Structural Change and Economic Growth: Production in the Short Run — A generalisation in terms of vertically hyper-integrated sectors

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Abstract Pasinetti's (1981) *Structural Change and Economic Growth* provides a complete and far reaching theoretical framework for the study of structural change, and therefore of economic development, rooted in in the Classical-Sraffian tradition.

Some attempts have been made, both in the '80s — for instance Siniscalco (1982) and Momigliano & Siniscalco (1986) — and more recently — e.g. Montresor & Vittucci Marzetti (2007a) and Montresor & Vittucci Marzetti (2008) — to use this framework for empirical purposes. However, all these attempts are based on Pasinetti's (1973) paper, i.e. on vertically integrated analysis. It is my contention that, as a consequence, they failed to recognise, and therefore to take advantage of, the main analytical feature of the 1981 book, namely vertical *hyper-integration*.

Actually, when trying to overcome the simplifying assumptions made by Pasinetti (1981) as regards the description of the technique, the starting point should be Pasinetti (1988), and not Pasinetti (1973), the latter being an intermediate step leading to the former.

The aim of the present paper is therefore, first of all, that of highlighting the key differences between Pasinetti (1973) and Pasinetti (1988), in order to show Pasinetti's (1981) vertically *hyper*-integrated character.

In the second place, the whole analytical framework provided by Pasinetti (1981) will be generalised by reintroducing inter-industry relations and allowing for more complex dynamics of economic magnitudes.

This conceptual clarification and analytical generalisation is intended to be the first step of a line of research aiming at using, and extending, the present framework to perform empirical analyses and study the behaviour of actual economic systems.

Keywords Natural system, vertically integrated sectors, vertically hyper-integrated sectors, functional income distribution, natural rates of profit, natural prices.

JEL classification B51,L16,O41

1 Introduction

Pasinetti started developing his multi-sectoral framework at the beginning of the Sixties, with his doctoral dissertation (see Pasinetti 1962). The development of such a framework went through different stages,[1] the milestones of which are Pasinetti (1973), Pasinetti (1981) and Pasinetti (1988).

If the latter work has provided us with a full and explicit generalisation of the notion of vertically integrated sector — namely, with the introduction of the concept of vertically *hyper*-integrated sector, or growing *subsystem* — Pasinetti's (1981) book, though being *naive* in some analytical respects (the very notion of vertically hyper-integrated sector was already *in pectore*, but not completely elaborated), touches upon a great deal of theoretical and practical issues, giving us a reading key to face many problems which have been left unsolved by former economic theory, and most of all many insights to go on working with the Classical/Sraffian approach, by overcoming its major shortcoming — the difficulty in dealing with dynamics and hence with *growth*, which is, without any doubt, the most important feature of all modern economic systems.

It is my contention, therefore, that such an approach to economic theory is a very important starting point to go "back to the future"[2] of Classical Political Economy.

In order to fruitfully do so, however, some preliminary work needs to be done, mainly to fill the gap between Pasinetti (1981) and Pasinetti (1988). This paper is intended to be one of the necessary building blocks.

After presenting, in section 2, the basic notation that will be used all throughout the paper, section 3 provides a brief presentation of the traditional industry-level framework at the basis of Modern Classical Economics. Such a summary is intended to be a reference point to fully understand the main innovations introduced by Pasinetti's work.

Section 4 then presents the main features and categories of vertically integrated (Pasinetti 1973) and vertically *hyper*-integrated (Pasinetti 1988) analysis, trying to stress and clarify the differences between the two, with particular attention to

[1] For details on the stages of development of the concept of vertically hyper-integrated sectors, see Garbellini & Wirkierman (2010, section 6).

[2] To cite Pasinetti himself: Pasinetti (2007, p. 329).

the way in which new investment is treated and therefore net output is defined.

Section 5 then goes to *Structural Change and Economic Growth*, and is divided into three subsections.

Section 5.1 presents the original formulation, though restated in matrix terms and solved as an eigenproblem.

