Endogenous Uncertainty and Market Volatility

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by

Mordecai Kurz and Maurizio Motolese
Stanford University
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For correspondence:

Mordecai Kurz, Joan Kenney Professor of Economics
Department of Economics
Serra Street at Galvez
Stanford University
Stanford, CA. 94305-6072
mordecai@leland.stanford.edu
http://www.stanford.edu/~mordecai/

Summary.

In this paper we advance the theory that the distribution of beliefs in the market is *the most important propagation mechanism of economic volatility*. Our model is based on the theory of Rational Beliefs (RB) and Rational Belief Equilibrium (RBE) developed by Kurz [1994], [1997]. The paper argues that most of the observed volatility in financial markets is generated by the beliefs of the agents and the diverse market puzzles which are examined in this paper, such as the equity premium puzzle, are all driven by the structure of market expectations. To make the case in support of our view, we present a single RBE model with which we study a list of phenomena that have been viewed as "anomalies" in financial markets. The model is able to predict the correct order of magnitude of:

(i) the long term mean and standard deviation of the price\dividend ratio;
(ii) the long term mean and standard deviation of the risky rate of return on equities;
(iii) the long term mean and standard deviation of the riskless rate;
(iv) the long term mean equity premium.

In addition, the model predicts

(v) the GARCH property of risky asset returns;
(vi) the observed pattern of the predictability of long returns on assets;
(vii) the Forward Discount Bias in foreign exchange markets.

The common economic explanation for these phenomena is the existence of heterogenous agents with diverse but correlated beliefs. Given such diversity, some agents are optimistic and some pessimistic about future capital gains. We develop a simple model which allows agents to be in these two states of belief but the identity of the optimists and the pessimists fluctuates over time since any agent may be in these two states of belief at any date. In this model there is a unique parameterization under which the model makes all the above predictions *simultaneously*. Any parameter choice in this small neighborhood requires the optimists to be in the majority but the rationality of belief conditions of the RBE require the pessimists to have a higher intensity level. This higher intensity has a decisive effect on the market: it increases the demand for riskless assets, decreases the equilibrium riskless rate and increases the equity premium. In simple terms, the large equity premium and the lower equilibrium riskless rate are the result of the fact that at any moment of time there are agents who hold extreme pessimistic beliefs and they have a relatively stronger impact on the market. The relative impact of these two groups of agents who are, at any date, in the two states of belief is a direct consequence of *the rationality of belief conditions and in that sense it is unique to an RBE.* The paper also studies the effect of correlation of beliefs among investors. It shows that the main effect of such correlation is on the dynamic patterns of asset prices and returns and is hence important for studying such phenomena as stochastic volatility.

JEL Classification Numbers: D5, D84, G12.
Key Words: Rational Expectations, Rational Beliefs, Rational Belief Equilibrium (RBE), Endogenous Uncertainty, states of belief, stock price, discount bond, equity premium, market volatility, GARCH, Forward Discount Bias.

Note: The RBE model developed in this paper and the associated programs used to compute it are available to the public on Mordecai Kurz's web page at http://www.stanford.edu/~mordecai/

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Mordecai Kurz and Maurizio Motolese
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The theory of Rational Belief Equilibrium (in short, RBE; see Kurz [1994], [1997]) was developed with the view of studying the effects of the beliefs of economic agents on the volatility of economic variables and on social risk. Application of the theory to various markets were reported by Kurz and Beltratti [1997], Kurz and Schneider [1996], Kurz [1997a], [1997b], [1998], Nielsen [1997], and Wu and Guo [1998]. Some of these papers advanced the idea that the "equity premium puzzle" due to Mehra and Prescott [1985] (in short, M&P [1985]), can be resolved by the theory of Rational Beliefs (in short, RB). This is in contrast with recent attempts to resolve the equity premium puzzle by the use of a "habit forming" utility function (see Abel [1999], Campbell and Cochrane [1999] and Constantinides [1990])[2].

Most of the work on the equity premium concentrated on the analysis of the premium as an isolated phenomenon and in this context researchers usually examine the riskless rate, the risky rate and their second moments. The fact is that there are other volatility phenomena which have puzzled students of financial markets. To our knowledge participants in the equity premium debate have not suggested that the question of excess stock price volatility raised by Shiller

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[2] Other approaches to the equity premium puzzle were reported by Brennan and Xia [1998], Epstein and Zin [1990], Cecchetti, Lam and Mark [1990],[1993], Heaton and Lucas [1996], Mankiw [1986], Reitz [1988], Weil [1989] and many others.

[1981] is intimately related to the equity premium puzzle. Indeed, the "calibration" literature has mostly ignored the comparison between the model's volatility of stock prices and the historical record although such a comparison is one more test of the model's ability to explain the data. Also, financial markets exhibit other dynamical patterns for which standard models have failed to give a satisfactory explanation. Examples include stochastic volatility, the GARCH phenomenon in asset returns, the "Forward Discount Bias" in foreign exchange markets and the various "smile curves" in derivative asset pricing. It is clear that the validity of any equilibrium theory should not be judged by its ability to match any specific market statistic but rather by the range and depth of market phenomena and "anomalies" that the theory is capable of explaining.

This paper is not another study of the equity premium. In scope it is broader than previous papers on RBE and it presents a *unified* framework for the study of market volatility. It argues that *the distribution of beliefs is the central volatility propagation mechanism in the market*. It thus claims that most volatility in financial markets is expectationally generated and that many market "anomalies" such as the excess volatility of asset prices and foreign exchange rates, the equity premium puzzle, the GARCH pattern of asset returns and the Forward Discount Bias in foreign exchange markets *are all driven by the structure of heterogenous beliefs in the market.* In support of this unified view of the volatility propagation mechanism we present a single, relatively simple, market model and show by simulations that the RBE of the model is able to explain a wide range of these phenomena. First, it predicts the correct order of magnitude of (i) the first and second moments of the price\dividend ratio, (ii) the first and second moments of the risky return, (iii) the first and second moments of the riskless rate and hence of the equity premium. Second, the time series of stock returns exhibit a GARCH phenomenon, and third, an extension of the model to a two countries model exhibits a high volatility of the foreign exchange rate and a "forward discount bias" in its foreign exchange market. Technically speaking our model drastically generalizes the approaches of Kurz and Beltratti [1997], Kurz and Schneider [1996] and Kurz [1997b] with a simplified parameter space which satisfies anonymity in accord with Kurz [1998]. In addition, important components of the paper are the integrated economic interpretations of the results and the implied testable implications of the theory. These are important conclusions of the paper and we shall devote Section 4 below to highlight them.

Before turning to the description of our OLG model in Section 2, we discuss the merits of the heterogenous belief paradigm and present a brief review of the RB theory. Section 3 presents the simulation results and Section 4 integrates our conclusions.

1. **A Paradigm of Heterogenous Beliefs**

The theory of RBE is motivated by the observation that intelligent economic agents hold diverse beliefs even when there is no difference in the information at their disposal. Indeed, the center of their disagreement is the diverse *interpretations* of this information. By adopting axioms which allow rational agents to hold diverse beliefs, our theory does not lead, in general, to a Rational Expectations Equilibrium (in short, REE). However, an REE is also an RBE since the theory of RBE is an extension of the theory of REE.

The search for an extension of the theory of REE is motivated by the widespread dissatisfaction with the REE model (see Sargent [1993]). This results from the fact that central implications of the REE theory are contradicted by the empirical evidence in many areas of Economics and Finance. One recent line of research has focused on alternative choice criteria such as Robustness (see Anderson, Hansen and Sargent [1999] or Hansen, Sargent and Tallarini [1999]) even if such criteria imply behavior that is viewed as irrational by the REE or the RBE theories. Instead of adopting such a “Bounded Rationality” approach, we follow Kurz [1974], [1994], [1998], the papers included in Kurz [1997], Garmaise [1998], Motolese [1998], Nielsen [1997] and Wu and Guo [1998] in studying the heterogeneity of beliefs as the key propagation mechanism of market volatility. We note the existence of ample empirical evidence to support the view that equally informed agents interpret differently the same information (see Frankel and Froot [1990], Frankel and Rose [1995], Kandel and Pearson [1995], Takagi [1991] and others). Moreover, the heterogeneity of beliefs persists regardless of the amount of past information available implying that agents use different probability beliefs which they condition on *the same public information*. Before proceeding with our development we discuss the alternative view which holds that the observed diversity of beliefs originates in the heterogeneity of information.

1a. Diversity of Information or Diversity of Beliefs?

Starting with financial markets, there is a significant REE based literature which holds that the observed heterogeneity of beliefs does not arise from the heterogeneity of prior probabilities but, rather, from the diversity of private information (see, for example, Kyle [1985], Wang [1993] [1994] and references there). This explanation is unsatisfactory from both theoretical as well as empirical perspectives. Theoretical considerations lead to the information revelation of REE (e.g. Grossman [1981], Radner [1979]) which implies that prices make public all private information and therefore the introduction of asymmetric information, *by itself*, is not sufficient. It simply transforms the problem into other paradoxes. These include the problem of explaining why under

REE agents trade at all (e.g. Milgrom and Stokey [1982]); why asset prices fluctuate more than could be explained by "fundamentals" (e.g. Shiller [1981]), indirectly generating an equity premium puzzle (see M&P [1985]); and why any resources are ever used for the production of information (see Grossman and Stiglitz [1980]). To explain the observed heterogeneity and avoid such paradoxes researchers had, therefore, to introduce some additional assumptions of market structure that would remove the information revelation property of REE. Consider the explicit introduction of uninformed noise traders or general "noise" which leads to a theory of "noisy rational expectations equilibrium." This is a negation of REE since the assumption of noise in prices explicitly introduces *irrationality* of uninformed traders into the theory. This artificial assumption of irrationality is then the one driving all the important conclusions.

Empirical considerations also suggest that the assumption of asymmetric information in financial markets is unsatisfactory. We have already noted the ample empirical evidence in support of the opposite view that equally informed agents interpret differently the same information. This leads to a simple question: is there any empirical evidence to support the assumption of widespread use of private information in financial markets? We think that the evidence is not there. Observe first that since it is illegal to trade on inside private information, are we to conclude that the high volatility of financial markets is a result of widespread and persistent *criminal* behavior by traders? The majority of firms whose securities are traded on public exchanges are monitored carefully by a professional community of regulators, brokers and financial managers. Hence there is ample evidence that, on the whole, the majority of firms avoid letting any market participant either obtain private information or trade on it if he has such information. Furthermore, since modern financial markets are dominated by large institutions with vast resources, elementary competitive behavior in the search for information should lead us to conclude that all will possess essentially the same information.

Turning to REE in macroeconomics, the critique of the Keynesian theory was associated with the rejection of the wage and price rigidities implicit in the Keynesian system. However, under the classical assumptions of price and wage flexibility and market clearing in equilibrium, a REE cannot explain the observed cyclical correlation among economic variables such as the positive correlation between the price level and aggregate output (the "inflation - output tradeoff"). In order to explain the data, the New Classical Theory introduced asymmetric information which became the driving force of the theory. More specifically, agents are assumed to be *unable to obtain information which is public in other parts of the economy*. This rigidity in the transmission of public information leads to diverse models of Phelpsian or Lucasian "islands" (see Phelps [1970] and Lucas [1973]). The important *Lucas supply curve* (Lucas [1973]) is then deduced from the artificial assumption that firms are not able to observe the aggregate price level which is normally an observable variable. Hence, the heterogeneity in the beliefs of agents is caused by an artificial informational assumptions. As proposed by Lucas [1982], the models are "rigged" to generate the heterogeneity which induces the desired empirical implication.

The arguments presented here highlight the fact that the implications of the common belief assumption in an REE - by itself - are counter factual. The crucial empirical implications of these models are generated by an *added set of assumptions*. These include asymmetry of information, rigidity in the transmission of information and outright irrational behavior of some agents. These added assumptions introduce "stories" with questionable theoretical and empirical foundations but *these questionable assumptions are the ones which drive the results!*

The theory of RBE provides the foundation for the use of heterogenous beliefs *as a substitute for the "additional" artificial assumptions which drive most of the REE based models*. The theory suggests that the paradigm of diverse beliefs is entirely plausible and generates a powerful propagation mechanism of social risk and market fluctuations. The paradigm is based on the hypothesis that agents do not know the true structural relationships in the economy. Consequently, *rational agents may have diverse beliefs about what they do not know.* The empirical evidence for these *two* components of our approach is substantial and this alternative paradigm offers useful economic insights with which we can answer difficult economic questions.

1b. Rational Beliefs

The central assumption of the RB theory (due to Kurz [1994]) is that economic agents do not know the exact demand or supply functions, equilibrium maps or true probability laws induced by an equilibrium. In the terminology of Kurz [1994], agents do not possess "*structural knowledge*" (as distinct from "empirical knowledge" or "information"). Lacking structural knowledge, rational agents develop their own theories about the underlying structure and use the available data to test the validity of such theories.

The second assumption which distinguishes the RB theory from the Bayesian perspective is that at each date an economic agent has at his disposal a vast amount of data about the past performance of the economy. Hence, instead of accepting Savage's [1954] axioms on preferences which imply an arbitrary prior belief, the agent's central point of reference is the empirical distribution derived from the frequency at which events occurred in the past. The availability of a large amount of past data may lead one to speculate that learning by the agents may cause heterogeneity of beliefs to vanish. Work along this line was inspired by martingale convergence theorems and lead to a heated debate under the heading of "Bayes Consistency" (see Diaconis and Freedman [1986]). The conclusion of the debate is that the convergence of the posterior to the true distribution is a rare occurrence. In two influential papers, Freedman [1963], [1965] shows that even when the statistician has a controlled experiment and the data is generated i.i.d., the convergence of the posterior is a rare event if the true distribution is complex. The problem is compounded in learning situations in markets where the data is generated by an unknown process which may be non-stationary and the convergence of the posterior to the true probability is even a less likely event (see Feldman [1991]).

