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UNDERIVABILITY RESULTS IN MIXED SYSTEMS OF MONADIC DEONTIC LOGIC

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The problem of the derivability in specific deontic calculuses of deontic formulae from sets of alethic formulae has been confronted in recent years by various authors (cf. F. von Kutschera [1973] and [1977], E. Morscher [1974] and [1984], P. Kaliba [1981] and [1983], R. Stuhlmann-Laeisz [1983]). These authors have not only sought to define the concept of deontic formula (1) as opposed to that of alethic formula within a rigorous framework of language; they have also obtained numerous underivability results. The fundamental result in this sense was achieved by von Kutschera, who [1977] demonstrated by means of a model-theoretical technique specifically elaborated for the purpose that it is not possible to derive in a specific deontic calculus (2) deontic propositions (which are not logically true) from any consistent set of alethic formulae. This finding was subsequently extended by Kaliba [1981] to calculuses where, as well as deontic operators, there also appear other intensional

(1) Given a propositional language where the deontic operator **0** occurs as a primitive symbol (and the other deontic operators are defined by means of it), a formula α is said to be "deontic" iff it takes one of the following forms: $\alpha \equiv \mathbf{0}\beta$ (and β is any formula of the language), $\alpha \equiv \neg\beta$ and β is a deontic formula, $\alpha \equiv \beta \mathbf{~o~} \gamma$ (where **o** designates any of the usual propositional connectives of degree 2) and both β and γ are deontic formulae. The term "alethic" is used for formulae where there is no occurrence of **0**.

(2) For the general definition of (pure) deontic calculus, see L. Åqvist [1984], pp. 665-675 and B.F. Chellas [1980], pp. 190-194. Various methods of designation are used in modal and deontic calculuses. For the sake of clarity, I have preferred to use the method (also used by Chellas) of employing the calculus abbreviation to refer to the set of axioms that characterize it. Thus, for example, the modal system S5 is designated by the abbreviation **KT5**, which expresses the fact that it results from the minimal normal modal system (characterized by the rule of necessitation N) plus the axioms **T**: $\Box\alpha \rightarrow \alpha$ and **5**: $\Diamond\alpha \rightarrow \Box\Diamond\alpha$. Since deontic calculuses derive from the deontic interpretation of their corresponding modal systems (syntactically, $\Box$ is replaced by **0**), it is convenient to use **0·A** to designate the deontic axiom deriving from its respective modal axiom **A**, and **0·C** to designate the deontic calculus deriving from the modal calculus **C**. Thus, for example, the deontic calculus **0·KD45** is given by the shared component of all the normal deontic systems (characterized by the rule of 0-necessitation **0·N**) plus the axioms **0·D**: $\mathbf{0}\alpha \rightarrow \mathbf{P}\alpha$, **0·4**: $\mathbf{0}\alpha \rightarrow \mathbf{00}\alpha$ and **0·5**: $\mathbf{P}\alpha \rightarrow \mathbf{0P}\alpha$.

operators as primitive terms (mixed calculuses). This made it possible to extend the thesis of underivability to include the case of consistent sets of modal alethic formulae (3). However, Kaliba's generalized conclusion of underivability depends closely on the fact that the mixed system that he considered did not contain any principle whereby a connection could be established between deontic modalities and alethic modalities (bridge-principle). This, in particular, excludes the possibility of immediately extending Kaliba's result to the realm of the alethic systems of deontic logic (4), where, as is widely known, there are various principles that establish a connection between alethic modalities and deontic modalities. Nevertheless, the intention of this essay is to demonstrate, first of all, that the general result of underivability holds for certain extensions of the mixed system used by Kaliba (cf. Theorems 2 and 4) that also include the bridge-principle **0□0** (5). Secondly, it will be shown that the independence result, suitably weakened, can be extended to even more powerful systems. In particular, proof will be provided of the underivability – in specific mixed systems that are also inclusive of the bridge-principle **0◇** (6) – of deontic formulae of obligation from any consistent set of alethic (and modal) formulae (cf. Theorems 1 and 3). Finally, proof will be given of the independence of an even more restricted class of formulae of obligation (axiologically important obligations) within the framework of the powerful alethic system of deontic logic **KT5Q** (cf. Theorem 5).

(3) Given that in the language of mixed calculuses there occur various intensional operators (in particular, **0** and □), with regard to the set of alethic formulae it makes sense to make the further distinction between non-modal alethic formulae and modal alethic formulae. Unlike the latter, the former contain no occurrence of □. Henceforth, by alethic formulae will be meant both modal and non-modal alethic formulae.

(4) For the general definition of the alethic systems of deontic logic, see L. Åqvist [1984], pp. 675-688. In accordance with the conventions established in note 2, also the alethic systems of deontic logic are designated here by abbreviations indicative of their respective axioms. Thus, for example, the alethic system that, in the pages cited above Åqvist refers to by means of $S5_Q{}^+$, corresponds here to the system **KT5Q**, in so far as it is obtained by adding to **KT5** (i.e. to S5) the further axiom **Q**: $\Diamond Q$ (which governs the prohairetic propositional constant Q characteristic of the language of all the alethic systems of deontic logic).

(5) The **0□0** principle is $\mathbf{0}\alpha \rightarrow \Box\mathbf{0}\alpha$

(6) The **0◇** principle is $\mathbf{0}\alpha \rightarrow \Diamond\alpha$.

1. Mixed system **K**$-$**0**$\Diamond$

It is an easily verifiable fact that, in systems of deontic logic where the bridge-principle **0**$\Diamond$ holds, deontic formulae can be derived from suitable sets *M* of alethic formulae. If, for example, in *M* there occurs the formula $\Box\alpha$, then the deontic formula **P**α can, trivially, be derived from *M*.

The intention of this first section is to examine whether this derivability relation holds in general for deontic formulae or whether, on the contrary, it is limited to a specific class of such formulae. We shall find that, within the mixed system under scrutiny, the bridge-principle does not permit the derivation of deontic formulae of obligation from sets of alethic formulae. This outcome seems to be of particular interest from the point of view of logical discussion of Hume's thesis: the derivability by **0**$\Diamond$ of *permissions* is not to be treated on the same level as the derivability by **0**$\Diamond$ of *obligations*. In fact, unlike obligation, permission derived from an appropriate set *M* of alethic formulae by means of the **0**$\Diamond$ principle (according to which, possibility is only a necessary condition for obligatoriness) is not such by virtue of the "non-badness" of its content but for the simple reason that its content is necessitated. In other words, since this is a matter of a permission determined by a state of necessity, its derivability from a set of alethic formulae cannot constitute a real violation of Hume's thesis. Of course, there are bridge-principles that enable the derivation of deontic formulae of obligation as well. And the following sections will have the task of examining whether these forms of derivation are, by contrast, a real violation of Hume's thesis.

Let, therefore, the reference system for the analysis conducted in this section be the mixed system **KT5**$-$**0**$\cdot$**KD4**$-$**0**$\Diamond$ (in brief, **K**$-$**0**$\Diamond$). Since this is a mixed system (and not an alethic system of deontic logic), the language of **K**$-$**0**$\Diamond$ will comprise $\Box$ and **0** as primitive symbols (and **0** is not defined in terms of $\Box$ and *prohairetic* propositional constant *Q*). The formulae of the **K**$-$**0**$\Diamond$ language may all be inductively constructed as follows: *p,q,r,...* are formulae; if α is a formula, then also $\neg\alpha$, $\Box\alpha$ and **0**α are formulae; if α and β are formulae, then also $\alpha\wedge\beta$, $\alpha\vee\beta$ and $\alpha\rightarrow\beta$ are formulae. The axioms and the rules of **K**$-$**0**$\Diamond$ are those of **KT5** and **0**$\cdot$**KD4** plus the axiom **0**$\Diamond$: **0**$\alpha\rightarrow\Diamond\alpha$.

A model for **K**$-$**0**$\Diamond$ (or **K**$-$**0**$\Diamond$-model) is constituted by the ordered quadruple $<W, R, S, I>$, where *W* and *I* are defined in the usual manner, and *R* and *S* are distinct relations, both defined on *W*, so that: (i)

R is reflexive and euclidean (i.e. reflexive, symmetrical and transitive), (ii) S is serial and transitive, (iii) R and S jointly satisfy the *intersection condition* $\forall u \exists v(uSv$ *et* $uRv)$. It should be noted that, while R performs the function of accessibility relation, S performs that of deontic alternativeness; hence, the truth of the necessitated formulae is defined with reference to R, while the truth of the formulae of obligation is defined with reference to S. For the same reason, the intersection condition meets the requirement that, given a world u, the intersection between its deontic alternatives and the set of worlds accessible from it should be non-empty.

It can be proved that $\mathbf{K}-\mathbf{0}\Diamond$ is sound and complete with respect to the concept of model presented here. The soundness of $\mathbf{K}-\mathbf{0}\Diamond$ follows immediately from the formal properties imposed on R and S. In fact, $\mathbf{K}-\mathbf{0}\Diamond$ is given by **KT5** plus $\mathbf{0}\cdot$**KD4** plus $\mathbf{0}\Diamond$.

Thus the reflexivity and euclidicity of R ensure the validity of the **KT5** axioms; the seriality and the transitivity of S ensure the validity of the $\mathbf{0}\cdot$**KD4** axioms; the intersection condition ensures the validity of $\mathbf{0}\Diamond$. One consequence of the soundness of $\mathbf{K}-\mathbf{0}\Diamond$ should be noted immediately: the principle $\Box\mathbf{0}$: $\Box\alpha \rightarrow \mathbf{0}\alpha$ is not a $\mathbf{K}-\mathbf{0}\Diamond$ theorem. In fact, a $\mathbf{K}-\mathbf{0}\Diamond$-model based on a frame where there exists a world u related by S to v_1 and v_2 and by R only to v_1 (as well as to itself) and such that α is true at u and at v_1 but not at v_2 satisfies at u $\Box\alpha$ but not $\mathbf{0}\alpha$.