Sections 5.2 and 5.3 are attempts at taking the frameworks developed, respectively, by Pasinetti (1973) and Pasinetti (1988) and restating them in terms analogous to those of Pasinetti (1981), introducing the same categories, magnitudes, and equilibrium conditions — first in vertically integrated and then in vertically *hyper*-integrated terms.

This restatement aims at making it clear that Pasinetti (1981) represents an intermediate stage towards the elaboration of the notion of *growing subsystems*, by stressing both the novelties with respect to Pasinetti (1973) and the analogies with Pasinetti (1988). At the same time, section 5.3 is intended to be the basis for further generalisation of Pasinetti's (1981) framework in vertically hyper-integrated terms and with a more realistic description of the technique in use.

Finally, section 6 is a note on the price system, section 7 discusses some relevant sectoral and aggregate economic magnitudes, and section 8 provides some final remarks.

The Appendices include some algebraic manipulations which I have left implicit in the paper not to take the reader's attention away from the development of the main arguments.

2 Basic notation

Consider an economic system in which m commodities, denoted by the subscript i $(i = 1, 2, \ldots, m)$ are produced. Such commodities can be used *either* as (pure) consumption goods *and/or* as intermediate commodities.

Moreover, make the simplifying assumption that those commodities used as means of production are completely used up in each period, and therefore have to be replaced entirely.[3]

The economic system can be described by:

[3] No treatment of fixed capital is made here. This simplification is intended to be a first step to be followed by a complete treatment of this issue too. However, since extending the description of the technology in use introduces many complications, I have decided to limit myself, for the time being, to consider circulating capital only.

$\mathbf{q}$	=	$[q_i]$:	vector of total quantities;
$\mathbf{x}$	=	$[x_i]$:	vector of per-capita (average) final demand for consumption goods;
$\mathbf{j}$	=	$[j_i]$:	vector of final per-capita (average) demand for investment goods;
$\mathbf{y}$	=	$[y_i]$:	vector of final per-capita (average) demand, with $y_i = x_i + j_i$, $i = 1, 2, \ldots, m$;
$\mathbf{A}$	=	$[a_{ij}]$:	matrix of inter-industry coefficients;
$\mathbf{a}_{ni}$	=	$[a_{ni}]$:	vector of direct labour requirements;
$\mathbf{a}_{in}$	=	$[a_{in}]$:	vector of demand coefficients for consumption goods: $x_i = a_{in}x_n$;
$\mathbf{a}_{k_in}$	=	$[a_{k_in}]$:	vector of demand coefficients for new investment: $j_i = a_{k_in}x_n$;
$\mathbf{s}$	=	$[s_i]$:	vector of intermediate commodities necessary for the production of quantities q_i;
$\mathbf{p}$	=	$[p_i]$:	vector of commodity prices;
		x_n:	total labour.
		g:	rate of growth of population;
		r_i:	rate of growth of per-capita (average) demand of commodity i as a final good; $(i = 1, \ldots, m)$

All throughout the paper, the following conventions will be observed:

- All vectors and matrices will be denoted by boldface symbols, while all scalar quantities by normal type ones;
- all matrices will be denoted by upper case letters, while all vectors by lower case ones;
- all vectors will be intended as column vectors; row vectors will be denoted by transposed vectors;
- a vector with a hat will denote a diagonal matrix with the element of the corresponding vector on the main diagonal.

3 Quantity and price system at the industry level

3.1 A stationary system

Let us suppose to start from a situation of *stationary equilibrium*, i.e. a situation in which the economic system produces, in each period, a total quantity of commodities equal to the final demand for consumption goods plus the productive capacity used up during the production process, in order to be able to satisfy, period after period, the same final demand for consumption goods.

Since there is no growth, there are no new investments, and therefore the net output is given only by final demand for consumption goods: $\mathbf{y} = \mathbf{x} = \mathbf{a}_{in} x_n$.