The third component of the RB theory is the observation that the economic life of any agent is short relative to the clock at which new data arrives. Thus, let $x_t \in X \subseteq \mathbb{R}^N$ be a vector of the N observables in the economy and let $x = (x_0 , x_1 , x_2 , ..)$ be the history of the observed data from date 0 to infinity. Define the history from date t on by $x^t = (x_t, x_{t+1}, x_{t+2},\ldots)$ and hence $x^0 = x$. The history *up to date* t is defined by $I_t = (x_0, x_1, x_2,\ldots, x_t)$. An agent forms a belief at t about the probability of events *in the future*. The theory assumes that t is very large so that an agent can construct the empirical distribution generated by the history. The agent's life L is the span of time in which he makes decisions and L is very short relative to t. By this we mean that investors, fund managers, CEO of a corporations etc. make decisions over periods ranging from 10 to 20 years which is very short relative to t. An agent's belief may be correct or not but the little data - $(x_t , x_{t+1} , x_{t+2} ,\ldots, x_{t+L})$- generated by the economy during his own economic life is much too small to provide a reliable test of his theory since most economic data flow at the very slow annual or quarterly rates. This is particularly true if the agent believes that the data is generated by a non-stationary process and he has little data on each regime which may be in place during an interval of time. One must then conclude that the *rationality* of a belief Q cannot be judged by the usual Bayesian learning criterion which insists on the compatibility of Q with the limit of the data *in the future*. Instead, the RB theory defines the rationality of belief in terms of the compatibility of that the belief *with the empirical distribution of past data.*

To explain the rationality conditions of the RB theory we start with the definition of *Statistical Stability*. Let X^∞ be the space of infinite sequences x and $\mathcal{B}(X^\infty)$ be the Borel σ-field of X^∞. For each finite dimensional set (cylinder set) $B \in \mathcal{B}(X^\infty)$ define the expression

$$m_n(B)(x) = \frac{1}{n}\sum_{k=0}^{n-1} 1_B(x^k) = \left\{\begin{array}{l}\text{The relative frequency that B occurred} \\ \text{among n observations since date 0}\end{array}\right\}$$

where

$$1_B(y) = \begin{cases} 1 & \text{if } y \in B \\ 0 & \text{if } y \notin B \end{cases}.$$

Although the set B is finite, it can be a very complicated set. For example

$$B = \left\{\begin{array}{l}\text{price of commodity 1 today} \le \$1, \text{price of commodity 6 next year} \ge \$3, \\ 2 \le \text{quantity of commodity 14 consumed five years later} \le 5\end{array}\right\}.$$

<u>Definition 1</u>: (Property 1) A stochastic process $\{x_t, t = 0, 1, 2, ...\}$ with true probability Π on $(X^\infty, \mathcal{B}(X^\infty))$ is said to be *Statistically Stable* if for each finite dimensional set $B \in \mathcal{B}(X^\infty)$

$$\lim_{n\to\infty} m_n(B)(x) = m(B)(x) \text{ exists} \qquad \Pi \text{ a.e.}$$

We assume that the data is generated by a stable process but to simplify the exposition, we also assume that the process is *Ergodic*. This carries the implication that

$$m(B)(x) = m(B) \text{ independent of } x, \quad \Pi \text{ a.e.}$$

Our agents do not know Π and start by computing the empirical frequencies. Although they have only finite data, we assume that they actually know the limits m(B) in Definition 1 for all cylinders. Again, this assumption is made for simplification[3]. Technical work implies that the agents deduce from the data a full probability measure m on the space $(X^\infty, \mathcal{B}(X^\infty))$. Indeed, we know (see Kurz [1994]) that

(i) m is unique.

(i) m is stationary and hence is called "*the stationary measure of* Π."

Since m is obtained from the data, there is no disagreement among the agents about it. The probability m is *their common empirical knowledge*.

Our agents who do not know the true probability Π discover from the data the probability m induced by the dynamics under Π. If the economy is stationary then $m = \Pi$ *but agents could not know this fact*. What are then the restrictions which the knowledge of m places on the beliefs of rational agents? To answer this question we introduce the concept of Weak Asymptotic Mean Stationary (WAMS) Dynamical System. For an event $B \in \mathcal{B}(X^\infty)$ let $B = B^{(0)}$ and

$$B^{(k)} = \text{the event } B \text{ occurring k periods later} \equiv \{x \mid x^k \in B\}.$$

<u>Definition 2</u>: (Property 2) A system $\{x_t, t = 1, 2, 3, ...\}$ with probability Π on $(X^\infty, \mathcal{B}(X^\infty))$ is said to be WAMS if for each finite dimensional (cylinder) event $B \in \mathcal{B}(X^\infty)$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Pi(B^{(k)}) = m^\Pi(B) \quad \text{exists.}$$

The collection of $m^\Pi(B)$ induces a unique probability m^Π on $(X^\infty, \mathcal{B}(X^\infty))$ which is

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[3]The assumption that the limit in Definition 1 is known to the agents is made to avoid the complexity of an approximation theory. Without this assumption the diversity of beliefs would be increased due to the diverse opinions about the approximation. The assumption of Ergodicity is also not needed and is not made in Kurz [1994].

stationary. For *any* WAMS probability Q, we shall then use the notation m^Q to denote the probability on $(X^\infty, \mathcal{B}(X^\infty))$ induced by Property 2 which Q must satisfy. The central result of the RB theory can now be stated.

Theorem 1: Properties 1 and 2 are equivalent and $m(A) = m^{\Pi}(A)$ for *all events* $A \in \mathcal{B}(X^\infty)$.

Agents compute m from the data and Theorem 1 leads to a natural definition of what it means for a probability belief Q to be "compatible with the data", which m represents:

Definition 3: A probability belief Q is said to be *compatible with the observed data m* if

(i) Q is a WAMS probability on $(X^\infty, \mathcal{B}(X^\infty))$,

(ii) $m^Q(A) = m(A)$ for all events $A \in \mathcal{B}(X^\infty)$.

Equality (ii) is the key implication of Theorem 1. Now consider a *rational* agent who holds the belief Q. If Q was the true probability it would have to be compatible with the data m since $m^Q \neq m$ would constitute a proof that Q is not the truth. This leads to our final definition.

Definition 4: A probability belief Q is said to be *a Rational Belief (RB) relative to m* if Q is compatible with the known data m and satisfies the rationality conditions

(1) $$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} Q(A^{(k)}) = m^Q(A) = m(A) \quad \text{for all cylinder sets } A \in \mathcal{B}((X)^\infty).$$

The rationality conditions in (1) are the central restrictions of the RB theory. To see an example for these conditions, let t be the current period and consider the following random variables:

$Z^{(t+k)}$ = the annualized rate of return on the S&P500 stock index k periods after date t.

Consider the expectation $E_Q[Z^{(t+k)} \mid I_t]$. The time average of $Z^{(t+k)}$ is approximately 8% and if some agents adopt m as their belief, $Z^{(t+k)}$ is essentially unpredictable and $E_m[Z^{(t+k)} \mid I_t] = 8\%$ for all k. Under a rational belief Q such that $Q \neq m$, the rationality conditions require that

$$\frac{1}{n}\sum_{k=0}^{n-1} E_Q[Z^{(t+k)} \mid I_t] \cong 8\% \quad \text{for large n.}$$

It is a fact that in any experiment in which agents are asked to predict Z^{t+1} there is *a wide distribution* of forecasts but the distribution of $Z^{(t+k)}$ narrows down as k increases.

The immediate implication of (1) is that under RB agents may disagree about probabilities of short term events but not about long term averages. Observe that (i) Π is a Rational Belief and hence REE is an RBE; (ii) m is an RB although it is possible that $m \neq \Pi$; (iii) RB Q and the true Π may disagree on timing or sequencing; (iv) RB Q and the true Π may put different probabilities on important rare events; (v) RB Q allow optimism/pessimism relative to m.

Several Important Observations

(i) We have already noted that if the agents believed that the economy is a stationary dynamical system they would all adopt the universal beliefs Q = m. Hence, the crucial feature of an RBE is that it is an equilibrium theory in which agents believe that the economy is a non-stationary process and *their main uncertainty is about the structure of the process*. Any disagreement at date t revolves around unknown parameters, such as the mean value function, of the stochastic process of prices and quantities which prevails in the economy at that time.

(ii) When agents disagree, the distribution of beliefs affects excess demands functions and hence in an economy in which *the beliefs of H agents matters* the equilibrium map takes the general form

$$p_t = \Phi(I_t, Q_t^1, Q_t^2, Q_t^3, \ldots, Q_t^H)$$

where $(Q_t^1, Q_t^2, Q_t^3, \ldots, Q_t^H)$ are date t conditional probabilities of the H agents. In such equilibria the distribution of beliefs is a propagation mechanism of price volatility. On the more fundamental level, the RBE theory rejects the formulation of uncertainty as being only an *exogenous* phenomenon. It insists that economic uncertainty and fluctuations have a large endogenous component which is propagated *within* the economy rather than being caused by exogenous shocks. Following Kurz [1974] we call it *Endogenous Uncertainty*. Our paper claims that *this uncertainty is the dominant form of uncertainty in our society*. Endogenous Uncertainty is, indirectly, the uncertainty about the beliefs and actions of other agents. Hence we define:

Definition 5: *Endogenous Uncertainty* is that component of price volatility which is caused by the distribution of beliefs.

(iii) Our final observation is simple but important: *disagreement among agents in an RBE imply that their conditional probabilities must fluctuate over time*. To be specific, consider a finite state Markov economy and suppose the conditional probabilities Q_t^k of agent k are represented by J possible Markov matrices G_j^k for j = 1, 2,...J and m is represented by a single Markov matrix Γ. If agents disagree then their beliefs are not represented by Γ and hence $G_j^k \neq \Gamma$. The rationality conditions imply that the *average* forecast of k, each of which is made by some G_j^k, must be the same as the forecast made under Γ. It would be irrational for agent k to use only *one* matrix, say G_1^k, since the mean forecast under G_1^k is not equal to the forecast under Γ. Hence *in a world with disagreement, rational agents must use varying matrices over time*.

Any application of the RBE theory requires a simplification of the very general rationality conditions in (1). We address this issue in the context of the model developed later.

2. The RBE of an OLG Stock Market Economy

Our stock market economy is a relatively standard two-agent, OLG, economy with a single, homogenous, consumption good. Each agent lives two periods, the first when he is "young" and the second when he is "old." Each young agent is a replica of the old agent who preceded him, where the term "replica" refers to *utilities* and *beliefs*, and hence this is a model of two infinitely lived "dynasties" denoted by k = 1, 2. One can think of k as the identity of the pair of young and old agents of the dynasty at date t. We often use the term "agent k" but the context should make it clear whether the agent is the young or the old of dynasty k. Only young agents receive an endowment Ω_t^k, t = 1, 2, ... of the single consumption good. We view Ω_t^k as the labor income of agent k at date t and the stochastic processes $\{\Omega_t^k, t = 1, 2, ...\}$ for k = 1, 2 will be specified below. Additional net output is supplied by a firm which produces exogenously, as in Lucas [1978], the strictly positive profit process $\{D_t , t = 1, 2, ...\}$ with no input. These net outputs are paid out to the shareholders of the firm as dividends at the date at which the output is produced. The ownership shares are traded on a stock market and their aggregate supply is 1.

The economy has three markets: (i) a market for the consumption good with an aggregate supply equaling the total endowment plus total dividends, (ii) a stock market with a total supply of 1, and (iii) a market for a zero net supply, short term riskless debt instrument which we call a "bill". Since the stochastic growth rate of dividends is Markovian with two states, *the economy has a complete financial structure* in the sense that the number of financial instruments equals the number of exogenous states. The financial sector is initiated at date 1 by distributing the unit supply of shares among the old of that date. Our notation is as follows: for k = 1, 2

C_t^{1k} - consumption of k when young at t;

C^{2k}_{t+1} - consumption of k when old at t + 1 (implying that the agent was born at t);

$d_{t+1} = \dfrac{D_{t+1}}{D_t}$ - the random growth rate of dividends;

θ^k_t - amount of stock purchases by young agent k at t;

B^k_t - amount of one period bill purchased by young agent k at t;

Ω^k_t - endowment of young agent k at t;

P_t - the price of the common stock at t and $p_t = \dfrac{P_t}{D_t}$ - the price/dividend ratio at t;

q_t - the price of a one period bill at t. This is a discount price;

2.1 The Markov Equilibrium Concept.

We normalize prices by using consumption as a numeraire. Given this, the optimization problem of agent k has the following structure at all t: given I_t - the history of all observables

(2a) $$\underset{(C^{1k}_t, \theta^k_t, B^k_t, C^{2k}_{t+1})}{\text{Max}} \; E_{Q^k_t}\left\{ u^k(C^{1k}_t, C^{2k}_{t+1}) | \; I_t \right\}$$

subject to

(2b) $$C^{1k}_t + P_t\theta^k_t + q_t B^k_t = \Omega^k_t$$

(2c) $$C^{2k}_{t+1} = \theta^k_t(P_{t+1} + D_{t+1}) + B^k_t.$$

Q^k is a probability belief of agent k on all future variables which he does not know. To enable us to compute equilibria we take the utility function agent k to be

$$u^k(C^{1k}_t, C^{2k}_{t+1}) = \frac{1}{1-\gamma_k}(C^{1k}_t)^{1-\gamma_k} + \frac{\beta_k}{1-\gamma_k}(C^{2k}_{t+1})^{1-\gamma_k}, \quad \gamma_k > 0, \quad 0 < \beta_k < 1.$$

With this specification the Euler equations for agent k are

(3a) $$-P_t(C^{1k}_t)^{-\gamma_k} + \beta_k E_{Q^k_t}((C^{2k}_{t+1})^{-\gamma_k}(P_{t+1} + D_{t+1}) | I_t) = 0$$

(3b) $$-q_t(C^{1k}_t)^{-\gamma_k} + \beta_k E_{Q^k_t}((C^{2k}_{t+1})^{-\gamma_k} | I_t) = 0.$$

The parameters of the model are selected to equal the values of the corresponding estimates for the real economy. We thus aim to calibrate the model and test its ability to generate solutions which are of the same order of magnitudes as the observed endogenous variables.

(2.1a) *__The dividend process and the equilibrium map__*. The simulation model is Markovian with an exogenous process of dividends as specified in M&P [1985]. It takes the following form

$$D_{t+1} = D_t d_{t+1}. \tag{4}$$

where $\{d_t, t = 1, 2, ...\}$ is a stationary and ergodic Markov process. The state space of the process is $J_D = \{d^H, d^L\}$ with $d^H = 1.054$ and $d^L = .982$ and a transition matrix

$$\begin{bmatrix} \phi, 1 - \phi \\ 1 - \phi, \phi \end{bmatrix} \tag{5}$$

with $\phi = .43$. Hence, over time agents experience a secular rise of dividends and it is therefore convenient to focus on growth rates. To do that let

$\omega_t^k = \dfrac{\Omega_t^k}{D_t}$ is the endowment/dividend ratio of agent k at date t;

$b_t^k = \dfrac{B_t^k}{D_t}$ is the bill/dividend ratio of agent k at date t;

$c_t^{1k} = \dfrac{C_t^{1k}}{D_t}$ is the ratio of consumption when young to aggregate capital income;

$c_{t+1}^{2k} = \dfrac{C_{t+1}^{2k}}{D_{t+1}}$ is the ratio of consumption when old to aggregate capital income;

We assume that $\omega_t^k = \omega^k$ for $k = 1, 2$ are constant, $\nu = \omega^1 + \omega^2$ then $(\Omega_t^1 + \Omega_t^2) = \nu D_t$ for all t. We do not consider random endowments in part because production and labor markets are not the focus of this paper and in part because of computational feasibility. Now divide (2b) by D_t, (2c) by D_{t+1}, equation (3a) by $D_t^{1-\gamma_k}$ and equation (3b) by $D_t^{-\gamma_k}$ to obtain, for k = 1, 2

(6a) $$c_t^{1k} = -p_t\theta_t^k - q_t b_t^k + \omega^k,$$

(6b) $$c_{t+1}^{2k} = \theta_t^k(p_{t+1}+1) + \frac{b_t^k}{d_{t+1}},$$

(6c) $$-p_t(c_t^{1k})^{-\gamma_k} + \beta_k E_{Q_t^k}((c_{t+1}^{2k}d_{t+1})^{-\gamma_k}(p_{t+1}+1)d_{t+1}) \mid I_t) = 0,$$

(6d) $$-q_t(c_t^{1k})^{-\gamma_k} + \beta_k E_{Q_t^k}((c_{t+1}^{2k}d_{t+1})^{-\gamma_k} \mid I_t) = 0 .$$

(6a) - (6d) imply demand functions which take the general time dependent form, for k = 1, 2

(7a) $$b_t^k = b_t^k(p_t, q_t, d_t, I_t)$$

(7b) $$\theta_t^k = \theta_t^k(p_t, q_t, d_t, I_t)$$

Equilibrium requires the market clearing conditions

(7c) $$\theta_t^1 + \theta_t^2 = 1$$

(7d) $$b_t^1 + b_t^2 = 0;$$

The equilibrium in (7a)-(7d) depends upon the beliefs of the agents and upon what they condition on. In this paper we restrict our attention to stable Markov equilibria.