The completeness of $\mathbf{K}-\mathbf{0}\Diamond$ raises more serious problems. This is obtained by the method of canonical models, which requires that proof be given that the $R_{\mathbf{K}-\mathbf{0}\Diamond}$ and $S_{\mathbf{K}-\mathbf{0}\Diamond}$ relations of the canonical model for $\mathbf{K}-\mathbf{0}\Diamond$ are reflexive and euclidean (the former) and serial and transitive (the latter), and that both satisfy the intersection condition. Now, since $\mathbf{K}-\mathbf{0}\Diamond$ contains **KT5**, the reflexivity and euclidicity of $R_{\mathbf{K}-\mathbf{0}\Diamond}$ is ensured by the completeness of the **KT5** system with respect to the reflexive and euclidean models. Likewise, since $\mathbf{K}-\mathbf{0}\Diamond$ also contains $\mathbf{0}\cdot$**KD4**, the seriality and transitivity of $S_{\mathbf{K}-\mathbf{0}\Diamond}$ is ensured by the completeness of $\mathbf{0}\cdot$**KD4** with respect to the serial and transitive models. All that remains, therefore, is to demonstrate that both relations satisfy the intersection condition, and this is achieved by extension of the technique in use for demonstration of the completeness of **S4.2**.

Let $< W_{\mathbf{K}-\mathbf{0}\Diamond}, R_{\mathbf{K}-\mathbf{0}\Diamond}, S_{\mathbf{K}-\mathbf{0}\Diamond} >$ be the frame of the canonical model for $\mathbf{K}-\mathbf{0}\Diamond$. This is therefore a matter of demonstrating that $(\forall u \in W_{\mathbf{K}-\mathbf{0}\Diamond})$

$(\exists z \in W_{K-0\Diamond})$ $(u\ S_{K-0\Diamond}z$ *et* $u\ R_{K-0\Diamond}z)$. For this purpose, it is sufficient to show that, given a generic element u of $W_{K-0\Diamond}$, the set Z of all the α formulae whose respective obligations or necessitations belong to u, i.e. such that $Z = \{\alpha : \mathbf{0}\alpha \in u$ *vel* $\Box\alpha \in u\}$ is $\mathbf{K}-\mathbf{0}\Diamond$-consistent. In such a case, in fact, by Lindenbaum's lemma there exists a $\mathbf{K}-\mathbf{0}\Diamond$-maximal extension of Z and therefore by the definition of $W_{K-0\Diamond}$, $(\exists z \in W_{K-0\Diamond})$ $(\forall\alpha)$ $(\mathbf{0}\alpha \in u$ *vel* $\Box\alpha \in u \Rightarrow \alpha \in z)$. But, then, by the usual laws of predicate calculus $(\exists z \in W_{K-0\Diamond})$ $(\forall\alpha(\mathbf{0}\alpha \in u \Rightarrow \alpha \in z)$ *et* $\forall\alpha(\Box\alpha \in u \Rightarrow \alpha \in z))$ holds; hence by definition of $S_{K-0\Diamond}$ and $R_{K-0\Diamond}$ and generalization one also obtains $(\forall u \in W_{K-0\Diamond})$ $(\exists z \in W_{K-0\Diamond})$ $(u\ S_{K-0\Diamond}z$ *et* $u\ R_{K-0\Diamond}z)$. Now, let Z be non $\mathbf{K}-\mathbf{0}\Diamond$-consistent. Thus, because of the finiteness of the derivability relation, there exists a finite sub-set of Z, from which the contradiction is derivable in $\mathbf{K}-\mathbf{0}\Diamond$. Let this be $F(Z)$. At this point, three cases arise: $F(Z)$ is constituted only by formulae whose obligation exists in u, only by formulae whose necessitation exists in u, by formulae of one or the other kind. It can be shown that all three cases are impossible. In the first case, one would have – using $\mathbf{0}$ $(F(Z))$ to indicate the set of formulae which results by $\mathbf{0}$-necessitation of the formulae belonging to $F(Z)$ – $\mathbf{0}(F(Z)) \vdash_{K-0\Diamond} \mathbf{0}\bot$. On the other hand, all the formulae of $\mathbf{0}(F(Z))$ belong to u; hence, given the $\mathbf{K}-\mathbf{0}\Diamond$-closure of u, $\mathbf{0}\bot$ would also belong to u. But this is impossible, since $\neg\mathbf{0}\bot$ holds in $\mathbf{K}-\mathbf{0}\Diamond$ and, therefore, given the $\mathbf{K}-\mathbf{0}\Diamond$-consistency of u, $\mathbf{0}\bot$ cannot belong to u. The second case is structurally identical with the first. Thus there remains only the third case to consider. Let β_0 be the conjunction of the formulae $\beta \in F(Z)$ such that $\mathbf{0}\beta \in u$ and $\beta_\Box$ the conjunction of the formulae $\beta \in F(Z)$ such that $\Box\beta \in u$. Given these definitions and the definition of $F(Z)$, one obtains therefore, by the propositional part of $\mathbf{K}-\mathbf{0}\Diamond$, $\beta_0 \vdash_{K-0\Diamond} \neg\beta_\Box$ and, by $\mathbf{0}$-necessitation, $\mathbf{0}\Diamond$ and def. of $\Diamond$, $o\beta_0 \vdash_{K-0\Diamond} \neg\nearrow\beta_\Box$. On the other hand, to u belong all the obligations of the members of the β_0 conjunction, and from these, in $\mathbf{K}-\mathbf{0}\Diamond$, $\mathbf{0}\beta_0$ is also derivable. By the $\mathbf{K}-\mathbf{0}\Diamond$-closure, this means that $\neg\Box\beta_\Box \in u$, and by the $\mathbf{K}-\mathbf{0}\Diamond$-consistency of u that $\Box\beta_\Box \notin u$. However, this result contradicts the fact that to u belong all the necessitations of the members of $\beta_\Box$ and, therefore, by the modal part of $\mathbf{K}-\mathbf{0}\Diamond$ and $\mathbf{K}-\mathbf{0}\Diamond$-closure of u, also $\Box\beta_\Box$. To sum up, the impossibility of these three cases implies the rejection of the hypothesis and therefore the $\mathbf{K}-\mathbf{0}\Diamond$-consistency of Z.

The elements of the $\mathbf{K}-\mathbf{0}\Diamond$ system that have been set out provide an adequate basis for approaching the problem of the derivability or otherwise in $\mathbf{K}-\mathbf{0}\Diamond$ of obligations from a **KT5**-consistent set of alethic formulae. The negative answer is obtained by the following theorem.

Theorem 1: *Let M be a* **KT5**-*consistent set of alethic formulae. Let* α *be an obligation, that is, a deontic formula taking the form* $\mathbf{0}\beta$, *where* β *is any formula* $\in \mathbf{L}(\mathbf{K}-\mathbf{0}\Diamond)$, *satisfying the unprovability condition* $\nvdash_{\mathbf{K}-\mathbf{0}\Diamond}$ $\mathbf{0}\beta$. *Then* $M \nvdash_{\mathbf{K}-\mathbf{0}\Diamond} \mathbf{0}\beta$.

Proof:

Given the soundness of the system, it is necessary to show that there exists a model for $\mathbf{K}-\mathbf{0}\Diamond$ $<W,R,S,I>$ and a $u \in W$ such that $<W,R,S,I> \models_u M$ and $<W,R,S,I> \not\models_u \mathbf{0}\beta$. Now, we know from the completeness of **KT5** that there exists a model for **KT5** $<W'',R'',I''>$ and a world $i \in W''$ such that $<W'',R'',I''> \models_i M$. Moreover, because of the unprovability condition, there exists a model for $\mathbf{K}-\mathbf{0}\Diamond$ $<W',R',S',I'>$ and a world $j \in W'$ such that $<W',R',S',I'> \not\models_j \mathbf{0}\beta$, i.e. $<W',R',S',I'> \models_j \mathbf{P}\neg\beta$. This is therefore a matter of constructing the required model on the basis of these two given items.

Construction of $<W,R,S,I>$

Since the $<W'',R'',I''>$ model is given, let a relation S'' be defined on W'' such that it satisfies the conditions of seriality, transitivity and intersection, as well as the further condition of *deontic inaccessibility* of i, consisting of the fact that the world $i \in W''$ is not a deontic alternative to any other world $\in W''$. Clearly, $<W'',R'',S'',I''$ is a model for $\mathbf{K}-\mathbf{0}\Diamond$ such that $(\forall\alpha$ alethic$)$ $(\forall u \in W'')$ $(<W'',R'',S'',I''> \models_u \alpha \Leftrightarrow <W'',R'',I''> \models_u \alpha)$ *. In fact, the sentence holds because of the coincidence of the alethic component of the two models. This is, moreover, a model for $\mathbf{K}-\mathbf{0}\Diamond$, since all the required formal properties are satisfied by construction. In particular, the satisfaction of the deontic inaccessibility condition of i does not raise difficulties, insofar as the models of $\mathbf{0}\cdot\mathbf{KD4}$ can contain worlds that are not deontic alternatives to any other world and, for the purposes of the simultaneous satisfaction of the intersection condition, it is possible to presuppose that, by R, i is related to worlds that are different from i.

