_r} \boldsymbol{v}_\alpha^2\, dx\right)^{1/2} + \int_{B_r} \boldsymbol{v}_\alpha^2\, dx} \geq \frac{\lambda_k - 2D\left(\int_{B_{r_0}} |\nabla \boldsymbol{u}_\alpha|^2\, dx\right)^{1/2} + D^2}{1 + 2C_d|B_{r_0}|^{1/d} D + C_d^2|B_{r_0}|^{2/d} D^2} \\[2mm]
&> \lambda_{k-l}\,,
\end{aligned} \quad (4.7)$$

where the last inequality is again true as soon as r_0 is small enough. Moreover, we also have

$$B + 2tb + t^2 \geq (1 - \delta)B \qquad \forall t \in \mathbb{R}\,. \quad (4.8)$$

Indeed, if $\int_{B_r} \boldsymbol{v}_\alpha^2 \leq 100$, then for r_0 small enough we have

$$|b| \leq \sqrt{\int_{B_r} u^2}\sqrt{\int_{B_r} \boldsymbol{v}_\alpha^2} \leq \delta\,,$$

thus $2tb + t^2 \geq -b^2 \geq -\delta^2 \geq -\delta B$ also by (4.6) and (4.8) holds. Instead, if $\int_{B_r} \boldsymbol{v}_\alpha^2 > 100$, then $b \leq \delta\sqrt{B}$ and thus again $2tb + t^2 \geq -b^2 \geq -\delta^2 B$ and (4.8) is again true.

We are finally in position to conclude. Indeed, if

$$|t| \leq \sqrt{\delta} D \qquad \text{and} \qquad D^2 \geq \lambda_k\,,$$

then $A \leq 3D^2$ and then, recalling (4.6) and (4.8), we have

$$\frac{A + 2ta + t^2\lambda_{k-l}}{B + 2tb + t^2} \leq \frac{A + 2\delta^{3/2}D^2 + \delta D^2\lambda_{k-l}}{B(1 - \delta)} \leq \frac{A}{B} + \eta D^2$$

as soon as δ is small enough with respect to η. Keeping in mind (4.5), this estimate gives (4.3). Instead, if

$$|t| \leq \sqrt{\delta} D \qquad \text{and} \qquad D^2 \leq \lambda_k\,,$$

then

$$\frac{A + 2ta + t^2\lambda_{k-l}}{B + 2tb + t^2} \leq \frac{A + 2\delta^{3/2}D^2 + \delta D^2\lambda_{k-l}}{B - 2\delta^{3/2}D^2} \leq \frac{A}{B} + \eta D^2$$

and we again deduce (4.3). Finally, if

$$|t| \geq \sqrt{\delta} D\,,$$

then by (4.6) $|at| \leq \sqrt{\delta}t^2$ and $|bt| \leq \sqrt{\delta}t^2$, which in view of (4.7) if $\delta \ll 1$ gives

$$\frac{A + 2ta + t^2\lambda_{k-l}}{B + 2tb + t^2} \leq \frac{A + t^2(\lambda_{k-l} + 2\sqrt{\delta})}{B + t^2(1 - 2\sqrt{\delta})} \leq \frac{A}{B} \leq \frac{A}{B} + \eta D^2\,.$$

We have then deduced (4.3) in any case, and the proof is concluded. $\qquad \square$

Remark 4.4. *The preceding lemma enlightens one of the main difficulties in the study of the regularity of spectral minimizers. Indeed, let Ω^* be a solution of a spectral optimization problem of the form (1.1) involving λ_k and such that (4.1) holds for some $l > 1$. Then every perturbation $\widetilde{u}_k = u_k + v$ of the eigenfunction $u_k \in \widetilde{\mathcal{H}}_0^1(\Omega^*)$ gives information on some linear combination $\boldsymbol{u}_\alpha$ of eigenfunctions $u_k, \dots, u_{k-l+1}$, and not simply on the function u_k. Recovering some information on u_k from estimates on $\boldsymbol{u}_\alpha$ is a difficult task, since the combination is a priori unknown, and anyway it depends on the perturbation v.*

Remark 4.5. *In case $\lambda_k(\Omega) > \lambda_{k-1}(\Omega)$, the estimate (4.3) reads as*

$$\lambda_k(\Omega \cup B_r(x)) \leq \frac{\int |\nabla(u_k + v)|^2 \, dx}{\int |u_k + v|^2 \, dx} + \eta \int |\nabla v|^2 \, dx \tag{4.9}$$

for every ball $B_r(x)$ with $r < r_0$ and every $v \in \mathcal{H}_0^1(B_r(x))$.

Lemma 4.6. *Let $\Omega \subseteq \mathbb{R}^d$ be a shape quasi-minimizer for λ_k such that $\lambda_k(\Omega) > \lambda_{k-1}(\Omega)$. Then every eigenfunction $u_k \in \widetilde{\mathcal{H}}_0^1(\Omega)$, normalized in L^2 and corresponding to the eigenvalue $\lambda_k(\Omega)$, is Lipschitz continuous on $\mathbb{R}^d$. Moreover, the Lipschitz constant depends only on $\lambda_k(\Omega)$, $|\Omega|$, d, and on the constant C in Definition 4.1, but not on u_k nor on Ω.*

Proof. Let u_k be a normalized eigenfunction corresponding to λ_k. Applying first the shape quasi-minimality of Ω and then the estimate (4.9) for $v \in \mathcal{H}_0^1(B_r(x))$, with $r \leq r_0 \leq 1$ and $\int |\nabla v|^2 \leq 1$, we obtain

$$\lambda_k(\Omega) \leq \lambda_k\big(\Omega \cup B_r(x)\big) + C|B_r| \leq \frac{\int |\nabla(u_k + v)|^2 \, dx}{\int |u_k + v|^2 \, dx} + \eta \int |\nabla v|^2 \, dx + C|B_r|. \tag{4.10}$$

We now observe that, using Poincaré inequality and the hypotheses $r \leq 1$, $\int |\nabla v|^2 \, dx \leq 1$, we have

$$\int |u_k + v|^2 \, dx \leq 2 \int u_k^2 \, dx + 2 \int v^2 \, dx \leq 2 + \frac{2}{\lambda_1(B_r)} \int |\nabla v|^2 \, dx \leq 4.$$

Then we multiply both sides of (4.10) by $\int |u_k + v|^2 \, dx$, so to get

$$-2 \int \nabla u_k \cdot \nabla v + 2\lambda_k(\Omega) \int u_k v \, dx + \lambda_k(\Omega) \int v^2 \, dx \leq (4\eta + 1) \int |\nabla v|^2 \, dx + 4C|B_r|,$$

from which we deduce

$$\left| \langle \Delta u_k + \lambda_k(\Omega) u_k, v \rangle \right| = \left| - \int \nabla u_k \cdot \nabla v \, dx + \lambda_k(\Omega) \int u_k v \, dx \right| \leq \frac{4\eta + 1}{2} \int |\nabla v|^2 \, dx + 2C|B_r|.$$

Hence the function u_k is a quasi-minimizer for the functional J_f, with $f = \lambda_k(\Omega) u_k$, thanks to (3.4) with $C_1 = \frac{4\eta+1}{2}$ and $C_2 = 2C$. Since u_k is bounded by (2.9), the claim follows directly by Theorem 3.4. $\qquad\square$

It is important to underline something: if Ω is a minimizer of λ_k, then one expects the eigenvalue not to be simple; nevertheless, in the next sections we will be able to extract some information on optimal sets by using the above result.

5. Shape supersolutions of spectral functionals

In this section we introduce the concept of shape supersolutions, and we use the result of the preceding sections to derive the existence of Lipschitz eigenfunctions for sets which are shape supersolutions of suitable spectral functionals.

Definition 5.1. *We say that the set $\Omega^* \subseteq \mathbb{R}^d$ is a* shape supersolution *for the functional $\mathcal{F} : \mathcal{A} \to \mathbb{R}$, defined on the class of Lebesgue measurable sets $\mathcal{A}$, if it satisfies*

$$\mathcal{F}(\Omega^*) \leq \mathcal{F}(\Omega), \qquad \forall \Omega \supseteq \Omega^*.$$

Let us list immediately some obvious but useful observations.

