Maps with vanishing denominator explained through applications in Economics

Fabio Tramontana

Department of Mathemtical Sciences, Mathematical Finance and Econometrics
Catholic University of Milan, Italy

E-mail: `fabio.tramontana@unicatt.it`

Abstract. We present the main results about plane maps with vanishing denominator by using as examples some economic applications where these results are used to investigate the global properties of the dynamic model. This kind of maps, originally brought to the attention of the researchers through their appearence in an economic application, are now well understood and the related results can now be applied to new models.

1. Introduction

Consider a plane map T of the following form:

$$
T : \begin{cases} x' = G(x,y) \\ y' = F(x,y) = \dfrac{N(x,y)}{D(x,y)} \end{cases}
\tag{1}
$$

with $(x,y) \in \mathbb{R}^2$, $G(x,y)$, $N(x,y)$ and $D(x,y)$ are continuously differentiable functions defined on $\mathbb{R}^2$. The set of points where $D(x,y)$ vanishes is called *set of nondefinition* of the map T:

$$
\delta_s = \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, D(x,y) = 0 \right\},
\tag{2}
$$

which is assumed to be the union of smooth curves of the plane.

An interesting question is related to what happens to the image by T of an arc γ, transverse to δ_s and crossing it in correspondence of the point (x_0, y_0). If we parameterize the arc as $\gamma(\tau)$, such that $\gamma(0) = (x_0, y_0)$, we have that in general:

$$
\lim_{\tau \to 0\pm} T(\gamma(\tau)) = (G(x_0, y_0), \infty),
$$

with ∞ either equal to $+\infty$ or $-\infty$. In other words, the image $T(\gamma)$ is the union of two disjoint unbounded arcs asymptotic to the line of equation $x = G(x_0, y_0)$. This result is correct provided that in the point (x_0, y_0) only $D(x,y)$ vanishes, but what would happen if also $N(x,y)$ becomes 0 in correspondence of (x_0, y_0)?

This is exactly the situation that came out in an economic model that dates back to 1997, studied by Bischi and Naimzada (1997). They obtained quite particular structures of the basins of attraction, founding that these phenomena were related to the presence of points where a component of the map assumes the form 0/0.

Moved by the necessity of solving the puzzle originated in the economic application, some authors (see Gardini & Bischi 1996, Bischi & Gardini 1997, Bischi et al. 1999, 2003, 2005, Gardini et al. 2007)[1] developed a theory that permits to explain not only the shape of the basins of attractions of the original model but also other interesting global phenomena arising in different fields of application such as Biology (Gu & Huang 2006, Gu 2009, Djellit et al. 2013) and Population Dynamics (Gu & Hao 2007). The theory developed to solve a particular problem has been extended and later applied to other models (see Bischi et al. 2001a for an early survey without reference to particular fields of application). So, it is interesting as in this case we can see in action the bidirectionality between theory and applications that characterize scientific research.

Recently, other economic models characterized by a map with vanishing denominator, have been studied by using the results of Bischi and collaborators. In this paper we summarize some of the most important theoretical results concerning this class of maps by using as examples the economic applications. In such a way we will also obtain a survey of the economic applications of maps with vanishing denominator.

2. Basic definitions

A point $Q = (x_0, y_0)$ such that $D(x_0, y_0) = N(x_0, y_0) = 0$ is called *focal point* if there exist smooth simple arcs $\gamma(\tau)$, with $\gamma(0) = Q$ and $\lim_{\tau \to 0} T(\gamma(\tau))$ is finite.

If we take different arcs $\gamma(\tau)$ through Q, the union of all the finite values of such a limit form the so-called *prefocal set* (or *curve*), denoted by δ_Q and belonging to $x = G(Q)$.

A focal point is called *simple* if:

$$\overline{N}_x \overline{D}_y - \overline{N}_y \overline{D}_x \neq 0, \tag{3}$$

where $\overline{N}_x = (\partial N / \partial x)(x_0, y_0)$ and the other partial derivatives are analoguous. Otherwise Q is called *non simple focal point*. Stated differently, a focal point is simple if it a simple root of the algebraic system made up by $N(x, y) = 0$ and $D(x, y) = 0$.

2.1. Main results and applications when the focal point is simple

One of the most important results related to the presence of a simple focal point is the following:

Proposition 1 *(Bischi et al., 1999). If Q is a simple focal point of a fractional rational map such as (1), then the following one-to-one relation between the slope m of an arc γ through Q and the point $(G(Q), y)$ in which $T(\gamma)$ crosses δ_Q exists:*

$$m \to (G(Q), y(m)),$$

with

$$y(m) = \frac{\overline{N}_x + m\overline{N}_y}{\overline{D}_x + m\overline{D}_y}$$

and

$$(F(Q), y) \to m(y)$$

with

$$m(y) = \frac{\overline{D}_x y - \overline{N}_x}{\overline{N}_y - \overline{D}_y y}.$$

---

[1] Some earlier work where maps with vanishing denominator were also considered are Mira (1981, 1996).

