

# The put-call symmetry for American options in the Heston stochastic volatility model

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## Abstract

We extend to the Heston stochastic volatility framework the parity result of McDonald and Schroder (1998) for American call and put options.

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## 1 Introduction

Several authors have studied American options within the Heston model (see for instance Broadie and Kaya (2006), Andersen (2008), Vellekoop and Nieuwenhuis (2009) and the references therein). This paper contributes to the literature on American options in the Heston model by providing the link between American put options and American call options in this framework. In the European case, the put-call parity relates the prices of European call and put options on the same underlying asset, with the same maturity and the same strike via the law of one price. Violations of the put-call parity lead to arbitrages that are eagerly

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exploited by investors. When the options are American, early exercise is possible before maturity. Buy-and-holding an American option therefore is not anymore a self-financing strategy. The put-call parity fails in the American case. However it is possible to derive a symmetry relation (see Carr and Chesney (1996) and Schroder (1999)) that is very significant, since American options are vastly traded. Moreover, the symmetry relation is useful for the analysis of optimal decision making for real option holders (see for instance Battauz and alii, 2012 and 2014). In the standard Black-Scholes model the American put-call symmetry relates the price of an American call option to the price of an American put option by swapping the initial underlying price with the strike price and the dividend yield with the interest rate. Similar symmetry results have been obtained by Fajardo and Mordecki (2008) when the underlying asset follows a Levy process. In this paper we provide a simple proof of the symmetry relation between American call and put options in the Heston stochastic volatility model. Our result is obtained by applying the change of numeraire (see Geman, et alii, (1995) for a discussion on the change of numeraire technique and A. Battauz (2002) for applications to American options). See also Meyer (2013) for an alternative proof based on partial differential equations.

## 2 The American put-call symmetry in the Heston model.

In the Heston model (see Heston (1993)) the stock price  $S$  is described by the following stochastic differential equation with respect to the risk-neutral measure  $Q$

$$\frac{dS(s)}{S(s)} = (r - q) ds + \sqrt{v(s)} dW_1(s), \quad S(0) = S_0 \text{ for any } s \geq 0 \quad (1)$$

$$dv(s) = k(\bar{v} - v(s)) ds + \xi \sqrt{v(s)} \left( \rho dW_1(s) + \sqrt{1 - \rho^2} dW_2(s) \right) \quad v(0) = v_0 \quad (2)$$

where  $W_1$  and  $W_2$  are two independent standard Brownian motions under the risk neutral measure  $Q$  and the filtration  $\mathcal{F}$ ;  $r$  is the riskless interest rate;  $q$  is the dividend yield of the stock;  $\sqrt{v(s)}$  is the stochastic volatility of  $S$  at time  $s$ ;  $\bar{v}$  is the long variance;  $k$  is the speed of mean reversion of  $v$  towards  $\bar{v}$ ;  $\xi$  is the *vol of vol*;  $\rho$  is the correlation between  $S$  and  $v$ . We assume that  $2k\bar{v} > \xi^2$ , to ensure that the volatility is always positive.

We denote by  $B(t) = e^{rt}$  the riskless bond at date  $t$ .

Consider now an American call option on  $S$ . Its no-arbitrage price is

$$c(t) = \text{ess sup}_{t \leq \tau \leq T} \mathbb{E} \left[ e^{-r(\tau-t)} (S(\tau) - K)^+ \middle| \mathcal{F}_t \right] \quad (3)$$

for any  $t \in [0, T]$ , where  $\mathbb{E}[\cdot]$  denotes the (conditional) risk neutral expectation, and  $\tau$  denotes a stopping time with respect to the filtration  $\mathcal{F}$ . It can be shown that  $c(t)$  is a deterministic function of  $t$ ,  $S(t)$  and

current levels of volatility  $\sqrt{v(t)}$ . With a small abuse of notations we write

$$c(t) = c(t, S(t), v(t)).$$

The function  $c$  depends on the values of the fundamental parameters. We denote such dependence by writing

$$c(t) = c(t, S(t), v(t); r, q, \bar{v}, k, \xi, \rho, K).$$

The no-arbitrage price of the American put option on  $S$  is

$$p(t) = \operatorname{ess\,sup}_{t \leq \tau \leq T} \mathbb{E} \left[ e^{-r(\tau-t)} (K - S(\tau))^+ \middle| \mathcal{F}_t \right] \quad (4)$$