Remark 5.2. • *If Ω^* is a shape supersolution for $\mathcal{F} + \Lambda| \cdot |$ and $\Lambda > 0$ then, for every $\Lambda' > \Lambda$, Ω^* is the unique solution of*

$$\min\left\{\mathcal{F}(\Omega) + \Lambda'|\Omega| : \ \Omega \ \textit{Lebesgue measurable}, \ \ \Omega \supseteq \Omega^*\right\}.$$

- *If $\mathcal{F}$ is non-increasing with respect to the inclusion, then every shape supersolution of the functional $\mathcal{F} + \Lambda| \cdot |$, where $\Lambda > 0$, is also a shape quasi-minimizer for the functional $\mathcal{F}$, with constant $C = \Lambda$ in Definition 4.1 (this immediately follows by Remark 4.2).*
- *If Ω^* is a shape supersolution for the functional $\sum_{i=1}^m c_i\lambda_i + \Lambda \cdot |$, then it is also a shape supersolution for the functional $\sum_{i=1}^m \tilde{c}_i\lambda_i + \widetilde{\Lambda}|\cdot|$ whenever $0 \leq \tilde{c}_i \leq c_i$ for every $1 \leq i \leq m$, and $\widetilde{\Lambda} \geq \Lambda \geq 0$ (this is immediate from the definition).*
- *If Ω^* minimizes λ_k among all the sets of given volume, then it is also a shape quasi-minimizer for the functional $\mathcal{F} = \lambda_k$, as well as a shape supersolution of $\lambda_k + \Lambda| \cdot |$ for some positive Λ (this follows just by rescaling).*

In Lemma 4.6 it was shown that the k-th eigenfunctions of the the shape quasi-minimizers for λ_k are Lipschitz continuous under the assumption $\lambda_k(\Omega) > \lambda_{k-1}(\Omega)$. In the next Theorem, we show that for shape supersolutions of $\lambda_k + \Lambda| \cdot |$ the later assumption can be dropped.

Theorem 5.3. *Let $\Omega^* \subseteq \mathbb{R}^d$ be a bounded shape supersolution for the functional $\lambda_k + \Lambda| \cdot |$, being $\Lambda > 0$. Then there is an eigenfunction $u_k \in \widetilde{\mathcal{H}}_0^1(\Omega^*)$, normalized in L^2 and corresponding to the eigenvalue $\lambda_k(\Omega^*)$, which is Lipschitz continuous on $\mathbb{R}^d$.*

Proof. The core of the proof of this theorem is the following claim.
<u>*Claim.*</u> *Let Ω^* be a bounded shape supersolution for $\lambda_j + \Lambda_j| \cdot |$, with some $\Lambda_j > 0$. Then, either there exists a Lipschitz eigenfunction u_j for $\lambda_j(\Omega^*)$, or $\lambda_j(\Omega^*) = \lambda_{j-1}(\Omega^*)$ and there exists some constant Λ_{j-1} such that Ω^* is also a shape supersolution for $\lambda_{j-1} + \Lambda_{j-1}| \cdot |$.*
We show now first that the claim implies the thesis, and then the validity of the claim.
<u>*Step I. The claim implies the thesis.*</u>
By hypothesis, we can apply the claim with $j = k$. If we find a Lipschitz eigenfunction u_k for $\lambda_k(\Omega^*)$ we are done; otherwise, we can apply the claim with $j = k - 1$. If we find a Lipschitz eigenfunction u_{k-1} for $\lambda_{k-1}(\Omega^*)$ we are again done, since if we are in this situation then $\lambda_{k-1}(\Omega^*) = \lambda_k(\Omega^*)$. Otherwise, we pass to $j = k - 2$ and so on, with a finite recursive argument (which surely concludes since at least for $j = 1$ the first alternative of the claim must hold true). Summarizing, there is always some $1 \leq \bar{j} \leq k$ such that a Lipschitz eigenfunction for $\lambda_{\bar{j}}(\Omega^*)$ exists, and by construction $\lambda_{\bar{j}}(\Omega^*) = \lambda_k(\Omega^*)$. Therefore, the thesis comes from the claim.
<u>*Step II. The claim holds true.*</u>
First of all, since λ_j is non-increasing with respect to the inclusion, then by Remark 5.2 we know that Ω^* is a shape quasi-minimizer for λ_j, with constant $C = \Lambda_j$ in Definition 4.1. As a consequence, if $\lambda_j(\Omega^*) > \lambda_{j-1}(\Omega^*)$, then Lemma 4.6 already ensures the Lipschitz property for any normalized eigenfunction u_j corresponding to $\lambda_j(\Omega^*)$, and the claim is already proved.

Let us instead assume that $\lambda_j(\Omega^*) = \lambda_{j-1}(\Omega^*)$ and, for every $\varepsilon \in (0, 1)$, consider the problem

$$\min\left\{(1 - \varepsilon)\lambda_j(\Omega) + \varepsilon\lambda_{j-1}(\Omega) + 2\Lambda_j|\Omega| : \ \Omega \supseteq \Omega^*\right\}. \tag{5.1}$$

It is well-known that a minimizer Ω_ε of this problem exists, and it is clear that any such minimizer is a shape supersolution of the functional $\lambda_j + 2(1 - \varepsilon)^{-1}\Lambda_j| \cdot |$.

Suppose then that, for some sequence $\varepsilon_n \to 0$, there is a corresponding sequence of solutions Ω_{ε_n} to (5.1) which satisfy $\lambda_j(\Omega_{\varepsilon_n}) > \lambda_{j-1}(\Omega_{\varepsilon_n})$. Again by Lemma 4.6, we deduce the existence of normalized eigenfunctions u_j^n for $\lambda_j(\Omega_{\varepsilon_n})$, which are Lipschitz with a constant depending only on d, $\lambda_j(\Omega_{\varepsilon_n})$, $|\Omega_{\varepsilon_n}|$, and on Λ_j. Since the sets Ω_{ε_n} are uniformly bounded (see for instance [9, Proposition 5.12]), a suitable subsequence γ-converges to some set

$\widetilde{\Omega} \supseteq \Omega^*$, which is then a minimizer of the functional $\lambda_j + 2\Lambda_j |\cdot|$ among sets containing Ω^*, and thus in turn it is $\widetilde{\Omega} = \Omega^*$ by Remark 5.2. Still up to a subsequence, the functions u_j^n uniformly and weakly-$\mathcal{H}_0^1$ converge to a function $u_j \in \mathcal{H}_0^1(\Omega^*)$; moreover, since for every $v \in \mathcal{H}_0^1(\Omega^*)$ we have

$$\int \nabla u_j \cdot \nabla v \, dx = \lim_{n \to \infty} \int \nabla u_j^n \cdot \nabla v \, dx = \lim_{n \to \infty} \lambda_j(\Omega_{\varepsilon_n}) \int u_j^n v \, dx = \lambda_j(\Omega^*) \int u_j v \, dx \,,$$

we deduce that u_j is a normalized Lipschitz eigenfunction for $\lambda_j(\Omega^*)$, and then the claim has been proved also in this case.

We are then left to consider the case when $\lambda_j(\Omega^*) = \lambda_{j-1}(\Omega^*)$ and, for some small $\bar{\varepsilon} > 0$, every solution $\Omega_{\bar{\varepsilon}}$ of (5.1) has $\lambda_j(\Omega_{\bar{\varepsilon}}) = \lambda_{j-1}(\Omega_{\bar{\varepsilon}})$. This implies that $\Omega_{\bar{\varepsilon}}$ minimizes also $\lambda_j(\Omega) + 2\Lambda_j |\Omega|$ for sets $\Omega \supseteq \Omega^*$, so that actually $\Omega_{\bar{\varepsilon}} = \Omega^*$ again by Remark 5.2. In other words, Ω^* itself is a solution of (5.1) for $\bar{\varepsilon}$. As an immediate consequence, Ω^* is a shape supersolution for the functional $\lambda_{j-1} + 2\Lambda_j \bar{\varepsilon}^{-1} |\cdot|$: indeed, for any $\Omega \supseteq \Omega^*$ one has

$$\bar{\varepsilon}\lambda_{j-1}(\Omega^*) + 2\Lambda_j |\Omega^*| \leq \bar{\varepsilon}\lambda_{j-1}(\Omega^*) + (1 - \bar{\varepsilon})\big(\lambda_j(\Omega^*) - \lambda_j(\Omega)\big) + 2\Lambda_j |\Omega^*| \leq \bar{\varepsilon}\lambda_{j-1}(\Omega) + 2\Lambda_j |\Omega|$$

by (5.1), and thus the claim has been proved also in this last case. $\qquad\square$

A particular case of the above theorem concerns the optimal sets for the k-th eigenvalue.

Corollary 5.4. *Let Ω^* be a minimizer of the k-th eigenvalue among all the quasi-open sets of a given volume. Then, there exists an eigenfunction $u_k \in \widetilde{\mathcal{H}}_0^1(\Omega^*)$, corresponding to the eigenvalue $\lambda_k(\Omega^*)$, which is Lipschitz continuous on $\mathbb{R}^d$.*

Proof. Since it is known that such a minimizer exists and is bounded (see [7, 23, 21]), and since we have already observed in Remark 5.2 that any such minimizer is also a shape quasi-minimizer for $\lambda_k + \Lambda |\cdot|$, the claim follows just by applying Theorem 5.3. $\qquad\square$

It is important to observe that, if Ω^* is a minimizer of the k-th eigenvalue and the k-th eigenvalue of Ω^* is not simple (which actually seems always to be the case, unless when $k = 1$), then the above corollary only states the existence of a Lipschitz eigenfunction for $\lambda_k(\Omega^*)$, but not that the whole eigenspace of λ_k in $\widetilde{\mathcal{H}}_0^1(\Omega^*)$ is done by Lipschitz functions.