Definition 6: Beliefs (Q^1,Q^2) and a stochastic process $\{(p_t, q_t, (\theta_t^1, b_t^1), (\theta_t^2, b_t^2), d_t), t = 1, 2, ...\}$ with initial portfolios $((\theta_0^1, b_0^1 = 0), (\theta_0^2, b_0^2 = 0))$ and with true probability Π constitute a *stable Markov competitive equilibrium* if

(i) $(p_t, q_t, \theta_t^1, b_t^1, \theta_t^2, b_t^2, d_t)$ satisfy conditions (7a) - (7d) at all dates t;

(ii) $b_t^k = b_t^k(p_t, q_t, d_t, I_t)$ and $\theta_t^k = \theta_t^k(p_t, q_t, d_t, I_t)$ are independent of the history I_{t-1}.

(iii) (Q^1, Q^2, Π) are stable measures in the sense of definition 1.

It follows from (7a)-(7d) that the price process $\{(p_t, q_t), t = 1, 2, ...\}$ of a stable Markov equilibrium is defined by an equilibrium sequence of maps

$$\text{(7e)} \qquad \begin{bmatrix} p_t \\ q_t \end{bmatrix} = \Phi_t(d_t)$$

where the time dependence of the equilibrium map represents the potential time dependence of the beliefs of agents. In an REE, $Q^1 = Q^2 = \Pi$ where Π is the true probability induced by (5) and by the stationary equilibrium map (7e). In an REE states of beliefs of agents have no effect on prices and all demand functions are time independent. We review this particular equilibrium first.

(2.1b) *Stable Rational expectations equilibria.* In a *stable* Markov REE $Q^1 = Q^2 = \Pi$ and, deduced from (5), the probabilities of $(p_{t+1}, q_{t+1}, d_{t+1})$ in (6c) - (6d) are conditioned only on the realized value of d_t. It then follows that the demand functions must take the form

$$\text{(8a)} \qquad b_t^k = b^k(p_t, q_t, d_t)$$

$$\text{(8b)} \qquad \theta_t^k = \theta^k(p_t, q_t, d_t).$$

(8a)-(8b) and the market clearing conditions (7a)-(7b) imply a stationary equilibrium map

$$\text{(9a)} \qquad \begin{bmatrix} p_t \\ q_t \end{bmatrix} = \hat{\Phi}(d_t).$$

In the special case postulated in (5) the growth rate of dividends takes two values. In this case Equation (9a) shows that a stable Markov equilibrium is, in fact, a stationary equilibrium with two prices and two optimal portfolios.

2.2 The Structure of Beliefs and Rational Belief Equilibrium (RBE)

Our brief review of the theory of RBE implies that under the Markov assumption and given an economy with two agents, the equilibrium map takes the form

(7e′) $$\begin{bmatrix} p_t \\ q_t \end{bmatrix} = \Phi(d_t, Q_t^1, Q_t^2)$$

and hence the joint distribution of (d_t, Q_t^1, Q_t^2) will define the true equilibrium probability Π. We aim, in the rest of this paper, to construct a Markov equilibrium with a Markov stationary measure m and since it will be a finite state process, it follows that the stationary measure m is fully characterized by a transition matrix of prices which we denote by Γ.

Our earlier review has also shown that if agent k adopts an RB Q^k which is different from the stationary measure m, he must believe that the economic environment is non-stationarity. However, a non-stationary probability of a Markov process with finite number of states is fully characterized by a time varying sequence of Markov matrixes $(F_1^k, F_2^k, F_3^k, ...)$ specifying that at date t the process is defined by the transition matrix F_t^k. If the set of *possible* Markov matrices is $\{G_1, G_2, ..., G_M\}$ the non-stationary probability Q^k is represented by a time function g_t^k taking values in {1, 2, ... , M} which defines the sequence of transition matrices $F_t^k = G^k_{g_t^k}$, t = 1 , 2 ,.... For a Markov Rational Belief Q^k which is represented by such a sequence of matrices it turns out that the complicating factor is the determination of the *rationality of belief* conditions which the sequence of matrices must satisfy. The method of "assessment variables" is our tractable tool to describe the non stationarity of such RB and to develop the rationality conditions.

(2.2a) *Assessment Variables and the State Space*. Assessment variables are privately perceived random variables $\{y_t^k$, t = 1, 2, ...} for k = 1, 2 *generated by the agents*. We assume here that $y_t^k \in Y = \{0, 1\}$ and Q^k is defined as a probability on the *joint* process $\{(p_t, q_t, d_t, y_t^k), t = 1, 2, ...\}$ which is a Markov process, and the *effective belief* is $Q^k_{y^k}$, the conditional probability of Q^k given

the sequence y^k. Hence, under Q^1 and Q^2, the assessment variables are jointly distributed with the real market variables and hence their distribution may depend upon observed economic variables. Hence, conditioning on y_t^k alters the predictions, by agent k, of future economic variables.

From an economic perspective, assessment variables are privately perceived parameters indicating *how an agent interprets current information* and hence are tools for the description of stable and non-stationary processes (see Kurz and Schneider [1996] pages 491-495 and Nielsen [1996] on this point). These variables have purely subjective meaning and should not be taken to be objective and transferable "information". An agent does not know the values of the assessment variables of other agents and *would not understands their meaning even if he knew them*. Their impact on the real economy arises from the fact that conditioning on them by the agents *alters their probability beliefs* about future values of economic variables. We explain this point now.

(i) Assessment variables and the equilibrium map. In (6c) - (6d) agent k uses the probability of $(p_{t+1}, q_{t+1}, d_{t+1}, y_{t+1}^k)$ *conditional* on (p_t, q_t, d_t, y_t^k). It follows from our Markov assumptions that the demands of agent k for stocks and bills are time-independent functions of the form

(10a) $$b_t^k = b^k(p_t, q_t, d_t, y_t^k)$$

(10b) $$\theta_t^k = \theta^k(p_t, q_t, d_t, y_t^k).$$

Consequently we can write the market clearing conditions as

(10c) $$\theta^1(p_t, q_t, d_t, y_t^1) + \theta^2(p_t, q_t, d_t, y_t^2) = 1$$

(10d) $$b^1(p_t, q_t, d_t, y_t^1) + b^2(p_t, q_t, d_t, y_t^2) = 0.$$

The system (10c)-(10d) implies that the equilibrium map of this economy takes the form

(11) $$\begin{bmatrix} p_t \\ q_t \end{bmatrix} = \Phi^*(d_t, y_t^1, y_t^2).$$

The equilibrium map (11) reveals that prices are determined by the exogenous shock d_t and by the "*state of belief*" represented by (y_t^1, y_t^2).

To clarify the role of the assessment variables in (11) note that in (10a) - (10b) we specified that the demand functions are not time dependent and hence the assessment variables (y_t^1, y_t^2) completely determine the conditional probabilities (Q_t^1, Q_t^2). From the assumption of a Markov equilibrium it follows that y_t^k determines completely the transition matrix from (p_t, q_t) to (p_{t+1}, q_{t+1}) which is used by agent k at date t. Moreover, $y_t^k \in \{0, 1\}$ implies that the agent has *at most two Markov matrices* and at each date the value taken by his assessment variable determines which of these two the agent uses. We shall later define the beliefs in such a manner that "1" is a state of *optimism relative to m* while "0" is a state of *pessimism relative to m*.

(11) implies that there are at most 8 distinct price vectors (p_t, q_t) that may be observed, corresponding to the 8 combinations of (d_t, y_t^1, y_t^2). The Markov assumption implies that the *true* equilibrium transition probability from the 8 prices (p_t, q_t) to the 8 prices (p_{t+1}, q_{t+1}) is determined by the transition probabilities from (d_t, y_t^1, y_t^2) to $(d_{t+1}, y_{t+1}^1, y_{t+1}^2)$. For simplicity we select the joint process $\{(d_t, y_t^1, y_t^2), t = 1, 2, ...\}$ to be a stationary Markov process with a transition matrix Γ. This implies[4] that the true equilibrium process of prices has a *fixed* transition probability from (p_t, q_t) to (p_{t+1}, q_{t+1}) defined by Γ. The agents discover Γ from the data and use it to construct the stationary measure. However, they do not know that Γ is the true equilibrium

[4] The choice of the equilibrium dynamics being generated by a fixed, stationary, matrix is a matter of convenience and simplicity in this paper. In general the process $\{(d_t, y_t^1, y_t^2), t = 1, 2, ...\}$ could have been selected to be any stable process with a Markov stationary measure induced by the empirical distribution. In such a case the fixed transition matrix Γ would characterize only the *stationary measure* of the equilibrium dynamics rather than be the matrix of the *true* probability of the equilibrium dynamics of prices. For simplicity we avoid this additional complication.

probability and for this reason they form RB relative to Γ. Indeed, the fact that they form RB using (y_t^1, y_t^2) *is what rationalizes* Γ *to be the equilibrium probability of the implied RBE.*

(ii) Assessment variables and the state space. The state space for prices is $(J_D \times Y \times Y)^\infty$ but one may also consider the state space to be S^∞ where S is the index set S = {1, 2, ..., 8}. We can then define a new equilibrium map Φ between the *indices* of prices (i.e., a number from 1 to 8 rather than by t) and the states of dividends and assessment variables by

$$\text{(12)} \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = \Phi \begin{bmatrix} d_1 = d^H, \ y_1^1 = 1, \ y_1^2 = 1 \\ d_2 = d^H, \ y_2^1 = 1, \ y_2^2 = 0 \\ d_3 = d^H, \ y_3^1 = 0, \ y_3^2 = 1 \\ d_4 = d^H, \ y_4^1 = 0, \ y_4^2 = 0 \\ d_5 = d^L, \ y_5^1 = 1, \ y_5^2 = 1 \\ d_6 = d^L, \ y_6^1 = 1, \ y_6^2 = 0 \\ d_7 = d^L, \ y_7^1 = 0, \ y_7^2 = 1 \\ d_8 = d^L, \ y_8^1 = 0, \ y_8^2 = 0 \end{bmatrix}.$$

d^H is the "high dividends" and d^L is the "low dividends" states. (11)-(12) highlight the idea of Endogenous Uncertainty showing that the volatility of prices depends upon the states of belief.

(iii) The exogenous variables. A belief Q^k was defined as a probability on the space of sequences $\{(p_t, q_t, d_t, y_t^k), t = 1, 2, ...\}$ but we have also seen that Q^k was defined by a selection of transition matrices from (p_t, q_t) to (p_{t+1}, q_{t+1}). This appears to ignore the probability of the exogenous variable d_t. To see that this is not so, consider the map Φ in (12). The probability of d^H equals the probability of prices {1, 2, 3 ,4} and the probability of d^L equals the probability of prices {5, 6, 7, 8}. Thus, the distribution of d_t is defined by the *partition* of the state space. Agents discover this partition and for simplicity *we have assumed that they believe it to be the truth*.[5]

[5] By studying the relationship between prices and d_t agents discover the partition in the long run data. This happens to be the truth at all dates but an agent may not believe it. Instead he may form a rational belief about this variable. The issue has little significance to our study and we chose the simpler assumption.

(2.2b) *The Rationality of Belief Conditions*. A *Rational Belief Equilibrium* (RBE) is a stable equilibrium in which agents hold RB. To establish an RBE we need to construct an equilibrium process which is stable, from which agents compute the empirical distribution. The rationality conditions require the beliefs of the agents to imply a stationary measure which is equal to the one computed from the data. We explain that in our model, these concepts are significantly simplified.

Although beliefs Q^k are probabilities on sequences $\{(p_t, q_t, d_t, y_t^k), t = 1, 2, \ldots\}$ the probabilities used in (6c)-(6d) are $Q^k((\bullet)| y^k)$, which are Q^k *conditional on* y^k *of agent k*. Hence, the rationality of belief conditions must apply to $Q^k((\bullet)| y^k)$. These conditions require that

(i) $Q^k((\bullet)| y^k)$ is a stable measure;

(ii) the stationary measure of $Q^k((\bullet)| y^k)$ equals the probability on sequences induced by Γ.

Since $Q^k((\bullet)| y^k)$ is represented by two Markov matrices, we need to specify the joint distribution of (p_t, q_t, y_t^k) and the rationality conditions which are consistent with these matrices. To that end we use the "Conditional Stability Theorem" (Kurz and Schneider [1996] pp. 492 - 494). It says that if the probability Q^k of the *joint* process$\{(p_t, q_t, y_t^k), t = 1, 2, ...\}$ is stable, then a *conditional* probability $Q^k((\bullet)| y^k)$ of Q^k on y_t^k, is a stable probability on $\{(p_t, q_t), t = 1, 2, \ldots\}$ and the stationary measure of $Q^k((\bullet)| y^k)$ is the *marginal* of Q^k on (p_t, q_t) obtained by integrating on y^k.

To simplify the above procedure assume that the *marginal* distribution of Q^k on y_t^k is i.i.d. with $Q^k\{y_t^k = 1\} = \alpha_k$ for k = 1, 2. By the Conditional Stability Theorem Q^1 and Q^2 are characterized by two pairs of matrices, (F_1, F_2) for agent 1 and (G_1, G_2) for agent 2, such that:

(13a) *Q^1 for agent 1:* adopt F_1 if $y_t^1 = 1$ *Q^2 for agent 2*: adopt G_1 if $y_t^2 = 1$

adopt F_2 if $y_t^1 = 0$ adopt G_2 if $y_t^2 = 0$.

(13b) $$\alpha_1 F_1 + (1 - \alpha_1) F_2 = \Gamma \quad , \quad \alpha_2 G_1 + (1 - \alpha_2) G_2 = \Gamma .$$

An intuitive interpretation starts by noting that these rational agents believe that the price-dividend process is not stationary and their beliefs are parameterized by (y_t^1, y_t^2). (13b) implies that the sequence of matrices which they adopt is compatible with the true price process (i.e. generating the same empirical distribution) which is a Markov process with transition Γ. α_1 is the frequency at which agent 1 uses matrix F_1 and α_2 is the frequency at which agent 2 uses matrix G_1. This leads to a formal definition of the equilibrium which we construct below:

Definition 7: *A Markov Rational Belief Equilibrium* (RBE) is a Markov Competitive Equilibrium in which the (Q^1, Q^2) are defined by (13a) and satisfy the rationality conditions (13b)[6].

We observe that an RBE is simply *an incomplete Radner [1972] equilibrium* in which (i) the agents do not know the equilibrium map, (ii) the agents hold Rational Belief rather than arbitrary beliefs, and (iii) the state space is endogenously expanded to include the states of belief.