In other words T maps different arcs γ_j through Q with different slopes m_j into bounded arcs $T(\gamma_j)$ crossing the prefocal curve δ_Q in different points $(G(Q), y(m_j))$.

Even more interesting is to look at the action of the inverse map T^{-1} (or one of the inverses if the map is noninvertible). If we have an arc ω moving towards the prefocal curve δ_Q, its preimage $\omega^{-1} = T^{-1}(\omega)$ moves towards Q. When ω is tangent to δ_Q then its preimage ω^{-1} is characterized by a cusp point in Q. By moving further ω until it crosses δ_Q in two points, then ω^{-1} forms a loop with a double point at Q.

Now if the arc we are considering is a portion of a basin boundary, we can state the following:

Proposition 2 *(Bischi et al., 1999). When a basin boundary has a contact with a prefocal curve, a particular type of basin bifurcation occurs, causing the creation of cusp points and, after the crossing, of loops, along the basin boundary.*

Let us introduce the first economic example to better understand these results.

2.1.1. The Bischi and Tramontana (2005) model In 2005 Bischi and Tramontana consider the problem of a boundedly rational consumer who adjusts her demand adaptively by updating the quantity of good consumed in the direction indicated by discrepancy the current price and the observed relative utility gain. They also consider the preference of the good not exogenous but endogenously determined by the consumption of the good itself.

The evolution in discrete steps of the quantity (x) consumed and the preference (α) for the good, is regulated by the following map of the plane:

$$T_1 : \begin{cases} x' = F(x, \alpha) = \frac{N(x,\alpha)}{D(x,\alpha)} = \frac{(1-\alpha)x^2 + \alpha\mu m - x\mu p}{x(1-\alpha)} \\ \alpha' = G(x) = \frac{1}{k_1 + k_2 k_3^x} \end{cases} , \tag{4}$$

where $\mu > 0$ measures how reactive are consumers in adjusting their demand, $m > 0$ is the income of the consumer, $p > 0$ is the exogenous price of the good, and the remaining parameters $k_1 > 1$, $k_2 > 0$ and $0 < k_3 < 1$ permit to shape the relation between consumption and preference.

Map T_1 has in its first component a denominator that vanishes along the lines:

$$x = 0 \quad \text{and} \quad \alpha = 1$$

which are the *set of non-definition* δ_S of the map. In particular, the map takes the form $0/0$ in the simple focal point $Q = (0, 0)$, whose prefocal line is

$$\delta_Q = \left\{ (x, \alpha) | \alpha = \frac{1}{k_1 + k_2} \right\}.$$

A consequence of Proposition 2 is that in the presence of a focal point the shape of the basins of attraction may be strongly influenced by it. We have an example by considering the following set of parameters: $\mu = 0.5$, $m = 400$, $p = 8$, $k_1 = 17$, $k_2 = 5$ and $k_3 = 0.1$. The basins of attractions of al fixed point corresponding to a rational decision (F_r, in light blue) and of diverging trajectories (F_∞, in grey) are represented in Fig.1. As we can see the border of F_r is close to the prefocal curve. In fact, the preimages of the border are close to the focal point. By increasing k_2 to a value of 15 we are at the tangency condition and the cusp in the focal point is created (Fig. 2). After the crossing, with $k_2 = 90$ for instance, it is clearly visible the loop created by the preimages of the portion of the basin boundary that have crossed δ_Q (Fig. 3). Loops like that are called *lobes*.