Our next aim is to improve Theorem 5.3 by considering functionals depending on more than just a single eigenvalue, hence of the form $F\big(\lambda_{k_1}(\Omega), \ldots, \lambda_{k_p}(\Omega)\big)$. To do so, we need the following preliminary result.

Lemma 5.5. *Let $\Omega^* \subseteq \mathbb{R}^d$ be a bounded shape supersolution for the functional*

$$\lambda_k + \lambda_{k+1} + \cdots + \lambda_{k+p} + \Lambda |\cdot| \,,$$

for some constant $\Lambda > 0$. Then there are L^2-orthonormal eigenfunctions $u_k, \ldots, u_{k+p} \in \widetilde{\mathcal{H}}_0^1(\Omega^*)$, corresponding to the eigenvalues $\lambda_k(\Omega^*), \ldots, \lambda_{k+p}(\Omega^*)$, which are Lipschitz continuous on $\mathbb{R}^d$.

Proof. For any $k \leq j \leq k + p$, the set Ω^* is a shape supersolution for $\lambda_j + \Lambda |\cdot|$, thus also a shape quasi-minimizer for λ_j with constant Λ, by Remark 5.2; hence, if $\lambda_j(\Omega^*) > \lambda_{j-1}(\Omega^*)$, by Lemma 4.6 we already know that the whole eigenspace corresponding to $\lambda_j(\Omega^*)$ is done by Lipschitz functions, and then for every $j \leq l \leq k + p$ such that $\lambda_j(\Omega^*) = \lambda_l(\Omega^*)$ we have orthogonal eigenfunctions $u_j, u_{j+1}, \ldots, u_l$ corresponding to the eigenvalues $\lambda_j(\Omega^*) = \lambda_{j+1}(\Omega^*) = \cdots = \lambda_l(\Omega^*)$.

Since eigenfunctions corresponding to different eigenvalues are always orthogonal, the above observation concludes the proof of the lemma if $\lambda_k(\Omega^*) > \lambda_{k-1}(\Omega^*)$.

Otherwise, we can use an argument very similar to that of the proof of Theorem 5.3: for every $\varepsilon \in (0, 1)$ we consider a solution Ω_ε of the problem

$$\min\left\{ \sum_{j=k+1}^{k+p} \lambda_j(\Omega) + (1 - \varepsilon)\lambda_k(\Omega) + \varepsilon\lambda_{k-1}(\Omega) + 2\Lambda |\Omega| : \ \Omega \supseteq \Omega^* \right\}, \tag{5.2}$$

which is in turn also a shape supersolution for the functional

$$\sum_{j=k}^{k+p} \lambda_j + \frac{2\Lambda}{1-\varepsilon} \, |\cdot| \, ,$$

again using Remark 5.2. If there is a sequence $\varepsilon_n \to 0$ such that $\lambda_k(\Omega_{\varepsilon_n}) > \lambda_{k-1}(\Omega_{\varepsilon_n})$, then we can apply the above argument to every set Ω_{ε_n} finding orthogonal eigenfunctions u_j^n for $k \leq j \leq k+p$ which are Lipschitz continuous, with constants not depending on ε. Then, exactly as in the proof of Theorem 5.3, one immediately obtains that Ω_{ε_n} γ-converges to Ω^*, and that weak-$\mathcal{H}_0^1$ limits u_j of the functions u_j^n are the desired Lipschitz eigenfunctions.

We must now only face the case that, for some small $\bar\varepsilon$, every solution $\Omega_{\bar\varepsilon}$ of (5.2) satisfies $\lambda_k(\Omega_{\bar\varepsilon}) = \lambda_{k-1}(\Omega_{\bar\varepsilon})$, and thus $\Omega_{\bar\varepsilon}$ actually coincides with Ω^*. Since this implies in particular that Ω^* is a shape supersolution for the functional

$$\lambda_{k-1} + \lambda_k + \lambda_{k+1} + \cdots + \lambda_{k+p} + \frac{2\Lambda}{\bar\varepsilon} \, |\cdot| \, ,$$

then we are in the same situation as at the beginning, with k replaced by $k-1$. With a finite recursion argument (which surely has an end, because we conclude when $\lambda_k > \lambda_{k-1}$, which emptily holds when $k=1$), we obtain the thesis. $\qquad\square$

Before stating the main result of this section, we fix the following terminology:
- given two points $x = (x_1, \ldots, x_p)$ and $y = (y_1, \ldots, y_p)$ in $\mathbb{R}^p$, we say that $x \geq y$ if $x_i \geq y_i$ for all $i = 1, \ldots, p$;
- a function $F : \mathbb{R}^p \to \mathbb{R}$ is said to be *increasing* if $F(x) \geq F(y)$ whenever $x \geq y$;
- we say that $F : \mathbb{R}^p \to \mathbb{R}$ is *increasingly bi-Lipschitz* if F is increasing, Lipschitz, and there is a constant $c > 0$ such that

$$F(x) - F(y) \geq c|x-y| \qquad \forall \, x \geq y \, .$$

- an increasing and locally Lipschitz function $F : \mathbb{R}^p \to \mathbb{R}$ is said *locally increasingly bi-Lipschitz* if for every x there is a constant $c(x)$ and a neighborhood $U \subseteq \mathbb{R}^p$ of x such that, for every $y_1 \geq y_2$ in U, one has $F(y_1) - F(y_2) \geq c(x)|y_1 - y_2|$.

Theorem 5.6. *Let $F : \mathbb{R}^p \to \mathbb{R}$ be a locally increasingly bi-Lipschitz function, and let $0 < k_1 < k_2 < \cdots < k_p \in \mathbb{N}$ and $\Lambda > 0$. Then for every bounded shape supersolution Ω^* of the functional*

$$\Omega \mapsto F\big(\lambda_{k_1}(\Omega), \ldots, \lambda_{k_p}(\Omega)\big) + \Lambda|\Omega| \, ,$$

there exists a sequence of orthonormal eigenfunctions $u_{k_1}, \ldots, u_{k_p}$, corresponding to the eigenvalues $\lambda_{k_j}(\Omega^*)$, $j = 1, \ldots, p$, which are Lipschitz continuous on $\mathbb{R}^d$. Moreover,
- *if for some k_j we have $\lambda_{k_j}(\Omega^*) > \lambda_{k_j-1}(\Omega^*)$, then the full eigenspace corresponding to $\lambda_{k_j}(\Omega^*)$ consists only on Lipschitz functions;*
- *if $\lambda_{k_j}(\Omega^*) = \lambda_{k_j-1}(\Omega^*)$, then there exist at least $k_j - k_{j-1} + 1$ orthonormal Lipschitz eigenfunctions corresponding to $\lambda_{k_j}(\Omega^*)$.*

Proof. Since the eigenspaces corresponding to different eigenvalues are orthogonal, we can restrict ourselves to consider the case when $\lambda_{k_1}(\Omega^*) = \lambda_{k_p}(\Omega^*)$. Moreover, the local bi-Lipschitz property ensures that Ω^* is also shape supersolution of the functional

$$\sum_{j=1}^{p} \lambda_{k_j} + \Lambda'|\cdot| \, ,$$

for a suitable positive constant Λ'. As a consequence, Ω^* is shape supersolution also for the functional

$$\left(\sum_{j=1}^{p-1} \frac{1}{k_{j+1}-k_j} \sum_{i=k_j}^{k_{j+1}-1} \lambda_i \right) + \lambda_{k_p} + \Lambda'|\cdot| \, ,$$

and then finally, using again Remark 5.2, also for the functional

$$\sum_{j=k_1}^{k_p} \lambda_j + \Lambda''|\cdot|\,.$$

The claim then directly follows from Lemma 5.5. $\qquad\qquad\square$

6. Optimal sets for functionals depending on the first k eigenvalues

In this last Section we will be able to show that, at least for some specific functionals, a minimizer is actually an open set, instead of a quasi-open set. The following results are, essentially, consequences of Theorem 5.6.

Theorem 6.1. *Let $F : \mathbb{R}^k \to \mathbb{R}$ be a locally increasingly bi-Lipschitz function. Then every solution Ω^* of the problem*

$$\min\left\{F\big(\lambda_1(\Omega),\dots,\lambda_k(\Omega)\big) : \ \Omega \subseteq \mathbb{R}^d \ measurable, \ |\Omega|=1\right\}, \qquad (6.1)$$

is essentially an open set. Moreover, the eigenfunctions of the Dirichlet Laplacian on Ω^*, corresponding to the eigenvalues $\lambda_1(\Omega^*),\dots,\lambda_k(\Omega^*)$, are Lipschitz continuous on $\mathbb{R}^d$.

