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# On consistency of optimal portfolio choice for state-dependent exponential utilities

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In an arbitrage-free simple market, we demonstrate that for a class of state-dependent exponential utilities, there exists a unique prediction of the random risk aversion that ensures the consistency of optimal strategies across any time horizon. Our solution aligns with the theory of forward performances, with the added distinction of identifying, among the infinite possible solutions, the one for which the profile is the actual optimizer of the system of preferences specified *a priori*.

*Keywords:* Time consistency; State-dependent utility; Portfolio choice; Forward performance; Exponential utility; Market price of risk

*JEL Classifications:* D81, C61, D11

## 1. Introduction

The problem of optimal portfolio choice stands out as one of the most significant streams of literature related to decision making under uncertainty (see Gilboa 2009, for an introduction to the theory of decisions). In this framework, an agent aims at maximizing her utility by investing in a financial market. Numerous approaches to optimal investment have been proposed since the seminal contributions of Samuelson (1969) and Merton (1969), which extended this theory beyond the Markowitz (1952) one-period model.

In a multi-period setting, an optimal strategy is an investment plan that ensures the agent to maximize a function of her terminal wealth resulting from trading in the market. Ideally, such a strategy is determined at the beginning of the trading period and is intended to be followed ‘consistently’ until the specified terminal date. However, the optimization problem may be *time-inconsistent*, meaning that the strategy currently considered optimal may not remain optimal in subsequent periods. Consequently, it is not universally clear what optimality should signify in such contexts.

The literature has addressed time inconsistency through two primary approaches. The first, known as the *precommitment* approach, resolves inter-temporal inconsistency by committing to a strategy evaluated as optimal at inception,

regardless of future deviations. This approach has been extensively studied in works such as Li and Ng (2000), Zhou and Li (2000), and Jin and Zhou (2008). The second approach adopts a game-theoretic perspective, treating future incentives to deviate as constraints on optimization. This perspective, pioneered by Ekeland and Lazrak (2010) and further developed by Basak and Chabakauri (2010), Björk *et al.* (2014), and Björk *et al.* (2017), often addresses time inconsistency arising from mean-variance utility formulations.

Classical studies, such as the Merton problem, focus on maximizing the expected utility  $\mathbb{E}[u(V_t)]$  at a terminal time  $t$  by deriving optimal strategies through backward induction. In contrast, more recent research (Musielà and Zariphopoulou 2007, 2008, 2009, Henderson and Hobson 2007) introduces an alternative, maturity-independent framework. In this setting, the utility function evolves stochastically and forwardly over time. Known as forward performance theory, this framework substitutes the classical indirect utility function with a forward utility criterion derived from the financial market. The stochastic utility is deduced in order to satisfy specific (super)martingale conditions to recover the Dynamic Programming Principle.

In this paper, we approach portfolio choice in a simple market model from a ‘forward’ perspective, focusing primarily on the issue of consistency. Unlike forward performance theory, our approach centers on deriving optimal strategies that can be consistently extended beyond a fixed terminal date in a

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self-financing manner. In Definition 2.1 below, we adapt and refine the seminal notion of consistency provided in Kydland and Prescott (1977, p. 475). A policy  $\alpha^*$  is consistent if, ‘for any time period  $t$ , it optimizes the objective functional, taking as given previous decisions and that future policy decisions are similarly selected’.

In Proposition 2.1, we demonstrate that inconsistency may arise in the classical Merton problem when the market price of risk exhibits sufficient stochasticity. The rationale is that constant risk aversion prevents the agent from effectively counterbalancing randomness in the investment set, forcing her to later deviate from the initially devised strategy (see Remark 2.3 for additional details). However, this drawback is mitigated when uncertainty about the agent’s future risk aversion is introduced. Our main result, stated in Theorem 3.1, establishes that consistency for exponential-type utilities is achieved if and only if the agent’s risk aversion evolves according to a unique dynamic, matching in this way the form obtained in Žitković (2009, Theorem 4.4). Similarly to our paper, Žitković (2009) focuses on random fields with an exponential structure and provides necessary and sufficient conditions for the self-generation of preferences over time. However, Proposition 3.1 demonstrates that, even within a simplified market model, an infinite class of forward performances can be readily constructed. Nonetheless, the  $u_t$  derived in Theorem 3.1 guarantees a strategy  $\alpha^*$  that remains optimal at all times  $t$  for the objective functional  $\mathbb{E}[u_t(\cdot)]$ . Thus, consistency emerges as a crucial criterion for inferring the agent’s preferences ‘adjusted’ to the temporal evolution of market information.

## 2. Preliminaries and problem formulation

The probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  is fixed throughout the paper and we shall denote by  $L^0(\Omega, \mathcal{F}, \mathbb{P})$  the space of  $\mathcal{F}$  measurable random variables that are  $\mathbb{P}$  a.s. finite. Similarly  $L^1(\Omega, \mathcal{F}, \mathbb{P})$  will denote the space of integrable random variables. In this paper a filtration  $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$  will describe the information available to an economic agent at any time  $t \in [0, \infty)$  and we shall always assume that the filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$  satisfies the usual conditions, that is the filtration  $\mathbb{F}$  is right-continuous and complete.

A decision maker aims at maximizing her utility by investing her initial endowment  $x \in \mathbb{R}$  in a market composed by a risky asset  $(S_t)_{t \geq 0}$  and a bond  $(B_t)_{t \geq 0}$  whose dynamics are as follows

$$dS_t = S_t (\mu_t dt + \sigma_t dW_t), \quad dB_t = rB_t dt$$

with  $S_0 > 0$ ,  $B_0 = 1$ ,  $W$  being a 1-dimensional Brownian motion and  $(\mu_t)_{t \geq 0}, (\sigma_t)_{t \geq 0}$   $\mathbb{F}$ -progressively measurable processes. We assume that the filtration  $\mathbb{F}$  is the completion of the natural filtration generated by  $(W_t)_{t \geq 0}$ , therefore the market is complete. Moreover, for the sake of the exposition of this decision problem, we also assume that

$$\mathbb{P} \left( \theta_t \neq 0 \text{ and } \sup_{0 \leq u \leq t} |\theta_u| < K_t \right) = 1$$

for any  $t > 0$  and some  $K_t \in [0, \infty)$ . (1)

As a result, the process

$$Z_t = \exp \left\{ -\frac{1}{2} \int_0^t \theta_s^2 ds + \int_0^t \theta_s dW_s \right\}$$

where  $\theta_t = - \left( \frac{\mu_t - r}{\sigma_t} \right)$

is a  $(\mathbb{F}, \mathbb{P})$ -martingale and for any  $0 < t < \infty$  the probability measure  $\mathbb{Q}_t$ , with  $Z_t = \frac{d\mathbb{Q}_t}{d\mathbb{P}}$ , defines a martingale measure for the discounted price process  $(e^{-ru} S_u)_{u \in [0, t]}$ . We observe that, for any choice of  $t, s$  such that  $t > s$ , the restriction of  $\mathbb{Q}_t$  to  $\mathcal{F}_s$  coincides with  $\mathbb{Q}_s$ . Hence, to avoid burdensome notation, we omit the subscript and denote by  $\mathbb{Q}$  the family of measures<sup>†</sup> induced by the process  $(Z_t)_{t \geq 0}$ . In particular we adopt the following convention, for an  $\mathcal{F}_t$ -measurable random variable  $X$  we define  $\mathbb{E}_{\mathbb{Q}}[X] := \mathbb{E}_{\mathbb{Q}_t}[X] = \mathbb{E}_{\mathbb{P}}[Z_t X]$  as well as  $\mathbb{E}_{\mathbb{Q}}[X | \mathcal{F}_s] := \mathbb{E}_{\mathbb{Q}_t}[X | \mathcal{F}_s] = \frac{\mathbb{E}_{\mathbb{P}}[Z_t X | \mathcal{F}_s]}{Z_s}$ , for  $s < t$ . Additionally, we shall refer to the process  $W^{\mathbb{Q}}$  as  $\mathbb{Q}$ -Brownian motion, with the interpretation that for any  $t > 0$  the process  $(W_s^{\mathbb{Q}})_{s \in [0, t]}$  is a Brownian motion under the measure  $\mathbb{Q}_t$ .

Throughout this paper, all the financial quantities of interest are to be considered in discounted terms. Hence,  $(V_t^\alpha)_{t \geq 0}$  denotes the discounted value of a self-financing portfolio and  $\alpha_t$  represents the proportion of  $V_t^\alpha$  invested in the risky asset at time  $t$ . The dynamics of the portfolio process are given by

$$dV_t^\alpha = V_t^\alpha [(\mu_t - r)\alpha_t dt + \sigma_t \alpha_t dW_t]. \quad (2)$$

For a fixed time horizon  $t \in (0, +\infty)$ , the agent is endowed with a preference relation  $\succeq^t$  that admits a numerical representation through a state-dependent utility  $u_t : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\}$  and the subjective probability of the agent coincides with the reference<sup>‡</sup> measure  $\mathbb{P}$ . With an abuse of notation we shall always denote by  $u_t(X)$  the random variable  $\omega \mapsto u_t(\omega, X(\omega))$  for any  $X \in L^0(\Omega, \mathcal{F}_t, \mathbb{P})$ . Given  $X, Y \in L^0(\Omega, \mathcal{F}_t, \mathbb{P})$  such that  $u_t(X), u_t(Y) \in L^1(\Omega, \mathcal{F}_t, \mathbb{P})$ , the preference ordering is given by

$$X \succeq^t Y \text{ if and only if } \mathbb{E}_{\mathbb{P}}[u_t(X)] \geq \mathbb{E}_{\mathbb{P}}[u_t(Y)]. \quad (3)$$

Wakker and Zank (1999) proved that this representation holds if the preference  $\succeq^t$  is monotone, pointwise continuous, and satisfies the Sure Thing principle. Furthermore, Berton *et al.* (2025) showed in addition that  $u_t$  can be chosen to be  $\mathcal{F}_t \otimes \mathbb{B}_{\mathbb{R}}$  measurable<sup>§</sup>, with  $u_t(\omega, \cdot)$  strictly increasing and continuous for every  $\omega \in \Omega$ .

<sup>†</sup> We remark that this is only a notational convention and does not imply the existence of a probability measure on  $\mathcal{F}$ , unless additional technical conditions are imposed (see Remark 3.2).

<sup>‡</sup> This can be assumed without loss of generality if the subjective probability is equivalent to the reference measure, as the function  $u_t$  is unique up to a change of measure, Wakker and Zank (see 1999, Theorem 12)

<sup>§</sup>  $\mathbb{B}_{\mathbb{R}}$  is the Borel  $\sigma$ -algebra on the real line, the  $\sigma$ -algebra generated by open sets of  $\mathbb{R}$

In the decision framework just depicted, the classical utility maximization problem thus takes the form

$$\text{maximize } \mathbb{E}_{\mathbb{P}} [\omega_t(V_t^\alpha)] \text{ constrained to } V_0^\alpha = x, \quad (4)$$

where  $\alpha$  belongs to a suitable family  $\mathcal{A}$  of admissible controls and  $x \in \mathbb{R}$  represents the initial wealth of the agent. In particular, we will always suppose that any admissible control  $\alpha \in \mathcal{A}$  guarantees the existence and uniqueness of a (weak) solution of (2). We shall denote by  $(\alpha_u^*)_{u \in [0, t]}$  the strategy that maximizes the expected utility of the agent in the time interval  $[0, t]$  and, therefore,  $V_t^{\alpha^*}$  denotes the maximizer of the objective functional  $\mathbb{E}_{\mathbb{P}} [\omega_t(\cdot)]$ .