for any  $t \in [0, T]$ . It can be shown that  $p(t)$  is a deterministic function of  $t$ ,  $S(t)$  and current levels of volatility  $\sqrt{v(t)}$ . With a small abuse of notations we write

$$p(t) = p(t, S(t), v(t)) = p(t, S(t), v(t); r, q, \bar{v}, k, \xi, \rho, K).$$

As we already anticipated, in the American it is possible to write  $c(t)$  in terms of a *symmetric American put option*, whose definition in the stochastic volatility setting is provided here follows:

**Definition 1 (The symmetric put option)** *The symmetric American put option associated to the American call option (3) is the American put option on a Heston underlying  $S_{put}$  driven by the following equations for  $s \geq t$*

$$\frac{dS_{put}(s)}{S_{put}(s)} = \mu_{put} ds + \sqrt{v_{put}(s)} dW_1(s), \quad (5)$$

$$dv_{put}(s) = k_{put} (\bar{v}_{put} - v_{put}(s)) ds + \xi_{put} \sqrt{v_{put}(s)} \left( \rho_{put} dW_1(s) + \sqrt{1 - \rho_{put}^2} dW_2(s) \right) \quad (6)$$

where the values for the fundamental parameters are:  $S_{put}(t) = K$ ,  $\mu_{put} = q - r$ ,  $v_{put}(t) = v(t)$ ,  $\bar{v}_{put} = \frac{k\bar{v}}{k - \xi\rho}$ ,  $k_{put} = (k - \xi\rho)$ ,  $\xi_{put} = \xi$ ,  $\rho_{put} = -\rho$ ,  $r_{put} = q$ , and  $K_{put} = S(t)$ .

In the next theorem we provide the fundamental symmetry result that relates the time- $t$  price of the American call option  $c(t)$  to the time- $t$  price of the *symmetric American put option*.

**Theorem 2 (American put-call symmetry)** *Consider the American call option defined in (3) whose value at time  $t \in [0; T]$  is denoted with  $c(t) = c(t, S(t), v(t); r, q, \bar{v}, k, \xi, \rho, K)$ .*

*Consider the symmetric American put option defined in Definition 1, whose value at time  $t \in [0; T]$  is denoted with*

$$p(t) = p(t, S_{put}(t), v_{put}(t); r_{put}, q_{put}, \bar{v}_{put}, k_{put}, \xi_{put}, \rho_{put}, K_{put}).$$

The value of the American call coincides with the value of the symmetric American put as defined in Definition 1. More precisely, for any  $0 \leq t \leq T$  we have

$$c(t, S(t), v(t); r, q, \bar{v}, k, \xi, \rho, K) = p(t, S_{put}(t), v_{put}(t); r_{put}, q_{put}, \bar{v}_{put}, k_{put}, \xi_{put}, \rho_{put}, K_{put}) \quad (7)$$

Moreover, given  $x = S(t)$ ,  $K$  and  $v = v(t)$ , for any  $\hat{x}_{put}, \hat{K}_{put}$  such that  $\frac{x}{K} = \frac{\hat{K}_{put}}{\hat{x}_{put}}$  we have that

$$c(t, x, v; r, q, \bar{v}, k, \xi, \rho, K) = \sqrt{xK} \frac{p\left(t, \hat{x}_{put}, v_{put}; r_{put}, q_{put}, \bar{v}_{put}, k_{put}, \xi_{put}, \rho_{put}, \hat{K}_{put}\right)}{\sqrt{\hat{x}_{put}\hat{K}_{put}}}, \quad (8)$$

where  $\hat{x}_{put}$  replaces  $S_{put}(t)$  and  $\hat{K}_{put}$  replaces  $K_{put}$  in Definition 1.