Proof. We first note that the existence of a solution of (6.1) follows by the results from [7] and [23]. Then, we claim that every solution Ω^* is a shape supersolution of the functional

$$\Omega \mapsto F\big(\lambda_1(\Omega),\dots,\lambda_k(\Omega)\big) + \Lambda|\Omega|, \qquad (6.2)$$

for some suitably chosen $\Lambda > 0$. Indeed, let us take a generic set $\Omega \supseteq \Omega^*$ and let us call $t := (|\Omega|/|\Omega^*|)^{1/d} > 1$; we can assume that t is as close to 1 as we wish, since otherwise the claim is emptily true, up to increase the constant Λ. Thus, calling L the Lipschitz constant of F in a neighborhood of $\big(\lambda_1(\Omega^*), \dots, \lambda_k(\Omega^*)\big)$, by the optimality of Ω^* we have

$$F\big(\lambda_1(\Omega^*),\dots,\lambda_k(\Omega^*)\big) \le F\big(\lambda_1(\Omega/t),\dots,\lambda_k(\Omega/t)\big) = F\big(t^2\lambda_1(\Omega),\dots,t^2\lambda_k(\Omega)\big)$$

$$\le F\big(\lambda_1(\Omega),\dots,\lambda_k(\Omega)\big) + L(t^2-1)\sum_{i=1}^{k}\lambda_i(\Omega)$$

$$\le F\big(\lambda_1(\Omega),\dots,\lambda_k(\Omega)\big) + L(t^d-1)\sum_{i=1}^{k}\lambda_i(\Omega^*)$$

$$= F\big(\lambda_1(\Omega),\dots,\lambda_k(\Omega)\big) + \frac{L}{|\Omega^*|}\sum_{i=1}^{k}\lambda_i(\Omega^*)\Big(|\Omega| - |\Omega^*|\Big).$$

Then, Ω^* is a shape supersolution for the functional (6.2), as claimed, and thus the Lipschitz continuity of an orthonormal set $\{u_1, \dots, u_k\}$ of eigenfunctions follows by Theorem 5.6.

The openness of the set Ω^* follows by the observation that the open set

$$\Omega^{**} := \bigcup_{i=1}^{k}\{u_k \ne 0\}$$

is essentially contained in Ω^* and has the same first k eigenvalues as Ω^*: indeed, these eigenvalues are smaller than those of Ω^* by the characterization (2.8) of the eigenvalues and thanks to the functions u_i, but also greater than those of Ω^* because Ω^{**} is essentially contained in Ω^*. By the optimality of Ω^* we deduce that $|\Omega^* \Delta \Omega^{**}| = 0$, i.e., Ω^{**} is equivalent to Ω^* and the proof is completed. $\qquad\square$

Remark 6.2. *In dimension 2, the continuity of the first k eigenfunctions, which is enough to deduce the openness in Theorem 6.1, can be obtained also by a more direct method involving only elementary tools (see [22]). Roughly speaking, using the argument from Remark A.4, one can prove that in each level set of an eigenfunction there cannot be small*

holes, since otherwise it is more convenient to "fill" them and rescale the set. More precisely, for every $\xi > 0$ there exists some $r > 0$ such that Ω^* contains the ball of radius r centered at any x such that $u_1^2(x) + \cdots + u_k^2(x) > \xi$. In particular, this fact provides an estimate on the modulus of continuity of the function $u_1^2 + \cdots + u_k^2$ on the boundary of Ω^*.

Observe that, by the definition of the open set Ω^{**} in the above proof, it follows that the first k eigenvalues defined on the space $\widetilde{\mathcal{H}}_0^1(\Omega^{**})$, and those defined on the classical Sobolev space $\mathcal{H}_0^1(\Omega^{**})$, coincide. Thus, we have a solution of the shape optimization problem (6.1) in its classical formulation.

Corollary 6.3. *Let $F : \mathbb{R}^k \to \mathbb{R}$ be a locally increasingly bi-Lipschitz function. Then there is a solution Ω^* of the problem*

$$\min\left\{ F\big(\lambda_1(\Omega), \ldots, \lambda_k(\Omega)\big) : \ \Omega \subseteq \mathbb{R}^d \ open, \ |\Omega| = 1 \right\}.$$

Moreover, the eigenfunctions of the Dirichlet Laplacian on Ω^*, corresponding to the eigenvalues $\lambda_1(\Omega^*), \ldots, \lambda_k(\Omega^*)$, are Lipschitz continuous on $\mathbb{R}^d$.

The openness can be obtained not only for sets minimizing (6.1), but also for shape supersolutions.

Proposition 6.4. *Let $F : \mathbb{R}^k \to \mathbb{R}$ be a locally increasingly bi-Lipschitz function, and let Ω^* be a shape supersolution for the functional*

$$\Omega \mapsto F\big(\lambda_1(\Omega), \ldots, \lambda_k(\Omega)\big) + \Lambda|\Omega|. \tag{6.3}$$

Then there is an open set $\Omega^{**} \subseteq \Omega^*$ such that $\lambda_i(\Omega^{**}) = \lambda_i(\Omega^*)$ for $i = 1, \ldots, k$, and which is still a supersolution for the functional (6.3). Moreover, there exists a sequence of Lipschitz orthonormal eigenfunctions corresponding to the first k eigenvalues in Ω^{**}.

Proof. Applying Theorem 5.6 to Ω^*, we find an orthonormal set of Lipschitz eigenfunctions $u_1, u_2, \ldots, u_k$ for Ω^*; then, as in Theorem 6.1, we define

$$\Omega^{**} := \bigcup_{i=1}^{k} \{u_i \neq 0\},$$

which is open since the functions u_i are Lipschitz continuous. As before, Ω^{**} is essentially contained in Ω^*, thus it has bigger eigenvalues, and on the other hand the definition of eigenvalues –together with the fact that each u_i is by definition in $\mathcal{H}_0^1(\Omega^{**})$– gives the opposite inequality. As a consequence, we conclude that $\lambda_i(\Omega^*) = \lambda_i(\Omega^{**})$ for every $i = 1, \ldots, k$. It is now immediate to show that Ω^{**} is also a shape supersolution for (6.3): indeed, for every $\Omega \supseteq \Omega^{**}$, we just compute

$$F\big(\lambda_1(\Omega^{**}), \ldots, \lambda_k(\Omega^{**})\big) + \Lambda|\Omega^{**}| = F\big(\lambda_1(\Omega^*), \ldots, \lambda_k(\Omega^*)\big) + \Lambda|\Omega^*| - \Lambda|\Omega^* \setminus \Omega^{**}|$$

$$\leq F\big(\lambda_1(\Omega \cup \Omega^*), \ldots, \lambda_k(\Omega \cup \Omega^*)\big) + \Lambda|\Omega \cup \Omega^*| - \Lambda|\Omega^* \setminus \Omega^{**}|$$

$$\leq F\big(\lambda_1(\Omega), \ldots, \lambda_k(\Omega)\big) + \Lambda|\Omega|.$$

Being then Ω^{**} a shape supersolution for (6.3), and being the functions u_i also eigenfunctions for Ω^{**}, the proof is concluded. $\qquad\square$

For functionals of the form

$$\Omega \mapsto F\big(\lambda_{k_1}(\Omega), \ldots, \lambda_{k_p}(\Omega)\big),$$

depending on some non-consecutive eigenvalues $\lambda_{k_1}, \ldots, \lambda_{k_p}$, it is still possible to obtain that an optimal sets Ω^* for the problem

$$\min\left\{ F\big(\lambda_{k_1}(\Omega), \ldots, \lambda_{k_p}(\Omega)\big) : \ \Omega \subseteq \mathbb{R}^d \ measurable, \ |\Omega| = 1 \right\}, \tag{6.4}$$

is open, provided that an additional condition on the eigenvalues of Ω^* is satisfied.

Proposition 6.5. *Let $F : \mathbb{R}^p \to \mathbb{R}$ be a locally increasingly bi-Lipschitz function, $0 < k_1 < k_2 < \cdots < k_p$ be natural numbers, and Ω^* be a minimizer for the problem (6.4). If for all $j = 1, \ldots, p$ one has $\lambda_{k_j}(\Omega^*) > \lambda_{k_j-1}(\Omega^*)$, then Ω^* is essentially open. Moreover all the eigenfunctions corresponding to $\lambda_{k_j}(\Omega^*)$, for $j = 1, \ldots, p$ are Lipschitz continuous on $\mathbb{R}^d$.*

Proof. First of all, we remind that a minimizer for the problem (6.4) exists and is bounded, and moreover it is also a shape supersolution of the functional $F\big(\lambda_{k_1}(\Omega), \ldots, \lambda_{k_p}(\Omega)\big)\big) + \Lambda|\Omega|$ for a suitable Λ, exactly as in the proof of Theorem 6.1; thus, the second part of the claim simply follows by Theorem 5.6, and it only remains to show that Ω^* is essentially open.