As illustrated in the introduction, we shall adopt throughout the paper the following notion of consistency, which rephrases the seminal notion in Kydland and Prescott (1977).

**DEFINITION 2.1** For  $s < t$ , let  $V_s^{\alpha^*}$  be the maximizer of  $\mathbb{E}_{\mathbb{P}} [\omega_s(\cdot)]$  and  $V_t^{\alpha^*}$  be the maximizer of  $\mathbb{E}_{\mathbb{P}} [\omega_t(\cdot)]$  both selected by solving (4). An optimization problem is consistent with respect to any time horizon if for any  $s < t$  it holds that<sup>†</sup>  $\mathbb{E}_{\mathbb{Q}} [V_t^{\alpha^*} | \mathcal{F}_s] = V_s^{\alpha^*}$ .

Moreover we make an abuse of notation, as a priori the strategies  $(\alpha_u^*)_{u \in [0, s]}$ ,  $(\alpha_u^*)_{u \in [0, t]}$  which define  $V_s^{\alpha^*}$ ,  $V_t^{\alpha^*}$  might be different on  $[0, s] \subset [0, t]$ . Nevertheless, assuming that consistency holds in the chosen market model, they must coincide over the common time interval.

This type of consistency naturally leads to strategies which are not affected by a fixed time horizon, which is the basis of the well known theory of forward performances, which we now recall, as it will play a crucial role in the discussion.

**DEFINITION 2.2** Given  $u_0 : \mathbb{R} \rightarrow \mathbb{R}$  a concave and increasing function, an  $\mathbb{F}$ -adapted process  $(\omega_t)_{t \geq 0}$  with  $\omega_t : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a forward performance if

- the mapping  $x \mapsto \omega_t(\omega, x)$  is increasing and concave for each  $\omega \in \Omega$  and  $t \geq 0$ ;
- it satisfies  $\omega_0(\omega, x) = u_0(x)$ ;
- for all  $t, s \in [0, \infty)$  such that  $t > s$  and for any  $\alpha \in \mathcal{A}$ ,  $\mathbb{E}_{\mathbb{P}} [\omega_t(V_t^\alpha) | \mathcal{F}_s] \leq \omega_s(V_s^\alpha)$  holds;
- for all  $t, s \in [0, \infty)$  such that  $t > s$  there exists  $\alpha^* \in \mathcal{A}$  such that  $\mathbb{E}_{\mathbb{P}} [\omega_t(V_t^{\alpha^*}) | \mathcal{F}_s] = \omega_s(V_s^{\alpha^*})$ .

**REMARK 2.1** Observe that Definition 2.1 is concerned with a notion of consistency which can be effectively interpreted as a self-financing constraint for the optimal portfolio, in order to guarantee optimality at any date  $t$  for a time dependent preference structure. Conversely, the concept of consistency that motivates Definition 2.2 pertains to a natural ‘martingale optimality principle’ (see e.g. Davis and Varaiya 1973) characterizing optimality in stochastic control problems.

### 2.1. On consistency for Von Neumann-Morgenstern type preferences

Consistency may fail already in classical optimization problems, when utilities are not state dependent, as described in

<sup>†</sup> We stress that this condition is equivalent to  $\mathbb{E}_{\mathbb{P}} [Z_t V_t^{\alpha^*} | \mathcal{F}_s] = Z_s V_s^{\alpha^*}$ .

the following proposition. In complete markets, the classical Merton problem can be solved by computing explicitly at time  $t$  the optimal discounted wealth  $\xi_t^*$  and then deriving optimal strategies backwards via a perfect hedging procedure, which intrinsically links them to the terminal date. It seems therefore appropriate to discuss this simple motivating example before proceeding to our main result. In fact the proposition implies that if the market price of risk fails to be deterministic, an agent with classical preferences may not be able to devise consistent optimal plans at all.

**PROPOSITION 2.1** Assume that (1) holds true for  $\theta_t = -(\frac{\mu_t - r}{\sigma_t})$ . For

$$u(x) = -\frac{1}{\gamma} e^{-\gamma x}, \quad \gamma > 0 \quad (5)$$

problem (4) is consistent if and only if the function  $t \mapsto \theta_t^2$  is deterministic.

**REMARK 2.2** The issue of inconsistency arises not only for utilities of the exponential type. Simple inspections show that the Merton problem is always consistent for  $u(x) = \ln(x)$ , but for power utilities  $u(x) = \frac{x^\gamma}{\gamma}$  with  $\gamma \in (0, 1)$  problem (4) is consistent if and only if  $t \mapsto \theta_t^2$  is a deterministic function (see Appendix A.6 for the proof), in analogy with the case treated in Proposition 2.1.

**REMARK 2.3** (What if consistency fails?) If we consider two times  $s < t$ , the respective optima  $V_s^{\alpha^*}$  and  $V_t^{\alpha^*}$  and the risk neutral price  $\Pi_s := \mathbb{E}_{\mathbb{Q}} [V_t^{\alpha^*} | \mathcal{F}_s]$  (i.e. the hedging price at time  $s$  of the optimum  $V_t^{\alpha^*}$ ), situations where  $\mathbb{P}(\Pi_s \neq V_s^{\alpha^*}) > 0$  may arise if consistency does not hold true. This issue is of little relevance whenever the agent cannot change her initially specified investment horizon  $t$ . We stress that often in practice an agent chooses to optimize up to time  $t$  with the option of reconsidering her initially set horizon, either with  $t_1 < t$  or  $t_2 > t$ . Usually short time horizons are considered and after that (and depending on the performance of the portfolio), the agent decides either to quit investing or to extend the final date. In this situation once the optimum is found, then the agent performs the strategy  $\alpha^*$  up to time  $s$  reaching the output  $V_s^{\alpha^*}$ . However, as the event  $A = \{V_s^{\alpha^*} < \Pi_s\} \in \mathcal{F}_s$  may have positive probability, if  $A$  occurs the agent will not be able to roll over her strategy from  $s$  to  $t$  in order to achieve the time  $t$  optimum  $V_t^{\alpha^*}$ .

## 3. Main results

Motivated by the previous discussion, we are now ready to illustrate the main contribution of this paper. We consider an agent who plans to invest in the market which has a stochastic market price of risk  $(\theta_t)_{t \geq 0}$  (see Assumption (A\*) below). The agent faces uncertainty on her future risk aversion and may be willing to adjust the terminal date of the investment over time. We assume that the preferences of the agent are represented by a state-dependent utility function of the exponential type  $\omega_t : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  for  $t \in [0, \infty)$ , such that  $\omega_t(\omega, x) = -\frac{1}{\gamma_t(\omega)} e^{-\gamma_t(\omega)x}$ . For our purposes, we model the

stochastic risk aversion  $(\gamma_t)_{t \geq 0}$  specifying the dynamics of the general  $\mathbb{F}$ -adapted process  $(1/\gamma_t)_{t \geq 0}$  as follows:

$$\begin{cases} d\left(\frac{1}{\gamma_t}\right) = \frac{1}{\gamma_t}(\eta_t dt + \beta_t dW_t^{\mathbb{Q}}), \\ \gamma_0 > 0, \end{cases}$$

where  $dW_t^{\mathbb{Q}} = dW_t - \theta_t dt$  is a  $\mathbb{Q}$ -Brownian motion. As is customary,  $1/\gamma_t$  will be referred to as risk tolerance at time  $t$ . We will be working with the following set of assumptions in addition to (1):

**ASSUMPTION (A)** *The processes  $(\eta_t)_{t \geq 0}, (\beta_t)_{t \geq 0}$  are progressively measurable and for any  $t > 0$  there exists a constant  $K_t > 0$  such that*

$$\mathbb{P}\left(\sup_{0 \leq u \leq t} |\eta_u| < K_t, \sup_{0 \leq u \leq t} |\beta_u| < K_t\right) = 1.$$

**ASSUMPTION (A\*)** *Assumption (A) is in force and for any  $t > 0$  we have  $\text{Var}(\theta_t^2) \neq 0$ .*

**REMARK 3.1** Assumption (A\*) restricts the financial market model to the case of a stochastic market price of risk. However, Proposition 3.2 below addresses the complementary scenario of a deterministic market price of risk with stochastic risk aversion, thereby providing a comprehensive perspective on the problem.

Assumption (A) is needed to formally justify a few technical steps in the proofs, although the reader will notice that the boundedness conditions imposed on  $(\eta_t)_{t \geq 0}$  and  $(\beta_t)_{t \geq 0}$  are not too restrictive in light of the results of the following theorem (i.e.  $\eta_t = 0$  and  $\beta_t = -\frac{\theta_t}{2}$ ).

**THEOREM 3.1** *Suppose Assumption (A\*) holds and consider  $u_t(\omega, x) = -\frac{1}{\gamma_t(\omega)} e^{-\gamma_t(\omega)x}$ , with  $(\gamma_t)_{t \geq 0}$  which solves*

$$d\left(\frac{1}{\gamma_t}\right) = \frac{1}{\gamma_t}(\eta_t dt + \beta_t dW_t^{\mathbb{Q}}), \quad \gamma_0 > 0.$$

*Problem (4) is consistent if and only if  $\eta_t = 0$  and  $\beta_t = -\frac{\theta_t}{2}$ .*

*Moreover, for this choice of  $(\eta_t)_{t \geq 0}$  and  $(\beta_t)_{t \geq 0}$ , the following hold:*

- *The optimal wealth process is given by*

$$\xi_t^* = \frac{1}{\gamma_t}(\gamma_0 x - \ln(Z_t)),$$

- *The optimal strategy  $\alpha_t^*$  is given by*

$$\alpha_t^* = -\frac{1}{\gamma_t \sigma_t \xi_t^*} \left( \theta_t + \frac{\theta_t \gamma_t \xi_t^*}{2} \right),$$

- *for all  $t \in [0, \infty)$ ,  $u_t(V_t^{\alpha^*}) \in L^1(\Omega, \mathcal{F}_t, \mathbb{P})$ , and for all  $s < t$*

$$\mathbb{E}_{\mathbb{P}}[u_t(V_t^{\alpha^*}) | \mathcal{F}_s] = u_s(V_s^{\alpha^*}).$$

We defer the proof of Theorem 3.1 to the appendix. Theorem 3.1 points out that, whenever the market price of

risk  $(\theta_t)_{t \geq 0}$  is stochastic, there is a unique choice of  $(\gamma_t)_{t \geq 0}$  that ensures the consistency of the problem is recovered.