(2.2c) The Stationary Measure. We now assemble the conditions which Γ must satisfy. The event $d_t = d^H$ is equivalent to the event that prices {1, 2, 3, 4} occur and the event $d_t = d^L$ is

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[6] An RBE is not a sunspot equilibrium. We note first that assessment variables are not observables, the joint distribution of (y_t^1, y_t^2) is not known and the agents do not have the structural knowledge needed to invert an equilibrium map. Hence an RBE cannot be a fully revealing equilibrium. But the issue is deeper. Even if we assumed, for the sake of discussion, that (y_t^1, y_t^2) is observable and that the agents know the equilibrium map, the RBE is not a sunspot equilibrium because the "sunspot" variable (y_t^1, y_t^2) *alters the real economy*. That is, for the RBE to be a sunspot equilibrium the different values of (y_t^1, y_t^2) must be associated with exactly the same fundamentals of the economy. This is not the case in an RBE. For example, in the "sunspot" state (1, 1) the von-Neumann Morgenstern preferences of the agents are defined by the probabilities (F_1, G_1) while in the "sunspot" state (0, 1) they are defined by (F_2, G_1). These changes in the fundamentals of the economy show that *endogenous uncertainty has real effects on the economy* induced by the states of belief (y_t^1, y_t^2). We also note that given each state of belief (y_t^1, y_t^2), the equilibrium at date t is unique.

equivalent to the event that prices {5, 6, 7, 8} occur. Now, (5) specified the dividend process and since prices in the RBE are functions of (d_t, y_t^1, y_t^2), the marginal of Γ with respect to d_t must equal the dividend matrix in (5). Similarly with respect to (y_t^1 , y_t^2) : the marginal of Γ with respect to each of the y_t^k must be i.i.d. with probability α_k.

Note that each agent has a marginal distribution on his own assessment variable, hence the i.i.d. requirement on the marginals of Γ with respect to each y_t^k is a consistency condition between the market observations and what each agent perceives. No such conditions apply to the *joint* distribution of the assessments. This joint effect of the assessment variables, as distinct from the individually perceived effect, is that part of Γ which describes *the externalities of beliefs in the market performance*. These externalities cannot be found in each of the marginal distributions of the beliefs Q^k. They are, however, reflected in the equilibrium process. They describe the interaction among the agents resulting from communication in society, the manner in which agents influence each other and how the real economic variables (i.e. dividends) affect this interaction.

In sum, the matrix Γ must satisfy the following:

(14a) the marginal on y_t^k is i.i.d. with $P\{y_t^k = 1\} = \alpha_k$ for $k = 1, 2$;

(14b) the marginal on d_t is Markov as specified by the dividend process (5);

The family of matrices which satisfy these conditions is limited. Our main criterion for selecting the following matrix Γ from this family is simplicity and flexibility in parameterization.

(15)
$$\Gamma = \begin{bmatrix} \phi A, & (1-\phi)A \\ (1-\phi)B, & \phi B \end{bmatrix}$$

where A and B are 4×4 matrices which are characterized by the 10 parameters α_1, α_2, and (a, b) where $a = (a_1, a_2, a_3, a_4)$, $b = (b_1, b_2, b_3, b_4)$:

$$(16) \qquad A = \begin{bmatrix} a_1, & \alpha_1 - a_1, & \alpha_2 - a_1, & 1 + a_1 - \alpha_1 - \alpha_2 \\ a_2, & \alpha_1 - a_2, & \alpha_2 - a_2, & 1 + a_2 - \alpha_1 - \alpha_2 \\ a_3, & \alpha_1 - a_3, & \alpha_2 - a_3, & 1 + a_3 - \alpha_1 - \alpha_2 \\ a_4, & \alpha_1 - a_4, & \alpha_2 - a_4, & 1 + a_4 - \alpha_1 - \alpha_2 \end{bmatrix}, \quad B = \begin{bmatrix} b_1, & \alpha_1 - b_1, & \alpha_2 - b_1, & 1 + b_1 - \alpha_1 - \alpha_2 \\ b_2, & \alpha_1 - b_2, & \alpha_2 - b_2, & 1 + b_2 - \alpha_1 - \alpha_2 \\ b_3, & \alpha_1 - b_3, & \alpha_2 - b_3, & 1 + b_3 - \alpha_1 - \alpha_2 \\ b_4, & \alpha_1 - b_4, & \alpha_2 - b_4, & 1 + b_4 - \alpha_1 - \alpha_2 \end{bmatrix}$$

If $A \neq B$ then the distribution of $(y^1_{t+1},\ y^2_{t+1})$ depends upon d_t. (16) implies that $P\{y^k_t = 1\} = \alpha_k$ for k = 1, 2 as required in (14a). Note, however, that although each process $\{y^k_t, t = 1, 2, ...\}$ for k = 1, 2 is very simple, the joint process $\{(d_t, y_t^1, y_t^2), t = 1, 2,...\}$ may be complex: it permits correlation among the three central variables and these effects are important. If we set $\alpha_1 = \alpha_2 = .5$ and $a_i = b_i = .25$ for i = 1, 2, 3, 4 then all correlations are eliminated. In this case the stationary distribution $(\pi_1, \pi_2, ... ,\pi_8)$ implied in (16) is $\pi_i = .125$ for all i. If, in addition, the agents adopt the stationary measure as their belief (i.e. $F_1 = G_1 = \Gamma$), then we have exactly an REE.

For simplicity of parameterization, we set in almost all simulations the parameter values $\alpha_1 = \alpha_2 = .57$, $a = (a_1 \neq a_2 = a_3 = a_4)$ and $b = (b_1 \neq b_2 = b_3 = b_4)$. It is clear, however, that there are natural restrictions which the parameters must satisfy and these will be discussed later. We specify now the family of RBE which we use in the simulations.

(2.2d) _*Rational Beliefs: the Family of Optimism/Pessimism Beliefs*_. We use two parameters λ and μ to select two pairs of matrices: (F_1, F_2) of agent 1 and (G_1, G_2) of agent 2 satisfying the rationality conditions (13b). To do that denote the row vectors of A and B by:

$$A^j = (a_j,\ \alpha_1 - a_j,\ \alpha_2 - a_j,\ 1 + a_j - (\alpha_1 + \alpha_2)) \qquad j = 1, 2, 3, 4$$

$$B^j = (b_j,\ \alpha_1 - b_j,\ \alpha_2 - b_j,\ 1 + b_j - (\alpha_1 + \alpha_2)) \qquad j = 1, 2, 3, 4.$$

With this notation we define the 4 matrix functions of a real number z as follows:

$$(17)\quad A_1(z) = \begin{bmatrix} zA^1 \\ zA^2 \\ zA^3 \\ zA^4 \end{bmatrix},\ A_2(z) = \begin{bmatrix} (1-\phi z)A^1 \\ (1-\phi z)A^2 \\ (1-\phi z)A^3 \\ (1-\phi z)A^4 \end{bmatrix},\ B_1(z) = \begin{bmatrix} zB^1 \\ zB^2 \\ zB^3 \\ zB^4 \end{bmatrix},\ B_2(z) = \begin{bmatrix} (1-(1-\phi)z)B^1 \\ (1-(1-\phi)z)B^2 \\ (1-(1-\phi)z)B^3 \\ (1-(1-\phi)z)B^4 \end{bmatrix}.$$

Finally we define

$$(18)\qquad F_1 = \begin{bmatrix} \phi A_1(\lambda)\ , & A_2(\lambda) \\ (1-\phi)B_1(\lambda), & B_2(\lambda) \end{bmatrix} \qquad G_1 = \begin{bmatrix} \phi A_1(\mu)\ , & A_2(\mu) \\ (1-\phi)B_1(\mu), & B_2(\mu) \end{bmatrix}.$$

By the rationality conditions (13b), $F_2 = \frac{1}{1-\alpha_1}(\Gamma - \alpha_1 F_1)$, $G_2 = \frac{1}{1-\alpha_2}(\Gamma - \alpha_2 G_1)$.

To motivate this construction, note that the parameters λ and μ are proportional revisions of the conditional probabilities of states (1, 2, 3, 4) and (5, 6, 7, 8) *relative to* Γ. $\lambda > 1$ and $\mu > 1$ imply *increased* probabilities of states (1, 2, 3, 4) in matrix F_1 of agent 1 and matrix G_1 of agent 2 where the first four prices are associated with the states when $d_t = d^H$. Since these are the states of the higher prices, $\lambda > 1$ implies that agent 1 is optimistic about high prices at $t + 1$. Similarly for $\mu > 1$. In all simulations below we set $\lambda \geq 1$ and $\mu \geq 1$ and hence the assessment variables y_t^k have a simple interpretation: when $y_t^k = 1$ agent k is optimistic (*relative to* Γ) at t about high prices at $t + 1$. The special case of $\lambda = 1$, $\mu = 1$ and $a_i = b_i = .25$ identifies an REE. Finally, it turns out that the concepts of "agreement" and "disagreement" between the agents are useful. We then say that *the agents agree if* $y_t^1 = y_t^2$ *and disagree if* $y_t^1 \neq y_t^2$.

(2.2e) <u>*Stable Markov RBE*</u>. Conditions (6c)-(6d) require am agent to forecast prices (p_{t+1}, q_{t+1}). A rational agent can perform this task since there is a set of 8 prices $\{(p_{t+1}, q_{t+1})\}$ that can occur at $t + 1$ and all agents know this set from past history. We use the index set S to define equilibrium consumptions, portfolios and prices in terms of the transitions from state s to state j

in the set S. To state the equilibrium conditions in these terms denote by $Q^k(j\,|\,s\,,y_s^k)$ agent k's probability of price state j given price state s and the value of y_s^k which he perceives at state s but *under the competitive assumption that k knows neither the map (12) nor the fact that he influences prices*. Conditions (6) - (7) are then restated for k = 1, 2 and j, s = 1, 2, ..., 8:

(19a) $$c_s^{1k} = \omega^k - \theta_s^k p_s - b_s^k q_s$$

(19b) $$c_{sj}^{2k} = \theta_s^k (p_j + 1) + \frac{b_s^k}{d_j}$$

(19c) $$-(c_s^{1k})^{-\gamma_k} p_s + \beta_k \sum_{j=1}^{8} (c_{sj}^{2k} d_j)^{-\gamma_k} (p_j + 1) d_j Q^k(j|s, y_s^k) = 0$$

(19d) $$-(c_s^{1k})^{-\gamma_k} q_s + \beta_k \sum_{j=1}^{8} (c_{sj}^{2k} d_j)^{-\gamma_k} Q^k(j|s, y_s^k) = 0\,.$$

(19e) $$\theta_s^1 + \theta_s^2 = 1 \quad \text{for all } s$$

(19f) $$b_s^1 + b_s^2 = 0 \quad \text{for all } s.$$

The constructed Markov RBE is then a solution of equations (19a)-(19f) for feasible parameters.

(i) The assumption of Competitive Equilibrium. (19a)-(19f) shows that in our RBE agents are assumed to act competitively. However, this assumption is subtle and needs to be clarified since from the equilibrium map (12) it is also clear that when we have a finite number of agents, the belief of each agent has an effect on equilibrium prices. Competitive behavior means that an agent is required to disregard his effect on prices. To see what this entails observe from (12) that agent 1 uses matrix F_1 when $y_t^1 = 1$ but during those dates only prices {1, 2, 5, 6} are realized contrary to the belief of agent 1 that all prices *could* be realized. If the agent is allowed to take into account his effect on prices, he would use this information in formulating his belief and that would be part of the rationality of belief conditions. In fact, the assumption of Competitive Behavior means that the agent is asked to ignores these facts. For example, it would be a violation of Competitive Behavior if the belief Q^1 of agent 1 incorporated the forecast at t that with

probability 1 only prices {1, 2, 5, 6} would be realized at date t + 1 if $y^1_{t+1} = 1$.

(ii) Feasibility conditions on the model parameters. There is a large number of restrictions on the model parameters. The parameters (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), α_1, α_2 and ϕ must satisfy

(20)
$$\begin{aligned} &a_i\,,\, b_i \le \alpha_1 < 1 \quad \text{for} \quad i = 1, 2, 3, 4 \\ &a_i\,,\, b_i \le \alpha_2 < 1 \quad \text{for} \quad i = 1, 2, 3, 4 \\ &0 \le \phi \le 1\,. \end{aligned}$$

Similarly, the selection of $(\lambda\,,\,\mu)$ is restricted by 10 inequality constraints:

(21)
$$\begin{array}{llll} \lambda \le \dfrac{1}{\phi} & \mu \le \dfrac{1}{\phi} & \lambda \le \dfrac{1}{1-\phi} & \mu \le \dfrac{1}{1-\phi} \\ \lambda \le \dfrac{1}{\alpha_1} & \mu \le \dfrac{1}{\alpha_2} & \lambda \ge \dfrac{\alpha_1 + \phi - 1}{\phi\,\alpha_1} & \mu \ge \dfrac{\alpha_2 + \phi - 1}{\phi\,\alpha_2} \end{array}$$
$$\lambda \ge \frac{\alpha_1 - \phi}{(1-\phi)\,\alpha_1} \qquad \mu \ge \frac{\alpha_2 - \phi}{(1-\phi)\,\alpha_2}\,.$$

The RBE's in our simulations are solutions of the 48 equations (19a) - (19f) in prices and quantities which satisfy the feasibility constraints (20) - (21). The particular family of RBE which we shall study in the simulations is drastically simplified by the following criteria:

(i) a *single* intensity variable $\lambda = \lambda_s = \mu = \mu_s$ for all $s = 1, 2, \ldots, 8$[7];

(ii) $\alpha = \alpha_1 = \alpha_2$;

(iii) $a = (c_1\,,\, c_2\,,\, c_2\,,\, c_2)$ and $b = (c_1\,,\, c_2\,,\, c_2\,,\, c_2)$ for two parameters $(c_1\,,\, c_2)$;

---

[7] It is useful to comment on the difference between our treatment and Kurz and Beltratti [1997]. The earlier paper introduced the vectors $\lambda = (\lambda_1, \lambda_2, .,.,., \lambda_8)$ and $\mu = (\mu_1, \mu_2, .,.,., \mu_8)$ and allowed the agents to select 16 parameters. This means that the agents select $(\lambda_s\,,\,\mu_s)$ to vary with prices. Since there are only two agents in the model they obviously have an effect on prices but are required to act competitively and ignore such effect. By allowing the agents to select different $(\lambda_s\,,\,\mu_s)$ for different s, we permit the agents to take into account their effect on prices and thus violate the condition of anonymity (see Kurz [1998] for more details). Our procedure of selecting a single parameter $\lambda = \mu$ ensures anonymity.

(iv) γ_1 and γ_2 in the realistic interval [2.5, 3.5] (see Kurz and Beltratti [1997], pp. 290 - 294) ; β_1 and β_2 are in the empirically plausible interval [.85 , .95].

We thus observe that once we select realistic values for γ_1, γ_2, β_1 and β_2, equilibrium is *uniquely* determined by the four parameters : λ, α , c_1 , c_2 .

(iii) The model unit of time. The selection of discount rates around .9 implies that we calibrate our model to annual data and we later discuss extensively whether the OLG environment is appropriate for our purposes. Here we note that since in our model the "state of belief" of an agent changes over time, a unit of time of one year implies that the fluctuations in the market distribution of beliefs is a rather slow process. The model thus aims to capture the changes over relatively long market swings of bull and bear markets. These changes do not occur over short periods of days or weeks but rather over years.