Let us fix an orthonormal set of eigenfunctions $\{u_i,\ 1 \leq i \leq k_p\}$ for the first k_p eigenvalues in Ω^*, and consider the family of indices

$$I := \left\{ i \leq k_p : \lambda_i(\Omega^*) = \lambda_{k_j}(\Omega^*), \text{ for some } j \right\}.$$

Recalling again Theorem 5.6, we know that for every $i \in I$ the eigenfunction u_i is Lipschitz continuous, thus the set

$$\Omega_A := \left\{ x \in \mathbb{R}^d \ : \ \sum_{i \in I} u_i(x)^2 > 0 \right\}$$

is open, and of course essentially contained in Ω^*. Our aim is then to prove that $N = \Omega^* \backslash \Omega_A$ is negligible. Suppose, by contradiction, that $|N| > 0$ and let $x \in N$ be a point of density one for N, i.e.

$$\lim_{\rho \to 0} \frac{|N \cap B_\rho(x)|}{|B_\rho(x)|} = 1 \,.$$

Since, for $\rho \to 0$, the sets $\Omega^* \setminus (N \cap B_\rho(x))$ γ-converge to Ω^* we have the convergence of the spectra $\lambda_k(\Omega^* \setminus (N \cap B_\rho(x))) \to \lambda_k(\Omega^*)$, for every $k \in \mathbb{N}$. Then, being $\lambda_{k_j}(\Omega^*) > \lambda_{k_j-1}(\Omega^*)$, we can choose ρ small enough such that the set $\widetilde{\Omega} = \Omega^* \setminus (N \cap B_\rho(x))$ satisfies

$$\lambda_{k_j-1}(\widetilde{\Omega}) < \lambda_{k_j}(\Omega^*), \qquad \forall\, j = 1, \ldots, p. \tag{6.5}$$

We note now that for $i \in I$ the eigenfunction u_i belongs to $\widetilde{\mathcal{H}}_0^1(\widetilde{\Omega})$, and since $\widetilde{\Omega} \subseteq \Omega^*$ we get that u_i satisfies the equation

$$-\Delta u_i = \lambda_{k_j}(\Omega^*)u_i, \qquad u_i \in \widetilde{\mathcal{H}}_0^1(\widetilde{\Omega}) \,.$$

Thus, for each $i \in I$ the number $\lambda_i(\Omega^*)$ is also in the spectrum of the Dirichlet Laplacian on $\widetilde{\Omega}$. Combined with (6.5) and with the fact that $\widetilde{\Omega} \subseteq \Omega^*$, this gives

$$\lambda_k(\widetilde{\Omega}) = \lambda_k(\Omega^*), \qquad \forall\, k = 1, \ldots, k_p \,.$$

Since for every $\rho > 0$ we have $|N \cap B_\rho(x)| > 0$, it follows that $|\widetilde{\Omega}| < |\Omega^*| = 1$; by the strict monotonicity of F, if we rescale $\widetilde{\Omega}$ till volume 1 we get a competitor strictly better than Ω^* in (6.4), which is a contradiction with the optimality of Ω^*. $\qquad\square$

Remark 6.6. *Unfortunately, Proposition 6.5 provides the openness of optimal sets only up to zero Lebesgue measure. Hence we have that $\widetilde{\mathcal{H}}_0^1(\Omega^*) = \widetilde{\mathcal{H}}_0^1(\Omega_A)$, but we do not know in general if $\mathcal{H}_0^1(\Omega^*) = \mathcal{H}_0^1(\Omega_A)$; thus, it is not clear whether an open "classical" solution exists, where by "classical" we refer to the case when the eigenvalues are considered in the standard $\mathcal{H}_0^1$ spaces, and not in the modified $\widetilde{\mathcal{H}}_0^1$ spaces. Keep in mind that this problem did not occur with the situation of Theorem 6.1, as noticed right after Remark 6.2.*

APPENDIX A. APPENDIX: PROOF OF THEOREM 3.3

For the sake of the completeness, we report here the proof of Theorem 3.3, given in [6]. We note that if the state function u, quasi-minimizer for the functional J_f, is positive, then the classical approach of Alt and Caffarelli (see [1]) can be applied to obtain the Lipschitz continuity of u. This approach is based on an external perturbation and on the following inequality (see [1, Lemma 3.2])

$$\frac{|B_r(x_0) \cap \{u = 0\}|}{r^2} \left(\fint_{\partial B_r(x_0)} u \, d\mathcal{H}^{d-1} \right)^2 \leq C_d \int_{B_r(x_0)} |\nabla(u - v)|^2 \, dx \,, \qquad (A.1)$$

which holds for every $x_0 \in \mathbb{R}^d$, $r > 0$, $u \in \mathcal{H}^1(\mathbb{R}^d)$, $u \geq 0$ and $v \in \mathcal{H}^1(B_r)$ that solves

$$\min \left\{ \int_{B_r(x_0)} |\nabla v|^2 \, dx : \ v - u \in \mathcal{H}_0^1(B_r(x_0)), \ v \geq u \right\} . \qquad (A.2)$$

Since for sign-changing state functions u, the inequality (A.1) is not known, one needs a more careful analysis on the common boundary of $\{u > 0\}$ and $\{u < 0\}$, which is based on the monotonicity formula of Alt, Caffarelli and Friedmann.

Theorem A.1. *Let $U^+, U^- \in \mathcal{H}^1(B_1)$ be continuous non-negative functions such that $\Delta U^{\pm} \geq -1$ on B_1 and $U^+ U^- = 0$. Then there is a dimensional constant C_d such that for each $r \in (0, \frac{1}{2})$*

$$\left(\frac{1}{r^2} \int_{B_r} \frac{|\nabla U^+(x)|^2}{|x|^{d-2}} \, dx \right) \left(\frac{1}{r^2} \int_{B_r} \frac{|\nabla U^-(x)|^2}{|x|^{d-2}} \, dx \right) \leq C_d \left(1 + \int_{B_1} |U^+ + U^-|^2 \, dx \right) . \quad (A.3)$$

For our purposes we will need the following rescaled version of this formula.

Corollary A.2. *Let $\Omega \subseteq \mathbb{R}^d$ be a quasi-open set of finite measure, $f \in L^{\infty}(\Omega)$ and $u : \mathbb{R}^d \to \mathbb{R}$ be a continuous function such that*

$$-\Delta u = f \quad in \quad \Omega, \qquad u \in \mathcal{H}_0^1(\Omega) . \qquad (A.4)$$

Setting $u^+ = \sup\{u, 0\}$ and $u^- = \sup\{-u, 0\}$, there is a dimensional constant C_d such that for each $0 < r \leq 1/2$

$$\left(\frac{1}{r^2} \int_{B_r} \frac{|\nabla u^+(x)|^2}{|x|^{d-2}} \, dx \right) \left(\frac{1}{r^2} \int_{B_r} \frac{|\nabla u^-(x)|^2}{|x|^{d-2}} \, dx \right) \leq C_d \left(\|f\|_{L^{\infty}}^2 + \int_{\Omega} u^2 \, dx \right) \leq C_m \,, \quad (A.5)$$

where $C_m = C_d \|f\|_{L^{\infty}}^2 \left(1 + |\Omega|^{\frac{d+4}{d}} \right)$.

Proof. We apply Theorem A.1 to $U^{\pm} = \|f\|_{L^{\infty}}^{-1} u^{\pm}$ and substituting in (A.3) we obtain the first inequality in (A.5). The second one follows, using the equation (A.4), since

$$\|u\|_{L^2}^2 \leq C_d |\Omega|^{2/d} \|\nabla u\|_{L^2}^2 = C_d |\Omega|^{2/d} \int_{\Omega} f u \, dx \leq C_d |\Omega|^{2/d+1/2} \|f\|_{L^{\infty}} \|u\|_{L^2} \,.$$

$\square$

The proof of the Lipschitz continuity of the quasi-minimizers for J_f needs two preliminary results, precisely in Lemma A.3 we prove the continuity of u and in Lemma A.5, we give an estimate on the Laplacian of u as a measure on the boundary $\partial\{u \neq 0\}$.

Lemma A.3. *If u satisfies the assumptions of Theorem 3.3, then it is continuous.*

Proof. Let $x_n \to x_\infty \in \mathbb{R}^d$ and set $\delta_n := |x_n - x_\infty|$. If for some n, $|B_{\delta_n}(x_\infty) \cap \{u = 0\}| = 0$, then $-\Delta u = f$ in $B_{\delta_n}(x_\infty)$ and so u is continuous in x_∞.