**REMARK 3.2** (On the implications of Theorem 3.1) Under the reference measure  $\mathbb{P}$  the dynamics of  $(1/\gamma_t)_{t \geq 0}$  are given by

$$d\left(\frac{1}{\gamma_t}\right) = \frac{1}{\gamma_t} \left( \frac{\theta_t^2}{2} dt - \frac{\theta_t}{2} dW_t \right),$$

and consequently

$$d\gamma_t = \gamma_t \left( -\frac{\theta_t^2}{4} dt + \frac{\theta_t}{2} dW_t \right).$$

First, we observe that  $(\gamma_t)_{t \geq 0}$  is a positive supermartingale and hence there exists a random variable  $\gamma_\infty \in L^1(\Omega, \mathcal{F}, \mathbb{P})$  such that  $\gamma_t \rightarrow \gamma_\infty$   $\mathbb{P}$ -a.s. as  $t \rightarrow \infty$  (see Revuz and Yor 1999, Corollary II.2.11). In particular, if  $\mathbb{P}(\gamma_\infty \neq 0) = 1$  then one obtains  $1/\gamma_t \rightarrow 1/\gamma_\infty$   $\mathbb{P}$ -a.s. as well. One can therefore think of  $1/\gamma_\infty$  as defining the agent's unknown true preferences  $(u, \mathbb{P})$  with  $u(\omega, x) = -\frac{1}{\gamma_\infty(\omega)} e^{-\gamma_\infty(\omega)x}$ . Since  $(1/\gamma_s)_{0 \leq s \leq t}$  is a  $\mathbb{Q}$ -martingale for all  $t$ , for every pair  $s < t$ , the process  $1/\gamma_s$  can be seen as the prediction at time  $s$  of the future risk tolerance  $1/\gamma_t$ . If, moreover, there existed an equivalent probability measure<sup>†</sup>  $\tilde{\mathbb{Q}}$  on  $\mathcal{F}$  we could interpret  $1/\gamma_t$  as the ‘most rational’ prediction of the real risk tolerance  $1/\gamma_\infty \in L^1(\Omega, \mathcal{F}, \mathbb{Q})$ , through the relation  $\frac{1}{\gamma_t} = \mathbb{E}_{\tilde{\mathbb{Q}}}\left[\frac{1}{\gamma_\infty} | \mathcal{F}_t\right]$ . Therefore, in devising her investment strategy, the agent is adopting a consistent estimate of her true risk aversion given the information available on the market. The novelty of this approach lies in the fact that the longer the agent engages with the market, the more she becomes aware of her true risk aversion. This leads to a strong interplay between the time evolution of the agent's strategy and the updating of her preference order. We present a few economic insights related to the previous theorem. As the process  $1/\gamma_t$  evolves with positive drift, on average, the agent tends to become more risk-tolerant, or equivalently less risk-averse, with time. This is coherent with the previous idea that the agent gradually uncovers her true risk aversion by interacting with the market. However, trajectory-wise the risk-aversion is influenced by the movements of the market through the Brownian motion. The correction term in  $\alpha^*$  further emphasizes the point:  $\gamma_t \xi_t^*$  depends on the whole trajectory of the Brownian motion  $W_t$  up to time  $t$  through  $\ln(Z_t)$ . Therefore in order to be consistent the agent may need to invest contrary to the market movements.

Conversely, there is no systematic bias in the foreseen dynamics of risk tolerance under the pricing measure  $\mathbb{Q}$ . Indeed, Theorem 3.1 points out that a biased perception of risk tolerance dynamics would rule out any possible consistency.

Moreover the results perfectly fits into the theory of forward performances, as the martingale property of  $(u_t(V_t^{\alpha^*}))_{t \geq 0}$  is recovered, and in Proposition 3.1 we shall also prove that

<sup>†</sup> Although intuitively appealing, this assertion requires a stronger integrability assumption on  $(\theta_t)_{t \geq 0}$  in order to ensure the existence of a suitable equivalent probability measure  $\tilde{\mathbb{Q}}$  on  $(\Omega, \mathcal{F})$  and  $(1/\gamma_t)_{t \geq 0}$  uniformly integrable. Under this assumptions the conclusion follows from a straightforward application of martingale convergence Theorem (see Revuz and Yor 1999, Theorem II.3.1).

for any strategy  $\alpha \in \mathcal{A}$ ,  $(\mathfrak{u}_t(V_t^\alpha))_{t \geq 0}$  is a supermartingale. This result is extremely interesting in view of the existence of infinitely many forward performances with shape  $\mathfrak{u}_t(\omega, x) = -\frac{1}{\gamma_t(\omega)} e^{-\gamma_t(\omega)x}$ , since the solution proposed in Theorem 3.1 is the unique that guarantees the optimality of the strategy  $\alpha^*$ .

**PROPOSITION 3.1** *Suppose Assumption (A) holds and let  $d(\frac{1}{\gamma_t}) = \frac{1}{\gamma_t}(\eta_t dt + \beta_t dW_t^Q)$ ,  $\gamma_0 > 0$  with  $u_0(x) = -\frac{1}{\gamma_0} e^{-\gamma_0 x}$ . Consider the process  $(\eta_t)_{t \geq 0}$  defined by*

$$\eta_t = \frac{\theta_t(\theta_t + 2\beta_t)}{2(\gamma_t V_t^* + 1)}. \quad (6)$$

where

$$dV_t^* = \frac{(\gamma_t V_t^* \beta_t - \theta_t)}{\gamma_t \sigma_t} ((\mu_t - r) dt + \sigma_t dW_t) \quad (7)$$

Then for every  $(\beta_t)_{t \geq 0}$  such that the previous SDEs admit a solution, the stochastic utility function  $\mathfrak{u}_t(\omega, x) = -\frac{1}{\gamma_t(\omega)} e^{-\gamma_t(\omega)x}$  is a forward performance and for  $\alpha_t^* = \frac{(\gamma_t V_t^* \beta_t - \theta_t)}{\gamma_t V_t^* \sigma_t}$  the process  $\mathfrak{u}_t(V_t^{\alpha_t^*})$  is a  $\mathbb{P}$ -martingale.

In particular, for the choice of  $(\eta_t)_{t \geq 0}, (\beta_t)_{t \geq 0}$  in Theorem 3.1 and setting  $\frac{1}{\gamma_0} = \mathbb{E}_Q \left[ \frac{1}{\gamma_t} \right]$ ,  $(\mathfrak{u}_t)_{t \geq 0}$  is a forward performance.

Given the relation (6) between  $(\beta_t)_{t \geq 0}$  and  $(\eta_t)_{t \geq 0}$  in the previous proposition and the intertwining of  $(1/\gamma_t)_{t \geq 0}$  and  $(V_t^*)_{t \geq 0}$ , it is not immediate that such processes may actually possess a unique strong solution. In the following example we provide a family of coefficients that guarantees that all processes are indeed well defined.

**EXAMPLE 3.1** (On the existence of an infinite family of Forward Performances) Consider the auxiliary process

$$\begin{cases} dH_t = \left( \frac{\theta_t}{2(1+H_t^2)} \delta - \frac{\theta_t^2}{2} \right) dt - \theta_t dW_t^Q, \\ H_0 = \gamma_0 x \in \mathbb{R}, \end{cases} \quad (8)$$

where  $\delta > 0$  is an arbitrary constant. Recalling that  $(\theta_t)_{t \geq 0}$  is bounded by assumption (1), the process in (8) has a unique strong solution  $(H_t)_{t \geq 0}$  (see Pham 2009, Theorem 1.3.15). Set

$$\beta_t = \frac{1+H_t}{2(1+H_t^2)} \delta - \frac{\theta_t}{2}, \quad \eta_t = \frac{\theta_t}{2(1+H_t^2)} \delta. \quad (9)$$

and note that the dynamics in (8) can be rewritten as  $dH_t = (\beta_t \theta_t - \eta_t H_t) dt - \theta_t dW_t^Q$ . In particular, for this choice of parameters the SDE  $d(\frac{1}{\gamma_t}) = \frac{1}{\gamma_t}(\eta_t dt + \beta_t dW_t^Q)$  admits a unique strong solution. Finally, we define  $V_t^* := \frac{1}{\gamma_t} H_t$  and we show that  $(V_t^*)_{t \geq 0}$  satisfies the portfolio dynamics of Proposition 3.1. The stochastic differential of  $(V_t^*)_{t \geq 0}$  under the reference measure  $\mathbb{P}$  is given by

$$dV_t^* = d\left(\frac{1}{\gamma_t} H_t\right) = \frac{1}{\gamma_t} H_t \left( \beta_t - \frac{\theta_t}{H_t} \right) [-\theta_t dt + dW_t].$$

Substituting  $V_t^*$  for  $\frac{1}{\gamma_t} H_t$ , observing that by definition  $H_t = \gamma_t V_t^*$  and recalling that  $\theta_t = -\frac{(\mu_t - r)}{\sigma_t}$  we recover the dynamics

in equation (7) and guarantee the existence of a solution for the SDEs in Proposition 3.1.

Ultimately, we note that once we substitute  $\gamma_t V_t^*$  for  $H_t$  in  $\beta_t$  and  $\eta_t$ , the coefficients satisfy condition (6), and in particular for  $\delta = 0$  we recover the parametrization of Theorem 3.1.

Within this framework, we offer the following interpretation of the distinction between the ‘consistent’ and the ‘forward performance’ approach. We recall that, under the historical measure  $\mathbb{P}$ , the process  $(1/\gamma_t)_{t \geq 0}$  evolves according to  $d(\frac{1}{\gamma_t}) = \frac{1}{\gamma_t}[(\eta_t - \theta_t \beta_t) dt + \beta_t dW_t]$ . Substituting  $\eta_t, \beta_t$  from (9), the drift coefficient reads as

$$-\frac{\delta \theta_t H_t}{2(1+H_t^2)} + \frac{\theta_t^2}{2}. \quad (10)$$

Although  $(\mathfrak{u}_t)_{t \geq 0}$  is a forward performance for any  $\delta \geq 0$ , it makes problem (4) consistent in the sense of Definition 2.1 only when  $\delta$  vanishes. Specifically, for  $\delta = 0$  the risk-tolerance is a submartingale, whereas for  $\delta > 0$  the sign of the drift is determined by the interaction of  $\theta_t$  and  $H_t$ . For instance, if we impose  $\mathbb{P}(\mu_t \geq r) = 1$  for every  $t \in (0, \infty)$ , then the sign of the term in (10) is determined ultimately by the portfolio oscillations, with  $\delta$  acting as a sensitivity parameter.

**REMARK 3.3** (On preference systems) The intricate relationship between the processes  $(\eta_t)_{t \geq 0}$  and  $(\beta_t)_{t \geq 0}$  carries both economic and technical implications. As Example 3.1 illustrates, the agent’s past investment decisions are incorporated in her present risk-tolerance. From an economic perspective, the mutual dependence between  $\mathfrak{u}_t$  (through the risk aversion parameter) and the optimal portfolio  $V_t^{\alpha^*}$  implies that the pair  $(\mathfrak{u}_t, \mathbb{P})_{t \geq 0}$  cannot be regarded as an *a priori* system of dynamic preferences.

An interesting aspect of the result in Theorem 3.1 is that  $1/\gamma_t$  is completely disentangled from the selected strategies, hence providing a criterion to select a regular dynamic preference structure for the agent, which admits consistent optima.

In Proposition 3.2, we complete Theorem 3.1 by highlighting the following somewhat counterintuitive result: whenever an agent endowed with a utility function with stochastic risk aversion (analogous to that in Theorem 3.1) invests in a financial market with deterministic market price of risk, the problem is consistent regardless of the choice of  $(\beta_t)_{t \geq 0}$ , as long as it is a deterministic function. Nevertheless, the unique parametrization of  $(\beta_t)_{t \geq 0}$  that makes  $\mathfrak{u}_t(\omega, x)$  a forward performance is the one resulting from Theorem 3.1.