**Proof.** Define the numeraire (see Battauz (2002))  $N(t) = S(t)e^{-(r-q)t}$ , which is a  $Q$ -martingale, since  $\frac{dN(t)}{N(t)} = \sqrt{v(t)}dW_1(t)$ . The numeraire  $N$  is associated to the equivalent martingale measure  $\mathbb{Q}^N$  whose density with respect to  $Q$  is  $L(T) = \frac{d\mathbb{Q}^N}{dQ} = \frac{N(T)}{N(0)}$ . Girsanov theorem ensures that

$$dW_1^N(t) = -\sqrt{v(t)}dt + dW_1(t), \quad dW_2^N(t) = dW_2(t) \quad (9)$$

are the differentials of two standard independent  $\mathbb{Q}^N$  Brownian motions.

We apply the change of numeraire to  $c(t)$  in (3).

To evaluate the American call option at any  $t$ , we consider a generic stopping time  $t \leq \tau \leq T$  and compute  $\mathbb{E}\left[e^{-r(\tau-t)}(S(\tau) - K)^+ \mid \mathcal{F}_t\right] = \frac{\mathbb{E}^{\mathbb{Q}^N}\left[\frac{1}{L(T)}e^{-r(\tau-t)}(S(\tau) - K)^+ \mid \mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{Q}^N}\left[\frac{1}{L(T)} \mid \mathcal{F}_t\right]} = \frac{\mathbb{E}^{\mathbb{Q}^N}\left[\mathbb{E}^{\mathbb{Q}^N}\left[\frac{1}{L(T)}e^{-r(\tau-t)}(S(\tau) - K)^+ \mid \mathcal{F}_\tau\right] \mid \mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{Q}^N}\left[\frac{1}{L(T)} \mid \mathcal{F}_t\right]}$ , where the first equation follows from Bayes theorem, and the second from the law of iterated conditional expectation. Since  $e^{-r(\tau-t)}(S(\tau) - K)^+$  is  $\mathcal{F}_\tau$ -measurable and  $\frac{1}{L(t)}$  is a  $\mathbb{Q}^N$ -martingale we get  $\mathbb{E}\left[e^{-r(\tau-t)}(S(\tau) - K)^+ \mid \mathcal{F}_t\right] = \frac{\mathbb{E}^{\mathbb{Q}^N}\left[e^{-r(\tau-t)}(S(\tau) - K)^+ \frac{1}{L(\tau)} \mid \mathcal{F}_t\right]}{\frac{1}{L(t)}} = L(t)\mathbb{E}^{\mathbb{Q}^N}\left[e^{-r(\tau-t)}(S(\tau) - K)^+ \frac{1}{L(\tau)} \mid \mathcal{F}_t\right]$ . Recalling the definition of  $L$  we obtain  $\mathbb{E}\left[e^{-r(\tau-t)}(S(\tau) - K)^+ \mid \mathcal{F}_t\right] = \frac{S(t)e^{-(r-q)t}}{S(0)}\mathbb{E}^{\mathbb{Q}^N}\left[e^{-r(\tau-t)}(S(\tau) - K)^+ \mid \mathcal{F}_t\right] = \frac{S(0)}{S(\tau)e^{-(r-q)\tau}}\mathbb{E}^{\mathbb{Q}^N}\left[e^{-q(\tau-t)}\left(S(0) - \frac{S(0)K}{S(\tau)}\right)^+ \mid \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}^N}\left[e^{-q(\tau-t)}\left(S(t) - \frac{S(t)K}{S(\tau)}\right)^+ \mid \mathcal{F}_t\right]$ .

Passing to the essential supremum over all stopping times  $t \leq \tau \leq T$  we get that

$$c(t) = \text{ess sup}_{t \leq \tau \leq T} \mathbb{E}^{\mathbb{Q}^N}\left[e^{-q(\tau-t)}\left(S(t) - \frac{S(t)K}{S(\tau)}\right)^+ \mid \mathcal{F}_t\right] \quad (10)$$

The argument of the  $\mathcal{F}_t$ -expectation under  $\mathbb{Q}^N$  in Equation (10) is the payoff at  $\tau \geq t$  of an American put option with maturity  $T$ , interest rate  $r_{put} = q$ , strike  $K_{put} = S(t) = x$  on the asset  $S_{put}(s) = \frac{xK}{S(s)}$ . Applying Ito formula we derive the stochastic differential of  $S_{put}$  for any  $s \geq t$ :  $dS_{put}(s) = xK \cdot d\left(\frac{1}{S(s)}\right) = \frac{xK}{S(s)} \cdot \left(- (r - q) ds - \sqrt{v(s)}dW_1(s) + v(s) ds\right) = S_{put}(s) \cdot \left(- (r - q) ds - \sqrt{v(s)}dW_1(s) + v(s) ds\right)$ . From Equation (9) we substitute  $dW_1(s) = \sqrt{v(s)}ds + dW_1^N(s)$  and get  $\frac{dS_{put}(s)}{S_{put}(s)} = - (r - q) ds - \sqrt{v(s)}$ .