At this point, it is possible to proceed with the construction of $<W,R,S,I>$. Let x,y,z be worlds $\in W''$, $r,s,t \in W'$ and $u,v,w \in W$ and let us assume:

(i) $W = W' \cup W''$ (where the two sets W' and W'' are presupposed as being disjunct)
(ii) $R = R' \cup R''$, i.e. $xRv \Leftrightarrow xR''v$ and $rRv \Leftrightarrow rR'v$
(iii) a) $rSv \Leftrightarrow rS'v$; b) 1. $x = i \Rightarrow (xSv \Leftrightarrow jS'v$ *vel* $iS''v))$; b) 2. $x \neq i \Rightarrow (xSv \Leftrightarrow xS''v)$
(iv) $I = I' \cup I''$, i.e. $I(p, x) = I''(p,x)$ and $I(p,r) = I'(p,r)$ for every propositional letter p.

Lemma 1: $<W,R,S,I>$ is a model for $\mathbb{K}-\mathbf{0}\Diamond$

Clearly, W is by construction a set of worlds, an I is a definite function for p and for all $u \in W$. Hence we may restrict ourselves to proving the required formal properties for R and S. *Ad R*: this is a matter of proving that R is reflexive and euclidean and that it satisfies the condition of intersection. As regards the former of these two requirements, R is by construction a reflexive and euclidean relation defined on W, since $W = W' \cup W''$, W' and W'' are presupposed as being disjunct, R' and R'' are, in turn, reflexive and euclidean. As regards the condition of intersection, proof will be provided jointly with S. *Ad S*: proof must be given of the seriality of S, its transitivity and the condition of intersection – i.e. a) $\forall u \exists v(uSv)$; b) $\forall u \forall v \forall w(uSv$ *et* $vSw \Rightarrow uSw)$; c) $\forall u \exists v(uSv$ *et* $uRv)$.

Ad a) three cases should be distinguished: 1. Case: $u = r$. Hence, the seriality of S follows from the seriality of S' and df. of S (clause (iii) a)). 2. Case: $u = x$ and $x \neq i$. Hence the seriality of S follows from the seriality of S'' and df. of S (clause (iii) b) 2.). 3. Case: $u = x$ and $x = i$. Hence, the seriality of S follows from the seriality of S' or seriality of S'' and df. of S (clause (iii) b) 1.).

Ad b) here, too, three cases should be distinguished: 1. Case: $u = r$. Let uSv *et* vSw be for generic v and w. Hence by df. of S (clause (iii) a)) $uS'v$ is the case and v represents an element of W'. But then by def. of S (same clause), $vS'w$ is also the case, and thus – by transitivity of S' – $uS'w$, and therefore, once again by def. of S (same clause), uSw. 2. Case: $u = x$ and $x \neq i$. Let uSv *et* vSw be for generic v and w. Hence by def. of S (clause (iii) b) 2.) $uS''v$ is the case and v is a generic element of W''. Moreover, because of the condition of deontic inaccessibility of i $v \neq i$. Therefore, by def. of S (same clause) $vS''w$ also holds, thus – by transitivity of S'' – $uS''w$, and therefore, once again by def. of S (same clause), uSw. 3. Case: $u = x$ and $x = i$. Let uSv *et* vSw be for generic v and w. By df. of S (clause (iii) b) 1.) one has, therefore, two sub-cases:

$jS'v$ or $iS''v$ and v represents an element of W' or of W'', respectively. In the former sub-case, one obtains by df. of S (clause (iii) a)) $vS'w$; in the latter, by df. of S (clause (iii) b) 2.) $vS''w$. In both sub-cases there therefore follows the same result uSw, either by transitivity of S' and df. of S (clause (iii) b) 1.) or by transitivity of S'' and df. of S (same clause).

Ad c) the condition of intersection is ensured by the fact that R and S have been constructed by maintenance or extension of the previous relations. Therefore, by virtue of the fact that R' together with S' and R'' together with S'' satisfy the intersection condition, the same condition is also satisfied by R together with S.

$<W,R,S,I>$ is thus a model for $\mathbf{K}-\mathbf{0}\Diamond$. The theorem may therefore be taken to be concluded if $<W,R,S,I> \models_i M$ and $<W,R,S,I> \not\models_i \mathbf{0}\beta$ can be obtained. For this purpose, the following two lemmas may be employed.

Lemma 2: ($\forall\alpha$ alethic) ($<W,R,S,I> \models_x \alpha \Leftrightarrow <W'',R'',I''> \models_x \alpha$)
By * we already know that if α is alethic $<W'',R'',I''> \models_x \alpha \Leftrightarrow <W'',R'',S'',I''> \models_x \alpha$. Therefore it is enough to prove that for alethic α $<W,R,S,I> \models_x \alpha \Leftrightarrow <W'',R'',W'',I''> \models_x \alpha$. The proof is by induction on the complexity of α. Let stand for $<W,R,S,I>$ and $''$ for $<W'',R'',S'',I''>$.

Basis: $\alpha \equiv p$: $\mathcal{M} \models_x p \Leftrightarrow I(p,x) = 1$ (def. $\models$) $\Leftrightarrow I''(p,x) = 1$ (clause (iv)) $\Leftrightarrow \mathcal{M}'' \models_x p$ (def. $\models$).

Step: the propositional cases are obtained in the usual manner by inductive hypothesis; $\alpha \equiv \Box\gamma$: $\mathcal{M} \models_x \Box\gamma \Leftrightarrow \forall v(xRv \Rightarrow \mathcal{M} \models_v \gamma)$ (df. $\models$) $\Leftrightarrow \forall v(xR''v \Rightarrow \mathcal{M} \models_v \gamma)$ (clause (ii)) $\Leftrightarrow \forall y(xR''y \Rightarrow \mathcal{M}''\ \alpha_y\ \gamma)$ (df. y and inductive hypothesis) $\Leftrightarrow \mathcal{M}'' \models_x \Box\gamma$ (df. $\models$).

Lemma 3: $\forall\alpha(<W,R,S,I> \models_r \alpha \Leftrightarrow <W',R',S',I'> \models_r \alpha$. Let $\mathcal{M}$ and $'$ stand for respectively for $<W,R,S,I>$ and for $<W',R',S',I>$.

Basis: $\alpha \equiv p$: $\mathcal{M} \models_r p \Leftrightarrow I(p,r) = 1$ (df. $\models$) $\Leftrightarrow I'(p,r) = 1$ (clause (iv)) $\Leftrightarrow \mathcal{M}' \models_r p$ (df. $\models$).

Step: the propositional cases obtained in the usual manner by inductive hypothesis are omitted; $\alpha \Leftrightarrow \Box\gamma$: $\mathcal{M} \models_r \Box\gamma \Leftrightarrow \forall v(rRv \Rightarrow \mathcal{M} \models_v \gamma)$ (df. $\models$) $\Leftrightarrow \forall v(rR'v \geq \mathcal{M} \models_v \gamma)$ (clause (ii)) $\Leftrightarrow \forall s(rR's \Rightarrow \mathcal{M}' \models_s \gamma)$ (df. s and inductive hypothesis) $\Leftrightarrow \mathcal{M}' \models_r \Box\gamma$ (df. $\models$); $\alpha \equiv \mathbf{0}\gamma$: $\models_r \mathbf{0}\gamma$ $\Leftrightarrow \forall v(rSv \Rightarrow \mathcal{M} \models_v \gamma)$ (def. $\models$) $\Leftrightarrow \forall v(rS'v \Rightarrow \mathcal{M} \models_v \gamma)$ (clause (iii) a)) $\Leftrightarrow \forall s(rS's \geq \mathcal{M}' \models_s \gamma)$ (df. s and inductive hypothesis) $\Leftrightarrow \mathcal{M}' \models_r \mathbf{0}\gamma$ (df. $\models$).

At this point, the theorem is obtained in a few operations. From lemma 2 and from $<W'',R'',I''> \models_i M$ follows, first of all, $<W,R,S,I> \models_i M$. Moreover, also $<W,R,S,I> \not\models_i \mathbf{O}\beta$ obtains. In fact, from Lemma 3 and from $<W',R',S',I'> \models_j \mathbf{P}\neg\beta$ follows $<W,R,S,I> \models_j \mathbf{P}\neg\beta$, which means that there exists at least some deontic alternative by S of j where $\neg\beta$ is true. On the other hand, the deontic alternatives by S of j are also deontic alternatives by S of i (clause (iii) b) 1.), and thus there exists at least some deontic alternative by S of i where $\neg\beta$ is true. But this means $<W,R,S,I> \models_i \mathbf{P}\neg\beta$ and therefore $<W,R,S,I> \not\models_i \mathbf{O}\beta$. ■

2. *Mixed systems* K−5 *and* K−5−0◇

Inspection of the crucial steps of the proof show that Theorem 1 holds for mixed systems obtained by weakening either the alethic or the deontic part of the system. However, it no longer holds for systems where the deontic part contains the axiom **0·5**. This is due to the fact that **0·5** implies the euclidicity of the S relation, for which reason the method of construction for the new $<W,R,S,I>$ model fails. Indeed, the euclidicity of S implies that if *iSr et iSx* is the case, then so is *rSx*. Hence, in the completed model, the deontic alternatives of i belonging to W' are no longer the same (i.e. they have different contents) with respect to those of j in the $<W',R',S',I'>$ model. This, however, finds syntactic explanation in the fact that **0·5** together with **0◇** provides syntactic counterexamples to the theorem. Let, for example, $M = \{\Box\alpha\}$. Then by **0◇**, $M \vdash \mathbf{P}\alpha$ is the case, and from this it follows by **0·5** also $M \vdash \mathbf{OP}\alpha$.