3. Endogenous Uncertainty and Volatility: Simulation Results

We now examine the model's success in simulating the real economy. For this reason we first review the empirical averages of the key seven variables of the model in the U.S.:

p - the long term price/dividend ratio. M&P [1985] used the data base compiled by Shiller [1981] for 1889-1978. We used the updated version of the same Shiller's data base for 1889 - 1998 and estimated this variable to be 22.84;

σ_p - the standard deviation of the price/dividend ratio p. For the period 1889 - 1998 we estimated it to be 6.48 using the updated version of Shiller's [1981] data base;

R - the average risky return on equities was estimated by M&P [1985] to be 6.98%. Using the updated Shiller [1981] data for 1989 - 1998 our estimate is 8.34% suggesting that 6.98%

Endogenous Uncertainty and Market Volatility

(PART II: PAGES 31-66)

by

Mordecai Kurz and Maurizio Motolese
Stanford University
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For correspondence:

Mordecai Kurz, Joan Kenney Professor of Economics
Department of Economics
Serra Street at Galvez
Stanford University
Stanford, CA. 94305-6072
mordecai@leland.stanford.edu
http://www.stanford.edu/~mordecai/

is on the low side. We thus record the mean risky rate to be around 8.00%;

σ_R - the standard deviation of R was estimated by M&P [1985] to be 16.67%. Using the updated data for 1989 - 1998 our estimate is 18.08%;

r^F - M&P [1985] estimated the mean riskless interest rate to be .80% for 1889 - 1978 based on the 90 day treasury bill rate for 1931 - 1978. For 1889 - 1931 one may use various alternate securities. We offer no independent estimate and accept the view that the evidence places the mean riskless rate around 1.00%. Some evidence suggest that this low rate has prevailed mostly since the Great Depression and that prior to 1931 the rate was higher (see Siegel [1994]);

σ_{r^F} - the standard deviation of r^F was estimated by M&P [1985] to have an average of 5.67% during the period of 1889 - 1978;

ρ - the premium of equity return over the riskless rate. With the mean value of r^F set at around 1.00% and with the mean value of R estimated at 8.00%, we conclude that the empirical evidence places the mean equity premium around 7.00%;

3.1 The Scaling Problem of OLG Models

Before proceeding we resolve the issue of scaling an RBE. The problem arises from the fact that in an OLG economy agents live two periods and the young purchase from the old the capital stock of the economy using their labor endowment. Hence, equilibrium $p_t = \frac{P_t}{D_t}$ depends upon the labor endowment of the young. Since in the real economy it takes a generation for the capital stock to change ownership from the old to the young, an OLG model has a problem. If the labor income of the young is of the same order of magnitude as dividend income in the economy, the model could not generate a price\dividend ratio of 23. Hence, the young's labor endowment must be a multiple of D_t *in any one year* in order to attain an equilibrium price\dividend ratio equal to the historical average of about 23. To highlight the point, Table 1 below presents the equilibrium values of (p, r^F, R, ρ) in a sequence of REEs in which $\omega = \omega^1 = \omega^2$ take different

Table 1: REE Solutions for Varying Values of $\omega = \omega^1 = \omega^2$

	$\omega = 12$	$\omega = 14$	$\omega = 18$	$\omega = 22$	$\omega = 23$	$\omega = 24$	$\omega = 25$	$\omega = 26$
p	11.39	13.35	17.26	21.17	22.15	23.13	24.11	25.09
r^F	10.24%	8.93%	7.21%	6.13%	5.92%	5.72%	5.54%	5.38%
R	10.75%	9.44%	7.71%	6.62%	6.41%	6.21%	6.04%	5.87%
ρ	.51%	.51%	.50%	.49%	.49%	.49%	.49%	.49%

values. Other parameter choices in Table 1 are: $a_i = b_i = .25$ for all i; $\lambda = \mu = 1$; $\alpha_1 = \alpha_2 = .5$; $\gamma_1 = \gamma_2 = 3.25$; $\beta_1 = \beta_2 = .90$. Table 1 shows that variations in the endowment of the young acts as a scaling factor which determines the *level* of prices (p , q) and hence the average returns on securities. When ω reaches the range of 24, p is close to 23 and the mean risky return is 6.21%. Both means are close to the historical average. Our procedure is then to select the value of the endowment which results in a price\dividend ratio of approximately $p = 23$. For the RBE below, this value is $\omega = 26$. We view this as a pure scaling of the OLG model and in this sense the model does not reproduce the empirical evidence of $p = 23$, *it is scaled to that level.*

The problem of scaling the OLG model raises a deeper question, which may have already occurred to the reader: why should we expect the unrealistic OLG model to be an appropriate model for the study of market volatility? Since the discount rate is around 10%, *the unit of time is a year* and hence the model length of life of an agent is not an approximation of real human work life. Our answer to this question consists of two parts. First, note the fact that once the model was scaled, the predictions of the REE of the model reproduced very closely the predictions of the M& P [1985] model of infinitely lived agents. This analytical fact is the main reason why we have postponed the discussion of the question at hand until this point.

Turning to the second answer we note first that the Euler equations of an OLG agent are *exactly* the same as the Euler equations of an infinitely lived agent. The differences between these two sets of equations are the definitions of the budget constraint, their consumption and wealth. Since our model assumptions imply that aggregate consumption is proportional to total dividends, it follows that the growth rates of dividends and aggregate consumption are identically the same in the OLG and in the M& P [1985] infinite horizon models, and obey the exogenous Markov process defined by (5). Given this fact we need to assess why might one expect the models to have different predictions. If the equity premium and other "anomalies" in financial market are determined by real factors such as the horizon of the agents' optimization or by the life cycle saving patterns over the very long horizon, then the OLG model and the infinite horizon models would yield drastically different results. Alternatively, if the characteristics of market volatility under study are *essentially* driven by expectations, then, given the Markov structure of the model, it would not make any difference whether the agents trade many times over their own life-time or only once: *their expectations for each date at a time will drive the results*. Hence, if our theory is right and the phenomena under study are primarily driven by the distribution of beliefs, then the OLG model is an entirely useful model for the study of market volatility.

3.2 REE Simulations: Matching the M&P [1985] Results

Focusing on the case $\omega^1 = \omega^2 = 24$ we study further the REE defined by the parameter values : $a_i = b_i = .25$ for all i ; $\lambda = \mu = 1$; $\alpha_1 = \alpha_2 = .5$; $\gamma_1 = \gamma_2 = 3.25$ and $\beta_1 = \beta_2 = .90$. The results in Table 2 represent what M&P [1985] introduced as "the equity premium puzzle." In the

Table 2: REE Results

variable	REE	Empirical Record
p	23.13	23
σ_p	.069	6.48
R	6.21%	8.00%
σ_R	4.12%	18.08%
r^F	5.72%	1.00%
σ_{r^F}	.88%	5.67%
ρ	.49%	7.00%

narrow sense, the puzzle is the observation that the model prediction of ρ is .49% while the historical average is 7.00%. As in M& P [1985], this REE predicts reasonably well the mean rate of return on equities but errs in predicting a riskless rate of 5.72% when the empirical average is 1.00%. An inspection of Table 2 reveals that our OLG model reproduces very well the M&P [1985] results. Note, that the equity premium is not the only problem which the REE of the model presents; *all volatility measures in the table are low relative to the historical record*. The empirical value of σ_p is 94 times larger than the REE prediction, the value of σ_R is more than 4 times larger than the REE prediction and the value of σ_{r^F} is over 6 times larger than the model prediction. One objective of Kurz [1994] and of the papers in Kurz [1997] was to demonstrate that the theory of RBE points to *Endogenous Uncertainty* as the explanation of high volatility *propagated within the market* by the beliefs of the agents. Before exploring the volatility of the RBE we make two additional observations about Table 2:

(i) The model prediction of σ_p is downward biased (in the REE and RBE) since we assume, with M&P [1985], that dividends and GNP are proportional. Under the realistic assumption that profits are more volatile than GNP the model predictions of σ_p would become larger, but not large enough to alter the general result for the REE in Table 2.

(ii) The historical value of σ_{r^F} estimated to be 5.67% is downward biased relative to the model assumptions since during the second half of the 20th century monetary policy tended to stabilize short term rates. Such a policy is not in the model. Indeed, there is some evidence that before the Great Depression σ_{r^F} was substantially higher than 5.67%.

3.3 A Family of RBE with Optimists/Pessimists

We study a family of "optimists\pessimists" RBE. For this family we scale the model by selecting $\omega^1 = \omega^2 = 26$ and the four parameters which characterize this family are as follows:

(i) $\lambda = \mu = 1.7542$. Agent k is optimistic when $y_t^k = 1$ at which time he adjusts the probabilities of high prices at $t+1$ by factor of 1.7542 which is approximately the maximal feasible value.

(ii) $\alpha_1 = \alpha_2 = .57$. In the majority of dates (57%) an agent is optimistic but only 43% of the time he is pessimistic. In a large economy this means that the optimists are always in the *majority* but this also means that the pessimists are *more intense* in their outlook than the optimists;

(iii) Correlation of belief: a = b with $a_1 = b_1 = c_1 = .50$ and $a_i = b_i = c_2 = .14$ for i = 2, 3, 4. These parameters regulate the correlation of the states of beliefs of the agents. A random variable which sums up the state of belief is L_t, taking three values: (i) $L_t = 1$ if $y_t^1 = y_t^2 = 1$ is the state OO when both agents are optimistic; (ii) $L_t = 0$ if $y_t^1 = y_t^2 = 0$ is the state PP when both agents are pessimistic and (iii) $L_t = 2$ if $y_t^1 \neq y_t^2$ is the state DIS when the agents disagree. The stochastic process $\{ L_t , t = 1, 2, ...\}$ is a Markov process with the transition matrix:

	$(OO)_{t+1}$	$(PP)_{t+1}$	$(DIS)_{t+1}$
$(OO)_t$	.50	.36	.14
$(PP)_t$	.14	0	.86
$(DIS)_t$	.14	0	.86

The condition $a_i = b_i = c_2 = .14$ for $i = 2, 3, 4$ means that if at t the state is PP or DIS, at t +1 the state must be OO or DIS; PP *cannot occur* at t + 1. $a_1 = b_1 = c_1 = .50$ implies that total optimism at t can be followed by any state at t+1. Hence, the correlation takes the form:

(i) unanimous optimism at t may leads to any state of belief at t + 1;

(ii) unanimous pessimism or disagreement at t *prevents total pessimism* at t + 1.

The emergence of asymmetries in an otherwise symmetric economy is the key to understanding the structure of volatility. Note that the transition matrix of the states of belief is not symmetric and to understand the results caused by this matrix recall that price movements are caused by the joint movement of d_t and (y_t^1, y_t^2). Asymmetry in the transition matrix of the states of belief will thus translate into asymmetry in the dynamics of stock prices. We explore the exact pattern later.

We now report the simulation results for $\gamma = \gamma_1 = \gamma_2$ from 2.5 to 3.5 and $\beta = \beta_1 = \beta_2$ from .85 to .95 hence these results apply to a reasonably wide range of values of β and γ. Table 3 shows that for this parameterization, the model predicts well the historical record. Comparing the results in Tables 3 with the empirical record, we note that the mean risky return R is close to the average of 8.00% and its standard deviation σ_R is close to 18.08%; the riskless rate is within range of the average of 1.00%, and the equity premium is close to the average of 7.00%. The two moments σ_p and σ_{r^F} exhibit small deviations from the historical record: (i) the record of σ_p is 6.48% while the model predictions are *smaller*, around 2.5% - 3.4%, and (ii) the record of σ_{r^F}

Table 3: Results for RBE with optimists\pessimists

		γ = 2.5	γ = 2.75	γ = 3.00	γ = 3.25	γ = 3.50
β = .85	p	23.06	23.12	23.19	23.26	23.34
	σ_p	2.53	2.78	3.00	3.20	3.36
	R	7.85%	8.19%	8.51%	8.80%	9.05%
	σ_R	18.76%	20.63%	22.27%	23.69%	24.89%
	r^F	2.36%	1.79%	1.22%	.66%	.12%
	σ_{r^F}	14.62%	16.12%	17.41%	18.48%	19.35%
	ρ	5.49%	6.40%	7.29%	8.14%	8.93%
β = .90	p	23.36	23.38	23.43	23.48	23.54
	σ_p	2.52	2.77	2.99	3.18	3.34
	R	7.75%	8.08%	8.39%	8.68%	8.93%
	σ_R	18.48%	20.32%	21.94%	23.35%	24.55%
	r^F	2.37%	1.81%	1.25%	.71%	.18%
	σ_{r^F}	14.40%	15.89%	17.17%	18.24%	19.11%
	ρ	5.38%	6.27%	7.14%	7.97%	8.75%
β=.95	p	23.64	23.63	23.66	23.69	23.74
	σ_p	2.51	2.76	2.97	3.16	3.28
	R	7.65%	7.98%	8.29%	8.57%	8.61%
	σ_R	18.22%	20.03%	21.64%	23.03%	23.40%
	r^F	2.37%	1.83%	1.29%	.75%	.04%
	σ_{r^F}	14.20%	15.67%	16.95%	18.02%	19.04%
	ρ	5.28%	6.15%	7.00%	7.82%	8.57%

is 5.67% while the model predictions are *higher*, around 14.2%- 19.4%. Both predictions are of the correct *order of magnitudes* of the record, and the sizes and signs of the deviations are explained by the two model biases noted at the end of the previous section 3.2.

3.4 Interpreting the Propagation Mechanism of the RBE

Why is the RBE able to explain the data? Since an RBE has a propagation mechanism for market volatility, what is the *economic interpretations* of the parameter choices and why do they enable the model to explain the record? A skeptical view could suggest that even the tight space of parameters specified in Section (2.2e) is sufficiently large for the success of the model to be a chance event. We revisit this issue in Section 4 when we discuss some testable implications of the RBE theory and experimental methods of doing so. Here we focus on three main facts:

(A) We use only *two* key parameters to explains *a long array* of moments and *diverse* market volatility phenomena. Moreover, only a very small neighborhood of parameters enables the RBE's prediction to matches the empirical record;

(B) There is no other neighborhood of feasible parameters defining RBE which match the data.

(C) The values of ($\alpha_1 = \alpha_2$, $\lambda = \mu$) entail a simple economic interpretation: in the RBE optimists are in the majority but the intensity of the pessimists is stronger than the intensity of the optimists.

We start by the examination of a small neighborhood mentioned in fact (A). In Table 4 we report the results of varying the values of the parameters α_1 and α_2 over the range of .56, .57 and .58. The results are sensitive to variations in the values of $\alpha_1 = \alpha_2$ and of $\lambda = \mu$. Note that the restrictions on parameters in (20) - (21) show that small changes in α_1 and α_2, require change of other parameters in accord with the feasibility conditions. For example, if α_1 is changed from .57 to .58, the *maximal* value of λ which is feasible changes to 1.7241, the value of c_2 to .15 but the value of c_1 remains equal to .50. In all cases $\beta = .90$ and $\gamma = 3.25$.