$(\sqrt{v(s)} ds + dW_1^N(s)) + v(s) ds = (q - r) ds - \sqrt{v(s)} dW_1^N(s)$ . Therefore the underlying of the American put option is driven under the evaluation measure  $\mathbb{Q}^N$  by the ‘‘Heston dynamics’’ of type (1)

$$\frac{dS_{put}(s)}{S_{put}(s)} = (q - r) ds - \sqrt{v(s)} dW_1^N(s),$$

with  $r_{put} = q$  and  $q_{put} = r$ . We verify now that the volatility term follows a dynamics of the same type of Equation (2). By Girsanov theorem (9),  $v(s)$  is driven by  $dv(s) = k(\bar{v} - v(s)) ds + \xi \sqrt{v(s)} (\rho dW_1(s) + \sqrt{1 - \rho^2} dW_2(s)) = (k - \xi\rho) \left( \frac{k\bar{v}}{k - \xi\rho} - v(s) \right) ds + \xi \sqrt{v(s)} \left( \rho dW_1^N(s) + \sqrt{1 - \rho^2} dW_2^N(s) \right)$ . Since  $d\widehat{W}_1^N(s) = -dW_1^N(s)$  defines a standard  $\mathbb{Q}^N$ -Brownian motion that is  $\mathbb{Q}^N$ -independent of  $W_2^N$  we have that

$$\begin{aligned} \frac{dS_{put}(s)}{S_{put}(s)} &= (q - r) ds + \sqrt{v(s)} d\widehat{W}_1^N(s) \text{ and} \\ dv(s) &= (k - \xi\rho) \left( \frac{k\bar{v}}{k - \xi\rho} - v(s) \right) ds + \xi \sqrt{v(s)} \left( (-\rho) d\widehat{W}_1^N(s) + \sqrt{1 - \rho^2} dW_2^N(s) \right). \end{aligned}$$

Therefore under  $\mathbb{Q}^N$  the underlying of the put option  $S_{put}$  follows an Heston dynamics with  $S_{put}(t) = K$ ,  $\bar{v}_{put} = \frac{k\bar{v}}{k - \xi\rho}$ ,  $k_{put} = (k - \xi\rho)$ ,  $\xi_{put} = \xi$ , and  $\rho_{put} = -\rho$ , as in Definition 1. We conclude that Equation (10) can be rewritten as  $c(t) = \text{ess sup}_{t \leq \tau \leq T} \mathbb{E}^{\mathbb{Q}^N} [e^{-q(\tau-t)} (K_{put} - S_{put}(\tau))^+ | \mathcal{F}_t] = p(t, S_{put}(t), v_{put}(t); r_{put}, q_{put}, \bar{v}_{put}, k_{put}, \xi_{put}, \rho_{put}, K_{put})$ , which is (4).

To prove (8), take a  $\beta > 0$  such that  $\widehat{K}_{put} = \frac{x}{\beta}$ , is an unconstrained strike for the put option, and let  $\widehat{x}_{put} = \frac{S_{put}(t)}{\beta} = \frac{K}{\beta}$ . The remaining parameters for the symmetric put are  $r_{put}, q_{put}, \bar{v}_{put}, k_{put}, \xi_{put}, \rho_{put}, K_{put}$  as before: for simplicity we omit them. By formula (7)  $c(t, x, \dots, K) = p(t, K, \dots, x) = \beta p\left(t, \frac{K}{\beta}, \dots, \frac{x}{\beta}\right) = \beta \cdot p\left(t, \widehat{x}_{put}, \dots, \widehat{K}_{put}\right)$ , where the second equality follows from the homogeneity property of the put option. Since  $\beta = \frac{x}{\widehat{K}_{put}} = \frac{K}{\widehat{x}_{put}}$ , writing  $\beta = \sqrt{\beta \cdot \beta} = \sqrt{\frac{x}{\widehat{K}_{put}} \cdot \frac{K}{\widehat{x}_{put}}}$ , we arrive at (8). ■