What might the significance of axiom **0·5** be for the problem of Hume's thesis? First of all, it should be pointed out that there is apparently a valid distinction to be made between the philosophical significance of the principle and its importance from the point of view of Hume's thesis. In general, it is recognized that axiom **0·5** is, together with **0·4** a principle endowed with manifest significance and plausibility as regards the characterization of an unconditional concept of normativeness. On the other hand, the derivability through **0·5** of the obligations set out above does not appear to constitute a philosophically serious violation of Hume's thesis, insofar as the obligations in question have permissions as their content. And these, although derivable from M, do not in turn display the feature of counter-examples that would be of philosophical importance for the law. As has already been said, **P**α is derivable from M by means of the **0**$\Diamond$ principle, according to which possibility is a necessary but not sufficient condition for obligatoriness.

But, then, just as it cannot be said that such permissions constitute a real violation of Hume's law, so the obligations derived from them by **0·5** cannot constitute significant counter-examples to the law. Let us, at this point, call an obligation derivable from some M in a mixed system by means of **0·5** and **0**$\Diamond$ an obligation induced by a permission forced by the necessitations derivable from M (for the sake of brevity – *an obligation induced by a forced permission*). What, in conclusion, is there of importance for Hume's thesis if **0·5** is also added to **K**$-$**0**$\Diamond$ as an axiom? In view of the fact that, in **KT5**$-$**0·KD45**$-$**0**$\Diamond$ obligations induced by forced permissions are derivable from suitable M but that they are not of philosophical importance for the issue that concerns us here, it is important that in such a system it should not be possible to derive obligations not induced by forced permissions. That this is in fact the case is a corollary of Theorem 1 and of Theorem 2 below relative to the mixed system **KT5**$-$**0·KD45** (in short **K**$-$**5**).

The language of the mixed system **K**$-$**5** matches that of **K**$-$**0**$\Diamond$, while its axioms match those of **KT5** and **0·KD45**. Of course, the axiom **0**$\Diamond$ is lacking. A model for **K**$-$**5** (or **K**$-$**5**-model) is constituted by the ordered quadruple $<W,R,S,I>$, where W and I are defined in the usual manner and R and S are distinct relations both defined on W, such that: (i) R is reflexive and euclidean (that is, reflexive, symmetrical and transitive); (ii) S is serial, transitive and euclidean. Clearly, **K**$-$**5** is sound and complete with respect to the concept of logical consequence based on the notion of model put forward here. The proofs follow from those for **KT5** and **0·KD45**, respectively.

If we draw on the concepts of alethic formula and deontic formula set out in note 1, we may derive the following extension of Kaliba's theorem (7):

Theorem 2: *Let M be a* **KT5**-*consistent set of alethic formulae. Let* α *be a deontic formula satisfying the unprovability condition* $\nvdash_{K-5}\alpha$. *Then M* $\nvdash_{K-5}\alpha$.

(7) See P. Kaliba [1981]. The technique employed for the proof of Theorem 2. is also an extension of Kaliba's technique.

Proof:

Analogously to the above independence proof, here proof must be provided that there exists a model for **K**$-$**5** $<W,R,S,I>$ and a world $u \in W$ such that $<W,R,S,I> \models_u M$ and $<W,R,S,I> \not\models_u \alpha$. Now, from the **KT5**-consistency of M it follows because of the completeness of **KT5** that there exists a model $<W'',R'',I''>$ and a world $i \in W''$ such that $<W'',R'',I''> \models_i M$. Moreover, on the basis of the condition of unprovability of α in **K**$-$**5**, there exists a model $<W',R',S',I'>$ and a world $j \in W'$ such that $<W',R',S',I'> \not\models_j \alpha$. What is required now, therefore, is the construction, on the basis of the given models, of the $<W,R,S,I>$ model and to show, by means of a number of lemmas, that this model verifies M but not α in a certain world.

Construction of $<W,R,S,I>$

Let $<W'',R'',S'',I''>$ be the model that is obtained by defining on W'' a relation S'' that satisfies all the formal properties (except the condition of intersection) possessed by S'' in the homonymous model $<W'',R'',S'',I''>$ introduced at the beginning of the demonstration of Theorem 1, and, moreover, which is euclidean. Clearly, $<W'',R'',S'',I''>$ is by construction a model for **K**$-$**5**. Moreover, insofar as $<W'',R'',S'',I''>$ and $<W'',R'',I''>$ are coincident in their alethic parts, it is the case that ($\forall\alpha$ alethic) ($\forall u \in W''$) ($<W'',R'',S'',I''> \models_u \alpha \Leftrightarrow <W'',R'',I''> \models_u \alpha$) *. $<W,R,S,I>$ is obtained at this point on the basis of $<W'',R'',S'',I''>$ and $<W',R',S',I'>$ employing the four defining clauses (i)-(iv) that made it possible to construct the homonymous model of the previous theorem, the only variation being that (iii) b) 1. is now the new defining clause $x = i \Rightarrow (xSv \Leftrightarrow jS'v)$ (replacing $x = i \Rightarrow (xSv \Leftrightarrow jS'v \textit{ vel } iS''v)$).

Lemma 1: $<W,R,S,I>$ *is a model for* **K**$-$**5**

Proof here differs from that provided for Lemma 1 of Theorem 1 only by virtue of certain variations required by the new clause (iii) b) 1. and by the euclidicity of S. The reader is therefore referred to the demonstration of this lemma on previous pages (of which, clearly, the part dealing with the condition of intersection is not relevant), and treatment here will be limited to examination of the variations between the two lemmas.

Firstly, it is immediately apparent that the proof of the seriality and the transitivity of S is obtained by employing the new clause (iii) b) 1., in terms of which $x = i \Rightarrow (xSv \Leftrightarrow jS'v)$. In particular the 3. Case of *ad* a) becomes: $u = x$ and $x = i$. Thus the seriality of S follows from the seriality of S' and df. of S (clause (iii) b) 1.); and the 3. Case of *ad* b) becomes: $u = x$ and $x = i$. Let uSv *et* vSw be for generic v and w. Thus, by definition of S (clause (iii) b) 1.) $jS'v$ is the case and v is an element of W'. But then, by df. of S (clause (iii) a)) $vS'w$, is also the case, and thus – by transitivity of S' – $jS'w$, and therefore, once again by df. of S (clause (iii) b) 1.), uSw.

Secondly, it is necessary to prove *ex novo* that S is euclidean – that is, $\forall u \forall v \forall w(uSv$ *et* $uSw \Rightarrow vSw)$. For this purpose, the usual three cases need to be distinguished: 1. Case: $u = r$. Let uSv *et* uSw be for generic v and w. Then, by def. of S (clause (iii) a)) $uS'v$ *et* $uS'w$ holds. But, by euclidicity of S', $vS'w$ also follows and hence, once again by df. of S (same clause), vSw. 2. Case: $u = x$ and $x \neq i$. Let uSv *et* uSw be for generic v and w. Then, by df. of S (clause (iii) b) 2.) $uS''v$ *et* $uS''w$ holds and therefore, by euclidicity of S'', also $vS''w$. On the other hand, because of the condition of deontic inaccessibility of i, $v \neq i$, and thus one finally obtains, by df. of S (clause (iii) b) 2.), vSw (8). 3. Case: $u = x$ and $x = i$. Let uSv *et* uSw be for generic v and w. By df. of S (clause (iii) b) 1.) $jS'v$ and $jS'w$ are obtained. But then, by euclidicity of S', $vS'w$ is obtained and therefore, by df. of S (clause (iii) a)) also vSw.

Lemma 2: $\forall \alpha(<W,R,S,I> \models_r \alpha \Leftrightarrow <W',R',S',I'> \models_r \alpha)$

The proof is by induction on the complexity of α. Let $\mathcal{M}$ and $\mathcal{M}'$ stand, respectively, for $<W,R,S,I>$ and for $<W',R',S',I'>$.

Basis: $\alpha \equiv p$: $\mathcal{M} \models_r p \Leftrightarrow I(p,r) = 1$ (df. $\models$) $\Leftrightarrow I'(p,r) = 1$ (clause (iv)) $\Leftrightarrow \mathcal{M}' \models_r p$ (df. $\models$).

(8) To be noted is the fundamental importance of the condition of deontic inaccessibility of i. This not only makes it possible to obtain the transitivity of S (see the 2. Case of *ad* b) in the proof of the 1. lemma relative to the previous theorem – to be used integrally for the purposes of the proof being developed here) and thus to extend Kaliba's [1981] result to the **K–5** system (which, compared with the deontic **S5** system under consideration here also contains the axiom **0–4**); it is also essential in order to obtain the euclidicity of S. As far as transitivity is concerned: let i be by hypothesis deontically accessible, starting, let us suppose, from x. Then xSi is the case. If, at this point, iSr is also the case, because of the transitivity of S, it should also be possible to obtain xSr, but this is impossible given that $xSv \Leftrightarrow xS''v$. Likewise, as regards the euclidicity of S: let us suppose in fact that xSi and xSy is the case. Then one should also obtain iSy, but this is impossible given that $iSv \Leftrightarrow jS'v$.