Moving on to fact (B) we observe that there is no other neighborhood in the parameter space yielding predictions which are <u>*simultaneously*</u> close to the empirical record. Many model parameters generate volatility of prices and returns. However, as we move away from the small neighborhood under discussion, the model fails to generate *some* predictions which are essential components of the empirical record. The reason for this fact is that the model's ability to explain the record is the result of specific asymmetries, discussed below, which are unique to this neighborhood. Given parameter values of (β_k , γ_k) for k = 1, 2 at reasonably realistic values,

Table 4: Results for the Parameter Neighborhood

		$\alpha_1 = .56$	$\alpha_1 = .57$	$\alpha_1 = .58$
$\alpha_2 = .56$	p	23.56	23.59	23.97
	σ_p	2.69	2.79	2.11
	R	7.95%	8.09%	7.19%
	σ_R	19.86%	20.60%	15.84%
	r^F	3.32%	1.56%	1.41%
	σ_{r^F}	17.09%	16.42%	12.12%
	ρ	4.63%	6.53%	5.78%
$\alpha_2 = .57$	p	23.59	23.48	23.90
	σ_p	2.79	3.18	2.22
	R	8.09%	8.70%	7.31%
	σ_R	20.60%	23.35%	16.52%
	r^F	1.56%	.71%	.92%
	σ_{r^F}	16.42%	18.24%	12.70%
	ρ	6.53%	7.97%	6.39%
$\alpha_2 = .58$	p	23.97	23.90	23.87
	σ_p	2.11	2.22	1.94
	R	7.19%	7.31%	7.00%
	σ_R	15.84%	16.52%	14.35%
	r^F	1.41%	.92%	1.89%
	σ_{r^F}	12.12%	12.70%	10.96%
	ρ	5.78%	6.39%	5.11%

the small size of the neighborhood of the other four parameter is a striking fact! This suggests that the RBE offers a unique explanation of the historical record[8] which we now explore.

We thus turn to (C), which is the *economic interpretation* of the family of RBE defined by the parameters α, λ, c_1 and c_2. Recall that $\alpha_1 = \alpha_2 = .57$ means that both agents are optimistic in 57% of the dates. The second parameter is $\lambda = 1.7542$, which is approximately the maximal ratio

[8] In reference to the discussion in Section 1 we note that some who oppose the use of heterogenous beliefs have argued that such models allow for too many equilibria and hence give a researcher too much freedom in explaining any empirical phenomena. The conclusion here shows that this is a superficial argument since the isolation of a small neighborhood in the parameter space which is compatible with the historical record acts exactly as identification in any econometric model. The *existence* of such a set of parameters arises directly from the rationality conditions of the RBE and this fact provides added support for the the RBE theory.

by which an optimist at t adjusts the probability of $((p_1, q_1), (p_2, q_2), (p_3, q_3), (p_4, q_4))$ at $t + 1$. To see the implication of this choice recall the transition matrix (5) for the growth rate of d_t and the feasibility conditions (21). In the neighborhood of $\alpha_1 = \alpha_2 = .57$, we have $\alpha = 1 - \phi = .57$ and the binding feasibility constraints are $\lambda \leq \frac{1}{1-\phi}$, $\lambda \leq \frac{1}{\alpha_1} = \frac{1}{\alpha_2}$. Suppose that agent 1 is an optimist using F_1. As λ in F_1 rises, the rationality conditions $\alpha F_1 + (1-\alpha) F_2 = \Gamma$ require a *downward* adjustment of the probability of $((p_1, q_1), (p_2, q_2), (p_3, q_3), (p_4, q_4))$ in the pessimistic matrix F_2. Although the changes of the probabilities in F_2 are made to correspond to the change of probabilities in F_1, the rationality conditions, which regulate the relation between them, induce a *fundamental asymmetry between the intensities* of the two.

To explain the asymmetry in intensities, note that the matrix in (5) implies that almost the maximal feasible value of λ is reached at 1.7542 when some probabilities in F_2 are close to 0. Symmetry appears to dictate a correspondence between the 0 entries in F_2 and the entries of 1 in F_1. At $(\alpha_1 = \alpha_2 = .57, \lambda = 1.7542)$ *this symmetry does not hold*. If f^1_{ij} is the (ij) entry of F_1, then $f^1_{ij} = \lambda \Gamma_{ij}$ for j = 1, 2, 3, 4 and if f^2_{ij} is the (ij) entry of F_2, then $f^2_{ij} = \frac{1}{1-\alpha}[\Gamma_{ij} - \alpha\lambda\Gamma_{ij}]$. In the neighborhood of $\alpha_1 = \alpha_2 = .57$ and $\lambda = 1.7542$ we have the following *asymmetric conclusion*:

(22a) For all i = 1, 2, ..., 8, $f^2_{ij} \approx 0$ for j = 1, 2, 3, 4.

(22b) Only for i = 5, 6, 7, 8, $f^1_{ij} \approx 0$ for j = 5, 6, 7, 8.

(22a) says that in the neighborhood, pessimistic agents are *almost certain* that a recession will occur at date t + 1. This extreme degree of pessimism holds *for all* states of the economy at date t. Now, (22b) says that optimistic agents at date t are *almost certain* that a recession will not occur at t + 1 *only if at t the economy is in a recession* (i.e. $d_t = d^L$) . If the economy is in an expansion mode at t, the optimistic agent thinks that the probability of a recession at t + 1 is about 25%. We thus view the pessimists in this configuration as being *more intensely pessimistic than the optimists* and because of this difference in intensity, they have a greater effect on the security markets. Observe that the asymmetry discussed here *results from the rationality of belief conditions* and hence it is an essential characteristic of an RBE.

We finally turn to the interpretation of the parameters a = b = (.50 , .14, .14, .14). These regulate the correlation between y_t^1 and y_t^2 defined by the transition matrix of the states of belief which impacts the *dynamics* of prices. The correlation of (y_t^1 , y_t^2) implies that bull and bear markets are asymmetric. For the market to transit from the lowest price of the crash states (in the recession $d = d^L$ and the state of belief in DIS) to the highest prices of the bull market states (which occur in PP) it needs to take several steps: it cannot go *directly* from the low to the high prices. The opposite, however, is possible since at the bull market states there is a positive probability of reaching the crash states *in one step*. Thus a bull markets which reaches the highest price must evolve in several steps but a crash can occur in one step.

To sum up this section, we offer a simple and intuitive reason why the RBE generates a low riskless rate and a high equity premium. *Relative to Γ* there are, at any time, optimists and pessimists in the population of investors but on average there are more optimists than pessimists. Since over the entire population the average belief must correspond to Γ, the rationality of belief conditions imply that the intensity level of the pessimists dominates and their high demand for the riskless asset raises its price, leading to a low equilibrium riskless rate and high equity premium.

3.5 The Dynamics of Asset Prices and Returns

We turn now to an examination of some *dynamic* characteristics of asset prices under the

RBE theory. We start the discussion with an evaluation of the structure of asset price volatility

(i) The Structure of Asset Price volatility. We have noted that insufficient attention has been paid in recent literature to the question of price volatility and the problem of evaluating price volatility in the model in relation to real market price volatility. Figures 1a, 1b present time series of model simulation. Each contains 200 realized price\dividend ratios (which we call "the" price) generated by the REE of Table 2 and the RBE of Table 3 with $\beta_1 = \beta_2 = .90$ and $\gamma_1 = \gamma_2 = 3.25$. The standard deviation of the price\dividend ratio is .069 in the REE and 3.18 in the RBE. There are two *distinct* prices in the REE: 23.20 and 23.06 with a mean of 23.13. In the RBE there are 6 *distinct* prices with a conditional mean of 25.82 given d^H, with a conditional mean of 21.14 given

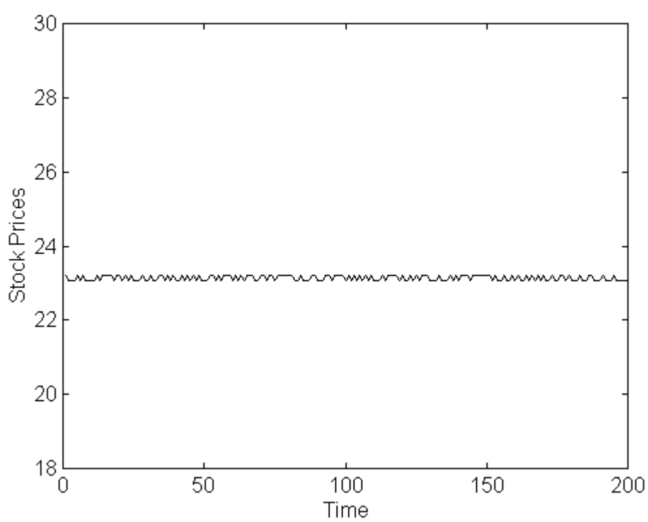

Fig. 1a: REE Simulation

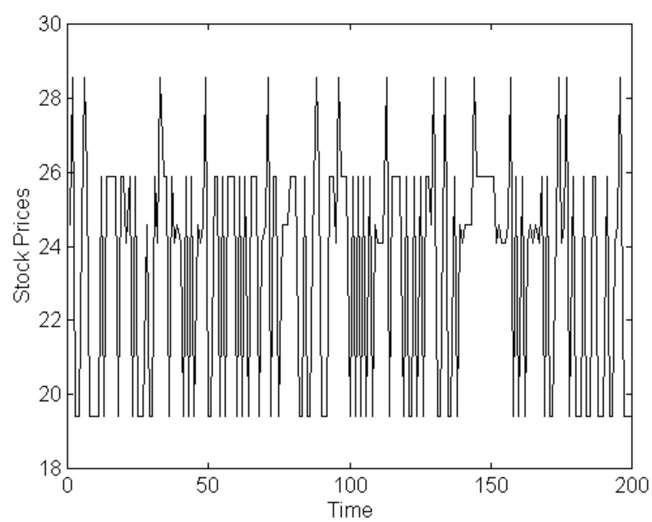

Fig. 1b: RBE Simulation

d^L and with an unconditional mean of 23.48. We decompose the standard deviation of prices in the RBE into two components. The first component, which is *overshooting*, or an *amplification* of the effect of d_t on prices, is measured by the standard deviation of a random variable which takes the values of 25.82 when $d_t = d^H$ and 21.14 when $d_t = d^L$. Hence, keeping the REE functional relation between prices and exogenous variables, amplification or overshooting increases the impact of exogenous variables on prices.

The second component of volatility is the pure effect which *the states of belief have on price volatility*. This component is uncorrelated with the exogenous dividend process and represents pure Endogenous Uncertainty which takes the form of additional prices induced by the states of beliefs and by the variability of the states of beliefs over time. To define this effect let $z_t^1 = 1$ when $d_t = d^H$ and 0 otherwise, and let $z_t^2 = 1$ when $d_t = d^L$ and 0 otherwise. Now define $e_t = p_t - 25.82\, z_t^1 - 21.14\, z_t^2$. In Figure 2 we exhibit 200 values of e_t computed from the simulated values of the RBE in Figure 1b. What is interesting about Figure 2 is the asymmetry in the distribution of e_t which is generated by the basic asymmetry in the causal structure of volatility in this model. We conclude by noting that if we take the volatility of the price\dividend ratio in the REE to be approximately the volatility that can be justified by the dividends, our analysis demonstrates that *most of the volatility of stock prices is generated by the beliefs of the agents* either in the form of price amplification or in the form of pure endogenous volatility. Thus, most of the volatility of asset prices is endogenously generated. However, an examination of the relative contribution of these two components of endogenous volatility shows that *price amplification or overshooting is the more important component*. We return to this

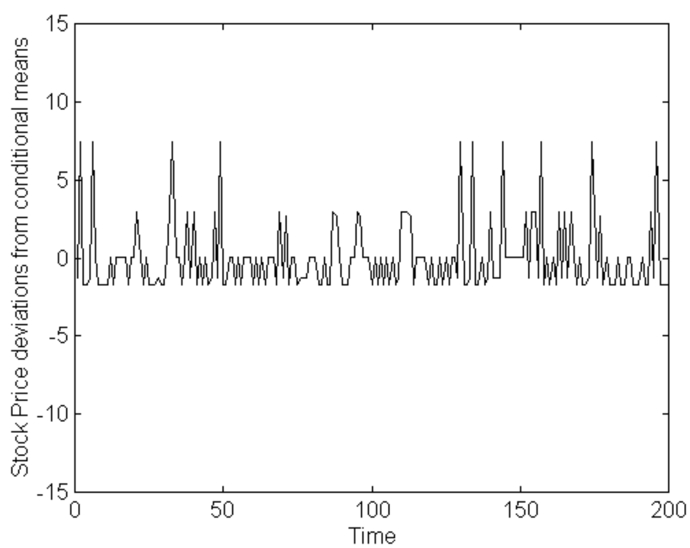

Fig. 2: RBE Simulation

important conclusion in Section 4 when we discuss the correlation of beliefs. Here we note that our result is consistant with the empirical evidence studied by Campbell and Shiller [1988].

(ii) *The GARCH Property of Asset Returns*. In Figure 3 we exhibit R_t^2 - the *square* of the risky returns - associated with the prices generated by the RBE of Figure 1b. Note that the bursts of price volatility in Figure 1b reappear as a GARCH property of asset returns. That is, Figure 3 shows that the variance of the risky rates of return is stochastic. Since the growth of dividends is a stationary Markov process, the stochastic volatility of the risky return is the result of the dynamical properties of the states of belief in the market. What is the cause for the GARCH property of the risky return? To answer this question recall the transition matrix of the state of beliefs which we reproduce here. We observe first that a regime of "agreement" (when $y_t^1 = y_t^2$ in states OO or PP) generates price variability which is *sharply different* from the price

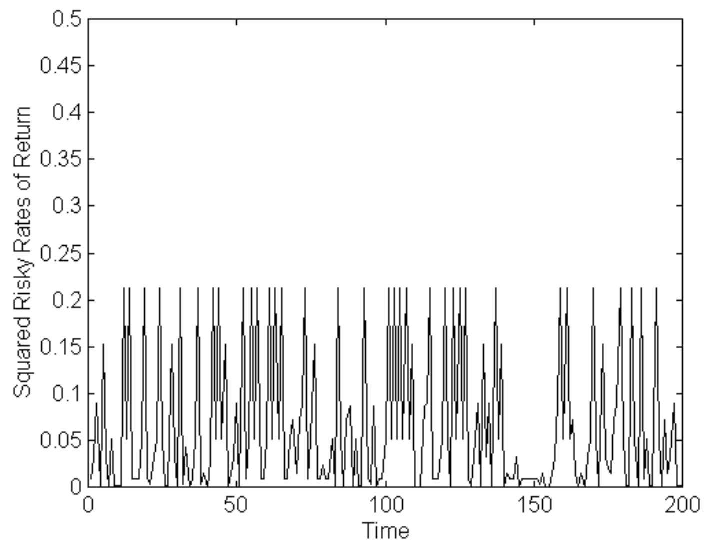

Fig. 3: RBE Simulation

	$(OO)_{t+1}$	$(PP)_{t+1}$	$(DIS)_{t+1}$
$(OO)_t$	.50	.36	.14
$(PP)_t$	.14	0	.86
$(DIS)_t$	.14	0	.86

variability in the regime of disagreement (when $y_t^1 \neq y_t^2$ in state DIS). Now suppose that at some date the state of belief is OO. From OO the economy can move to all states of beliefs. If it moves to PP it remains in the regime of agreement and if from PP it moves back to OO the market completes a cycle within the regime of agreement. If, however, it moves from PP to DIS, a regime of disagreement is started with sharply different price volatility characteristics. Note the sharp spikes in Figure 2. The highest price occurs only in the regime of agreement when the state of belief is PP while the lowest "crash" price occurs in the recession when $d_t = d^L$ and beliefs are in DIS. As the states of belief change over time, returns move among different volatility regimes. Indeed, the stochastic volatility of returns is a Markov process with varying degrees of persistence since the state of belief is a Markov process with varying degrees of persistence. Hence the GARCH property of asset return is caused by *the dynamic properties of the regimes of belief*.