Assume then that, for all n, $|B_{\delta_n}(x_\infty) \cap \{u = 0\}| \neq 0$, and consider the function $u_n : \mathbb{R}^d \to \mathbb{R}$ defined by $u_n(\xi) = u(x_\infty + \delta_n \xi)$. Since $\|u_n\|_{L^{\infty}} = \|u\|_{L^{\infty}}$ for any n –and in turn $\|u\|_{L^{\infty}} < \infty$ by (2.5)– we can assume, up to a subsequence, that u_n converges weakly-$*$

in L^∞ to some function $u_\infty \in L^\infty(\mathbb{R}^d)$. We will prove that $u_\infty \equiv 0$ and that $u_n \to u_\infty$ uniformly on B_1, and this will of course imply the continuity of u in x_∞.

Step I. u_∞ is a constant.
For all $R \geq 1$ and $n \in \mathbb{N}$, we introduce the function $v_{R,n}$ such that:

$$\begin{cases} -\Delta v_{R,n} = f, & \text{in } B_{R\delta_n}(x_\infty), \\ v_{R,n} = u, & \text{on } \partial B_{R\delta_n}(x_\infty). \end{cases}$$

Setting $v_n(\xi) := v_{R,n}(x_\infty + \delta_n\xi)$ and calling C_b the constant as in (3.5), for $\delta_n \leq r_0$ we get

$$\int_{B_R} |\nabla(u_n - v_n)|^2 d\xi = \delta_n^{2-d} \int_{B_{R\delta_n}(x_\infty)} |\nabla(u - v_{R,n})|^2 dx$$

$$= \delta_n^{2-d} \int_{B_{R\delta_n}(x_\infty)} \nabla u \cdot \nabla(u - v_{R,n}) dx - \delta_n^{2-d} \int_{B_{R\delta_n}(x_\infty)} f(u - v_{R,n}) dx$$

$$\leq C_b \delta_n^{2-d} \left(\int_{B_{R\delta_n}(x_\infty)} |\nabla(u - v_{R,n})|^2 dx \right)^{1/2} R^{d/2} \delta_n^{d/2}$$

$$\leq C_b R^{d/2} \delta_n \left(\int_{B_R} |\nabla(u_n - v_n)|^2 d\xi \right)^{1/2},$$

which implies

$$\fint_{B_R} |\nabla(u_n - v_n)|^2 d\xi \leq C_b^2 \delta_n^2.$$

In particular, $u_n - v_n \to 0$ in $\mathcal{H}^1(B_R)$ for any $R \geq 1$. On the other hand, calling $f_n(\xi) = f(x_\infty + \delta_n\xi)$, we have that

$$\begin{cases} -\Delta v_n = \delta_n^2 f_n, & \text{in } B_R, \\ v_n \leq \|u\|_{L^\infty}, & \text{on } \partial B_R. \end{cases}$$

Since $\|f_n\|_{L^\infty} = \|f\|_{L^\infty}$, the maximum principle implies that the v_n are equi-bounded in B_R; hence, the estimate (2.7) implies that they are also equi-Lipschitz, thus equi-continuous, in $B_{R/2}$. So, up to a subsequence, v_n uniformly converges to some function which is harmonic on $B_{R/2}$. Since $v_n - u_n \to 0$ in $\mathcal{H}^1(B_R)$ and u_n converges weakly-* in $L^\infty(\mathbb{R}^d)$ to u_∞, we deduce that v_n converges uniformly to u_∞ on B_R, and that u_∞ is harmonic on $B_{R/2}$ for every $R \geq 1$. Therefore, u_∞ is a bounded harmonic function on $\mathbb{R}^d$, and this finally implies that u_∞ is constant.

Step II. $u_n \to u_\infty$ in $\mathcal{H}^1_{loc}(\mathbb{R}^d)$.
In fact, for the functions $\widetilde{v}_n = v_n - u_\infty$, we have that

$$\begin{cases} -\Delta \widetilde{v}_n = \delta_n^2 f_n, & \text{in } B_R, \\ \widetilde{v}_n \leq 2\|u\|_{L^\infty}, & \text{on } \partial B_R, \end{cases}$$

and $\widetilde{v}_n \to 0$ uniformly on $B_{R/2}$. Again by (2.7), we have that $\|\nabla \widetilde{v}_n\|_{L^\infty(B_{R/4})} \to 0$, and so $v_n \to u_\infty$ in $\mathcal{H}^1(B_{R/4})$. Since $v_n - u_n \to 0$ in $\mathcal{H}^1(B_R)$, we conclude also this step.

Step III. If $u_\infty \geq 0$, then $u_n^- \to 0$ uniformly on balls.
Since on $\{u_n < 0\}$ the equality $-\Delta u_n^- = -\delta_n^2 f_n$ holds, on the whole $\mathbb{R}^d$ we have that $-\Delta u_n^- \leq -\delta_n^2 f_n I_{\{u_n<0\}} \leq \delta_n^2 |f_n|$. Thus, it is enough to prove that for each $R \geq 1$, $\widetilde{u}_n \to 0$ uniformly on $B_{2R/3}$, where

$$\begin{cases} -\Delta \widetilde{u}_n = \delta_n^2 |f_n|, & \text{in } B_R, \\ \widetilde{u}_n = u_n^-, & \text{on } \partial B_R. \end{cases}$$

Since $u_n^- \to 0$ in $\mathcal{H}^1(B_R)$, we have that $\fint_{\partial B_R} u_n^- \to 0$, thus the claim comes once again from the estimate (2.7).

Step IV. $u_\infty \equiv 0$.
Suppose, without loss of generality, that $u_\infty \geq 0$. Let $y_n = x_\infty + \delta_n\xi_n$, with $\xi_n \in B_1$, be a

Lebesgue point for u with $u(y_n) = 0$. For each $s > 0$ consider a function $\phi_s \in C_c^\infty(B_{2s}(y_n))$ such that $0 \le \phi_s \le 1$, $\phi_s \equiv 1$ on $B_s(y_n)$, and $\|\nabla\phi_s\|_{L^\infty} \le 2/s$. Thus, we have that

$$|\langle \Delta u + f, \phi_s \rangle| \le C_d C_b s^{d-1},$$

where C_b is the constant from (3.5). Denote with μ_1 and μ_2 the positive Borel measures $\Delta u^+ + f I_{\{u>0\}}$ and $\Delta u^- - f I_{\{u<0\}}$. Then, we have

$$\mu_1(B_s(y_n)) \le \langle \mu_1, \phi_s \rangle = \langle \mu_1 - \mu_2, \phi_s \rangle + \langle \mu_2, \phi_s \rangle \le C_d C_b s^{d-1} + \mu_2(B_{2s}(y_n)).$$

As a consequence, we have

$$
\begin{aligned}
\Delta u^+(B_s(y_n)) &\le C_d C_b s^{d-1} + \Delta u^-(B_{2s}(y_n)) + C_d\|f\|_{L^\infty} s^d \\
&\le C_d(C_b + \|f\|_{L^\infty})s^{d-1} + \Delta u^-(B_{2s}(y_n)),
\end{aligned}
\tag{A.6}
$$

where the last inequality holds for every $s \le 1$. Recall now the standard estimate

$$\frac{\partial}{\partial s} \fint_{\partial B_s(y_n)} u^+ = \fint_{\partial B_s(y_n)} \frac{\partial u^+}{\partial \nu} = \frac{1}{d\omega_d s^{d-1}} \Delta u^+(B_s),$$

and observe that since y_n is a Lebesgue point for u with $u(y_n) = 0$ then

$$\lim_{s \to 0} \fint_{\partial B_s(y_n)} u^+ = 0.$$

Thus, integrating the above estimate and using (A.6), we obtain

$$\fint_{\partial B_{\delta_n}(y_n)} u^+ \, d\mathcal{H}^{d-1} \le C_d(C_b + \|f\|_{L^\infty})\delta_n + \frac{1}{2} \fint_{\partial B_{2\delta_n}(y_n)} u^- \, d\mathcal{H}^{d-1}$$

or, equivalently,

$$\fint_{\partial B_1} u_n^+(\xi_n + \cdot) \, d\mathcal{H}^{d-1} \le C_d(C_b + \|f\|_{L^\infty})\delta_n + \frac{1}{2} \fint_{\partial B_2} u_n^-(\xi_n + \cdot) \, d\mathcal{H}^{d-1}.$$

By Step III we know that the right-hand side goes to zero as $n \to \infty$, hence so does also the left-hand side. Up to a subsequence we may assume that $\xi_n \to \xi_\infty$ and so, $u_n(\xi_n + \cdot) \to u_\infty(\xi_\infty + \cdot) = u_\infty$ in $\mathcal{H}_{loc}^1(\mathbb{R}^d)$. Thus $u_\infty \equiv 0$.