In the constant volatility framework, the optimal exercise policy for an American call option is the first time the underlying asset exceeds the critical price. The critical price is time-varying, and its graph in the plane  $(t, S)$  separating the continuation region from the immediate exercise region is called the free boundary. In the Heston model, the free boundary is a surface in the space  $(t, S, v)$ . The free boundary of the American call option is linked to the free boundary of the symmetric American put option via the following theorem:

**Theorem 3 (The free boundary)** *Consider the American call option defined in (3) whose value at time  $t \in [0; T]$  is denoted with  $c(t) = c(t, S(t), v(t); r, q, \bar{v}, k, \xi, \rho, K) = c(t, x, v; \dots, K)$ . The free boundary for the American call option at  $t$  and  $v = v(t)$  is*

$$fb(t, v) = \inf \{x \geq 0 : c(t, x, v; \dots, K) = (x - K)^+\}.$$

Let  $\widehat{K}_{put} = 1$  and consider the symmetric American put option where  $\widehat{x}_{put}$  replaces  $S_{put}(t)$  and  $\widehat{K}_{put} = 1$  replaces  $K_{put}$  in Definition 1 as for (8). The free boundary of the symmetric American put option  $v_{put}(t, \widehat{x}_{put}, v_{put}; r_{put}, q_{put}, \bar{v}_{put}, k_{put}, \xi_{put}, \rho_{put}, 1) = v_{put}(t, \widehat{x}_{put}, v_{put}; \dots, 1)$  is

$$fb_{put}(t, v_{put}) = \sup \left\{ \widehat{x}_{put} \geq 0 : v_{put}(t, \widehat{x}_{put}, v_{put}; \dots, 1) = (1 - \widehat{x}_{put})^+ \right\}.$$

Then

$$fb(t, v) = K \cdot fb_{put}(t, v_{put})$$

**Proof.** The parameters  $x, K$ , and  $\widehat{x}_{put}$  are constrained by the equality  $\frac{x}{K} = \frac{1}{\widehat{x}_{put}}$ . It follows that  $fb(t, v) = \inf \left\{ \frac{K}{\widehat{x}_{put}} \geq 0 : \sqrt{xK} \frac{p(t, \widehat{x}_{put}, v_{put}; \dots, 1)}{\sqrt{\widehat{x}_{put} \widehat{K}_{put}}} = \left( \frac{K}{\widehat{x}_{put}} - K \right)^+ \right\} = K \sup \left\{ \widehat{x}_{put} \geq 0 : \sqrt{xK} \frac{v_{put}(t, \widehat{x}_{put}, v_{put}; \dots, 1)}{\sqrt{\widehat{x}_{put}}} = \frac{K}{\widehat{x}_{put}} (1 - \widehat{x}_{put})^+ \right\} = K \sup \left\{ \widehat{x}_{put} \geq 0 : \sqrt{\frac{K}{\widehat{x}_{put}}} K \frac{v_{put}(t, \widehat{x}_{put}, v_{put}; \dots, 1)}{\sqrt{\widehat{x}_{put}}} = \frac{K}{\widehat{x}_{put}} (1 - \widehat{x}_{put})^+ \right\}$ , since  $x = \frac{K}{\widehat{x}_{put}}$ . Therefore  $fb(t, v) = K \cdot \sup \left\{ \widehat{x}_{put} \geq 0 : v_{put}(t, \widehat{x}_{put}, v_{put}; \dots, 1) = (1 - \widehat{x}_{put})^+ \right\} = K \cdot fb_{put}(t, \widehat{x}_{put}, v_{put}; \dots, 1)$ . ■

### 3 Conclusions

In this paper we provide a simple proof for the symmetry between American call and put options in the Heston stochastic volatility framework, relying on the change of numeraire technique. We supply also the link between the free boundaries of the symmetric American options.

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