To formally examine the GARCH property of asset returns we simulated 100,000 observations of R_t^2 in the RBE. Estimating the regression $R_t^2 = \xi_0 + \xi_1 d_t + \varepsilon_t$, we report in Table 5 the first 10 terms of the autocorrelation function of the residual of R_t^2. Note that the first three terms are large and the majority of terms are positive but decline rapidly, a result which

Table 5: The Autocorrelation Function of the Residuals of the Squared Return Regression

lag	1	2	3	4	5	6	7	8	9	10
	.026 (.003)	.044 (.003)	.016 (.003)	.007 (.003)	-.003 (.003)	-.005 (.003)	.0007 (.003)	.0003 (.003)	.001 (.003)	.004 (.003)

is compatible with the evidence (see Brock and LeBaron [1996]). We have explored several models that may best describe the behavior of the data over time. Following the Akaike Information Criterion, we found that the following E-GARCH(1, 1) model fits the data best:

$$R_t^2 = \underset{(.0003)}{-.3192} + \underset{(.0002)}{.3541}\, d_t + \varepsilon_t \,, \qquad \varepsilon_t \sim N(0\,, h_t)$$

where

$$\log(h_t) = \underset{(.0216)}{-5.8139} - \underset{(.0040)}{.2873}\log(h_{t-1}) - \underset{(.0064)}{1.6924}\left|\frac{\varepsilon_{t-1}}{\sqrt{h_{t-1}}}\right| + \underset{(.0038)}{.4938}\frac{\varepsilon_{t-1}}{\sqrt{h_{t-1}}}.$$

(iii) Predictability of long returns. The issue of predictability of returns generated a significant literature which we cannot review here (see, for example, Fama and French [1988], Poterba and Summers [1988], Campbell and Shiller [1988]). Although there is disagreement on the details of what the empirical record is, it seems to us that the stylized facts are as follows:

(i) short returns of one day or one month are too noisy and are not predicatable;

(ii) Long returns exhibit mean reversion but the effect declines with the returns' length;

(iii) The moving average of past returns and the price\dividend ratio (which, in our model, is the same as the price\earning ratio) are the best predictors of long returns.

In order to test the predictions of our model we shall consider two regression models. To do that let $R_t^k = R_{t-k+1}^1 \ldots\ldots R_{t-4}^1 R_{t-3}^1 R_{t-2}^1 R_{t-1}^1 R_t^1$ be the return of length k from $t - k + 1$ to t. Fama and French [1988] and Poterba and Summers [1988] allow both a mean reversion as well as random walk effects to be present in their models. Hence, in a regression of the form

$$\ln R_{t+k}^k = A_k + B_k \ln R_t^k + \epsilon_{t,k}$$

the parameters B_k measure the dominance of the two effects: $B_k = -.5$ implies the dominance of the mean reversion effect while $B_k = 0$ implies the dominance of the random walk component.

To test the above we generated a random sample of 30,000 observations and estimated the above regression for k = 1, 2, 3, ... , 20. Before presenting these results we note that the unit of time in our model is one year hence the model does not generate very short returns. Therefore, our resuls apply only to long returns starting with k = 1 which is one year. Table 6 records the estimated values $\hat{B}^k$ and the R^2 of the regressions. With 30,000 observations there is no point in reporting the standard errors $\sigma_{\hat{B}^k}$. The table shows that for small k the estimates are close to -.5 and hence the mean reversion effect dominates. For larger k this effect declines and a random walk effect become noticable. This qualitative result is the same as the results of Fama and

Table 6: Estimated Values of $\hat{B}_k$ and R^2

k	$\hat{B}^k$	R^2
1	-.529	.280
2	-.471	.222
3	-.469	.220
4	-.469	.220
5	-.462	.213
6	-.464	.215
8	-.444	.198
10	-.428	.183
12	-.412	.170
14	-.407	.166
16	-.393	.155
18	-.379	.143
20	-.376	.141

French [1988] and of Poterba and Summers [1988]. However, Fama and French's [1988] find that $\hat{B}^k$ are in the interval of (-.1 , -.4) only for horizons of up to k = 6 and decline sharply after that. They also report that the mean reversion effect is weak after 1941 and the results are strong only for 1926 - 1941[9]. Unfortunately, these conclusions of Fama and French [1988] are not confirmed by other studies. Poterba and Summers [1988] find a significant mean reversion in

[9] Fama and French [1988] seem to argue that the high volatility and pronounced mean reversion was unique to the 1926-1941 era whereas during the recent era of 1941-1985 the markets reverted to functioning more in accord with the random walk theory of stock returns. To evaluate this argument assume, for the moment, that the empirical findings are correct. It seems equally plausible to suggest that the *unusual* period was the era of 1941-1985 which included the long period of low productivity from the late 1960s to 1981. The low rate of economic growth resulted in the dramatic decline in real stock prices from 1967 to 1981 which was associated with low volume and low volatility. The recent high volatility of stock returns during 1985-1999 provides further ground to view Fama and French's [1988] argument less than compelling. More broadly, since a regression model estimates the parameters of the stationary measure there is no reason to select any particular period for exclusion from the data: the crash period of the 1930's is as important to the long term average as the roaring bull market of the 1990s or as the slow growth period of the 1970s.

returns of up to 96 months which is the maximal returns' length which they study. Campbell and Shiller [1988] show that (i) the predictability of returns is significant both for 1 year as well as for long horizons; indeed the most significant results are obtained for 10 year returns which is the longest horizon which they consider and, (ii) the moving average of past returns and the price\dividend ratio are the best predictors of long returns. To compare the prediction of our model with those of Campbell and Shiller [1988] we compute the following regression model proposed by Campbell and Shiller [1988] for k = 1, 2, 3, ... , 20

$$\ln R_{t+k}^{k} = \Upsilon_k^0 + \Upsilon_k^1 \ln R_t^{30} + \Upsilon_k^2 \ln p_t + \varepsilon_{t,k}.$$

Here we use the 30 period moving average of past returns and the current price\dividend ratio to predict returns for k = 1, 2, ... , 20. The results are reported in Table 7 .

Table7: Estimated Values of $\hat{\Upsilon}_k^1$, $\hat{\Upsilon}_k^2$ and R^2

k	$\hat{\Upsilon}_k^1$	$\hat{\Upsilon}_k^2$	R^2
1			
2			
3			
4			
5			
6			
8			
10			
12			
14			
16			
18			
20			

TO BE WRITTEN AFTER MAURIZIO'S COMPUTATIONS OF REGRESSIONS

(vi) *The Forward Discount Bias in Foreign Exchange Markets*. Kurz [1997b] and Black [1997] developed a model which is similar to ours except for the addition of a second country and two more short term nominal debt instruments. To define the problem that was addressed in these papers suppose that you estimate a regression of the form

$$\frac{ex_{t+1} - ex_t}{ex_t} = c + \zeta(r_t^D - r_t^F) + \varepsilon_{t+1} \tag{23}$$

where $(ex_{t+1} - ex_t)$ is the change of the exchange rate between date t and date t + 1 while $(r_t^D - r_t^F)$ is the difference between the short term *nominal* interest rates in the domestic and the foreign economies. Under rational expectations the differential of the nominal interest rates at t should provide an unbiased predictor of the depreciation of the currency between date t and date t + 1. This means that apart from a technical correction for risk aversion, the parameter ζ should be close to 1. In 75 empirical studies the estimates of the parameter ζ are significantly less than 1. Indeed, in many studies this parameter was estimated to be *negative* (see Froot [1990], Engel [1996] for an extensive survey). The failure of this parameter to exhibit estimated values close to 1 is known as the "Forward Discount Bias" in foreign exchange markets. Applying the RBE theory to this market, Kurz [1997b] and Black [1997] estimated ζ to be .152. However, the specifications in their models were different from ours and violated the condition of anonymity which we have imposed on our model. We have thus reformulated the model so as to satisfy our narrow parameter specification. There are, however, several issues that need to be evaluated first.

If we think of the first agent as the "domestic U.S." and the second agent as a "foreign economy" then we need to reformulate the model so as to allow the introduction of two nominal interest rates, two different monetary policies and a different stochastic structure. We thus assume that there is only one stock market in the home currency and the stochastic process of dividends is as in (5). As in our model above we also assume that the endowment\dividend ratio of the domestic agent is a constant ω and the domestic economy has a real bill which is traded by both agents. But then, how should we model the second country? What is the meaning of an exogenous shock in the foreign country? With such difficulties we (along with Kurz [1997b] and Black [1997]) model a *hypothetical* foreign economy which is characterized as follows:

(i) the endowment\dividend ratio ω* of the foreign agent is a random variable with two states $(\omega^{*H}, \omega^{*L})$ which is i.i.d. with the probability of $\omega^* = \omega^{*H}$ being .8;

(ii) the shocks to endowment are small, say of 2% - 3% hence in the REE $\omega^{*H} = 24.6$ and $\omega^{*L} = 23.4$ and in the RBE $\omega^{*H} = 26.6$ and $\omega^{*L} = 25.4$. Monetary policy in the home economy is responsive to the dividend shocks and monetary policy in the foreign country is responsive to the endowment shock in the foreign economy. The main reason for the endowment shock in the foreign economy is to allow the determination of the exchange rate in any REE;

(iii) an RBE requires a selection of a Γ* matrix to generate the stationary measure of the equilibrium dynamics. A matrix that satisfies the requirements specified is

$$\text{(24)} \qquad \Gamma^* = \begin{bmatrix} .8\phi A & .8(1-\phi)A & .2\phi A & .2(1-\phi)A \\ .8(1-\phi)B & .8\phi B & .2(1-\phi)B & .2\phi B \\ .8\phi C & .8(1-\phi)C & ,2\phi C & .2(1-\phi)C \\ .8(1-\phi)D & .8\phi D & .2(1-\phi)D & .2\phi D \end{bmatrix}.$$

where A, B, C, and D are matrices of the form (16). The crucial ingredient of the Explanatory Neighborhood is the assumption $\alpha_1 = \alpha_2 = .57$ and $\lambda_s = \lambda = \mu = \mu_s = 1.7542$ *and we shall continue to maintain this assumption.*

(iv) in our basic domestic model we set $A = B$ and $a = b = (.50, .14, .14, .14)$ which we shall continue to assume. Given that the probability of $\omega^* = \omega^{*H}$ is .8, it follows from the structure of the matrix Γ^* that 80% of the time, the international economy will look very much like our domestic economy when the second agent has endowment of ω^{*H}. But now, how should we select C and D? What about the other 20% of the time when the lower part of Γ^* is realized? To consider this point note that the *arbitrary* stochastic structure introduced by the i.i.d. process of $\{\omega_t^*, t = 1, 2, \ldots\}$ introduces into Γ^* a new and arbitrary element which may have nothing to do with the way the international economy *actually* works. This change must have some effect on the dynamics of the states of beliefs. The effect that we found was entirely minimal and is represented by the simple specification c = a = b = (.50, .14, .14, .14) but d = (.57, .14, .57, .14). Hence we can view the international model as a proper extension of our earlier model.

<u>Summary of specification</u>: $\phi = .43$, $\beta_1 = \beta_2 = .90$, $\gamma_1 = \gamma_2 = 3.25$, $\alpha_1 = \alpha_2 = .57$, $\lambda_s = \lambda = \mu = \mu_s = 1.7542$, a = b = c = (.50, .14, .14, .14), d = (.57, .14, .57, .14). In the REE ($\omega = 24$, $\omega^{*H} = 24.6$, $\omega^{*L} = 23.4$) ; in the RBE ($\omega = 26$, $\omega^{*H} = 26.6$, $\omega^{*L} = 25.4$).

Table 8 presents the simulation results for the REE and the RBE of the specified international model. In Table 6 "ex" denotes the "exchange rate" and σ_{ex} is the standard deviation of the exchange rate. Note first that the results for the REE are essentially the same as the results in Table 2 and the parameter ζ is computed to be .95, as is expected. From the point of view of comparing the RBE with the REE the only new result is the much larger variance of the foreign exchange rate in the RBE relative to the REE. Since the foreign economy is hypothetical we do not suggest any particular value for ex and σ_{ex}. Turning finally to the RBE, we observe that the results here are essentially the same as in Tables 3 or 4 but the new result is the simulated equilibrium value of $\zeta = .47$ which is significantly less than 1. We can thus conclude that the

Table 8: Results for the Reformulated International Model

variable	REE	RBE	Empirical Record
p	23.31	23.94	23
σ_p	.37	2 .70	6.48
R	6.21%	7.80%	8.00%
σ_R	4.72%	19.34%	18.08%
r^F	5.64%	1.52%	1.00%
σ_{r^F}	1.89%	16.37%	5.67%
ρ	.57%	6.28%	7.00%
ex	.68	.67	----
σ_{ex}	1.29%	9.93%	----
ζ	.95	.47	diverse < 1

Forward Discount Bias is one more anomaly which is explained *by the same RBE model*. We conjecture that sharper results for the parameter ζ could be obtained by formulating the foreign sector in a more realistic way.

Why does the RBE predict a value for ζ which is much lower than 1? Start by recalling the REE argument in favor of ζ close to 1. If $\zeta < 1$ then in an REE agents can make an *expectational arbitrage*: they can borrow in one currency and invest in the other, *expecting* that the net return on their investment will be larger than the depreciation of the currency. Note that in world of securities (rather than an Arrow-Debreu world of contingent claims) this is not an arbitrage in the strict sense of the term since the trades *do not take place at the same time*. However, in a stationary world in which all agents hold the same rational expectations the possibility of such an expectational riskless arbitrage cannot be an equilibrium.

In an RBE agents hold diverse beliefs and borrow and invest based on their own beliefs. In such a world, a differential nominal interest rates across countries offers an investment opportunity but now such investment is subjected to endogenous uncertainty. This results in a true, equilibrium, process of the exchange rate which exhibits excessive fluctuations in part due to variability in the states of belief of the agents. Hence, at almost all dates the nominal interest differential between the two countries is not an estimate of the rate of depreciation of the exchange rate one period later. Why should we expect that under $\zeta < 1$? To see why, consider first an REE in which the difference between the domestic and foreign nominal rates is z%. In that equilibrium you do not need to form expectations on currency depreciation. It is sufficient for you to believe that other investors or currency arbitragers know the true probability of currency depreciation and they have already induced the interest differential to be equal to the average rate of currency depreciation *which will then be z%*. Now consider an RBE. All agents know that no one knows the true probability distribution of the exchange rate and therefore the exchange rate is subject to endogenous uncertainty. Risk averse foreign currency investors would demand a risk premium on endogenous uncertainty and, on average, the difference $(1-\zeta)$ is the *proportional premium on nominal interest differential* demanded by currency investors for being willing to carry foreign currency positions. For a positive premium it follows that $\zeta < 1$.