In such a case, the physical quantity system can be written as:

$$\mathbf{q} = \mathbf{A}\mathbf{q} + \mathbf{y} = \mathbf{A}\mathbf{q} + \mathbf{x} \tag{3.1}$$

and therefore:

$$\mathbf{q} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{x} \tag{3.2}$$

The physical quantities to be produced in the economic system as a whole are given by the direct and indirect physical requirements for the production of the goods entering the vector of final demand $\mathbf{x}$.

Since we aim at describing a situation of equilibrium,[4]we want labour force to be fully employed; we can therefore add a further equation, namely $x_n = \mathbf{a}_{ni}^T \mathbf{q}$, to the physical quantity system, which thus becomes:

$$\begin{bmatrix} \mathbf{I} - \mathbf{A} & -\mathbf{a}_{in} \\ -\mathbf{a}_{ni}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ x_n \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \end{bmatrix} \tag{3.3}$$

or, as an eigenproblem:[5]

$$\begin{cases} (\lambda_q \overline{\mathbf{I}} - \overline{\mathbf{A}}_q)\overline{\mathbf{q}} = \overline{\mathbf{0}} \\ \lambda_q^* = 1 \\ \lambda_q^* = \lambda_q^{max} \end{cases} \tag{3.5}$$

The solution vector, $\overline{\mathbf{q}}$, is the right-hand eigenvector of matrix $\overline{\mathbf{A}}_q$, associated with the eigenvalue $\lambda_q = \lambda_q^* = 1$ which, for $\overline{\mathbf{q}}$ to have all real and non-negative elements, must also be the maximum eigenvalue. In fact, since all elements of matrix $\overline{\mathbf{A}}_q$ are non-negative, we can exploit the Perron-Frobenius theorems, saying that the a non-negative matrix has only one non-negative eigenvector, i.e. the one associated to its maximum eigenvalue.[6]

[4]What the word 'equilibrium' means, in this context, has been already explored in Garbellini & Wirkierman (2010). Suffice here to recall Pasinetti's own words: a single period equilibrium is "a situation in which there is full employment of the labour force and full utilisation of the existing productive capacity" (Pasinetti 1981, pp. 48-49).

[5]Where:

$$\overline{\mathbf{A}}_q = \begin{bmatrix} \mathbf{A} & \mathbf{a}_{in} \\ \mathbf{a}_{ni}^T & 0 \end{bmatrix} \quad \text{and} \quad \overline{\mathbf{q}} = \begin{bmatrix} \mathbf{q} \\ x_n \end{bmatrix} \tag{3.4}$$

[6]For a synthetic exposition of Perron-Frobenius theorems for non-negative matrices, see Pasinetti (1977, pp. 267-276).

The characteristic polynomial associated to this eigenproblem is:

$$|\lambda_q \mathbf{I} - \mathbf{A}| \left(-\lambda_q + \mathbf{a}_{ni}^T (\lambda_q \mathbf{I} - \mathbf{A})^{-1} \mathbf{a}_{in}\right) \tag{3.6}$$

In order to find the $2m+1$ eigenvalues of matrix $\overline{\mathbf{A}}_q$, we have to find the solutions to the characteristic equation, i.e. those values of λ_q making the characteristic polynomial equal to zero. The first factor of the polynomial, i.e. the determinant of matrix $(\lambda_q \mathbf{I} - \mathbf{A})$, cannot be zero, or the inverse $(\lambda_q \mathbf{I} - \mathbf{A})^{-1}$ would fail to exist.[7] Therefore, we concentrate our attention on the part in brackets:

$$\mathbf{a}_{ni}^T (\lambda_q \mathbf{I} - \mathbf{A})^{-1} \mathbf{a}_{in} - \lambda_q = 0 \tag{3.7}$$