**PROPOSITION 3.2** *Suppose Assumption (A) holds, let the function  $t \mapsto \theta_t$  be deterministic and consider  $\mathfrak{u}_t(\omega, x) = -\frac{1}{\gamma_t(\omega)} e^{-\gamma_t(\omega)x}$ , with  $(\gamma_t)_{t \geq 0}$  which solves*

$$d\left(\frac{1}{\gamma_t}\right) = \frac{1}{\gamma_t}(\eta_t dt + \beta_t dW_t^Q), \quad \gamma_0 > 0.$$

*Problem (4) is consistent if and only if  $\eta_t = 0$  and  $\beta_t = f(t)$  for some deterministic function  $f: [0, \infty) \rightarrow \mathbb{R}$ . Moreover  $(\mathfrak{u}_t)_{t \geq 0}$  is a forward performance if and only if  $\beta_t = -\frac{\theta_t}{2}$ .*

**EXAMPLE 3.2 (Multiplicative stochastic noise)** We conclude by mentioning an alternative class of stochastic utilities, where a deterministic utility is perturbed by a multiplicative martingale noise. While the expected utility of the agent on deterministic quantities is unaffected (and so are her preferences), the randomness of the market interacts with the martingale noise, distorting the agent's perceived utility from a portfolio. We find, however, that the problem is consistent only if the parameter of the noise process suitably counterbalances the randomness of the market. A further unappealing feature is that the consistency condition by itself is not enough to pin down a unique choice of parametrization of the noise, and additionally requiring that  $\mathfrak{u}_t(\omega, x)$  be a forward performance results in the agent not investing at all in the risky asset.

**PROPOSITION 3.3** *Let  $\mathfrak{u}_t(\omega, x) = u(x) \cdot X_t(\omega)$  where  $u : \mathbb{R} \rightarrow \mathbb{R}$  is a utility function of the type of equation (5) and  $X_t$  a solution of  $dX_t = X_t \beta_t dW_t$ ,  $X_0 = 1$  with  $(\beta_t)_{t \geq 0}$  a progressively measurable process such that for any  $t > 0$  there exists a constant  $K_t > 0$  such that  $\mathbb{P}(\sup_{0 \leq u \leq t} |\beta_u| < K_t) = 1$ .*

*Problem (4) is consistent if and only if  $(\theta_t - \beta_t)^2 = k(t)$  with  $k : [0, \infty) \rightarrow \mathbb{R}$  an arbitrary deterministic function. The optimal strategy is given by*

$$\alpha_t^* = \frac{k(t)}{\gamma \sigma \xi_t^*}.$$

*Furthermore,  $(\mathfrak{u}_t)_{t \geq 0}$  is a forward performance if and only if  $k(t) = 0 \forall t \geq 0$ , in which case  $\alpha_t^* = 0$ .*

**REMARK 3.4** Similarly to what we pointed out in Remark 2.2, it is possible to prove an analogous result assuming a power utility  $u(x) = \frac{x^\gamma}{\gamma}$ , with  $\gamma \in (0, 1)$ .

**REMARK 3.5** The results described so far naturally lead to the question of whether an optimal random time exists to exit the investment plan. We now present a heuristic argument to illustrate why consistency offers the advantage of ensuring no regret, regardless of the exit time chosen by the agent.

Consider the family  $\mathcal{T}$  of  $\mathbb{P}$  almost surely finite stopping times  $\tau : \Omega \rightarrow (0, +\infty)$ , then the process  $\xi_{t \wedge \tau}^*$  is a  $\mathbb{Q}$ -martingale for every  $\tau \in \mathcal{T}$  and it optimizes  $\mathbb{E}_{\mathbb{P}}[\mathfrak{u}_{t \wedge \tau}(\cdot)]$ . Under some technical assumption it is possible to infer that the agent is indifferent among all the possible terminal dates, as the expected reward of optimizing over all possible stopping rules does not exceed the one obtained without stopping, i.e.

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{P}}[\mathfrak{u}_{\tau}(\xi_{\tau}^*)] = \mathbb{E}_{\mathbb{P}}[\mathfrak{u}_t(\xi_t^*)] \quad \forall t > 0.$$

Furthermore, one can find that  $\xi_{\tau}^*$  maximizes  $\mathbb{E}_{\mathbb{P}}[\mathfrak{u}_{\tau}(\cdot)]$  over all the  $\mathcal{F}_{\tau}$ -measurable random variables  $\xi$  such that  $\mathbb{E}_{\mathbb{P}}[|\mathfrak{u}_{\tau}(\xi)|] < \infty$  and  $\mathbb{E}_{\mathbb{Q}}[\xi] = x$ .

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## Author contributions

**CRedit:** **Edoardo Berton:** Investigation, Writing – original draft, Writing – review & editing; **Marzia De Donno:** Investigation, Writing – original draft, Writing – review & editing; **Marco Maggis:** Investigation, Writing – original draft, Writing – review & editing

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## Appendices

### Appendix 1. Proofs

This appendix is entirely devoted to the proofs of previous section. The proof of Theorem 3.1 builds on the extension to state-dependent utilities of a well-known result on the characterization of the maximizer of an optimization problem. The detailed statement and proof of this result are provided in Appendix 2. In addition, we prove an auxiliary lemma that will contribute to the proof of the main result. The proof of Proposition 3.1 complements that of Theorem 3.1 and shows that the utility function we derived previously is actually a forward performance. We then conclude the section with the proofs of Propositions 3.2 and 3.3, which are at this point merely an application of the previous results.

#### A.1. Proof of Proposition 2.1

Consider  $t > 0$  fixed. Since the market is assumed to be complete, we can recast Problem (4) as the following infinite dimensional problem

$$\sup \left\{ \mathbb{E}_{\mathbb{P}} [u(\xi)] \mid \xi \in L^1(\Omega, \mathcal{F}_t, \mathbb{Q}) \text{ and } \mathbb{E}_{\mathbb{Q}} [\xi] = x \right\}. \quad (\text{A1})$$

The application of Lemma A.2 then yields the optimum

$$\xi_t^* = x + \frac{1}{\gamma} \mathbb{E}_{\mathbb{Q}} [\ln(Z_t)] - \frac{1}{\gamma} \ln(Z_t),$$

with  $\mathbb{P}(|\xi_t^*| < k) = 1$  for some  $k \in \mathbb{R}$ , as a consequence of (1). Considering some time  $s < t$  and mimicking the argument in Björk (2009, Proposition 20.11), we find

$$\mathbb{E}_{\mathbb{Q}} [\xi_t^* \mid \mathcal{F}_s] = x + \frac{1}{\gamma} [E_0^t - E_s - \ln(Z_s)],$$

where  $E_0^t = \mathbb{E}_{\mathbb{Q}} \left[ \frac{1}{2} \int_0^t \theta_u^2 du \right]$  and  $E_s = \mathbb{E}_{\mathbb{Q}} \left[ \frac{1}{2} \int_s^t \theta_u^2 du \mid \mathcal{F}_s \right]$ . In order to ensure the problem is time consistent we require

$$\mathbb{E}_{\mathbb{Q}} [\xi_t^* \mid \mathcal{F}_s] = \xi_s^*, \text{ i.e.}$$

$$x + \frac{1}{\gamma} [E_0^t - E_s - \ln(Z_s)] = x + \frac{1}{\gamma} \mathbb{E}_{\mathbb{Q}} [\ln(Z_s)] - \frac{1}{\gamma} \ln(Z_s), \quad (\text{A2})$$

where the dynamics of process  $\ln(Z_t)$  under  $\mathbb{Q}$  is given by  $d \ln(Z_t) = \frac{1}{2} \theta_t^2 dt + \theta_t dW_t^{\mathbb{Q}}$ . Observing that

$$E_0^t = \mathbb{E}_{\mathbb{Q}} \left[ \frac{1}{2} \int_0^t \theta_u^2 du \right] = \underbrace{\mathbb{E}_{\mathbb{Q}} \left[ \frac{1}{2} \int_0^s \theta_u^2 du \right]}_{\mathbb{E}_{\mathbb{Q}}[\ln(Z_s)]} + \mathbb{E}_{\mathbb{Q}} \left[ \frac{1}{2} \int_s^t \theta_u^2 du \right],$$

equation (A2) boils down to

$$\mathbb{E}_{\mathbb{Q}} \left[ \frac{1}{2} \int_s^t \theta_u^2 du \mid \mathcal{F}_s \right] = \mathbb{E}_{\mathbb{Q}} \left[ \frac{1}{2} \int_s^t \theta_u^2 du \right]. \quad (\text{A3})$$

Because the choice of  $s$  is arbitrary, condition (A3) must hold for all  $s < t$ . Whenever  $t \mapsto \theta_t^2$  is deterministic the latter is clearly satisfied. We conclude by showing the converse implication. First, applying Fubini-Tonelli Theorem we can rewrite (A3) as

$$\frac{1}{2} \int_s^t \mathbb{E}_{\mathbb{Q}} [\theta_u^2 \mid \mathcal{F}_s] du = \frac{1}{2} \int_s^t \mathbb{E}_{\mathbb{Q}} [\theta_u^2] du,$$

which in turn implies that  $\mathbb{E}_{\mathbb{Q}} [\theta_u^2 \mid \mathcal{F}_s] = \mathbb{E}_{\mathbb{Q}} [\theta_u^2]$  for almost every  $u \in (s, t)$ . Furthermore, for  $u < t$  fixed we also have that  $\mathbb{E}_{\mathbb{Q}} [\theta_u^2 \mid \mathcal{F}_s] = \mathbb{E}_{\mathbb{Q}} [\theta_u^2]$  for almost every  $0 \leq s < u$ . Define the stochastic process  $I(s) := \mathbb{E}_{\mathbb{Q}} [\theta_u^2 \mid \mathcal{F}_s]$  for  $0 \leq s < u$ . Since (1) holds true,  $\theta_u^2 \in L^1(\Omega, \mathcal{F}_u, \mathbb{Q})$ , then  $I(s)$  is a martingale that admits a representation of the form  $I(s) = I_0 + \int_0^s \psi_t dW_t^{\mathbb{Q}}$  where  $\psi_t$  is a predictable, square integrable process and  $I_0 \in \mathbb{R}$ . Without loss of generality we consider the continuous version of  $I(s)$  and hence, letting  $s \rightarrow u$  we obtain  $I(s) \rightarrow I(u) = \mathbb{E}_{\mathbb{Q}} [\theta_u^2 \mid \mathcal{F}_u] = \theta_u^2$ . Recalling that  $\mathbb{E}_{\mathbb{Q}} [\theta_u^2 \mid \mathcal{F}_s] = \mathbb{E}_{\mathbb{Q}} [\theta_u^2]$  for all  $s < u$  we conclude that  $\mathbb{E}_{\mathbb{Q}} [\theta_u^2] = \theta_u^2$  and, since the choice of  $u \in (s, t)$  was arbitrary, the function  $t \mapsto \theta_t^2$  must be a deterministic function.

#### A.2. Proof of Theorem 3.1

Before proving the main result we need the following auxiliary lemma.

LEMMA A.1 *Suppose that (1) and Assumption (A) hold. Consider  $u_t(\omega, x) = -\frac{1}{\gamma_t(\omega)} e^{-\gamma_t(\omega)x}$  with  $(\gamma_t)_{t \geq 0}$  being the solution of the SDE*

$$d \left( \frac{1}{\gamma_t} \right) = \frac{1}{\gamma_t} \beta_t dW_t^{\mathbb{Q}}, \quad \gamma_0 > 0, \quad (\text{A4})$$

then problem (4) is consistent if and only if one of the following is verified:

- $\beta_t = f(t)$  and  $\theta_t = g(t)$  with  $f, g$  deterministic functions of time;
- $\beta_t = -\frac{\theta_t}{2}$ .