Step: the propositional cases are obtained in the usual manner by inductive hypothesis; $\alpha \equiv \Box\beta : \mathcal{M} \models_r \Box\beta \Leftrightarrow \forall v(rRv \Rightarrow \mathcal{M} \models_v \beta)$ (df. $\models$) $\Leftrightarrow \forall v(rR'v \Rightarrow \mathcal{M} \models_v \beta)$ (clause (ii)) $\Leftrightarrow \forall s(rR's \Rightarrow \mathcal{M}' \models_s \beta)$ (df. s and inductive hypothesis) $\Leftrightarrow \mathcal{M}' \models_r \Box\beta$ (df. $\models$); $\alpha \equiv \mathbf{0}\beta : \mathcal{M} \models_r \mathbf{0}\beta \Leftrightarrow \forall v(rSv \Rightarrow \mathcal{M} \models_v \beta)$ (df. $\models$) $\Leftrightarrow \forall v(rS'v \Rightarrow \mathcal{M} \models_v \beta)$ (clause (iii) a)) $\Leftrightarrow \forall s(rS's \Rightarrow \mathcal{M}' \models_s \beta)$ (df. s and inductive hypothesis) $\Leftrightarrow \mathcal{M}' \models_r \mathbf{0}\beta$ (df. $\models$).

Lemma 3: ($\forall\alpha$ deontic) ($<W,R,S,I> \models_i \alpha \Leftrightarrow <W',R',S',I'> \models_j \alpha$)

The proof is as usual by induction on the complexity of α. Let $\mathcal{M}$ and $\mathcal{M}'$ stand, respectively, for $<W,R,S,I>$ and for $<W',R',S',I'>$.

Basis: $\alpha \equiv \mathbf{0}\beta$ (for any β): $\mathcal{M} \models_i \mathbf{0}\beta \Leftrightarrow \forall v(iSv \Rightarrow \mathcal{M} \models_v \beta)$ (df. $\models$) $\Leftrightarrow \forall v(jS'v \Rightarrow \mathcal{M} \models_v \beta)$ (clause (iii) b) 1.) $\Leftrightarrow \forall r(jS'r \Rightarrow \mathcal{M}' \models_r \beta)$ (df. r and Lemma 2) $\Leftrightarrow \mathcal{M}' \models_j \mathbf{0}\beta$ (df. $\models$).

Step: The step comprises only propositional cases obtained in the usual manner by inductive hypotheses.

Lemma 4: ($\forall\alpha$ alethic) ($<W,R,S,I> \models_x \alpha \Leftrightarrow <W'',R'',S'',I''> \models_x \alpha$)

The proof is by induction on the complexity of α. Let $\mathcal{M}$ and $\mathcal{M}''$ stand, respectively, for $<W,R,S,I>$ and $<W'',R'',S'',I''>$.

Basis: $\alpha \equiv p$: $\mathcal{M} \models_x p \Leftrightarrow I(p,x) = 1$ (df. $\models$) $\Leftrightarrow I''(p,x) = 1$ (clause (iv)) $\Leftrightarrow \mathcal{M}'' \models_x p$ (df. $\models$).

Step: the propositional cases are obtained in the usual manner by inductive hypothesis; $\alpha \equiv \Box\beta$: $\mathcal{M} \models_x \Box\beta \Leftrightarrow \forall v(xRv \Rightarrow \mathcal{M} \models_v \beta)$ (df. $\models$) $\Leftrightarrow \forall v(xR''v \Rightarrow \mathcal{M} \models_v \beta)$ (clause (ii)) $\Leftrightarrow \forall y(xR''y \Rightarrow \mathcal{M}'' \; \beta)$ (df. y and inductive hypothesis) $\Leftrightarrow \mathcal{M}'' \models_x \Box\beta$.

The conclusion of the theorem is obtained from the fact that since it is the case that by hypothesis $<W'',R'',I''> \models_i M$, then it holds by * $<W'',R'',S'',I''> \models_i M$ and therefore by Lemma 4 $<W,R,S,I> \models_i M$. And since it is the case that by hypothesis $<W',R',S',I'> \not\models_j \alpha$, by Lemma 3 $<W,R,S,I> \not\models_i \alpha$.

■

Let us now consider the mixed system $\mathbf{KT5-0 \cdot KD45-0\Diamond}$ (in short $\mathbf{K-5-0\Diamond}$) and adapt to it the definition given above of obligation induced by a forced permission. Let the condition of unprovability of obligation $\mathbf{0}\beta$ in $\mathbf{K-5-0\Diamond}$ be presupposed and assume that: $\mathbf{0}\beta$ is an abligation induced in $\mathbf{K-5-0\Diamond}$ by a permission forced by $M =_{\text{def}} M \vdash_{\mathbf{K-5-0\Diamond}} \mathbf{0}\beta$

et $M \nvdash_{\mathbf{K-0\Diamond}} \mathbf{0}\beta$ *et* $M \nvdash_{\mathbf{K-5}} \mathbf{0}\beta$. Then theorems 1 and 2 give the following corollary:

Corollary 1: *Let a* **KT5**-*consistent set M of alethic formulae be taken as given. Then the obligations that can be derived from M in* **K−5−0◇** *and that satisfy the condition of unprovability in* **K−5−0◇** *are only obligations induced by forced permissions.*

Proof:

Let us suppose that the obligation $\mathbf{0}\beta$ derivable from M in **K−5−0◇** is not an obligation induced by a forced permission. This means that $\mathbf{0}\beta$ is obtained without employment of **0◇** together with **0·5**. The possible cases are therefore three in number: $\mathbf{0}\beta$ is obtained without **0◇** and without **0·5**; $\mathbf{0}\beta$ is obtained with **0◇** and without **0·5**; $\mathbf{0}\beta$ is obtained with **0·5** and without **0◇**. Now the first and the second of these cases contrast with Theorem 1, according to which, by virtue of the theorem's hypotheses, in **K−0◇** no obligation is derivable from *M*. The third case is excluded by Theorem 2, on the basis of which, according to its hypotheses, in **K−5** no deontic formula and hence no obligation is derivable from *M*.

■

3. *Mixed systems* **K+0◇**, **K+5** *and* **K+5+0◇**

The above theorems preserve their validity even if the axiomatic basis of the respective systems is extended to include a further axiom **0□0**: $\mathbf{0}\alpha \rightarrow \Box\mathbf{0}\alpha$. I shall use **K+0◇** and **K+5** to refer respectively to the **KT5−0·KD4−0□0−0◇** and **KT5−0·KD45−0□0** systems. This is therefore a matter of proving that the methods used to obtain Theorem 1 and Theorem 2 are still applicable after the addition of **0□0**. For this purpose, firstly, two new definitions of model will be provided for **K+0◇** and **K+5** respectively. It will then be shown how it is possible to construct the two models (i.e. one for **K+0◇** and one for **K+5**) functional to the respective independence proofs.

A model for **K+0◇** (or **K+0◇**-model) is a model for **K−0◇** where *R* and *S* jointly satisfy the condition of *mixed euclidicity*: $\forall u \forall v \forall w$ (uRv *et* $uSw \Rightarrow vSw$). **K+0◇** is sound and complete with respect to the notion of **K+0◇**-model. The soundness follows from the soundness of **K−0◇**

with respect to the **K**−**0**◇-models and from the fact that, given mixed euclidicity, the models for **K**+**0**◇ satisfy the axiom **0**□**0**. Let in fact $<W,R,S,I> \models_u \mathbf{0}\alpha$ be assumed. For all w, therefore, it holds that $uSw \Rightarrow <W,R,S,I> \models_w \alpha$. On the other hand, by virtue of mixed euclidicity, uSw is implied by *vRu et vSw* and, by virtue of the symmetry of R also by *uRv et vSw*. By chain rule and subsequent generalization on w and v one finally obtains $\forall v(uRv \Rightarrow \forall w(vSw \Rightarrow <W,R,S,I> \models_w \alpha))$, i.e. $<W,R,S,I> \models_u \Box\mathbf{0}\alpha$. As far as completeness is concerned, it is sufficient to show that the relations $R_{\mathbf{K}+\mathbf{0}\Diamond}$ and $S_{\mathbf{K}+\mathbf{0}\Diamond}$ of the canonical model satisfy the property of mixed euclidicity, on the basis of the fact that they already satisfy the other properties required for the completeness of **K**−**0**◇ with respect to **K**−**0**◇-models. Let us assume $uR_{\mathbf{K}+\mathbf{0}\Diamond}v$ *et* $uS_{\mathbf{K}+\mathbf{0}\Diamond}w$. Proof is therefore required that $vS_{\mathbf{K}+\mathbf{0}\Diamond}w$. Given the symmetry of $R_{\mathbf{K}+\mathbf{0}\Diamond}$, the first hypothesis is first of all equivalent to $vR_{\mathbf{K}+\mathbf{0}\Diamond}u$. From this it therefore follows by df. of $R_{\mathbf{K}+\mathbf{0}\Diamond}$ and instantiation $\Box\mathbf{0}\alpha \in v \Rightarrow \mathbf{0}\alpha \in u$. Let us suppose at this point that $\mathbf{0}\alpha \in v$. This means by the **K**+**0**◇-closure also that $\Box\mathbf{0}\alpha \in v$ and therefore $\mathbf{0}\alpha \in u$. However, by developing the second hypothesis one obtains by df. of $S_{\mathbf{K}+\mathbf{0}\Diamond}$ and instantiation $\mathbf{0}\alpha \in u \Rightarrow \alpha \in w$. By applying chain rule one therefore obtains $\alpha \in w$. Finally, by discharging the assumption introduced during the demonstration and by generalization on α, one obtains $\forall\alpha(\mathbf{0}\alpha \in v \Rightarrow \alpha \in w)$, i.e. $vS_{\mathbf{K}+\mathbf{0}\Diamond}w$.