We now want to find the conditions for $\lambda_q = \lambda_q^*$ to be an eigenvalue of matrix $\overline{\mathbf{A}}_q$, i.e. a solution of equation (3.7). In order to do so, we substitute $\lambda_q = \lambda_q^*$ into (3.7) itself:

$$\mathbf{a}_{ni}^T (\mathbf{I} - \mathbf{A})^{-1} \mathbf{a}_{in} = 1 \tag{3.8}$$

To see that matrix $\overline{\mathbf{A}}_q$ has no eigenvalues greater than λ_q^*, let us suppose that there exists an eigenvalue $\mu > 1$; this would imply that:

$$\mathbf{a}_{ni}^T (\mu \mathbf{I} - \mathbf{A})^{-1} \mathbf{a}_{in} = \mu \tag{3.9}$$

By Perron-Frobenius theorems, all elements of matrix $(\mu \mathbf{I} - \mathbf{A})^{-1}$ are decreasing functions of μ; therefore, since $\mu > 1$, then $(\mu \mathbf{I} - \mathbf{A})^{-1} < (\mathbf{I} - \mathbf{A})^{-1}$, and hence:

$$\mathbf{a}_{ni}^T (\mu \mathbf{I} - \mathbf{A})^{-1} \mathbf{a}_{in} < 1 < \mu$$

which clearly leads to a contradiction.

Since $\lambda_q^* = \lambda_q^{max}$, and therefore the solution vector for physical quantities is real and non-negative for all possible vectors $\mathbf{a}_{ni}^T$ and $\mathbf{a}_{in}$, in order to completely determine it we have to fix arbitrarily one component, giving us the *scale* of the solution. For the physical quantity system case, the choice is quite obvious, since we have one magnitude — namely total population x_n — which is determined outside the economic system, and which therefore can be taken as given. By setting $x_n = \overline{x}_n$, we can write the solution vector as:

$$\begin{bmatrix} \mathbf{q} \\ x_n \end{bmatrix} = \begin{bmatrix} (\mathbf{I} - \mathbf{A})^{-1} \mathbf{a}_{in} \overline{x}_n \\ \overline{x}_n \end{bmatrix} \tag{3.10}$$

[7] This simply means that if matrix $\mathbf{A}$ has the same eigenvalues as matrix $\overline{\mathbf{A}}_q$, the inverse does not exist. As we will see later on, the maximum eigenvalue of matrix $\mathbf{A}$ must be smaller than one for gross quantities to be non-negative, while we will show that the maximum one of matrix $\overline{\mathbf{A}}_q$ is precisely one.

In conclusion, if λ_q^* is the maximum eigenvalue of matrix $\overline{\mathbf{A}}$, and if condition (3.8) is satisfied, then $\overline{\mathbf{q}}$ is a vector of real and non-negative quantities[8], the solution to our eigenproblem.

Mathematically, expression (3.8) is a condition for our eigenproblem to have non-trivial solutions. From an economic point of view, it is a *macroeconomic condition* which, once satisfied, ensures full employment of the labour force.

As to the price system, it can be written as:

$$\mathbf{p}^T = w\mathbf{a}_{ni}^T + \mathbf{p}^T\mathbf{A} + \mathbf{p}^T\mathbf{A}\pi \tag{3.11}$$

i.e.:

$$\mathbf{p}^T\left(\mathbf{I} - \mathbf{A}(1+\pi)\right) - w\mathbf{a}_{ni}^T = 0 \tag{3.12}$$

We can now follow the same procedure adopted above for the physical quantity system — namely that of characterising a situation of equilibrium — and add a further equation describing a situation of *full expenditure* of total income:

$$wx_n + \mathbf{p}^T\mathbf{A}\pi\mathbf{q} = \mathbf{p}^T\mathbf{y} \tag{3.13}$$

i.e.:

$$-\mathbf{p}^T\left(\mathbf{I} - \mathbf{A}(1+\pi)\right)\left(\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{a}_{in} + w = 0 \tag{3.14}$$

Total wages and total profits must be completely spent. Since we are in a stationary system, in which no new investments are made, the only expenditure recipient is represented by consumption goods.