*Proof of Lemma A.1* Let  $t > 0$  be fixed, and observe that since the market is assumed to be complete, Problem (4) is equivalent to the following infinite dimensional problem

$$\sup \left\{ \mathbb{E}_{\mathbb{P}} [u_t(\xi)] \mid \xi \in L^1(\Omega, \mathcal{F}_t, \mathbb{Q}) \text{ and } \mathbb{E}_{\mathbb{Q}} [\xi] = x \right\}.$$

Applying Lemma A.2 we can show that the optimal profile is

$$\xi_t^* = \frac{1}{\gamma_t} \left\{ c \left( x + \mathbb{E}_{\mathbb{Q}} \left[ \frac{1}{\gamma_t} \ln(Z_t) \right] \right) - \ln(Z_t) \right\}, \quad (\text{A5})$$

where  $c = \frac{1}{\mathbb{E}_{\mathbb{Q}}[1/\gamma_t]}$  (which does not depend on  $t$  by assumption) and  $Z_t = \frac{d\mathbb{Q}}{d\mathbb{P}}$ . Notice that  $\xi_t^* \in L^1(\Omega, \mathcal{F}, \mathbb{Q})$  since  $\ln(Z_t)$  is bounded

from (1) and  $\frac{1}{\gamma_t} = \exp(-\frac{1}{2} \int_0^t \beta_u^2 du + \int_0^t \beta_u dW_u^{\mathbb{Q}})$  is in  $L^1(\Omega, \mathcal{F}, \mathbb{Q})$  from Assumption (A).

By definition  $(1/\gamma_t)_t$  is a strictly positive martingale in the filtration generated by the  $\mathbb{Q}$ -Brownian motion  $dW_t^{\mathbb{Q}} = dW_t - \theta_t dt$ . Recalling that  $d \ln(Z_t) = \frac{1}{2} \theta_t^2 dt + \theta_t dW_t^{\mathbb{Q}}$  and given the dynamics of  $1/\gamma_t$  in (A4) we have

$$d\left(\frac{1}{\gamma_t} \ln(Z_t)\right) = \frac{\theta_t}{\gamma_t} \left(\frac{\theta_t}{2} + \beta_t\right) dt + \frac{1}{\gamma_t} (\theta_t + \beta_t \ln(Z_t)) dW_t^{\mathbb{Q}},$$

and hence the conditional expectation under  $\mathbb{Q}$  with respect to  $\mathcal{F}_s$  reads

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} \left[ \frac{1}{\gamma_t} \ln(Z_t) \mid \mathcal{F}_s \right] \\ = \frac{1}{\gamma_s} \ln(Z_s) + \mathbb{E}_{\mathbb{Q}} \left[ \int_s^t \frac{\theta_u}{\gamma_u} \left(\frac{\theta_u}{2} + \beta_u\right) du \mid \mathcal{F}_s \right]. \end{aligned}$$

We now move to the consistency condition  $\mathbb{E}_{\mathbb{Q}}[\xi_t^* \mid \mathcal{F}_s] = \xi_s^*$  and collecting previous computations we obtain that it is in fact equivalent to

$$\begin{aligned} c \left( x + E_{\mathbb{Q}} \left[ \frac{1}{\gamma_t} \ln(Z_t) \right] \right) \mathbb{E}_{\mathbb{Q}} \left[ \frac{1}{\gamma_t} \mid \mathcal{F}_s \right] - \mathbb{E}_{\mathbb{Q}} \left[ \frac{1}{\gamma_t} \ln(Z_t) \mid \mathcal{F}_s \right] \\ = c \left( x + E_{\mathbb{Q}} \left[ \frac{1}{\gamma_s} \ln(Z_s) \right] \right) \frac{1}{\gamma_s} - \frac{1}{\gamma_s} \ln(Z_s) \end{aligned}$$

and a few calculations then yield

$$\begin{aligned} \frac{c}{\gamma_s} \left( \mathbb{E}_{\mathbb{Q}} \left[ \int_s^t \frac{\theta_u}{\gamma_u} \left(\frac{\theta_u}{2} + \beta_u\right) du \right] \right) \\ = \mathbb{E}_{\mathbb{Q}} \left[ \int_s^t \frac{\theta_u}{\gamma_u} \left(\frac{\theta_u}{2} + \beta_u\right) du \mid \mathcal{F}_s \right]. \end{aligned} \quad (\text{A6})$$

We define  $L_t := \frac{c}{\gamma_t}$  and observe it is a positive martingale with  $\mathbb{E}_{\mathbb{Q}}[L_t] = 1$  and hence we can specify a new measure  $\mathbb{Q}_L$  through  $L_t = \frac{d\mathbb{Q}_L}{d\mathbb{Q}}$ . Let us denote  $\kappa_u := \theta_u(\frac{\theta_u}{2} + \beta_u)$ , then equation (A6) can be rewritten as

$$\mathbb{E}_{\mathbb{Q}_L} \left[ \frac{1}{L_t} \int_s^t L_u \kappa_u du \right] = \mathbb{E}_{\mathbb{Q}_L} \left[ \frac{1}{L_t} \int_s^t L_u \kappa_u du \mid \mathcal{F}_s \right]. \quad (\text{A7})$$

The linearity of the expectation allows to rearrange the above equation as follows

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_L} \left[ \frac{1}{L_t} \int_0^t L_u \kappa_u du \right] - \mathbb{E}_{\mathbb{Q}_L} \left[ \frac{1}{L_s} \int_0^s L_u \kappa_u du \right] \\ = \mathbb{E}_{\mathbb{Q}_L} \left[ \frac{1}{L_t} \int_0^t L_u \kappa_u du \mid \mathcal{F}_s \right] - \frac{1}{L_s} \int_0^s L_u \kappa_u du. \end{aligned} \quad (\text{A8})$$

We notice that

$$d\left(\frac{1}{L_t} \int_0^t L_u \kappa_u du\right) = \underbrace{\int_0^t L_u \kappa_u du \cdot d\left(\frac{1}{L_t}\right)}_{=: dM_t} + \frac{1}{L_t} L_t \kappa_t dt,$$

where  $M_t$  is a  $\mathbb{Q}_L$ -martingale. In integral notation the latter reads

$$\frac{1}{L_t} \int_0^t L_u \kappa_u du = M_t + \int_0^t \kappa_u du,$$

and substitution into (A8) yields

$$\mathbb{E}_{\mathbb{Q}_L} \left[ \int_s^t \kappa_u du \right] = \mathbb{E}_{\mathbb{Q}_L} \left[ \int_s^t \kappa_u du \mid \mathcal{F}_s \right]. \quad (\text{A9})$$

We point out that the above is trivially verified whenever  $\kappa_t$  is a deterministic function. In turn, this holds when either:

- $\beta_t = f(t)$  and  $\theta_t = g(t)$  with  $f, g$  deterministic functions of time, or
- $\beta_t = -\frac{\theta_t}{2}$ .

We now prove the converse implication. Consider the process

$$Y_t := \int_0^t \kappa_u du - \mathbb{E}_{\mathbb{Q}_L} \left[ \int_0^t \kappa_u du \right],$$

computing its conditional expectation and using (A9) we find that it is a  $\mathbb{Q}_L$ -martingale. Simple inspections show that  $Y_t$  is of finite variation and hence, by Revuz and Yor (1999, Proposition IV.1.2), we conclude  $Y_t$  is constant and therefore  $\theta_u(\frac{\theta_u}{2} + \beta_u)$  must be a deterministic function of time. This occurs when both  $\theta_t, \beta_t$  are deterministic, or when  $\beta_t = -\frac{\theta_t}{2}$ . ■

*Proof of Theorem 3.1* Fix  $t > 0$ , and consider Problem (4) for the state-dependent utility function  $\mathfrak{u}_t(\omega, x) = -\frac{1}{\gamma_t(\omega)} e^{-\gamma_t(\omega)x}$ . We reformulate the maximization problem as

$$\sup \left\{ \mathbb{E}_{\mathbb{P}}[\mathfrak{u}_t(\xi)] \mid \xi \in L^1(\Omega, \mathcal{F}_t, \mathbb{Q}) \text{ and } \mathbb{E}_{\mathbb{Q}}[\xi] = x \right\}.$$

Applying Lemma A.2 to  $\mathfrak{u}_t(\omega, x)$  yields the  $t$ -optimal discounted profile

$$\xi_t^* = -\frac{1}{\gamma_t} \ln(\lambda Z_t). \quad (\text{A10})$$

One can then recover  $\lambda$  by plugging (A10) into the budget constraint  $\mathbb{E}_{\mathbb{Q}}[\xi_t^*] = x$

$$\ln(\lambda) = \frac{1}{\mathbb{E}_{\mathbb{Q}}[1/\gamma_t]} \left( -x - \mathbb{E}_{\mathbb{Q}} \left[ \frac{1}{\gamma_t} \ln(Z_t) \right] \right).$$

Substituting in (A10) and setting  $c_t = \frac{1}{\mathbb{E}_{\mathbb{Q}}[1/\gamma_t]}$  we finally obtain

$$\xi_t^* = \frac{1}{\gamma_t} \left\{ c_t \left( x + \mathbb{E}_{\mathbb{Q}} \left[ \frac{1}{\gamma_t} \ln(Z_t) \right] \right) - \ln(Z_t) \right\}. \quad (\text{A11})$$

Notice that  $\xi_t^* \in L^1(\Omega, \mathcal{F}, \mathbb{Q})$  since  $\ln(Z_t)$  is bounded from (1) and  $\frac{1}{\gamma_t} = \exp(\int_0^t (\eta_u - \frac{1}{2} \beta_u^2) du + \int_0^t \beta_u dW_u^{\mathbb{Q}})$  is in  $L^1(\Omega, \mathcal{F}, \mathbb{Q})$  from Assumption (A). Consider now some  $s < t$  and denote by  $\xi_s^*$  the optimal discounted profile for the investment horizon  $s$ , the problem is consistent whenever the condition  $\xi_s^* = \mathbb{E}_{\mathbb{Q}}[\xi_t^* \mid \mathcal{F}_s]$  is verified.