In similar fashion it is possible to introduce the concept of model for **K**+**5** and demonstrate the soundness and completeness of **K**+**5** with respect to such a notion of model. In fact, a model for **K**+**5** (or **K**+**5**-model) is a model for **K**−**5** where R and S jointly satisfy the condition of mixed euclidicity, and it is for this reason that the proofs of soundness and completeness of **K**+**0**◇ can also be transferred to the case of **K**+**5**.

We may now proceed with the extension of theorems 1 and 2.

Theorem 3: *Let M be a* **KT5**-*consistent set of alethic formulae. Let* α *be an obligation, that is, a deontic formula of the form* $\mathbf{0}\beta$, where β *is any formula* $\in$ **L**(**K**+**0**◇) *that satisfies the condition of unprovability* $\nvdash_{\mathbf{K}+\mathbf{0}\Diamond} \mathbf{0}\beta$. *Then* $M \nvdash_{\mathbf{K}+\mathbf{0}\Diamond} \mathbf{0}\beta$.

Proof

Treatment will be given only of the new aspects of the proof, since the rest is structurally identical with that of Theorem 1. Apart from the fact that the models on which proof is based – $<W',R',S',I'>$ and $<W'',R'',S'',I''>$ (this latter as an extension of $<W'',R'',I''>$) – are now $\mathbf{K}+\mathbf{0}\Diamond$-models (and it is not required, in particular, that the condition of deontic inaccessibility of i should be satisfied by S'' in $<W'',R'',S'',I''>$), the main variation lies in the construction of $<W,R,S,I>$ and the proof that this is a $\mathbf{K}+\mathbf{0}\Diamond$-model. The defining clauses (i), (ii), (iii) a) and (iv) of $<W,R,S,I>$ are the same as those that applied to Theorem 1; but clauses (iii) b) 1. and (iii) b) 2. have been replaced by a single new clause (iii) b): $xSv \Leftrightarrow xS''v$ *vel* $jS'v$, on the basis of which the deontic alternatives of the $\in W''$ elements (which include i) are the original ones plus all the deontic alternatives of j. This variation, of course, requires that $<W,R,S,I>$ should be shown to be a $\mathbf{K}+\mathbf{0}\Diamond$-model, and for this purpose it is sufficient to show that: a) S is transitive, and b) S together with R satisfies the condition of mixed euclidicity. In fact, the proof of seriality and of the condition of intersection differs very little from the one employed for Theorem 1.

Ad a) treatment will be given only of the second case: 2. Case: $u = x$. Let uSv *et* vSw be for generic v and w. Then, by df. of S (clause (iii) b)), one obtains two sub-cases: $jS'v$ *vel* $uS''v$ and v represent respectively an element of W' or of W''. Let us take the first sub-case. By df. of S (clause (iii) a)) one obtains from the general hypothesis $vS'w$ and therefore, by transitivity of S', $jS'w$. But then, by df. of S (clause (iii) b)) one also obtains uSw. In the second subcase, however, one obtains – still from the general hypothesis and by def. of S (clause (iii) b)) $vS''w$ *vel* $jS'w$. Let now $vS''w$ be assumed. As a result of the transitivity of S'', one thus obtains $uS''w$ and therefore, by df. of S (clause (iii) b)) uSw. If instead one assumes $jS'w$, uSw is obtained immediately by df. of S (same clause). In both sub-cases, one may therefore conclude uSw.

Ad b) two cases should be distinguished. 1. Case: $u = r$. Let uRv *et* uSw. By df. of R (clause (ii)) and df. of S (clause (iii) a)) one obtains $uR'v$ *et* $uS'w$. But then, by the mixed euclidicity of S' one also obtains $vS'w$. On the other hand, v represents an element $\in W'$; hence, on the basis of the def. of S (clause (iii) a)) one also obtains vSw. 2. Case: $u = x$. Let uRv *et* uSw be assumed. By df. of R (clause (ii)) one then obtains $uR''v$, where v represents an element of W'', and, by df. of S (clause (iii) b)) $uS''w$ *vel* $jS'w$. Now let $uS''w$ be assumed. S'' satisfies the condition of mixed euclidicity, thus one obtains $vS''w$ and therefore, by df. of S ((clause iii) b)), also vSw. If instead one assumes $jS'w$, vSw is then immediately obtained solely by df. of S (clause (iii) b)). In both subcases one therefore concludes vSw. ■

Theorem 4: *Let M be a* **KT5**-*consistent set of alethic formulae. Let* α *be a deontic formula satisfying the condition of unprovability* $\nvdash_{K+5} \alpha$. Then $M \nvdash_{K+5} \alpha$.

Proof:

Compared with the proof of Theorem 2, there are two variations. Firstly, the construction of $<W,R,S,I>$ is carried out directly on the basis of $<W'',R'',I''>$ and $<W',R',S',I'>$ (which, in this case, is a **K**+5-model) without the mediation of the $<W'',R'',S'',I''>$ model. Secondly, clauses (iii) b) 1. and (iii) b) 2. are both replaced by a single clause (iii) b): $xSv \Leftrightarrow jS'v$. Thus there are no structural differences between the general articulation of this proof and that of Theorem 2. Apart from the non-essential modifications required by the new clause and the reformulation of Lemma 4 in terms of $<W,R,S,I>$ and $<W'',R'',I''>$, all that remains to be dealt with is the proof that $<W,R,S,I>$ is a **K**+5-model. For this purpose proof will be given of a) the seriality of S, b) the transitivity of S, c) the euclidicity of S, d) the mixed euclidicity of R and S.

Ad a) I shall consider only the case where $u = x$. On the basis of the seriality of S' one obtains $\exists v(jS'v)$. But then, by the def. of S (clause (iii) b)) one also obtains $\exists v(xSv)$.

Ad b) I shall consider only the case where $u = x$. Let uSv *et* vSw be. By df. of S (clause (iii) b)) one thus obtains $jS'v$. Moreover, v represents an element of W'. Therefore, $vS'w$ follows from vSw by df. of S (clause (iii) a)). Consequently, one obtains $jS'w$ by transitivity of S', and therefore, still by df. of S (clause (iii) b)), uSw.

Ad c) I shall consider only the case where $u = x$. Let uSv *et* uSw be. By df. of S (clause (iii) b)) one obtains $jS'v$ and v represents an element of W'. Likewise, $jS'w$ is also the case and w represents an element of W'. But then, by the euclidicity of S', one also obtains $vS'w$ and therefore by df. of S (clause (iii) a)) vSw.

Ad d) here, too, I shall only consider the case where $u = x$. Let uRv *et* uSw be. By df. of R (clause (ii)) one obtains $uR''v$, and v therefore represents an element of W''. On the other hand, by df. of S (clause (iii) b)) $jS'w$ also holds, where w represents an element of W'. By df. of S

(same clause) xSw therefore holds for any element x of W''. In particular, therefore, vSw holds. ■

Just as theorems 3 and 4 are a generalization of theorems 1 and 2 so too is it possible to generalize Corollary 1. Let the mixed system **KT5**$-$**0**$\cdot$**KD45**$-$**0**$\Box$**0**$-$**0**$\Diamond$ (in short **K**+**5**+**0**$\Diamond$) be taken as given. Now, by adapting the definition of obligation induced by a forced permission to the new system one has, as a consequence of theorems 3 and 4,

Corollary 2: *Let a* **KT5**-*consistent set M of alethic formulae be taken as given. Then the obligations that can be derived in* **K**+**5**+**0**$\Diamond$ *from M and that satisfy the condition of unprovability in* **K**+**5**+**0**$\Diamond$ *are only obligations induced by forced permissions.*

4. *Mixed system* **K**$-\Box$**0** *and alethic system of deontic logic* **KT5Q**

Whereas **0**$\Diamond$ only permits the immediate derivation of permissions from suitable sets of alethic formulae, the principle $\Box$**0** immediately generates obligations as well. Hence, in all the systems in which $\Box$**0** holds (in particular, in all the alethic systems of deontic logic where not only **0**$\Diamond$ but also $\Box$**0** holds), there is a systematic violation of Hume's thesis, insofar as, in these systems, deontic formulae which also take the form of obligation are derivable from specific sets of alethic formulae. However, this raises the question, as has been noted by many critics of the principle in question, whether $\Box$**0** provides those guarantees of plausibility that are given by other bridge-principles, e.g. **0**$\Diamond$. In effect, one is quite justified in arguing that there is a substantial difference of meaning between **0**$\Diamond$ and $\Box$**0**. Indeed, it should be remembered that $\Box$**0** is a consequence of a very broad interpretation of the concept of obligation (including, that is, the extreme case where the content of the obligation is necessitated), which in the context of the alethic systems of deontic logic, can only be avoided at the expense of a considerable complication of the definition, in alethic terms, of the obligation. Thus the $\Box$**0**-principle cannot be assigned any importance of content. The obligations that derive from it are vacuous, so to speak. That is to say, they are *trivial obligations* unrepresentative of any axiological content. In other words, for a state of affairs to be obligatory in the proper sense of the term, it is not enough that it should be true in all good worlds (since this, in fact, may be the case of states of affairs that, insofar as they are necessary, are true in all worlds and therefore, trivially, also in good worlds); there is the further requirement that they should represent something that is axiologically positive. The status of the **0**◇ principle is different. This establishes the connexion between obligatoriness and possibility; a connexion with deep roots in the concept of duty, one which alethic systems of deontic logic have sought to give semantic form to. The possibility of β is indeed a necessary condition for $\mathbf{0}\beta$. Naturally, **0**◇ implies the existence of forced permissions – that is to say, permissions not determined by axiological reasons. However, the existence of forced permissions is not as problematic as the existence of necessitated obligations.