Step V. The convergence $u_n \to 0$ is uniform on the ball B_1.

Since $u_\infty \equiv 0$, this follows just applying twice Step III, once to u and once to $-u$. □

Remark A.4. *In $\mathbb{R}^2$, the continuity of the state function u in Theorem 3.3 can be deduced by the classical Alt-Caffarelli argument, which one can apply after reducing the problem to the case when u is positive. For example, if $u \in \mathcal{H}^1(\mathbb{R}^2)$ is a function satisfying*

$$J_{\lambda u}(u) + c|\{u \ne 0\}| \le J_{\lambda u}(v) + c|\{v \ne 0\}|, \qquad \forall v \in \mathcal{H}^1(\mathbb{R}^2),$$

for some $\lambda > 0$, then u is continuous. Indeed, let $x_0 \in \mathbb{R}^2$ be such that $u(x_0) > 0$ and let $r_0 > 0$ and $\varepsilon > 0$ be small enough such that, for every $x \in \mathbb{R}^2$ and every $r \le r_0$, we have $\int_{B_r(x)} |\nabla u|^2 \, dx \le \varepsilon$. As a consequence, for every $x \in \mathbb{R}^2$ there is some $r_x \in [r_0/2, r_0]$ such that $\int_{\partial B_{r_x}(x)} |\nabla u|^2 \, dx \le 2\varepsilon/r_0$ and

$$osc_{\partial B_{r_x}(x)} u \le \int_{\partial B_{r_x}(x)} |\nabla u| \, d\mathcal{H}^1 \le \sqrt{2\pi r_0}\sqrt{2\varepsilon/r_0} \le \sqrt{4\pi\varepsilon}.
\tag{A.7}$$

On the other hand, the positive part $u^+ = \sup\{u, 0\}$ of u satisfies $\Delta u^+ + \lambda\|u\|_{L^\infty} \ge 0$ on $\mathbb{R}^2$, and so there is a constant $C > 0$ such that

$$u(x_0) \le \fint_{\partial B_{r_{x_0}}(x_0)} u \, d\mathcal{H}^1 + Cr_{x_0}^2,$$

which together with (A.7) gives that, choosing $r_0 > 0$ small enough, we can construct a ball $B_r(x_0)$ of radius $r \le r_0$ such that $u \ge u(x_0)/2 > 0$ on $\partial B_r(x_0)$.

We then notice that the set $\{u < 0\} \cap B_r(x_0)$ has measure 0. Indeed, if this is not the case, then the function $\widetilde{u} = \sup\{-u, 0\}I_{B_r(x_0)} \in \mathcal{H}_0^1(B_r(x_0))$ is such that $J_{\lambda u}(u) = J_{\lambda u}(-\widetilde{u}) + J_{\lambda u}(u + \widetilde{u})$. By the maximum principle $\|\widetilde{u}\|_{L^\infty} \leq Cr_0^2$ and so, for some constant $C > 0$, we have

$$\left| J_{\lambda u}(-\widetilde{u}) \right| \leq Cr_0^2 \left| \{u < 0\} \cap B_r(x_0) \right| < c \left| \{u < 0\} \cap B_r(x_0) \right|,$$

for r_0 small enough. Hence we have $J_{\lambda u}(-\widetilde{u}) + c|\{u < 0\} \cap B_r(x_0)| > 0$, which contradicts the quasi-minimality of u.

We conclude the proof by showing that the set $\{u = 0\} \cap B_r(x_0)$ has measure 0. Comparing u with the function $w = I_{B_r^c(x_0)}u + I_{B_r(x_0)}v$, being v the function from (A.2), we have

$$c\left|\{u = 0\} \cap B_r(x_0)\right| \geq J_{\lambda u}(u) - J_{\lambda u}(w)$$

$$= \frac{1}{2} \int_{B_r(x_0)} \left(|\nabla u|^2 - |\nabla v|^2\right) dx - \int_{B_r(x_0)} \lambda u(u - v)\, dx$$

$$\geq \frac{1}{2} \int_{B_r(x_0)} |\nabla(u - v)|^2\, dx$$

$$\geq \frac{C_2}{r^2}\left|\{u = 0\} \cap B_r(x_0)\right| \left(\fint_{\partial B_r(x_0)} u\, d\mathcal{H}^1 \right)^2,$$

where the last inequality is due to (A.1). If we suppose that $|\{u = 0\} \cap B_r(x_0)| > 0$, then for some constant $C > 0$, we would have $u(x_0) \leq Cr_0^2$, which is absurd choosing $r_0 > 0$ small enough.

Lemma A.5. *Let $u \in \mathcal{H}^1(\mathbb{R}^d)$ satisfy the assumptions of Theorem 3.3, and in particular let r_0 and C_b be as in (3.5). Then, for each $x_0 \in \mathbb{R}^d$ for which $u(x_0) = 0$ and for every $0 < r < r_0/4$, one has*

$$\left|\Delta|u|\right|(B_r(x_0)) \leq C\, r^{d-1},$$

where C depends only on d, $|\Omega|$, $\|f\|_{L^\infty}$ and C_b.

Proof. Without loss of generality we can suppose $x_0 = 0$. For each $r > 0$, consider the functions

$$v^r := v_+^r - v_-^r, \qquad w^r := w_+^r - w_-^r,$$

where $v_\pm^r$ and $w_\pm^r$ are the solutions of the following equations on B_r

$$\begin{cases} -\Delta v_\pm^r = f^\pm & \text{in } B_r, \\ v_\pm^r = u^\pm & \text{on } \partial B_r, \end{cases} \qquad \begin{cases} -\Delta w_\pm^r = f^\pm & \text{in } B_r, \\ w_\pm^r = 0 & \text{on } \partial B_r. \end{cases}$$

Thus we have that $v_\pm^r - w_\pm^r$ is harmonic in B_r, and so we estimate

$$\int_{B_r} |\nabla(v_\pm^r - w_\pm^r)|^2\, dx \leq \int_{B_r} |\nabla u^\pm|^2\, dx. \tag{A.8}$$

Since $u^\pm - v_\pm^r + w_\pm^r \in \mathcal{H}_0^1(B_r)$ and $v_\pm^r - w_\pm^r$ is harmonic, we have

$$\int_{B_r} |\nabla(u^\pm - v_\pm^r + w_\pm^r)|^2\, dx = \int_{B_r} \nabla u^\pm \cdot \nabla(u^\pm - v_\pm^r + w_\pm^r)\, dx$$

$$= \int_{B_r} |\nabla u^\pm|^2\, dx + \int_{B_r} \nabla u^\pm \cdot \nabla(w_\pm^r - v_\pm^r)\, dx \leq 2 \int_{B_r} |\nabla u^\pm|^2\, dx,$$

where the last inequality is due to (A.8). Thus, for $r \leq 1/2$ we obtain from the monotonicity formula (A.5) the estimate

$$\left(\fint_{B_r} |\nabla(u^+ - v_+^r + w_+^r)|^2\, dx \right) \left(\fint_{B_r} |\nabla(u^- - v_-^r + w_-^r)|^2\, dx \right)$$

$$\leq 4 \left(\fint_{B_r} |\nabla u^+|^2\, dx \right) \left(\fint_{B_r} |\nabla u^-|^2\, dx \right) \leq 4C_m. \tag{A.9}$$

On the other hand, for $0 < r \leq r_0 \leq 1$, we have, also by (3.5),

$$
\begin{aligned}
\int_{B_r} |\nabla(u - v^r + w^r)|^2 \, dx &\leq 2 \int_{B_r} |\nabla(u - v^r)|^2 \, dx + 2 \int_{B_r} |\nabla w^r|^2 \, dx \\
&= 2 \int_{B_r} \left(\nabla u \cdot \nabla(u - v^r) - f(u - v^r) \right) dx + 2 \int_{B_r} |\nabla w^r|^2 \, dx \qquad \text{(A.10)} \\
&\leq C_b r^d + C_b \int_{B_r} |\nabla(u - v^r)|^2 + 2 \int_{B_r} |\nabla w^r|^2 \, dx \leq C r^d \,,
\end{aligned}
$$

where the constant C depends on d, $|\Omega|$, $\|f\|_{L^\infty}$ and C_b. Using (A.9) and (A.10), we have

$$
\begin{aligned}
\int_{B_r} |\nabla(u^+ - v_+^r &+ w_+^r)|^2 \, dx + \int_{B_r} |\nabla(u^- - v_-^r + w_-^r)|^2 \, dx \\
&\leq 2 \left(\int_{B_r} |\nabla(u^+ - v_+^r + w_+^r)|^2 \, dx \right)^{1/2} \left(\int_{B_r} |\nabla(u^- - v_-^r + w_-^r)|^2 \, dx \right)^{1/2} \\
&\quad + \int_{B_r} |\nabla(u - v^r + w^r)|^2 \, dx \leq C r^d \,,
\end{aligned}
$$

where the constant C might have increased but has the same dependences as before (recall that C_m depends on d, $|\Omega|$ and $\|f\|_{L^\infty}$!). Putting together this last estimate with (A.10), we finally get

$$
\int_{B_r} |\nabla(u^\pm - v_\pm^r)|^2 \, dx \leq C r^d \,. \qquad \text{(A.11)}
$$