The price system can thus be stated, in matrix form, as:

$$\begin{bmatrix} \mathbf{p}^T & w \end{bmatrix} \begin{bmatrix} \mathbf{I} - \mathbf{A}(1+\pi) & -\left(\mathbf{I} - \mathbf{A}(1+\pi)\right)\left(\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{a}_{in} \\ -\mathbf{a}_{ni}^T & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{0}^T & 0 \end{bmatrix} \tag{3.15}$$

or as an eigenproblem:[9]

$$\begin{cases} \overline{\mathbf{p}}^T(\lambda_p\overline{\mathbf{I}} - \overline{\mathbf{A}}_p) = \overline{\mathbf{0}} \\ \lambda_p^* = 1 \end{cases} \tag{3.17}$$

[8] This also implies that $(\mathbf{I} - \mathbf{A})^{-1}$ is non-negative, i.e. that its maximum eigenvalue, λ_A^{max}, satisfies $\lambda_A^{max} < 1$

[9] Where:

$$\overline{\mathbf{A}}_p = \begin{bmatrix} \mathbf{A}(1+\pi) & \left(\mathbf{I} - \mathbf{A}(1+\pi)\right)\left(\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{a}_{in} \\ \mathbf{a}_{ni}^T & 0 \end{bmatrix} \tag{3.16}$$

In this case, matrix $\overline{\mathbf{A}}_p$ has some non-positive elements, i.e. off-diagonal elements of matrix $(\mathbf{I} - \mathbf{A}(1+\pi))$. Therefore, we will proceed stating the conditions for $\lambda_p^* = 1$ to be an eigenvalue of matrix $\overline{\mathbf{A}}_p$. Then we will compute the associated eigenvector, and we will derive the conditions for it to be real and non-negative.

The characteristic equation associated to this eigenproblem is:

$$|\mathbf{A}(1+\pi)-\lambda_p\mathbf{I}|\left(-\lambda_p+\mathbf{a}_{ni}^T(\lambda_p\mathbf{I}-\mathbf{A}(1+\pi))^{-1}(\mathbf{I}-\mathbf{A}(1+\pi))(\mathbf{I}-\mathbf{A})^{-1}\mathbf{a}_{in}\right)=0$$

i.e.:

$$\mathbf{a}_{ni}^T(\lambda_p\mathbf{I}-\mathbf{A}(1+\pi))^{-1}(\mathbf{I}-\mathbf{A}(1+\pi))(\mathbf{I}-\mathbf{A})^{-1}\mathbf{a}_{in}=\lambda_p \tag{3.18}$$

When $\lambda_p=\lambda_p^*$, expression (3.18) reduces to:

$$\mathbf{a}_{ni}^T(\mathbf{I}-\mathbf{A}(1+\pi))^{-1}(\mathbf{I}-\mathbf{A}(1+\pi))(\mathbf{I}-\mathbf{A})^{-1}\mathbf{a}_{in}=1 \tag{3.19}$$

i.e.

$$\mathbf{a}_{ni}^T(\mathbf{I}-\mathbf{A})^{-1}\mathbf{a}_{in}=1 \tag{3.20}$$

which is precisely the same condition as the one previously found for the quantity system. Mathematically, it is again a condition for non-trivial solutions to exist. Economically, it is a *macroeconomic condition* for full expenditure (and, from the quantity system, for full employment of the labour force).

Also in this case, in order for the solution vector to be completely determined, we have to fix arbitrarily one component. Since here no magnitude is exogenously given, as it was the case for total population within the quantity system, determining the scale of the solution means choosing a *numéraire* for the price system. Clearly, such a *numéraire* can be any commodity, or composite commodity, whose price has to be taken as given. In this case, we choose labour as the *numéraire* commodity for the price system, therefore setting $w=\overline{w}$.