The shape of the basins of attraction can be even more complicated when a map with vanishing denominator is characterized by a multiplicity of focal points or when basins of different coexisting attractors are intermingled. Two good examples are provided by the next economic models.
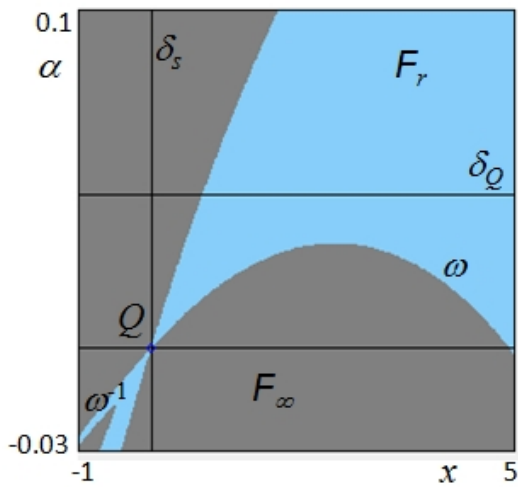
Figure 1. The basin boundary is close to the prefocal line.
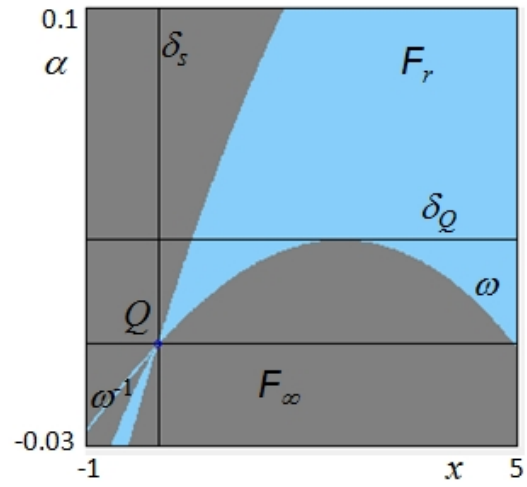
Figure 2. The basin boundary is tangent to the prefocal line.
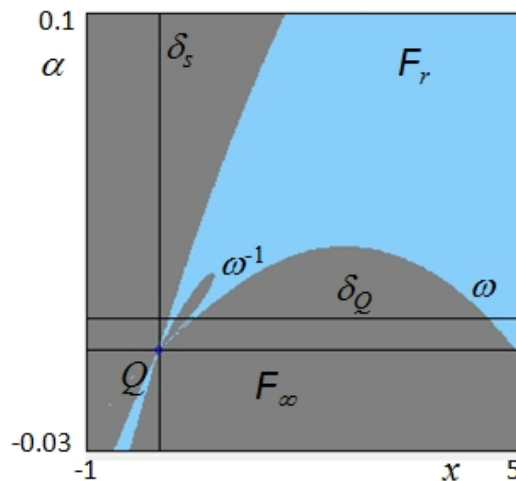
Figure 3. The basin boundary has crossed the prefocal line and a lobe issues from the focal point.

2.1.2. The Bischi and Naimzada (1997) model This model is of particular relevance because it is the one that originated the study of maps with vanishing denominator. Bischi and Naimzada (1997) were studying a model where agents are endowed with a fading memory and try to forecast the next realization of a particular market price by using the information concerning past prices, when they found unusual shapes of the basins of attractions and decided to deepen the mathematical analysis of this map. They realized that the particular basins' structure was a consequence of the presence of a denominator that can vanish in the two-dimensional dynamical system and its dynamic properties were studied in Gardini & Bischi (1996) and Bischi & Gardini (1997). The map regulates the evolution of the expected price (z) and another dynamic variable

(W) useful to model the fading memory of economic agents. This is the map:

$$T_2 : \begin{cases} z' = F(z, W) = \frac{N(z,W)}{D(z,W)} = \frac{\rho W z + \beta z^2 + \delta}{1 - \rho W} \\ W' = G(W) = 1 + \rho W \end{cases} , \qquad (5)$$

with the restriction: $\rho \in [0, 1]$.

Map T_2 in its first component presents a denominator that vanishes in the line of equation:

$$\delta_s : W = -\frac{1}{\rho}.$$

In this case $F(z, W)$ may assume the form $0/0$ in two points:

$$Q_1 \left(\frac{1+\sqrt{1-4\beta\delta}}{2\beta}, -\frac{1}{\rho} \right) \quad \text{and} \quad Q_2 \left(\frac{1-\sqrt{1-4\beta\delta}}{2\beta}, -\frac{1}{\rho} \right) ,$$

both characterized by the same prefocal line:

$$\delta_Q : W = 0,$$

so when an invariant set, such as a basin boundary, crosses the prefocal line, two lobes emerge simultaneously from the focal points, as it is clearly visible in Fig.4 (obtained with $\rho = 0.6$, $\delta = 3$ and $\beta = -1.9$). By reducing the value of β also these lobes cross the prefocal line and other secondary lobes emerge from the focal points and if they cross the prefocal line the structure of the basins may become more complicated, as so the one represented in Fig.5 (where $\beta = -4.1$).
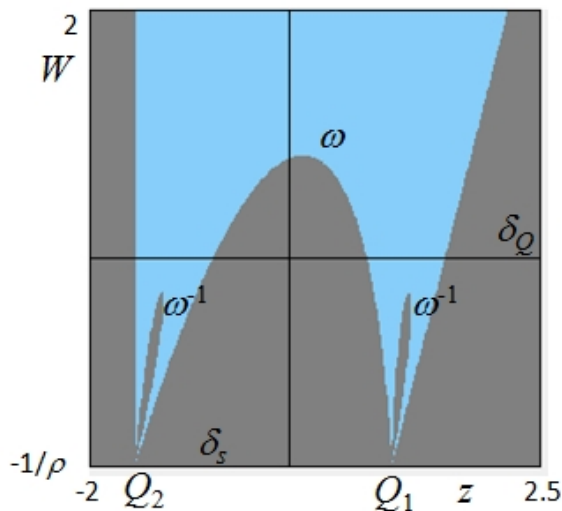
Figure 4. The light blue basin denotes convergence to a fixed point. Starting from the grey region trajectories diverge.
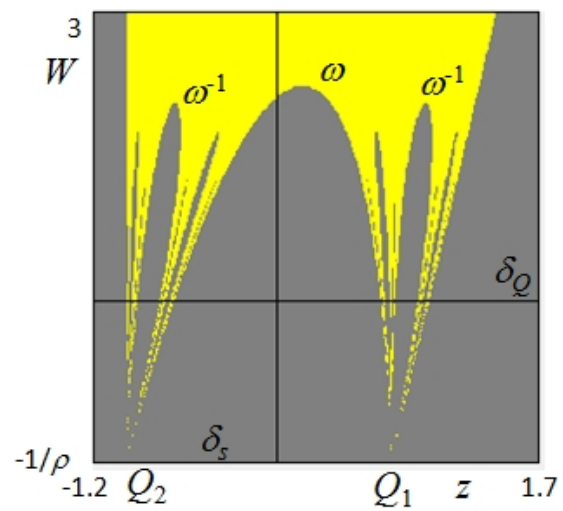
Figure 5. The yellow basin denotes convergence to a cycle of period 3. Starting from the grey region trajectories diverge.

2.1.3. The Naimzada and Tramontana (2009) model Naimzada and Tramontana (2009) study a model of boundedly rational consumer that is a variation of Bischi and Tramontana (2005) that we have already seen. In particular they introduce a different decisional mechanism characterized

by an adjustment towards the direction indicated by the marginal utility currently achieved by the consumer. The dynamical system in the consumption (x) and preference (α) is the following:

$$T_3 : \begin{cases} x' = F(x,y) = x + \gamma \left[\frac{N(x,y)}{D(x,y)} \right] = x + \gamma \left[\frac{\alpha(m-px)x^{1-2\alpha} - p(1-\alpha)x}{x^{1-\alpha}(m-px)^{\alpha}} \right] \\ \alpha' = G(x) = \frac{1}{k_1 + k_2 k_3^x} \end{cases} , \qquad (6)$$

where $\gamma > 0$ is the speed of adjustment and the meaning of the other parameters is the same of Bischi and Tramontana (2005). The set of non-definition of this map with denominator is given by:

$$\delta_S : x = 0 \cup x = \frac{m}{p},$$

and we can identify two focal points:

$$Q_1(0,0) \quad \text{and} \quad Q_2\left(\frac{m}{p}, 1\right) ,$$

whose prefocal lines are respectively:

$$\delta_{Q_1} : \alpha = \frac{1}{k_1 + k_2} \quad \text{and} \quad \delta_{Q_2} : \alpha = \frac{1}{k_1 + k_2 k_3^{m/p}} .$$