By assumption  $d(\frac{1}{\gamma_t}) = \frac{1}{\gamma_t} (\eta_t dt + \beta_t dW_t^{\mathbb{Q}})$  for  $\eta_t, \beta_t$   $\mathbb{F}$ -adapted processes and with  $W_t^{\mathbb{Q}}$  a  $\mathbb{Q}$ -Brownian motion defined by  $dW_t^{\mathbb{Q}} = dW_t - \theta_t dt$ . Observing that  $d \ln(Z_t) = \frac{1}{2} \theta_t^2 dt + \theta_t dW_t^{\mathbb{Q}}$ , stochastic Leibniz rule yields

$$\begin{aligned} d\left(\frac{1}{\gamma_t} \ln(Z_t)\right) = \frac{1}{\gamma_t} \left( \frac{\theta_t^2}{2} + \eta_t \ln(Z_t) + \theta_t \beta_t \right) dt \\ + \frac{1}{\gamma_t} (\theta_t + \beta_t \ln(Z_t)) dW_t^{\mathbb{Q}}. \end{aligned} \quad (\text{A12})$$

Recall that  $\theta, \eta, \beta$  satisfy the boundedness conditions in Assumption (A). Then in particular we can rewrite the latter in integral form and take the expectation conditional on  $\mathcal{F}_s$  to obtain

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} \left[ \frac{1}{\gamma_t} \ln(Z_t) \mid \mathcal{F}_s \right] \\ = \frac{1}{\gamma_s} \ln(Z_s) + \mathbb{E}_{\mathbb{Q}} \left[ \int_s^t \frac{1}{\gamma_u} \left( \frac{\theta_u^2}{2} + \eta_u \ln(Z_u) + \theta_u \beta_u \right) du \mid \mathcal{F}_s \right]. \end{aligned} \quad (\text{A13})$$

Plugging (A11) into the time consistency condition we obtain

$$c_t \left( x + \mathbb{E}_{\mathbb{Q}} \left[ \frac{1}{\gamma_t} \ln(Z_t) \right] \right) \mathbb{E}_{\mathbb{Q}} \left[ \frac{1}{\gamma_t} \mid \mathcal{F}_s \right] - \mathbb{E}_{\mathbb{Q}} \left[ \frac{1}{\gamma_t} \ln(Z_t) \mid \mathcal{F}_s \right]$$

$$= c_s \left( x + \mathbb{E}_{\mathbb{Q}} \left[ \frac{1}{\gamma_s} \ln(Z_s) \right] \right) \frac{1}{\gamma_s} - \frac{1}{\gamma_s} \ln(Z_s). \quad (\text{A14})$$

Let us set  $k_t = c_t \left( x + \mathbb{E}_{\mathbb{Q}} \left[ \frac{1}{\gamma_t} \ln(Z_t) \right] \right)$  and observing that

$$\mathbb{E}_{\mathbb{Q}} \left[ \frac{1}{\gamma_t} \middle| \mathcal{F}_s \right] = \frac{1}{\gamma_s} + \mathbb{E}_{\mathbb{Q}} \left[ \int_s^t \frac{1}{\gamma_u} \eta_u \, du \middle| \mathcal{F}_s \right],$$

equation (A14) boils down to

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[ \frac{1}{\gamma_t} \ln(Z_t) \middle| \mathcal{F}_s \right] - \frac{1}{\gamma_s} \ln(Z_s) \\ &= (k_t - k_s) \frac{1}{\gamma_s} + k_t \mathbb{E}_{\mathbb{Q}} \left[ \int_s^t \frac{1}{\gamma_u} \eta_u \, du \middle| \mathcal{F}_s \right]. \end{aligned}$$

Substituting (A13) in the latter we find

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[ \int_s^t \frac{1}{\gamma_u} \left( \frac{\theta_u^2}{2} + \eta_u \ln(Z_u) + \theta_u \beta_u \right) \, du \middle| \mathcal{F}_s \right] \\ &= (k_t - k_s) \frac{1}{\gamma_s} + k_t \mathbb{E}_{\mathbb{Q}} \left[ \int_s^t \frac{1}{\gamma_u} \eta_u \, du \middle| \mathcal{F}_s \right], \end{aligned}$$

and a simple algebraic rearrangement then provides

$$\frac{k_t - k_s}{\gamma_s} = \mathbb{E}_{\mathbb{Q}} \left[ \int_s^t \frac{1}{\gamma_u} \left( \frac{\theta_u^2}{2} + \eta_u \ln(Z_u) + \theta_u \beta_u - k_t \eta_u \right) \, du \middle| \mathcal{F}_s \right], \quad (\text{A15})$$

which has to hold for any couple  $s, t \in [0, \infty)$  such that  $s < t$ .

Observe that, setting  $\eta_t = 0$  and  $\beta_t = -\frac{\theta_t}{2}$  for all  $t \geq 0$  we have  $k_t = k_s = x \cdot \gamma_0$  with  $1/\gamma_0 = \mathbb{E}_{\mathbb{Q}} [1/\gamma_t]$  and therefore equation (A15) is trivially verified.

We now show the reverse implication. For  $s < t$  we define the following  $\mathbb{Q}$ -martingale:

$$M_s := \mathbb{E}_{\mathbb{Q}} \left[ \int_0^t \frac{1}{\gamma_u} \left( \frac{\theta_u^2}{2} + \eta_u \ln(Z_u) + \theta_u \beta_u - k_t \eta_u \right) \, du \middle| \mathcal{F}_s \right].$$

In light of equation (A15) we may write

$$M_s = \int_0^s \frac{1}{\gamma_u} \left( \frac{\theta_u^2}{2} + \eta_u \ln(Z_u) + \theta_u \beta_u - k_t \eta_u \right) \, du + \frac{k_t - k_s}{\gamma_s}. \quad (\text{A16})$$

Observe that

$$d \left( \frac{k_t - k_s}{\gamma_s} \right) = (k_t - k_s) \left( \frac{1}{\gamma_s} (\eta_s \, ds + \beta_s \, dW_s^{\mathbb{Q}}) \right) - \frac{1}{\gamma_s} \, dk_s,$$

with

$$\begin{aligned} dk_s &= \left( x + \mathbb{E}_{\mathbb{Q}} \left[ \frac{1}{\gamma_s} \ln(Z_s) \right] \right) \left( -c_s^2 \mathbb{E}_{\mathbb{Q}} \left[ \frac{\eta_s}{\gamma_s} \right] \right) \, ds \\ &\quad + c_s \, d \mathbb{E}_{\mathbb{Q}} \left[ \frac{1}{\gamma_s} \ln(Z_s) \right] \end{aligned}$$

and

$$d \mathbb{E}_{\mathbb{Q}} \left[ \frac{1}{\gamma_s} \ln(Z_s) \right] = \mathbb{E}_{\mathbb{Q}} \left[ \frac{1}{\gamma_s} \left( \frac{\theta_s^2}{2} + \eta_s \ln(Z_s) + \beta_s \theta_s \right) \right] \, ds.$$

Since  $M_s$  is a martingale it has null drift and volatility  $\frac{(k_t - k_s) \beta_s}{\gamma_s}$ . Therefore we can write  $M_s$  as

$$M_0 + \int_0^s \frac{(k_t - k_u) \beta_u}{\gamma_u} \, dW_u^{\mathbb{Q}},$$

with  $M_0 = \mathbb{E}_{\mathbb{Q}} [M_s]$ . Letting  $s \rightarrow t$  one would get

$$M_0 + \int_0^t \frac{(k_t - k_u) \beta_u}{\gamma_u} \, dW_u^{\mathbb{Q}}$$

$$= \int_0^t \frac{1}{\gamma_u} \left( \frac{\theta_u^2}{2} + \eta_u \ln(Z_u) + \theta_u \beta_u - k_t \eta_u \right) \, du, \quad (\text{A17})$$

Let  $\mathbb{B}_{[0, \infty)}$  denote the Borel<sup>†</sup>  $\sigma$ -algebra on  $[0, \infty)$  and  $Leb$  the Lebesgue measure on  $\mathbb{B}_{[0, \infty)}$ . equation (A17) holds true if and only if both the left- and right-hand sides are null. This occurs if and only if

$$\begin{aligned} & \mathbb{P} \left( (k_t - k_u) \beta_u = 0 \text{ and } \eta_u = \frac{-\frac{\theta_u^2}{2} - \theta_u \beta_u}{\ln(Z_u) - k_t} \right) \\ &= 1 \quad \text{for every } u \in [0, \infty) \setminus A \text{ with } Leb(A) = 0. \end{aligned}$$

Indeed, up to a modification of the processes, we can assume that the previous property holds for every  $u \in [0, \infty)$ . On the other hand the choice of  $t \in [0, \infty)$  is arbitrary and the definition of the process  $\eta$  cannot depend on  $t$ . This implies that  $k_t = k$  for every  $t \in [0, \infty)$ .

Since  $k = k_t = c_t(x + \mathbb{E}_{\mathbb{Q}} [\frac{1}{\gamma_t} \ln Z_t])$ , this implies

$$0 = (c_t - c_s)x + c_t \mathbb{E}_{\mathbb{Q}} \left[ \frac{1}{\gamma_t} \ln Z_t \right] - c_s \mathbb{E}_{\mathbb{Q}} \left[ \frac{1}{\gamma_s} \ln Z_s \right] \quad \forall x \in \mathbb{R}.$$

The previous identity leads necessarily to  $c_t = \gamma_0$  and  $\mathbb{E}_{\mathbb{Q}} [\frac{1}{\gamma_t} \ln Z_t] = 0$ , for every  $t \in [0, \infty)$ . Given the dynamics  $d(\frac{1}{\gamma_t}) = \frac{1}{\gamma_t} (\eta_t \, dt + \beta_t \, dW_t^{\mathbb{Q}})$ , this is the case if and only if  $\mathbb{P}(\eta_t = 0) = 1$  for every  $t \in [0, \infty)$  (again up to a modification of the process). By Lemma A.1 jointly with the assumption that  $\text{Var}(\theta_t^2) \neq 0$  we have that  $\beta_t = -\frac{\theta_t}{2}$ .  $\blacksquare$

In order to compute the optimal wealth  $\xi_t^*$ , we first notice that for  $\eta_t = 0$  and  $\beta_t = -\frac{\theta_t}{2}$ , equation (A12) collapses to

$$d \left( \frac{1}{\gamma_t} \ln(Z_t) \right) = \frac{1}{\gamma_t} \left( \theta_t - \frac{\theta_t}{2} \ln(Z_t) \right) \, dW_t^{\mathbb{Q}},$$

so that  $\mathbb{E}_{\mathbb{Q}} [\frac{1}{\gamma_t} \ln(Z_t)] = 0$ . Furthermore,  $1/\gamma_t$  is a  $\mathbb{Q}$ -martingale, with  $\mathbb{E}_{\mathbb{Q}} [\frac{1}{\gamma_t}] = \frac{1}{\gamma_0}$  and in particular  $c_t = \gamma_0$ . Substituting these latter results into (A11) yields

$$\xi_t^* = \frac{1}{\gamma_t} (\gamma_0 x - \ln(Z_t)).$$

Finally, since the market is complete by assumption, the optimal strategy  $\alpha^*$  is the hedging strategy for the above  $t$ -claim. The stochastic differential of  $\xi_t^*$  reads

$$d\xi_t^* = - \left( \frac{\theta_t}{2} \xi_t^* + \frac{\theta_t}{\gamma_t} \right) \, dW_t^{\mathbb{Q}}.$$

Equating the coefficient above with the diffusion coefficient in the portfolio dynamics (2) yields the optimal strategy

$$\alpha_t^* = - \frac{1}{\gamma_t \sigma_t \xi_t^*} \left( \theta_t + \frac{\theta_t}{2} \gamma_t \xi_t^* \right).$$

We conclude the proof by showing that for this choice of  $(\eta_t, \beta_t)$  and the optimal strategy  $\alpha^*$  the process  $\omega_t(V_t^{\alpha^*})$  is a  $\mathbb{P}$ -martingale. Firstly, as  $V_t^{\alpha^*}$  is the replicating portfolio for  $\xi_t^*$  we have that  $\mathbb{P}$ -a.s.  $V_t^{\alpha^*} = \xi_t^*$ , therefore we plug  $\xi_t^*$  into  $\omega_t$  and we obtain

$$\omega_t(\xi_t^*) = - \frac{1}{\gamma_t} Z_t e^{-\gamma_0 x}. \quad (\text{A18})$$

We recall that the process  $\frac{1}{\gamma_t}$  has  $\mathbb{P}$ -dynamics given by  $d(\frac{1}{\gamma_t}) = \frac{1}{\gamma_t} (\frac{\theta_t^2}{2} \, dt - \frac{\theta_t}{2} \, dW_t)$ . Observe that, by virtue of (1), we have  $Z_t \in$

<sup>†</sup> i.e. the  $\sigma$ -algebra generated by the open subsets of  $[0, \infty)$ .