The solutions for commodity prices therefore are:

$$\begin{bmatrix}\mathbf{p}^T & w\end{bmatrix}=\begin{bmatrix}\overline{w}\mathbf{a}_{ni}^T(\mathbf{I}-\mathbf{A}(1+\pi))^{-1} & \overline{w}\end{bmatrix} \tag{3.21}$$

The condition for them to be non-negative is:

$$\pi^{max}\leq\frac{1-\lambda_A^{max}}{\lambda_A^{max}}$$

where λ_A^{max} is the maximum eigenvalue of matrix $\mathbf{A}$.

3.2 A growing system

Let us now make the assumption that population grows at the constant, exogenous rate $g\geq 0$, and that per-capita demand for commodity i *as a consumption good* is growing at the rate $r_i \lesseqgtr 0$, $(i=1,2,\ldots,m)$. At the aggregate level, therefore, demand for commodity i as a consumption good grows at the rate $(g + r_i)$, $(i = 1, 2, \ldots, m)$.

The total quantities to be produced in period t must now satisfy final demand for consumption goods, replace worn out productive capacity *and* expand it through *new investments*.

In this case, thus, the net output is given by both demand for consumption and demand for new investments:

$$\mathbf{y} = \mathbf{x} + \mathbf{j}$$

where $x_i = a_{in}x_n$ and $j_i = a_{k_in}x_n$.

The quantity system, in this case, is given by:

$$\begin{bmatrix} \mathbf{I}-\mathbf{A} & -(\mathbf{a}_{in}+\mathbf{a}_{k_in}) \\ -\mathbf{a}_{ni}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ x_n \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \end{bmatrix} \tag{3.22}$$

and therefore expression (3.8) becomes:

$$\mathbf{a}_{ni}^T(\mathbf{I}-\mathbf{A})^{-1}(\mathbf{a}_{in}+\mathbf{a}_{k_in}) = 1 \tag{3.23}$$

the solutions being:

$$\begin{bmatrix} \mathbf{q} \\ x_n \end{bmatrix} = \begin{bmatrix} (\mathbf{I}-\mathbf{A})^{-1}(\mathbf{a}_{in}+\mathbf{a}_{k_in})\overline{x}_n \\ \overline{x}_n \end{bmatrix} \tag{3.24}$$

The price system can be written as:

$$\begin{bmatrix} \mathbf{p}^T & w \end{bmatrix} \begin{bmatrix} \mathbf{I}-\mathbf{A}(1+\pi) & -\left(\mathbf{I}-\mathbf{A}(1+\pi)\right)(\mathbf{I}-\mathbf{A})^{-1}(\mathbf{a}_{in}+\mathbf{a}_{k_in}) \\ -\mathbf{a}_{ni}^T & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{0}^T & 0 \end{bmatrix} \tag{3.25}$$

and expression (3.20) becomes:

$$\mathbf{a}_{in}^T(\mathbf{I}-\mathbf{A})^{-1}(\mathbf{a}_{in}+\mathbf{a}_{k_in}) = 1 \tag{3.26}$$

the solutions being:

$$\begin{bmatrix} \mathbf{p}^T & w \end{bmatrix} = \begin{bmatrix} \overline{w}\mathbf{a}_{ni}^T(\mathbf{I}-\mathbf{A}(1+\pi))^{-1} & \overline{w} \end{bmatrix} \tag{3.27}$$

As it can be seen, while gross quantities are different with respect to the stationary case, having to include new investments too, prices are still the same.

4 Vertically integrated and hyper-integrated sectors

When introducing growth in the picture, a crucial role is played by new investments, which are part of the net output in the *current* period, and re-enter the circular flow, as intermediate commodities to be used up by the production process, in the *following* one.

As we are going to see in a moment, the way of treating new investments — and therefore of defining the *net output* — is the key difference between Pasinetti's (1973) and Pasinetti's (1988) approach, i.e. between vertically integrated and vertically *hyper*-integrated analysis.