Let us now define

$$
U := u^+ - v_+^r \,, \qquad \mu_1 := \Delta u^+ + f I_{\{u>0\}} \,, \qquad \mu_2 := \Delta u^- - f I_{\{u<0\}} \,.
$$

Since $U \in \mathcal{H}_0^1(B_r)$ by definition, and it is sub-harmonic because

$$
\Delta U = \Delta(u^+ - v_+^r) = \Delta u^+ + f^+ \geq \Delta u^+ + f I_{\{u>0\}} = \mu_1 \geq 0 \,, \qquad \text{(A.12)}
$$

we obtain $U \leq 0$, and then also by (A.11)

$$
\int_{B_r} v_+^r \, d\mu_1 = \int_{B_r} (v_+^r - u^+) \, d\mu_1 \leq \int_{B_r} |\nabla U|^2 \, dx \leq C r^d \,. \qquad \text{(A.13)}
$$

Recalling that U is negative, for each $z \in B_{r/4}$ we find

$$
\fint_{\partial B_{3r/4}(z)} U \, d\mathcal{H}^{d-1} \leq 0 \leq u^+(z) = U(z) + v_+^r(z) \,.
$$

Applying then (2.6) to $U \in \mathcal{H}^1(B_r)$ (which is admissible because every signed distribution is a measure) and using (A.12), we obtain

$$
\begin{aligned}
v_+^r(z) \geq -U(z) &= -\fint_{\partial B_{3r/4}(z)} U \, d\mathcal{H}^{d-1} + \frac{1}{d\omega_d} \int_{s=0}^{3r/4} s^{1-d} \Delta U(B_s(z)) \, ds \\
&\geq \frac{1}{d\omega_d} \int_0^{3r/4} s^{1-d} \Delta U(B_s(z)) \, ds \geq \frac{1}{d\omega_d} \int_0^{3r/4} s^{1-d} \mu_1(B_s(z)) \, ds \,.
\end{aligned}
$$

By (A.13) we get then

$$
\begin{aligned}
C(r/4)^d &\geq \int_{B_{r/4}} v_+^r(z) \, d\mu_1(z) \geq \frac{1}{d\omega_d} \int_{B_{r/4}} d\mu_1(z) \int_0^{3r/4} s^{1-d} \mu_1(B_s(z)) \, ds \\
&\geq \frac{1}{d\omega_d} \int_{B_{r/4}} d\mu_1(z) \int_{r/2}^{3r/4} s^{1-d} \mu_1(B_s(z)) \, ds \\
&\geq \frac{1}{d\omega_d} \int_{B_{r/4}} d\mu_1(z) \int_{r/2}^{3r/4} s^{1-d} \mu_1(B_{r/4}) \, ds \geq C_d r^{2-d} \left(\mu_1(B_{r/4}) \right)^2 \,,
\end{aligned}
$$

i.e., $\mu_1(B_r) \leq Cr^{d-1}$ as soon as $0 < r < r_0/4$. Since the very same claim clearly holds for μ_2, and since $\big|\Delta|u|\big| \leq \mu_1 + \mu_2 + f$, recalling that $f \in L^\infty$ we get the thesis. $\qquad\square$

We are finally in position to give the proof of Theorem 3.3.

Proof of Theorem 3.3. By Lemma A.3 we know that u is continuous, so we can assume that Ω coincides with the open set $\{u \neq 0\}$. Thanks to Lemma A.5, we already know the validity of (3.6) for x such that $u(x) = 0$ and $0 < r < r_0/4$, hence to prove (2) of Theorem 3.3 we only need to check that $\Delta|u|$ is a Borel measure on $\mathbb{R}^d$. Since $\Delta|u| \equiv 0$ outside of $\overline{\Omega}$, we have only to take care of $\overline{\Omega}$. But $\Delta|u|$ coincides with $\pm f \in L^\infty$ inside Ω, thus just covering the compact set $\partial\Omega$ with finitely many balls of radius $r_0/5$ centered at points of $\partial\Omega$ we immediately obtain that $\Delta|u|$ is a Borel measure on the whole $\mathbb{R}^d$.

Let us now prove (1). For any $r > 0$, denote with $\Omega_r \subseteq \Omega$ the set $\{x \in \Omega : d(x, \Omega^c) < r\}$. Choose $x \in \Omega_{r_0/12}$ and let $y \in \partial\Omega$ be such that $R_x := d(x, \Omega^c) = |x - y|$. We claim now that

$$\|u\|_{L^\infty(B_{2R_x}(y))} \leq \frac{9R_x^2}{2d}\|f\|_{L^\infty(B_{3R_x}(y))} + C_d \fint_{\partial B_{3R_x}(y)} |u|\, d\mathcal{H}^{d-1}. \qquad (A.14)$$

Notice that this is exactly the second gradient estimate (2.7) applied to u in the ball $B_{3R_x}(y)$, but actually we cannot apply this estimate because on that ball the equation $-\Delta u = f$ is not satisfied. To prove the validity of (A.14), assume then without loss of generality that $\|u\|_{L^\infty(B_{2R_x}(y))} = \|u^+\|_{L^\infty(B_{2R_x}(y))}$, and define v^+, as in Lemma A.5, the solution of

$$\begin{cases} -\Delta v^+ = f^+ & \text{in } B_{3R_x}(y), \\ v^+ = u^+ & \text{on } \partial B_{3R_x}(y). \end{cases}$$

As already observed during the proof of Lemma A.5, in (A.12), the function $u^+ - v^+$ is sub-harmonic hence, since it belongs to $\mathcal{H}^1_0\big(B_{3R_x}(y)\big)$, it is negative in $B_{3R_x}(y)$. By this observation, and applying (2.7) in $B_{3R_x}(y)$ to the function v^+, which is admissible, we get

$$\begin{aligned} \|u\|_{L^\infty(B_{2R_x}(y))} = \|u^+\|_{L^\infty(B_{2R_x}(y))} &\leq \|v^+\|_{L^\infty(B_{2R_x}(y))} \\ &\leq \frac{9R_x^2}{2d}\|f^+\|_{L^\infty(B_{3R_x}(y))} + C_d \fint_{\partial B_{3R_x}(y)} |v^+|\, d\mathcal{H}^{d-1} \\ &\leq \frac{9R_x^2}{2d}\|f\|_{L^\infty(B_{3R_x}(y))} + C_d \fint_{\partial B_{3R_x}(y)} |u^+|\, d\mathcal{H}^{d-1} \\ &\leq \frac{9R_x^2}{2d}\|f\|_{L^\infty(B_{3R_x}(y))} + C_d \fint_{\partial B_{3R_x}(y)} |u|\, d\mathcal{H}^{d-1}, \end{aligned}$$

thus the validity of (A.14) is established. Hence, applying the first gradient estimate (2.7) to u in the ball $B_{R_x}(x)$, using (A.14), and then applying the estimate (2.6), which is possible because $\Delta|u|$ is a measure (mind also that $u(y) = 0$), we get

$$\begin{aligned} |\nabla u(x)| &\leq C_d\|f\|_{L^\infty} + \frac{2d}{R_x}\|u\|_{L^\infty(B_{R_x}(x))} \leq C_d\|f\|_{L^\infty} + \frac{2d}{R_x}\|u\|_{L^\infty(B_{2R_x}(y))} \\ &\leq (C_d + r_0)\|f\|_{L^\infty} + \frac{C_d}{R_x}\fint_{\partial B_{3R_x}(y)} |u|\, d\mathcal{H}^{d-1} \\ &\leq (C_d + r_0)\|f\|_{L^\infty} + \frac{C_d}{R_x}\int_0^{3R_x} s^{1-d}|\Delta|u||(B_s(y))\, ds \leq (C_d + r_0)\|f\|_{L^\infty} + 3C_dC, \end{aligned}$$

where C is the constant from Lemma A.5. Since for $x \in \Omega \setminus \Omega_{r_0/12}$ we have, still by (2.7), that

$$|\nabla u(x)| \leq C_d\|f\|_{L^\infty} + \frac{24d}{r_0}\|u\|_{L^\infty},$$

we obtain that u is Lipschitz and its Lipschitz constant can be estimated as desired. $\qquad\square$

Acknowledgments

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