Having granted this, it is therefore reasonable to argue that the derivability, by □**0**, of the obligations in question does not constitute a significant violation of Hume's thesis. It is, therefore, an equally reasonable undertaking to ascertain whether there also exist genuinely significant counter-examples to the thesis in the context of alethic systems of deontic logic. In this section we shall examine the problem in terms of the system **KT5Q**. Firstly, proof will be given of the equivalence, under a specific function of translation, between the mixed system **K**– □**0** (for the def. of **K**– □**0**, see below) and **KT5Q**. Secondly, we shall advance the proposal that obligations which are not trivial and not induced by forced permissions are to be regarded as *axiologically important obligations*. Finally, given this definition, the conclusive result of the underivability of axiologically important obligations for **KT5Q** will be achieved.

4.1 Taken as given shall be the mixed system **KT5 – 0 · KD – 0□0 –** □**0** (in short **K**– □**0**) obtainable from **KT5 – 0 · KD – 0□0** by adding the further axiom □**0**: $\Box\alpha \rightarrow \mathbf{0}\alpha$. Let the alethic system of deontic logic **KT5Q** be also taken as given, as well as the following definition of the translation function from **L**(K– □**0**) into **L(KT5Q)**: $\phi\ (p) \equiv p$, $\phi\ (\neg\alpha) \equiv \neg\phi\ (\alpha)$, $\phi\ (\alpha \mathbf{o} \beta) \equiv \phi\ (\alpha)\ \mathbf{o}\ \phi\ (\beta)$ (where **o** stands for any connective of degree 2), $\phi\ (\Box\alpha) \equiv \phi\ \Box(\alpha)$, $\phi\ (\mathbf{0}\alpha) \equiv \Box(Q \rightarrow \phi\ (\alpha))$. It can thus be shown that, under this translation, **KT5Q** and **K**– □**0** are equivalent, i.e.: 1. $X \vdash_{K-\Box 0} \alpha \Rightarrow \phi\ (X) \vdash_{KT5Q} \phi\ (\alpha)$ and 2. $\phi\ (X) \vdash_{KT5Q} \phi\ (\alpha) \Rightarrow X \vdash_{K-\Box 0} \alpha$ (where $\phi\ (X)$ designates the set of all the ϕ-translations of the $\in X$ formulae). The first part of the equivalence is obtained in the usual syntactic manner, by employing the derivability in **KT5Q** of the translations of all the axioms of $\mathbf{K}-\Box\mathbf{0}$. In order to obtain the second part, it is instead necessary to prove the soundness and completeness of $\mathbf{K}-\Box\mathbf{0}$ with respect to the following definition of model. $<W,R,S,I>$ is a model for $\mathbf{K}-\Box\mathbf{0}$ (or $\mathbf{K}-\Box\mathbf{0}$-model) iff W and I are defined in the usual manner; R and S are distinct relations both defined on W, such that: (i) R is reflexive and euclidean (i.e. it is an equivalence relation), (ii) S is serial, (iii) R and S jointly satisfy the following condition of *quasi-equivalence*: $\forall v(\exists x(xSv) \Rightarrow \forall u(uRv \Leftrightarrow uSv))$, according to which if v is a deontic alternative to some world, then it is a deontic alternative to all worlds of which it is an alethic alternative and vice versa.

The soundness of $\mathbf{K}-\Box\mathbf{0}$ with respect to the $\mathbf{K}-\Box\mathbf{0}$-models follows from the formal properties of R and S. This is achieved in immediate fashion because of the properties of reflexivity and euclidicity of R and the seriality of S, which respectively ensure the satisfiability of axioms **T**, **5** and $\mathbf{0}\cdot\mathbf{D}$. The condition of quasi-equivalence then guarantees the satisfiability of $\mathbf{0}\Box\mathbf{0}$ and $\Box\mathbf{0}$. In fact, from the condition of quasi-equivalence there follow a) the mixed euclidicity of R and S, b) the condition of *inclusion* for R and S: $\forall u\forall v(uSv \Rightarrow uRv)$.

Ad a) let uSw be assumed. Because of the condition of quasi-equivalence this implies uRw. Let the further hypothesis uRv now be assumed. By the euclidicity of R one therefore obtains vRw, and this implies, once again because of the condition of quasi-equivalence, vSw.

Ad b) let uSv be assumed. Thus, by virtue of the condition of quasi-equivalence, one also obtains uRv. The argument, let it be noted, is not conclusive if one starts from the assumption uRv and if one seeks to obtain uSv.

The completeness of $\mathbf{K}-\Box\mathbf{0}$ is obtained by proving that $R_{\mathbf{K}-\Box\mathbf{0}}$ and $S_{\mathbf{K}-\Box\mathbf{0}}$ satisfy the condition of quasi-equivalence, i.e. by df. of $R_{\mathbf{K}-\Box\mathbf{0}}$ and $S_{\mathbf{K}-\Box\mathbf{0}}$: $\forall v[\exists x\forall\alpha(\mathbf{0}\alpha \in x \Rightarrow \alpha \in v) \Rightarrow \forall u(\forall\alpha(\Box\alpha \in u \Rightarrow \alpha \in v) \Leftrightarrow \forall\alpha(\mathbf{0}\alpha \in u \Rightarrow \alpha \in v))]$. On the basis of the $\mathbf{K}-\Box\mathbf{0}$-closure of the elements belonging to the set of the canonical model, one obtains, first of all, by $\Box\mathbf{0}$ and strengthening antecedents, $\exists x\forall\alpha(\mathbf{0}\alpha \in x \Rightarrow \alpha \in v) \Rightarrow (\forall\alpha(\mathbf{0}\alpha \in u \Rightarrow \alpha \in v) \Rightarrow \forall\alpha(\Box\alpha \in u \Rightarrow \alpha \in v))$. It is therefore necessary to prove the "if" part of the consequent. By virtue of inclusion in $\mathbf{K}-\Box\mathbf{0}$ of **KT5** and the completeness of **KT5** with respect to the models where R is an equivalence relation, it is the case that *: $uR_{\mathbf{K}-\Box\mathbf{0}}v$ *et* $xR_{\mathbf{K}-\Box\mathbf{0}}v \Rightarrow uR_{\mathbf{K}-\Box\mathbf{0}}x$, i.e. $\forall\alpha(\Box\alpha \in u \Rightarrow \alpha \in v)$ *et* $\forall\alpha(\Box\alpha \in x \Rightarrow \alpha \in v) \Rightarrow \forall\alpha(\Box\alpha \in u \Rightarrow \alpha \in x)$. Now, let $\forall\alpha(\Box\alpha \in u \Rightarrow \alpha \in x)$ and $\Box\mathbf{0}\alpha \in$

u be assumed. One therefore obtains $\mathbf{0}\alpha \in x$ and, by also assuming $\forall\alpha(\mathbf{0}\alpha \in x \Rightarrow \alpha \in v)$, one also obtains $\alpha \in v$. On the other hand, because of the $\mathbf{K}-\Box\mathbf{0}$-closure of u, the hypothesis $\Box\mathbf{0}\alpha \in u$ may be replaced, on the basis of $\mathbf{0}\Box\mathbf{0}$, by $\mathbf{0}\alpha \in u$. By discharging the final hypothesis and by generalization, one therefore reaches as a first conclusion that from $\forall\alpha(\Box\alpha \in u \Rightarrow \alpha \in x)$ and $\forall\alpha(\mathbf{0}\alpha \in x \Rightarrow \alpha \in v)$ there follows $\forall\alpha(\mathbf{0}\alpha \in u \Rightarrow \alpha \in v)$. At this point, by *, the first hypothesis may be replaced by $\forall\alpha(\Box\alpha \in u \Rightarrow \alpha \in v)$ together with $\forall\alpha(\Box\alpha \in x \Rightarrow \alpha \in v)$. On the other hand, because of the closure of x under $\Box\mathbf{0}$, $\forall\alpha(\mathbf{0}\alpha \in x \Rightarrow \alpha \in v)$ implies $\forall\alpha(\Box\alpha \in x \Rightarrow \alpha \in v)$, in such a way that $\forall\alpha(\mathbf{0}\alpha \in u \Rightarrow \alpha \in v)$ is obtainable only on the basis of $\forall\alpha(\mathbf{0}\alpha \in x \Rightarrow \alpha \in v)$ and $\forall\alpha(\Box\alpha \in u \Rightarrow \alpha \in v)$. Therefore, by assuming that $\exists x\forall\alpha(\mathbf{0}\alpha \in x \Rightarrow \alpha \in v)$ one obtains $\forall\alpha(\Box\alpha \in u \Rightarrow \alpha \in v) \Rightarrow \forall\alpha(\mathbf{0}\alpha \in u \Rightarrow \alpha \in v)$ and therefore, by generalization, $\exists x\forall\alpha(\mathbf{0}\alpha \in x \Rightarrow \alpha \in v) \Rightarrow \forall u(\forall\alpha(\Box\alpha \in u \Rightarrow \alpha \in v) \Rightarrow \forall\alpha(\mathbf{0}\alpha \in u \Rightarrow \alpha \in v))$.