The presence of two focal points with different prefocal lines associated with the coexistence of several locally stable attractors for the map T_3 may give rise to quite complicated basins of attraction like those represented in Fig.s 6 and 7, obtained by keeping fixed $m = 18$, $\gamma = 0.9$ and $k_3 = 0.307$.
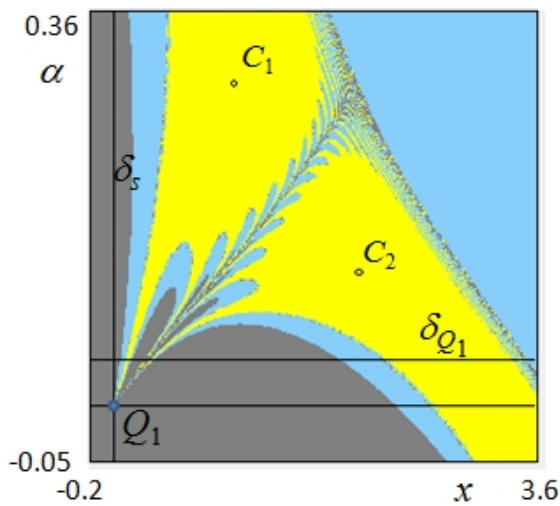
Figure 6. The light blue basin denotes convergence to the fixed point E, while the yellow one convergence to a cycle of period 2 (C_1 and C_2). Starting from the grey region trajectories diverge. We used $p = 3.525$, $k_1 = 1.5$ and $k_2 = 22.555$.
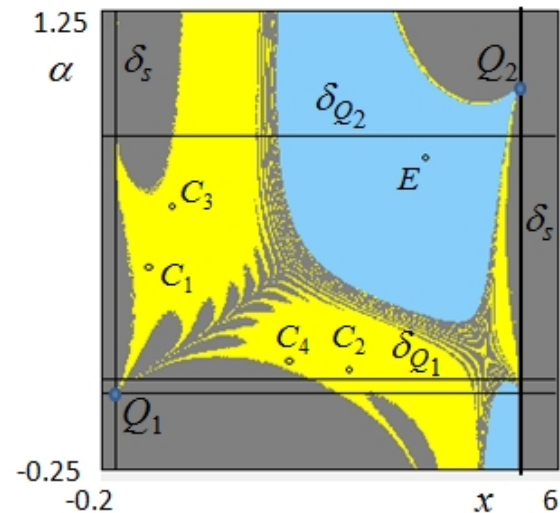
Figure 7. The yellow basin here denotes convergence to a cycle of period 4. We used $p = 3.29$, $k_1 = 1.157$ and $k_2 = 20.4$.

2.1.4. The Foroni et al. (2003) model Three focal points characterize the map studied by Foroni et al. (2003). They consider a cobweb market for the price of a perishable-good. The authors focus on the case of fish but it can be adapted for any other renewable resource market. The producers make their choices on the basis of the expected price of the fish, with a fading memory procedure similar to the one already seen in the Bischi and Naimzada (1997) model (see sec. 2.1.2). By imposing the market clearing conditions they obtain a two-dimensional system regulating the dynamics of the expected price (z) and the memory variable (W):

$$T_4 : \begin{cases} z' = F(z,W) = \frac{N(z,W)}{D(z,W)} = \frac{\rho Wz + H(z)}{1 - \rho W} \\ W' = G(W) = 1 + \rho W \end{cases}, \tag{7}$$

where $\rho \in [0,1]$ regulates how fast the memory decays. The set of nondefinition is given by:

$$\delta_s : W = -\frac{1}{\rho}$$

and a focal point of the map T_4 must necessarily verify the condition:

$$H(z) = z \tag{8}$$

Foroni et al. select the following specification of the function $H(z)$:

$$H(z) = \frac{A - S(z)}{B},$$

with

$$S(z) = rg(z)\left(1 - \frac{g(z)}{k}\right)$$

and

$$g(z) = \frac{k}{4}\left\{1 + \frac{c}{qkz} - \frac{\delta}{r} + \sqrt{\left(1 + \frac{c}{qkz} - \frac{\delta}{r}\right)^2 + \frac{8c\delta}{qkrz}}\right\}$$

where $S(z)$ derives from the assumption of logistic intrinsic growth rate of the fish and $g(z)$ comes from the result of an optimization problem for the fisher. We refer the reader to Foroni et al. (2003) for a complete explanation of the meaning of the parameters.

For our pourposes it is important to specifiy that the particular specification of $H(z)$ leads to three roots of the equation (8), that is to three focal points whose coordinates can only be found numerically. The prefocal line is the same for all the three focal points, given by:

$$\delta_Q : W = 0.$$

It is possibile to find a combination of parameters such that two attractors coexist: a fixed point and a periodic cycle. The structure of their basins of attraction is highly influenced by the presence of three focal points. An example is provided in Fig. 8.

*2.2. Main results and applications when the focal point is not simple*
By using the definition of simple focal point (3) we can easily define the condition that must be verified in order to have a *non simple focal point*:

$$\overline{N}_x\overline{D}_y - \overline{N}_y\overline{D}_x = 0. \tag{9}$$

Bischi et al. (2005) distinguish among various subcases in which (9) holds, considering the number of zeroes in the matrix of first-order partial derivatives of the map.