4. Some Concluding Comments on Testable Implications of the RBE Theory and the Dynamics of Prices

In this paper we advance the proposition that most of the observed volatility of asset prices and returns *is propagated by the beliefs of the agents*. We suggest that exogenous shocks, central to the REE based explanation of market fluctuations, are insufficient to explain the structure of market volatility. The theory of RBE predicts the emergence of Endogenous

Uncertainty which is propagated within the economy by the beliefs of agents. By implication, the RBE theory proposes that many REE "anomalies" in financial markets such as the equity premium puzzle, the GARCH property of asset returns, the Forward Discount Bias in foreign exchange markets, the "smile curves" in derivative pricing, are all *propagated by the dynamics of beliefs in the markets*. In support of this claim we presented here a single RBE model which is calibrated to the U.S. economy and whose behavior is compatible with the list of REE anomalies. Monte Carlo studies of the model generate statistics which are compatible with the empirical record and the model predicts and explains the reasons for (i) the emergence of Stochastic Volatility in asset returns and, (ii) the presence of Forward Discount Bias in foreign exchange markets.

In our discussion in the text we have touched upon several problems which need further clarifications. In this concluding section we return to some of these important issues.

(i) Should Beliefs be Rational Ex-Ante or Ex-Post?

The typical definition of an equilibrium with endogenous beliefs requires the beliefs to be “self - fulfilling”. This *ex-post* rationality condition of REE is based on the idea of a stationary environment. In such an environment wrong beliefs are statistically rejected by the data generated by an equilibrium and by the time we reach a large date t, such beliefs disappear from the market. We reject this idea and require any RB to satisfy *ex-ante* rationality conditions which stipulate that at each date t the agent’s belief be compatible with the large record of past data available to him. The OLG model obscures this important issue which needs an explanation.

Our use of an OLG model was motivated by the feasibility of computations since in an OLG context agents can hold simple RB. To explain this point, suppose that agents were infinitely lived and hence had many periods to observe future data and examine the outcomes of their choices. In that case an agent could examine the *subsequence* of dates t in which he was optimistic about higher prices at $t + 1$. He would compute the frequency at which higher prices were realized at $t + 1$, compare it with the probability which he assigned to this event at date t and discover that he was wrong. Hence with a long sequence of future data, the agent would falsify the simple RB used in this paper. Such a conclusion may lead us to question whether the simple Markov belief with two states should be considered "rational" when an agent has sufficient future data to enable an ex-post rejection of his theory (i.e. his belief). We put this question forward in order to stress that the ex-ante rationality axiom in Definitions 3 and 4 is justified by two realistic perspectives which are central to the RBE theory: (i) the environment is non-stationary hence complex and, (ii) an infinite horizon optimization means only that each agent has a bequest motive but actually lives a finite life. Moreover, the length of his life is short relative to the rate at which economic data arrives. Let us explain these points.

A realistic description of the economy (see Kurz [1997a]) would view it as a non-stationary process described by an infinite sequence of "regimes" each of which occurs only once. Moreover, the duration of each regime is relatively short so that the data arrival "clock" (typically annual or quarterly) of key variables such as profits, GNP, productivity, inflation rate is slow relative to the clock at which technology and organization change. In such environments, an agent's non-stationary beliefs would be a far more complicated probability measure, allowing for infinite number of possible regimes of technology and financial structure. Under these conditions even if an agent knew the true starting and ending dates of a regime, the data available for each regime is usually insufficient to establish with high degree of confidence what the parameters of that regime were. The empirical identification of the switching dates is even a harder problem.

Equally important is the finite decision life of economic agents. Corporate executives, fund managers, officers of financial institutions etc. typically have a decision life of 10-20 years and even ordinary investors have a relatively short life of *significant* asset management. Given this fact, suppose that a sequence of regimes occurring during his own life provide an agent data with which he tests his belief statistically. Suppose also that he concluded that there is high likelihood that during this period his belief was wrong. Such a conclusion has two implications to the RBE framework. First, in a non-stationary environment an RB has the property that even if the belief was wrong during a regime under consideration, it does not constitute a proof that it will continue to be wrong in the future and for this reason *it continues to be an RB*. Second, by the time an agent has sufficient data to statistically test his own belief (i.e. his market theory) he is close to the end of his decision life. A conclusion that his belief may have been wrong is simply too late; all the important investment and consumption decisions have been made and cannot be reversed.

In sum, we reject ex-post rationality as inappropriate for a non-stationary economy since in this environment there is no objective procedure to generate a consensus on the "correct" belief no matter how large date t is. Equally so, in this environment past data is the only basis to judge if a belief is rational or not. Since ample data is available at the time an agent forms a belief, the rationality of belief must be an ex-ante concept. This also explains why an OLG model with private, *unobserved*, assessment variables is a convenient environment for our analysis: the agent has only one observation to test his belief and only past publicly *observed* data to evaluate the rationality of his belief. It should also be clear to the reader that computational feasibility has prevented us from taking advantage of the full mathematical generality of the RBE framework.

(ii) Comments on an Experimental Approach to Testing the Predictions of the RBE Theory

We now return to the question whether there are other ways to test the validity of the RBE theory, apart from the simulation methods used in this paper. Starting with econometric methods we note that Kurz [1997a] and Chernozhukov and Morozov [1999] study econometric restrictions on market data implied by the rationality conditions. For a parametric specification, Chernozhukov and Morozov [1999] show how to deduce from market data on consumption and portfolio choices the implied belief parameters of an agent and test if such a belief is rational.

Experimental approach offers an alternative way to test the implications of the theory. We have briefly touched upon this approach when we discussed the rationality conditions in (1). To provide more details, let t be the current period and consider the occurrence of any random variable $Z^{(t+k)}$ at k dates after date t. In our earlier example $Z^{(t+k)}$ = the annualized rate of return on the S&P500 stock index k periods after date t. Now find a sample of H sophisticated investors and ask each h to provide forecasts $E_{Q^h}[Z^{(t+k)} \mid I_t]$ for k = 1, 2, 3,..., K. The question is then how could such data allow us to distinguish among the following three hypotheses:

<u>Hypothesis I</u>: The agents hold Rational Expectations;

<u>Hypothesis II</u>: The agents hold Rational Beliefs which *are not* Rational Expectations;

<u>Hypothesis III</u>: The agents are not rational.

To examine these, observe that using all past data agents can estimate the regression model which best fits the data and make, with this empirical model, two forecasts: $E_m[Z^{(t+1)}]$ and $E_m[Z^{(t+1)}|I_t]$. Since we do not know the true data generating process, we need to define the term “holding Rational Expectations”. But, for an REE theory to make sense, one makes the standard REE assumption that the economy is stationary and agents know that it is stationary. Given this, the agents will accept the empirically based forecast as the truth. This leads us to a sequence of tests.

Test A: k = 1

Under REE all agents will make *the same* conditional forecast of $E_m[Z^{(t+1)}|I_t]$;

Under RBE the agents will make *diverse* conditional forecasts of $E_{Q^h}[Z^{(t+1)}\,|\,I_t]$, h = 1, 2, ..., H.

Test B: k = 1, 2, 3, ..., K for large K

Under REE all agents will make *the same* conditional forecast of $E_m[Z^{(t+k)}|I_t]$;

Under RBE the agents will make *diverse* conditional forecasts of $E_{Q^h}[Z^{(t+k)}\,|\,I_t]$, k = 1, 2, ..., K

and for each h = 1, 2, ..., H the sequence of forecasts satisfies the rationality condition

(25) $$\frac{1}{K}\sum_{k=1}^{K} E_{Q^h}[Z^{(t+k)}\,|\,I_t] \cong E_m[Z^t].$$

Interpretation of Test Results. For Test A the crucial difference between REE and RBE is simple:

Under REE agents agree that the *one correct* forecast is $E_m[Z^{(t+1)}|I_t]$;

Under RBE we have a *distribution* of forecasts even if agents have the same information.

However, one cannot define irrational behavior under the single forecast for t + 1.

For Test B we can make the following inference:

Under REE agents agree that the correct forecasts are $E_m[Z^{(t+k)}|I_t]$; this satisfies (25).

Under RBE agents make diverse forecast which must satisfy (25);

Agents are irrational if their forecast do not satisfy (25).

In several informal experiments we conducted Tests A and B for the case of $Z^{(t+k)}$ = the annualized rate of return on the S&P500 stock index k periods after date t[10]. In this case the REE forecast is approximately 8% for all k but all samples of investors, professional money

[10] We could not think of a better example of a random variable with respect to which no agent in the market has any private information. Hence, the wide diversity of forecasts of the S&P500 is a striking example of market and experimental observations where Tests A and B results go against the REE.

managers or doctoral students exhibit wide distribution of forecasts and, violating the REE prediction, *no sample ever provided a uniform forecast of 8%*[11]. In contrast with the violation of the REE requirement, in almost all cases the forecasts fulfilled condition (25) implying that agents satisfied the RBE rationality conditions. However, in some cases the stronger version of the rationality condition was satisfied. This condition stipulates that

$$E_{Q^h}[Z^{(t+k)} \mid I_t] \rightarrow E_m[Z^t] \quad \text{as} \quad k \rightarrow \infty . \tag{26}$$

Condition (26) is a strong rationality condition requiring not only the convergence of the *mean* forecast but the actual convergence of the agent's long run forecast to the unconditional forecast under m. We briefly explain the conditions which lead to (26).

Recall that the definitions of Stability (Definition 1) and WAMS (Definition 2) apply only to *finite* dimensional (cylinder) sets $B \in \mathcal{B}(X^\infty)$. This restriction is based on the assumption that agents have *finite* data with which they can determine the occurrence or non-occurrence of finite dimensional sets only. Suppose we reformulate Properties 1 and 2 by changing Definitions 1 and 2 to stipulate the stronger requirement that Stability holds for *all measurable sets* $B \in \mathcal{B}(X^\infty)$ and correspondingly define SAMS (strong asymptotic mean stationarity) to be Property 2, applicable to *all measurable sets* $B \in \mathcal{B}(X^\infty)$ including infinite dimensional sets. A modified Theorem 1' then says that the two properties are equivalent. However, the SAMS condition

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Pi(B^{(k)}) = m^{\Pi}(B) \quad \text{exists for all } B \in \mathcal{B}(X^\infty)$$

can be proved to require that for large t, we also have an approximate convergence condition

$$\lim_{k\to\infty} E_{\Pi}[1_{B^{(k)}} \mid I_t] \cong m^{\Pi}(B) = m(B) \quad \text{for all } B \in \mathcal{B}(X^\infty).$$

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[11] The reader may perform this same experiment. In practice we gave subjects the data of the annual returns on the S&P500 for 1889-1994 and computed for them values of the basic moments of the data.

This stronger condition (which satisfies condition (25)) appears to apply to some subjects.

(ii) The Dynamics and Correlation of Beliefs

A rather unique characteristic of the RBE studied earlier is the fact that there are 4 market states of beliefs (i.e. 4 values of (y_t^1, y_t^2)) and over time, the market state of belief fluctuates. This fact might lead one to conclude that the fluctuations of the market state of belief are *essential* to the volatility conclusions of the paper. Related to this is the fact that in formulating the model we assumed that the beliefs of the two agents could be correlated. Indeed, we introduced specific parameters which regulate this correlation and argued that the correlation plays a role in the dynamics of prices[12]. Hence it is important to explain what is the role played by the fluctuations of the market state of belief and of the correlation among agents. We show first that both the fluctuations of the market state of belief as well as the correlation between the individual states of belief *have little to do with the volatility moments of Table 4.*

To explain the claim above, consider an OLG economy as before except that now we introduce a continuum of agents with assessment variables which are identically distributed across agents and are independent across any countable collection of agents. Assume also that each assessment variable is the same as in the model presented earlier: $y_t^k \in \{0, 1\}$ and is i.i.d. over time with $P(y_t^k = 1) = .57$. It follows that in this economy there is *only one market state of belief* which is (.57 , .43). This is the fixed distribution of beliefs in the economy: at any date t,

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[12] The presence of correlation may also suggest that there could be a relationship between an RBE with correlation and a "Correlated Equilibrium" (see Aumann [1987]) of a game. To see why this is incorrect note that if we think of an RBE as a game, then the strategy of the auctioneer is to select prices. The strategy of the players is to select consumptions and portfolios so as to maximize expected utility *given* observed prices at t and given their rational beliefs about prices and dividends in future dates. In no sense does an RBE entail the agents selecting beliefs as strategies and hence correlation of beliefs has nothing to do with a correlated equilibrium in which agent's strategies are correlated.

57% of the agents are optimistic and .43 are pessimistic about capital gains at date t+1. Consider now the time series of this economy, compute the moments discussed earlier and compare with the moments in, say, the middle box (.57, .57) of Table 4. Table 9 presents the results. There are

Table 9: Volatility Comparison of Models With and Without Correlation of Beliefs

variable	RBE with a Single Market State of Belief	RBE with Correlation and Four Market States of Belief	The Empirical Record
σ_p	4.10%	3.18%	6.48%
R	9.55%	8.68%	8.00%
σ_R	31.00%	23.35%	18.08%
r^F	.43%	.71%	1.00%
σ_{r^F}	24.30%	18.24%	5.67%
ρ	9.98%	7.97%	7.00

two surprising results reported in this table. The first one is that the model with a single market state of belief generates about the same volatility (measured by the moments) as the model reported in Table 4, in which the market states of belief fluctuate over time and individual states of belief are correlated[13]. The second surprise is that the moments predicted by the model with a single market state of belief are reasonably close to the empirical record.

The two results reported in the last paragraph have one explanation which is also the factor which enables the model to generate the moments reported in Table 4. This factor is the interplay between the amplification or overshooting property of the model and the asymmetry between the intensities of the pessimists and of the optimists in the model. That is, the facts that enable both models to generate moments close to the empirical record are that in both models we

[13] We observe that this result is supported by the empirical evidence reported by Campbell and Shiller [1988] who show that amplification of the effect of the fluctuations of dividends is the dominant cause of excess volatility of returns.

have (i) $\alpha_1 = \alpha_2 = .57$ and (ii) $\lambda = \mu = 1.754$. These two imply that in both models the optimists are in the majority but the RBE rationality conditions require the pessimists to have a higher intensity level. This asymmetry has a decisive effect on the financial markets of both economies.

The natural question is then why should we consider models with multiple states of belief which fluctuate over time and why should we be concerned with the correlation among individual states of beliefs of agents in the economy? The reason is that the model with a single market state of belief generates results which are counter-factual with regard to the *dynamics* of prices. Examples of such results are:

(i) It implies that the variations in prices are *perfectly* correlated with the observed exogenous shocks and hence are completely explainable by these exogenous changes.

(ii) It is a fact that major market declines are associated with economic recessions. However, it is also a fact that a fraction of major market declines forecast recessions which never materialize. An implication of (i) is that all market declines are associated with recessions and the market never predicts recessions which do not occur. Technically speaking, a model with a single state of belief implies that all Endogenous Uncertainty is an amplification of exogenous shocks.

(iii) It implies that there are no extreme market price increases and no market crashes.

(iv) It fails to generate the stochastic volatility property of asset returns.

In short, there are two central reasons for our analysis of an RBE model in which the market state of belief (i.e. the distribution of beliefs) fluctuates over time and individual states of belief are correlated. First, because available measures of the distribution of beliefs such as the distribution of price and earning forecasts on Wall Street, published forecast distributions of inflation and GNP, all exhibit significant fluctuations over time. Second, if the model is to explain the empirical record it must also exhibit price dynamics which is compatible with the characteristics of price dynamics in the market. We think that the model with fluctuating state of belief and some correlation among beliefs is well suited for that goal.

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