$L^2(\Omega, \mathcal{F}_t, \mathbb{P})$  and  $\frac{1}{\gamma_t} \in L^2(\Omega, \mathcal{F}_t, \mathbb{P})$ . Cauchy-Schwartz inequality then implies that  $\mathfrak{u}_t(\xi_t^*) \in L^1(\Omega, \mathcal{F}_t, \mathbb{P})$ . The stochastic differential of (A18) reads

$$d(\mathfrak{u}_t(\xi_t^*)) = -e^{-\gamma_0 x} \frac{1}{\gamma_t} Z_t \frac{\theta_t}{2} dW_t,$$

which in turn implies  $\mathfrak{u}_t(V_t^{\alpha^*})$  is a martingale under the reference probability measure  $\mathbb{P}$ .

### A.3. Proof of Proposition 3.1

Observe that, given the dynamics of  $\frac{1}{\gamma_t}$  under the martingale measure  $\mathbb{Q}$  and recalling that  $dW_t^{\mathbb{Q}} = dW_t - \theta_t dt$ , we can write

$$\begin{cases} d\left(\frac{1}{\gamma_t}\right) = \frac{1}{\gamma_t} [(\eta_t - \theta_t \beta_t) dt + \beta_t dW_t] \\ \gamma_0 > 0 \end{cases}$$

and consequently we have that  $d\gamma_t = -\gamma_t [(\eta_t - \theta_t \beta_t - \beta_t^2) dt + \beta_t dW_t]$ . Let the portfolio process  $V_t^\alpha$  be as in equation (2) and observe that

$$d(-\gamma_t V_t^\alpha) = \gamma_t V_t^\alpha \left\{ (\theta_t \sigma_t \alpha_t + \eta_t - \theta_t \beta_t - \beta_t^2 + \beta_t \sigma_t \alpha_t) dt - (\sigma_t \alpha_t - \beta_t) dW_t \right\}.$$

Applying Ito's formula to  $\exp(-\gamma_t V_t^\alpha)$  then yields

$$\begin{aligned} d\left(e^{-\gamma_t V_t^\alpha}\right) &= e^{-\gamma_t V_t^\alpha} \gamma_t V_t^\alpha \\ &\times \left\{ \left[ (\sigma_t \alpha_t - \beta_t) \left( \theta_t + \beta_t + \frac{1}{2} \gamma_t V_t^\alpha (\sigma_t \alpha_t - \beta_t) \right) + \eta_t \right] dt \right. \\ &\left. - (\sigma_t \alpha_t - \beta_t) dW_t \right\} \end{aligned}$$

Collecting the results above, the stochastic differential of  $\mathfrak{u}_t(V_t^\alpha) = -\frac{1}{\gamma_t} e^{-\gamma_t V_t^\alpha}$  reads

$$\begin{aligned} d(\mathfrak{u}_t(V_t^\alpha)) &= -\frac{1}{\gamma_t} e^{-\gamma_t V_t^\alpha} \left\{ \left[ \gamma_t V_t^\alpha (\sigma_t \alpha_t - \beta_t) \left( \theta_t + \frac{1}{2} \gamma_t V_t^\alpha (\sigma_t \alpha_t - \beta_t) \right) \right. \right. \\ &\quad \left. \left. + \eta_t (\gamma_t V_t^\alpha + 1) - \theta_t \beta_t \right] dt \right. \\ &\quad \left. + [\beta_t - \gamma_t V_t^\alpha (\sigma_t \alpha_t - \beta_t)] dW_t \right\}. \end{aligned} \quad (\text{A19})$$

To show that  $\mathfrak{u}_t(\omega, x)$  is a forward performance, we need to impose conditions on  $(\eta_t, \beta_t)$  to ensure that  $\mathfrak{u}_t(\omega, V_t^\alpha)$  is a supermartingale for all  $\alpha \in \mathcal{A}$ . Considering that  $-\frac{1}{\gamma_t} e^{-\gamma_t V_t^\alpha} < 0$  for any  $\alpha$ , the latter condition is equivalent to requiring

$$\begin{aligned} \alpha_t^2 \left( \frac{1}{2} (\gamma_t V_t^\alpha)^2 \sigma_t^2 \right) + \alpha_t \left( \gamma_t V_t^\alpha \theta_t \sigma_t - (\gamma_t V_t^\alpha)^2 \sigma_t \beta_t \right) + \frac{1}{2} (\gamma_t V_t^\alpha)^2 \beta_t^2 \\ - \gamma_t V_t^\alpha \theta_t \beta_t + \eta_t (\gamma_t V_t^\alpha + 1) - \theta_t \beta_t \geq 0. \end{aligned} \quad (\text{A20})$$

We consider the parabola  $x \mapsto f(x, y, \eta_t)$  defined as

$$\begin{aligned} f(x, y, \eta_t) &= x^2 \left( \frac{1}{2} (\gamma_t y)^2 \sigma_t^2 \right) + x \left( \gamma_t y \theta_t \sigma_t - (\gamma_t y)^2 \sigma_t \beta_t \right) \\ &\quad + \frac{1}{2} (\gamma_t y)^2 \beta_t^2 - \gamma_t y \theta_t \beta_t + \eta_t (\gamma_t y + 1) - \theta_t \beta_t. \end{aligned}$$

If we compute the  $x$ -coordinate of the vertex of the parabola we have that  $\alpha_t^y = \frac{(\gamma_t y \beta_t - \theta_t)}{\gamma_t y \sigma_t}$ . Imposing that the vertex lies on the  $x$ -axis leads to the condition

$$\eta_t(y) = \frac{\theta_t(\theta_t + 2\beta_t)}{2(\gamma_t y + 1)}.$$

Indeed for any fixed  $y \in \mathbb{R}$  we have  $f(x, y, \eta_t(y)) \geq 0$  for all  $x \in \mathbb{R}$  and  $f(\alpha_t^y, y, \eta_t(y)) = 0$ , which in particular implies  $f(\alpha_t^*, V_t^*, \eta_t^*) = 0$  for

$$\alpha_t^* = \frac{(\gamma_t V_t^* \beta_t - \theta_t)}{\gamma_t V_t^* \sigma_t} \quad \eta_t^* = \frac{\theta_t(\theta_t + 2\beta_t)}{2(\gamma_t V_t^* + 1)}$$

and  $V_t^*$  satisfying

$$dV_t^* = \frac{(\gamma_t V_t^* \beta_t - \theta_t)}{\gamma_t \sigma_t} ((\mu_t - r) dt + \sigma_t dW_t).$$

Moreover, for every other strategy  $\alpha \in \mathcal{A}$  we have the inequality  $f(\alpha_t, V_t^*, \eta_t^*) \geq 0$ . By setting  $\mathfrak{u}_t(\omega, x) = -\frac{1}{\gamma_t(\omega)} e^{-\gamma_t(\omega)x}$  with  $\beta_t$  arbitrary and  $\eta_t = \eta_t^*$  defined as above, then necessarily  $\mathfrak{u}_t(\omega, V_t^{\alpha^*})$  is a  $\mathbb{P}$ -martingale and for any other strategy  $(\alpha_t)$  we have  $\mathfrak{u}_t(\omega, V_t^\alpha)$  is a  $\mathbb{P}$ -supermartingale. In fact once chosen  $\eta_t$  as  $\eta_t^*$  the quantity in (A20) is necessarily positive for every  $\alpha$  and therefore the drift in (A19) is negative, and annihilates only for the choice  $\alpha = \alpha^*$ .

In order to show that the choice  $\eta_t = 0$ ,  $\beta_t = -\frac{\theta_t}{2}$  results in  $\mathfrak{u}_t(\omega, x)$  being a forward performance, it suffices to plug  $\beta_t = -\frac{\theta_t}{2}$  into  $\frac{\theta_t(\theta_t + 2\beta_t)}{2(\gamma_t V_t^* + 1)}$  and observe that the couple satisfies the supermartingality condition. Moreover, setting the left-hand side of (A20) equal to 0, substituting  $\eta_t = 0$ ,  $\beta_t = -\frac{\theta_t}{2}$  and solving for  $\alpha_t$  yields the optimal strategy

$$\alpha_t^* = -\frac{1}{\gamma_t \sigma_t V_t^*} \left( \theta_t + \frac{\theta_t}{2} \gamma_t V_t^* \right).$$

### A.4. Proof of Proposition 3.2

If  $\theta_t = g(t)$  for some function  $g : [0, \infty) \rightarrow \mathbb{R}$ , imposing  $\eta_t = 0$  and  $\beta_t = f(t)$  for some deterministic function  $f : [0, \infty) \rightarrow \mathbb{R}$  then applying Lemma A.1 we have that Problem (4) is consistent.

Conversely, by repeating the same argument as that of Theorem 3.1, we can obtain the relation in equation (A15), which we report below for simplicity

$$\frac{k_t - k_s}{\gamma_s} = \mathbb{E}_{\mathbb{Q}} \left[ \int_s^t \frac{1}{\gamma_u} \left( \frac{\theta_u^2}{2} + \eta_u \ln(Z_u) + \theta_u \beta_u - k_t \eta_u \right) du \mid \mathcal{F}_s \right]. \quad (\text{A21})$$

An identical argument to that in the proof of Theorem 3.1 shows that equation (A21) implies  $\eta_t = 0$  for all  $t \geq 0$ . Subsequently, by Lemma A.1 together with the assumption that  $\theta_t$  is deterministic we conclude that  $\beta_t = f(t)$  for some deterministic function  $f : [0, \infty) \rightarrow \mathbb{R}$ .

### A.5. Proof of Proposition 3.3

First, we observe that the process  $X_t$ , defined as a solution to  $dX_t = X_t \beta_t dW_t$ ,  $X_0 = 1$ , defines a new probability measure  $\mathbb{P}^*$  such that  $X_t = \frac{d\mathbb{P}^*}{d\mathbb{P}}$ . Consequently, fixing some investment horizon  $t > 0$  and considering  $\mathfrak{u}_t(\omega, v) = u(v) \cdot X_t(\omega)$  we have that

$$\sup_{\substack{\alpha \in \mathcal{A} \\ V_0^\alpha = x}} \mathbb{E}_{\mathbb{P}} [\mathfrak{u}_t(V_t^\alpha)] = \sup_{\substack{\alpha \in \mathcal{A} \\ V_0^\alpha = x}} \mathbb{E}^* [\mathfrak{u}_t(V_t^\alpha)], \quad (\text{A22})$$

where the expectation on the right-hand side is taken with respect to the new probability measure  $\mathbb{P}^*$ . Similarly to the previous proofs, the

problem above can be dealt with from the perspective of an infinite dimensional maximization problem. We then rewrite (A22) as

$$\sup \left\{ \mathbb{E}^* [u(\xi)] \mid \xi \in L^1(\Omega, \mathcal{F}_t, \mathbb{Q}) \text{ and } \mathbb{E}_{\mathbb{Q}} [\xi] = x \right\}.$$

Observe that, whenever the function  $u : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\}$  is of exponential form, as in equation (5), the above problem is analogous to the one of Proposition 2.1. In particular, defining the  $\mathbb{P}^*$ -martingale  $\varphi_t := \frac{Z_t}{X_t}$  which has dynamics  $d\varphi_t = \varphi_t(\theta_t - \beta_t) dW_t^*$ , we can directly apply the results of Proposition 2.1 to obtain that the optimization problem is consistent if and only if  $(\theta_t - \beta_t)^2 = k(t)$ , where  $k : \mathbb{R} \rightarrow \mathbb{R}$  is some deterministic function.