A subcase that is particularly important for the shape of invariant sets crossing prefocal curves is verified whenever all the first-order partial derivatives vanish in the focal point. In this case we have the next result:
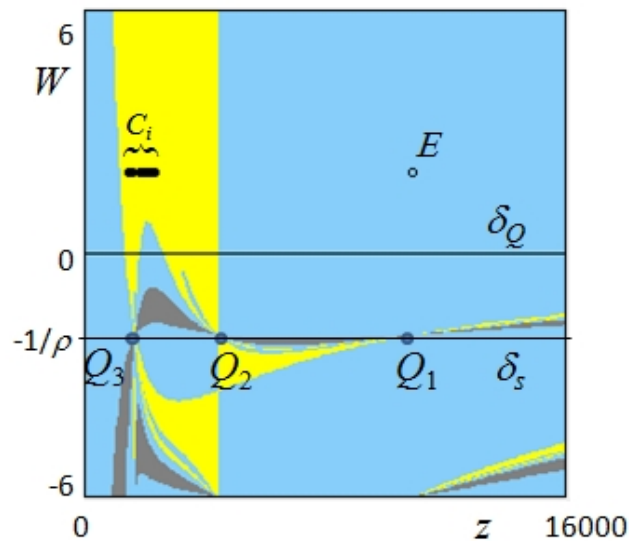
Figure 8. The light blue basin correspondes to convergence to the fixed point E, while the yellow basin is associated to a periodic cycle whose points are labeled C_i. Starting from the grey region trajectories diverge. We used $A = 5241$, $B = 0.28$, $c = 5000$, $k = 400000$, $q = 0.000014$, $r = 0.05$, $\rho = 0.5$ and $\delta = 0.12$.

Proposition 3 *(Bischi et al., 2005). When a map with vanishing denominator (1) has a non simple focal point Q such that $\overline{N}_x = \overline{D}_x = \overline{N}_y = \overline{D}_y = 0$, then:*

- *the prefocal set δ_Q associated with Q belongs to the line $x = G(Q)$*

- *if the second-order partial derivatives of $D(x, y)$ in Q are not all vanishing, then the correspondence between the slope m of an arc γ through Q and the y-coordinate of the point $(G(Q), y(m))$ is generally two-to-one.*

So we can conclude that strucures such as lobes arise by twos and a corollary of such a proposition is the following:

Corollary. *(Bischi et al. 2001b) If the focal point Q belongs to its prefocal curve δ_Q then any arc transverse to δ_Q has infinitely many preimages which are arcs through Q with slopes 0 and ∞ in Q.*

If we are in the situation where Proposition 3 holds and the focal point belongs to its prefocal curves, the loops issuing from the focal point after the global bifurcation are infinitely many. When this arc crossing the prefocal curve is the border of a basin of attraction the basins' structure can be extremely complicated, as it happens in the next economic application.

2.2.1. The Bischi-Kopel-Naimzada (2001) model   Bischi et al. (2001b) analyze a so-called *rent-seeking game*, that is a situation in which some agents (or players) compete for obtaining a prize (such as a patent or a contract with the government). In their paper the contenders are two boundedly rational firms that choose production efforts to get the highest profit. The firms are boundedly rational in the sense that they are not able to solve the maximization problem (either because of a lack of information or a lack of computational skills) and dynamically adjust their quantities by using the current marginal profit as a signal.

The two-dimensional dynamical system determining the quantities of the firms is the following:

$$T_5 : \begin{cases} q_1' = F(q_1, q_2) = q_1 \left[1 - c_1 v_1 + v_1 \frac{q_2}{(q_1+q_2)^2} \right] \\ q_2' = G(q_1, q_2) = q_2 \left[1 - c_2 v_2 + v_2 \frac{q_1}{(q_1+q_2)^2} \right] \end{cases}, \tag{10}$$

where the positive parameters c_i $(i = 1, 2)$ represent the marginal costs of the two firms and $v_i (i = 1, 2)$ the speeds of adjustment.

Map T_5 has both the components with a denominator that vanishes along the line:

$$\delta_s : q_2 = -q_1.$$

The map assumes the form $0/0$ in correspondence of the focal point $Q(0,0)$. Bischi et al. (2001) proved that the prefocal set δ_Q is the half-line:

$$\delta_Q : q_2 = \frac{v_2}{v_1} q_1 \text{ where } q_1 \le \frac{v_1}{4} \text{ and endpoint given by } C\left(\frac{v_1}{4}, \frac{v_2}{4}\right).$$