We may now pass to proof of 2. $\phi(X) \vdash_{\mathbf{KT5Q}} \phi(\alpha) \Rightarrow X \vdash_{\mathbf{K}-\Box\mathbf{0}} \alpha$. 2. may first of all be transformed into $\phi(X) \Vdash_{\mathbf{KT5Q}} \phi(\alpha) \Rightarrow X \Vdash_{\mathbf{K}-\Box\mathbf{0}} \alpha$. Let, therefore, $X \Vdash_{\mathbf{K}-\Box\mathbf{0}} \alpha$. It must be shown that $\phi(X) \Vdash_{\mathbf{KT5Q}} \phi(\alpha)$. In other words, it is necessary to transform the $\mathbf{K}-\Box\mathbf{0}$-model $<W',R',S',I'>$ such that $<W',R',S',I'> \vDash_x X$ and $<W',R',S',I'> \nvDash_x \alpha$ (for $x \in W'$) in a **KT5Q**-model $<W,R,b,I>$ such that $<W,R,b,I> \vDash_u \phi(X)$ and $<W,R,b,I> \nvDash_u \phi(\alpha)$ (for $u \in W$). The proof begins with the construction of $<W,R,b,I>$ on the basis of $<W',R',S',I'>$, it continues with proof that $<W,R,b,I>$ is a **KT5Q**-model and concludes with two lemmas.

$<W,R,b,I>$ is defined by means of the following clauses: (i) $W = W'$, (ii) $R = R'$, (iii) $I = I'$, (iv) $b = \{v: \exists x(xS'v)\}$, i.e. $v \in b \Leftrightarrow \exists x(xS'v)$. Clearly, $<W,R,b,I>$ is a model for **KT5Q**. In fact, it is a b-model where R is reflexive, euclidean and b-serial. By (ii), the reflexivity and euclidicity of R are immediate. We shall therefore only consider b-seriality: $\forall u\exists v(uRv \textit{ et } v \in b)$. Note that, given the coincidence between W and W' in the following proofs the variables all vary over the same domain. By the seriality of S', $\exists v(uS'v)$ is the case. Let, therefore, $uS'v$ be for generic u. Thus, because of the condition of quasi-equivalence, one obtains $uR'v$ *et* $\exists x(xS'v)$ and, because of clauses (ii) and (iv), uRv *et* $v \in b$. Therefore $\forall u\exists v(uRv \textit{ et } v \in b)$.

Lemma 1: $uS'v \Leftrightarrow uRv$ *et* $v \in b$

The "only if" part is obtained as above. As regards the "if" part let uRv *et* $v \in b$ be assumed. Thus, by clauses (ii) and (iv) one also obtains $uR'v$ *et* $\exists x(xS'v)$. Because of the condition of quasi-equivalence one also reaches the conclusion $uS'v$.

Lemma 2: $\forall\alpha(<W',R',S',I'> \models_u \alpha \Leftrightarrow <W,R,b,I> \models_u \phi(\alpha))$

The proof proceeds by induction on the complexity of α. Its only element of interest is the step where $\alpha \equiv \mathbf{0}\beta$. Using $\mathcal{M}'$ and $\mathcal{M}$ to indicate, respectively, $<W',R',S',I'>$ and $<W,R,b,I>$ one therefore obtains $\mathcal{M}' \models_u \mathbf{0}\beta \Leftrightarrow \forall v\,(uS'v \Rightarrow \mathcal{M}' \models_v \beta)$ (df. $\models$) $\Leftrightarrow \forall v(uRv$ *et* $v \in b \Rightarrow \mathcal{M}' \models_v \beta)$ (Lemma 1) $\Leftrightarrow \forall v(uRv$ *et* $v \in b \Rightarrow \mathcal{M} \models_v \phi(\beta))$ (inductive hypotheses) $\Leftrightarrow \forall v(uRv$ *et* $\mathcal{M} \models_v Q \Rightarrow \mathcal{M} \models_v \phi(\beta))$ (df. $\models$) $\Leftrightarrow \mathcal{M} \models_u \Box(Q \to \phi(\beta))$ (df. $\models$ and elementary steps) $\Leftrightarrow \mathcal{M} \models_u \phi(\mathbf{0}\beta)$ (df. ϕ).

At this point, 2. is an immediate consequence of Lemma 2.

4.2 The equivalence under ϕ of **KT5Q** and **K**$-\Box\mathbf{0}$ obtained in 4.1 enables us to translate the results of derivability in **KT5Q** into corresponding results in **K**$-\Box\mathbf{0}$. This means, in particular, that for all alethic M, if $M \vdash_{\mathbf{KT5Q}} \phi(\alpha)$ then also $M \vdash_{\mathbf{K}-\Box\mathbf{0}} \phi(\alpha)$; thus any results of underivability obtainable for **K**$-\Box\mathbf{0}$ can also be extended to **KT5Q**. The question whether, in **KT5Q**, axiologically significant obligations are derivable from suitable alethic M may therefore be answered by setting the same problem for **K**$-\Box\mathbf{0}$ and then seeking to provide an answer for the problem thus defined. But what is meant by the term "axiologically important obligation"? First of all, in order for an obligation to be axiologically important it should not be trivial. But, then, what is the meaning in rigorous terms of the expression "non trivial obligation"? Following the summary treatment given above, it might be proposed that an obligation derivable from some M in **K**$-\Box\mathbf{0}$ is non-trivial if and only if it is derivable from M in **K**$-\Box\mathbf{0}$ without employing $\Box\mathbf{0}$. That is, if and only if it is derivable in **KT5-0·KD**$-\mathbf{0}\Box\mathbf{0}$. On the other hand, in **K**$-\Box\mathbf{0}$, the axioms **0·4**, **0·5** e $\mathbf{0}\Diamond$(9), are also provable by employment of just $\Box\mathbf{0}$ and they are

(9) *Ad* $\mathbf{0}\Diamond$: $\Box\neg\alpha \vdash \mathbf{0}\neg\alpha$ ($\Box\mathbf{0}$), $\mathbf{P}\alpha \vdash \Diamond\alpha$ (contraposition), $\mathbf{0}\alpha \vdash \mathbf{P}\alpha$ (**0·D**), $\mathbf{0}\alpha \vdash \Diamond\alpha$ (chain rule). *Ad* **0·4**: this derives immediately from $\mathbf{0}\Box\mathbf{0}$ and $\Box\mathbf{0}$. *Ad* **0·5**: this derives immediately from $\Box\mathbf{0}$ and from the principle $\mathbf{P}\Box\mathbf{P}$ ($\mathbf{P}\alpha \vdash \Box\mathbf{P}\alpha$) derivable in its turn from $\mathbf{0}\Box\mathbf{0}$. In fact, $\mathbf{0}\neg\alpha \vdash \Box\mathbf{0}\neg\alpha$ ($\mathbf{0}\Box\mathbf{0}$), $\Diamond\mathbf{0}\neg\alpha \vdash \Diamond\Box\mathbf{0}\neg\alpha$ (introduction of $\Diamond$), $\Diamond\Box\mathbf{0}\neg\alpha \vdash \Box\mathbf{0}\neg\alpha$ (contraposition of 5), $\Diamond\mathbf{0}\neg\alpha \vdash \Box\mathbf{0}\neg\alpha$ (chain rule), $\Diamond\mathbf{0}\neg\alpha \vdash \mathbf{0}\neg\alpha$ (by **T**), $\mathbf{P}\alpha \vdash \Box\mathbf{P}\alpha$ (contraposition).

legitimable on the basis of their content and not for purely technical reasons as in the case of $\Box\mathbf{0}$. It does not therefore suit our purposes to stipulate that non-trivial obligations are only those derivable in $\mathbf{KT5-0\cdot KD-0\Box 0}$ because this would also exclude those that can be derived by means of $\mathbf{0\cdot 4}$, $\mathbf{0\cdot 5}$ or $\mathbf{0\Diamond}$. However, there exists a way of getting round the excessively damaging effect of this restriction. It is sufficient to extend the relation of derivability from $\mathbf{KT5-0\cdot KD-0\Box 0}$ to $\mathbf{KT5-0\cdot KD45-0\Box 0-0\Diamond}$ (i.e. to $\mathbf{K+5+0\Diamond}$), thereby arriving at the following definition (where it is presupposed that $\mathbf{0}\beta$ is unprovable in $\mathbf{K-\Box 0}$): $\mathbf{0}\beta$ is a non-trivial obligation derivable in $\mathbf{K-\Box 0}$ from $M =_{\text{def}} M \vdash_{\mathbf{K+5+0\Diamond}} \mathbf{0}\beta$. Now, non-triviality is certainly a necessary condition for axiological importance. However, it is not a sufficient one, since obligations induced by forced permissions are not determined by axiological considerations. Therefore, these latter are also to be excluded from the class of axiologically important obligations. If we take up again the definition of obligation induced by a forced permission set out in 2. and extended in 3., this leads us to the following conclusive definition (where it is still presupposed that $\mathbf{0}\beta$ is unprovable in $\mathbf{K-\Box 0}$): $\mathbf{0}\beta$ is an axiologically important obligation derivable in $\mathbf{K-\Box 0}$ from $M =_{\text{def}} M \vdash_{\mathbf{K+5+0\Diamond}} \mathbf{0}\beta$ *et* ($M \vdash_{\mathbf{K+0\Diamond}} \mathbf{0}\beta$ *vel* $M \vdash_{\mathbf{K+5}} \mathbf{0}\beta$).

4.3 With the defining clarifications of 4.2, we now have all the elements at our disposal to obtain the following conclusive result for **KT5Q**:

Theorem 5: Let a **KT5**-*consistent set M of alethic formulae be taken as given. There therefore does not exist any obligation derivable from M in* **KT5Q**, *that satisfies the condition of unprovability in* **KT5Q** *and that is axiologically important.*

The proof follows from the def. of axiologically important obligation, equivalence under the translation function ϕ between $\mathbf{K-\Box 0}$ and **KT5Q**, and Corollary 2. ■

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