Lastly, assuming  $u(v) = -\frac{1}{\gamma} e^{-\gamma v}$ , we show that  $u_t(\omega, v) = u(v) \cdot X_t(\omega)$  with initial datum  $u_0(v) = -\frac{1}{\gamma} e^{-\gamma v}$  is a forward performance if and only if  $k(t) = 0$ . Observe that, given the dynamics of  $V_t^\alpha$  in (2) we have that

$$\begin{aligned} d(u(V_t^\alpha)) &= V_t^\alpha e^{-\gamma V_t^\alpha} \left[ \left( (\mu_t - r)\alpha_t - \frac{1}{2}\gamma V_t^\alpha \sigma_t^2 \alpha_t^2 \right) dt + \sigma_t \alpha_t dW_t \right] \end{aligned}$$

and hence the stochastic differential of  $u_t(V_t^\alpha) = u(V_t^\alpha) \cdot X_t$  is given by

$$\begin{aligned} du_t(V_t^\alpha) &= X_t e^{-\gamma V_t^\alpha} \left[ \left( -\frac{1}{2}\gamma (V_t^\alpha \sigma_t \alpha_t)^2 - V_t^\alpha \sigma_t \alpha_t (\theta_t - \beta_t) \right) dt \right. \\ &\quad \left. + \left( V_t^\alpha \sigma_t \alpha_t - \frac{1}{\gamma} \beta_t \right) dW_t \right]. \end{aligned} \quad (\text{A23})$$

Notice that the drift coefficient of the latter describes a downward parabola. In particular, requiring that the drift be non-positive for all  $\alpha \in \mathcal{A}$  is equivalent to imposing that the  $y$ -coordinate of the vertex of the parabola be 0. This latter condition then implies that  $(\theta_t - \beta_t)^2 = k(t) = 0$ .

Finally, for this choice of  $k(t)$  a simple inspection of (A23) shows that the unique strategy  $\alpha^* \in \mathcal{A}$  such that  $(u_t(V_t^{\alpha^*}))_{t \geq 0}$  is a  $\mathbb{P}$ -martingale is  $\alpha_t^* = 0$ .

## Appendix 2. Additional material

**LEMMA A.2** *Let  $u : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\}$ , be  $\mathcal{F} \times \mathbb{B}_{\mathbb{R}}$ -measurable, with  $u(\omega, \cdot)$  strictly increasing and strictly concave for any  $\omega \in \Omega$ . If for any  $\omega \in \Omega$ ,  $u(\omega, \cdot)$  is differentiable and its derivative is invertible, then for  $\mathbb{Q} \sim \mathbb{P}$  the problem*

$$\sup \left\{ \int_{\Omega} u(\omega, \xi(\omega)) d\mathbb{P} \mid \xi \in L^1(\Omega, \mathcal{F}_t, \mathbb{Q}) \text{ and } \mathbb{E}_{\mathbb{Q}} [\xi] = x_0 \right\}.$$

has a unique solution  $\xi^*(\omega) = F(\omega, \lambda^* Y(\omega))$  where  $Y = \frac{d\mathbb{Q}}{d\mathbb{P}}$ ,  $F(\omega, y) = (u')^{-1}(\omega, y)$  and  $\lambda^*$  solves the equation  $\mathbb{E}_{\mathbb{Q}} [\xi^*] = x_0$ .

*Proof* The Lagrangian function associated to the optimization problem reads

$$\int_{\Omega} u(\xi) d\mathbb{P} - \lambda \left( \int_{\Omega} \xi d\mathbb{Q} - x_0 \right).$$

and by a can change of probability measure we obtain

$$\int_{\Omega} (u(\xi) - \lambda \cdot Y(\xi - x_0)) d\mathbb{P}.$$

Indeed the pointwise first order condition with respect to  $\xi$  becomes

$$u'(\omega, \xi) - \lambda Y(\omega) = 0,$$

so that the candidate for the optimum is  $F(\omega, \lambda Y(\omega))$ . Simple inspections show that for any  $\xi \in L^1(\mathbb{Q})$  with  $\mathbb{E}_{\mathbb{Q}} [\xi] = x_0$  we have

$$\int_{\Omega} u(\xi) d\mathbb{P} \leq \int_{\Omega} u(F(\cdot, \lambda Y)) d\mathbb{P} - \lambda \left( \int_{\Omega} F(\cdot, \lambda Y) d\mathbb{Q} - x_0 \right)$$

and therefore if  $\lambda^*$  solves the equation  $\mathbb{E}_{\mathbb{Q}} [F(\cdot, \lambda^* Y)] = x_0$  then  $F(\omega, \lambda^* Y(\omega))$  is optimal. Uniqueness follows from strict monotonicity and concavity.  $\blacksquare$

### A.6. Proof of Remark 2.2

Fix  $t > 0$ . Analogously to the previous proof we reformulate Problem (4) as its infinite dimensional counterpart as in equation (A1). Considering a power utility function, the optimal profile  $\xi_t^*$  takes the form

$$\xi_t^* = \frac{x \cdot Z_t^\beta}{\mathbb{E}_{\mathbb{Q}} [Z_t^\beta]},$$

where  $\beta = -\frac{1}{1-\gamma}$  and  $Z_t = \frac{d\mathbb{Q}}{d\mathbb{P}}$ . Following a standard argument (see e.g. Björk 2009) let us define

$$Z_t^0 := \exp \left\{ -\frac{1}{2} \int_0^t \theta_u^2 (\beta + 1)^2 du + \int_0^t \theta_u (\beta + 1) dW_u \right\}$$

and observe that  $Z_t^{\beta+1} = Z_t^0 \exp \left\{ \frac{1}{2} \int_0^t \theta_u^2 \beta^2 \gamma du \right\}$ . Denote by  $\mathbb{Q}_0$  the probability measure induced by  $Z_t^0 = \frac{d\mathbb{Q}_0}{d\mathbb{P}}$  and by

$$H_t = \mathbb{E}_{\mathbb{Q}} [Z_t^\beta] = \mathbb{E}_0 \left[ \exp \left\{ \frac{1}{2} \int_0^t \theta_u^2 \beta^2 \gamma du \right\} \right].$$

For some time  $s < t$  let  $\Pi_s$  denote the discounted no-arbitrage price of the optimum  $\xi_t^*$ . The problem is thus time-consistent whenever  $\Pi_s = \xi_s^*$  that is, whenever

$$\frac{x}{H_t} \mathbb{E}_{\mathbb{Q}} [Z_t^\beta | \mathcal{F}_s] = \frac{x \cdot Z_s^\beta}{H_s}. \quad (\text{A24})$$

On the left hand side of (A24) we have that

$$\mathbb{E}_{\mathbb{Q}} [Z_t^\beta | \mathcal{F}_s] = \frac{\mathbb{E}_{\mathbb{P}} [Z_t^{\beta+1} | \mathcal{F}_s]}{\mathbb{E}_{\mathbb{P}} [Z_t | \mathcal{F}_s]} = \frac{\mathbb{E}_{\mathbb{P}} [Z_t^{\beta+1} | \mathcal{F}_s]}{Z_s},$$

Moreover simple computations leads to

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} [Z_t^{\beta+1} | \mathcal{F}_s] &= Z_s^0 \exp \left\{ \frac{1}{2} \int_0^s \theta_u^2 \beta^2 \gamma du \right\} \mathbb{E}_0 \left[ \exp \left\{ \frac{1}{2} \int_s^t \theta_u^2 \beta^2 \gamma du \right\} \mid \mathcal{F}_s \right], \\ &\quad \underbrace{\hspace{10em}}_{Z_s^{\beta+1}} \end{aligned}$$

and therefore (A24) becomes

$$\frac{H_t}{H_s} = \mathbb{E}_0 \left[ \exp \left\{ \frac{1}{2} \int_s^t \theta_u^2 \beta^2 \gamma du \right\} \mid \mathcal{F}_s \right]. \quad (\text{A25})$$

We define the following process

$$M_t := \frac{\exp \left\{ \frac{1}{2} \int_0^t \theta_u^2 \beta^2 \gamma du \right\}}{\mathbb{E}_0 \left[ \exp \left\{ \frac{1}{2} \int_0^t \theta_u^2 \beta^2 \gamma du \right\} \right]}$$

and observe that

$$\begin{aligned} \mathbb{E}_0 [M_t | \mathcal{F}_s] &= \frac{\mathbb{E}_0 \left[ \exp \left\{ \frac{1}{2} \int_0^t \theta_u^2 \beta^2 \gamma du \right\} \mid \mathcal{F}_s \right]}{\mathbb{E}_0 \left[ \exp \left\{ \frac{1}{2} \int_0^t \theta_u^2 \beta^2 \gamma du \right\} \right]} \\ &= \frac{\mathbb{E}_0 \left[ \exp \left\{ \frac{1}{2} \int_s^t \theta_u^2 \beta^2 \gamma du \right\} \mid \mathcal{F}_s \right] \exp \left\{ \frac{1}{2} \int_0^s \theta_u^2 \beta^2 \gamma du \right\}}{\mathbb{E}_0 \left[ \exp \left\{ \frac{1}{2} \int_0^s \theta_u^2 \beta^2 \gamma du \right\} \right] \mathbb{E}_0 \left[ \exp \left\{ \frac{1}{2} \int_s^t \theta_u^2 \beta^2 \gamma du \right\} \mid \mathcal{F}_s \right]} \end{aligned} \quad (\text{A26})$$

Assuming equation (A25) holds we have that  $\mathbb{E}_0 \left[ \exp \left\{ \frac{1}{2} \int_s^t \theta_u^2 \beta^2 \gamma \, du \right\} \mid \mathcal{F}_s \right] \in \mathbb{R}$  and hence we can take it out of the expectation to obtain

$$\mathbb{E}_0 [M_t \mid \mathcal{F}_s] = \frac{\exp \left\{ \frac{1}{2} \int_0^s \theta_u^2 \beta^2 \gamma \, du \right\}}{\mathbb{E}_0 \left[ \exp \left\{ \frac{1}{2} \int_0^s \theta_u^2 \beta^2 \gamma \, du \right\} \right]} = M_s$$

and we conclude  $M_t$  is a  $\mathbb{Q}_0$ -martingale. A simple inspection show that  $M_t$  is of finite variation and by Revuz and Yor (1999, Proposition IV.1.2) it follows that it must be constant. In particular then  $M_t = M_0 = 1$  which entails

$$\exp \left\{ \frac{1}{2} \int_0^t \theta_u^2 \beta^2 \gamma \, du \right\} = \mathbb{E}_0 \left[ \exp \left\{ \frac{1}{2} \int_0^t \theta_u^2 \beta^2 \gamma \, du \right\} \right],$$

for any  $t \in [0, \infty)$ . Observing that we can differentiate with respect of time  $t$  under the expectation operator, taking the derivative on both sides we find that

$$\mathbb{E}_0 \left[ \theta_t^2 \right] = \theta_t^2$$

and since the above holds for any  $t \in [0, \infty)$ , we conclude that  $t \mapsto \theta_t^2$  is a deterministic function.