The focal point Q is not simple and in particular all the first-order partial derivates calculated in Q are equal to zero. Moreover it is easy to see that Q belongs to δ_Q. So we are exactly in the case described by Proposition 3 and in the realm of application of its corollary. We expect to see infinitely many lobes whenever the boundary of a basin of attraction crosses the pre-focal line. In fact, this is what we can see in Fig.9 where it is possible to see the so-called *crescents* (i.e. lobes merged with other lobes) and in its enlargment in Fig. 10 where new lobes are visible between other lobes. We know that they are infinitely many.
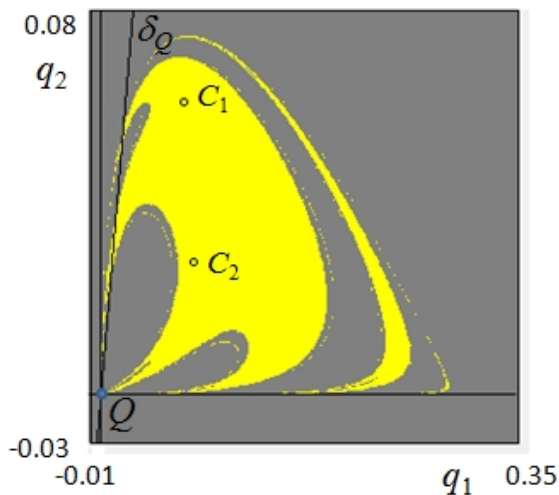
Figure 9. The yellow basin denotes convergence to a cycle of period 2. Starting from the grey region trajectories diverge. We used $c_1 = 3$, $c_2 = 5$, $v_1 = 0.2$ and $v_2 = 0.6$.
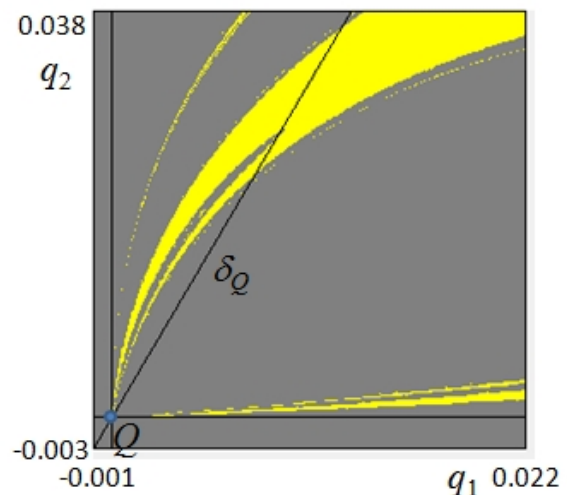
Figure 10. Enlargment of the region around the focal point Q.

2.3. The role of noninvertibility

Until now we have not taken into consideration the role played by the noninvertibility of a map on the structure of invariant sets and basins of attraction. We know that if a map T is noninvertible,

then some points of the phase plane (or more generally phase space) are characterized by a number of rank-1 preimages higher than one. This feature often determines a strcture of the basins of attraction that can be quite complicated (white *holes,* or non connected portions, for instance), and if it is combined with the presence of a denominator that can vanish the results can be even more peculiar. The last example, which is also the most recent, belongs to this case.

2.3.1.   The Cavalli et al.   (2015) model Cavalli et al.   (2015) recently proposed a two-dimensional map that describes the evolution of the quantities produced by two boundedly rational and heterogeneous duopolists. This is one of the first attempts to study an oligopoly model with different kinds of decisional mechanisms, without considering any rational player. One firm adopts a gradient-like mechanism of adjustment similar to those already seen in previous examples, while the second adopts a Local Monopolistic Approximation (i.e. it behaves as if it were a monopolist facing a linear demand function, see Bischi et al. (2007)).

The map they obtain is the following:

$$T_6 : \begin{cases} q_1' = F(q_1, q_2) = q_1 + \alpha q_1 \left[\frac{q_2}{(q_1 + q_2)^2} - c_1 \right] \\ q_2' = G(q_1, q_2) = \frac{1}{2} q_2 + \frac{1}{2} (q_1 + q_2) [1 - c_2 (q_1 + q_2)] \end{cases} , \qquad (11)$$

where $c_i > 0$ ($i = 1, 2$) are the marginal costs and $\alpha > 0$ the speed of adjustment of the first duopolist.

The denominator of the first difference equation vanishes along the line:

$$\delta_s : q_2 = -q_1$$

and there exists a focal point $Q(0, 0)$ belonging to its prefocal line, given by:

$$\delta_Q : q_2 = 0.$$

We are in the scenario described in the subsection (2.2), so whenever a lobe appears, infinitely many appear as well. Moreover the focal point may be located in a region of the phase plane whose points have many rank-1 preimages, leading to a repetition of the structure that emerges around the focal point. This is what occurs in Fig.11, where the focal point is in a region whose points have four preimages. Two of the rank-1 preimages (Q_a^{-1} and Q_b^{-1}) are the vertex of the pseudo-triangle where the structure of lobes characterizing the focal point is replicated.

3. Conclusions

In this work we have surveyed some economic applications where plane maps with vanishing denominator regulate the dynamics of the relevant variables. This permitted us on the one hand to summarize the main results obtained through the deep study of this class of maps, and on the other hand to emphasize how important can be for the application to global bifurcations related to the appearence of strucutres related to the so-called focal points of the map. We omitted, for lack of space, to mention the papers such as those of Billings & Curry (1996), Gardini et al. (1999), Gu & Ruan (2005) and Fischer & Gillis (2006), that, dealing with problems like finding the roots of a rational function, have important implications for applications, including those in economics. We hope our survey will be useful to those researchers that build a dynamical system characterized by a map with denominator and want to deepen the global consequences of such a feature.
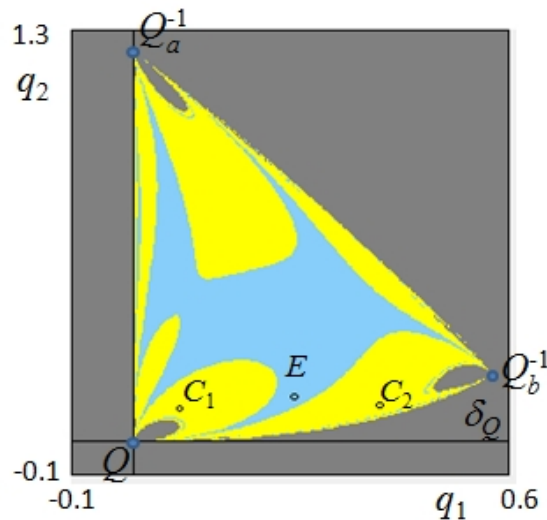
Figure 11. The light blue basin correspondes to convergence to the fixed point E, while the yellow basin is associated to a cycle of period 2. Starting from the grey region trajectories diverge. We used $c_1 = 0.9$, $c_2 = 1.62$ and $\alpha = 1.8$.

References

[1] Billings L and Curry JH 1996 *Chaos* **6** 108-119
[2] Bischi GI and Gardini L 1997 *Int. J. Bif. Chaos* **7** 1555-1577
[3] Bischi GI, Gardini L and Mira C 1999 *Int. J. Bif. Chaos* **9** 119-153
[4] Bischi GI, Gardini L and Mira C 2001a *Nonlinear Anal.* **47** 2171-2185
[5] Bischi GI, Gardini L and Mira C 2003 *Int. J. Bif. Chaos* **13** 2253-2277
[6] Bischi GI, Gardini L and Mira C 2005 *Int. J. Bif. Chaos* **15** 451-496
[7] Bischi GI, Kopel M and Naimzada A 2001b *Nonlinear Anal.* **47** 5309-5324
[8] Bischi GI and Naimzada A 1997 *Econ. Notes* **26** 445-476
[9] Bischi GI, Naimzada A, Sbragia L 2007 *J. Econ. Behav. Organ.* **62** 371-388
[10] Bischi GI and Tramontana F 2005 *Pure Math. Applications* **16** 345-363
[11] Cavalli F, Naimzada A and Tramontana F 2015 *Commun. Nonlinear Science Numer. Simul.* **18** 245-262
[12] Djellit I, Sahari ML and Hachemi A 2013 *J. Applied Anal. Comput.* **3** 11-20
[13] Fischer P and Gillis D 2006 *Int. J. Bif. Chaos* **16** 179-183
[14] Foroni I, Gardini L and Barkley Rosser Jr. J 2003 *Math. Comput. Simul.* **63** 541-567
[15] Gardini L and Bischi GI 1996 *Grazer Mathematische Berichte* 225-231
[16] Gardini L, Bischi GI and Fournier-Prunaret D 1999 *Chaos* **9** 367-380
[17] Gardini L, Bischi GI and Mira C 2007 *Scholarpedia* **2**(9):3277
[18] Gu EG 2009 *Commun. Nonlinear Science Numer. Simul.* **14** 2720-2727
[19] Gu EG and Hao YD 2007 *Chaos Solitons Fractals* **34** 1272-1284
[20] Gu EG and Huang J 2006 *Int. J. Bif. Chaos* **16** 2601-2613
[21] Gu EG and Ruan J 2005 *Int. J. Bif. Chaos* **15** 1625-1639
[22] Mira C 1981 *Comptes Rendus Acad. Sci. Paris Serie I* **292** 146-151
[23] Mira C 1996 *Proceedings of European Conference on iteration theory,* ECIT96, Urbino (8-14 Sept. 1996)
[24] Naimzada A and Tramontana F 2009 *Int. J. Bif. Chaos* **19** 2059-2071