4.1 Vertically integrated sectors — Pasinetti (1973)

Following Pasinetti (1973), let us define the notion of *vertically integrated* sectors.

The net product of the economy is given by

$$\mathbf{y} = \mathbf{x} + \mathbf{j} \tag{4.1}$$

where $\mathbf{x}$'s i-th element is the quantity of commodity i demanded as a consumption good, and $\mathbf{j}$'s i-th element is the quantity of commodity i demanded as *net* investment. Each vertically integrated sector therefore has, as its final output, a quantity y_i of commodity i, sold both for consumption (x_i) and for new investment (j_i) purposes. Such investment is considered as *exogenous* with respect to technology, and therefore investment goods are treated in the same way as consumption goods.

For each particular y_i, we can write:

$$\mathbf{q}^{(i)} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{y}^{(i)} \tag{4.2}$$

$$\mathbf{s}^{(i)} = \mathbf{A}\mathbf{q}^{(i)} = \mathbf{A}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{y}^{(i)} = \mathbf{H}\mathbf{y}^{(i)} \tag{4.3}$$

$$x_n^{(i)} = \mathbf{a}_{ni}^T\mathbf{q}^{(i)} = \mathbf{a}_{ni}^T(\mathbf{I} - \mathbf{A})^{-1}\mathbf{y}^{(i)} = \mathbf{v}^T\mathbf{y}^{(i)} \tag{4.4}$$

where $\mathbf{y}^{(i)} = \widehat{\mathbf{y}}\mathbf{e}^{(i)}$.

Matrix $\mathbf{H} = [\mathbf{h_i}]$, in expression (4.3), is the matrix of the units of *vertically integrated productive capacity*, i.e. of *direct and indirect* intermediate requirements for the production of the net product $\mathbf{y}$. The i-th column $\mathbf{h}_i$ of such a matrix therefore is a *unit of vertically integrated productive capacity* for vertically integrated sector i, i.e. a *composite commodity* made up by *all* the intermediate commodities directly and indirectly required in the whole economic system for the production of one unit of commodity i as net product.

In the same way, row vector $\mathbf{v}^T$, in expression (4.4), is the vector of *vertically integrated labour coefficients*, i.e. the vector of the quantities of labour directly and indirectly employed for the production of one unit of each good entering the net product $\mathbf{y}$.

As $i = 1, \ldots, m$, we have defined m vertically integrated sectors — or *sub-systems*, using Sraffa's terminology — which add up to the complete economic system, and composed by the ith element of vector $\mathbf{y}$, the ith column of matrix $\mathbf{H}$ and the ith element of vector $\mathbf{v}^T$:

> Consider a system of industries (each producing a different commodity) which is in a self-replacing state.
>
> The commodities forming the gross product [...] can be unambiguously distinguished as those which go to replace the means of production and those which together form the net product of the system.
>
> Such a system can be subdivided into as many parts as there are commodities in its net product, in such a way that each part forms a smaller self-replacing system the net product of which consists of only one kind of commodity. These parts we shall call 'sub-systems'.
>
> [...] Although only a fraction of the labour of a sub-system is employed in the industry which directly produces the commodity forming the net product, yet, since all other industries merely provide replacements for the means of production used up, the whole of the labour employed can be regarded as directly or indirectly going to produce that commodity.
>
> (Sraffa 1960, p. 89)

The *gross* quantities produced during the time period by each vertically integrated sector i ($i = 1, 2, \ldots, m$) are given by its net output $y_i = x_i + j_i$ and by a set of intermediate commodities which go to replace those used up during the production process. That part of the net output constituting new investments, j_i, will re-enter the circular flow the following period as part of the productive capacity, being distributed to *all* the m vertically integrated sectors according to their — *technologically given* once the rate of growth of demand for consumption goods is known — additional production requirements.

Hence, each vertically integrated sector i, in addition to the net product y_i, produces the quantities $\mathbf{Aq}^{(i)}$, i.e. the stock of capital goods necessary *at the beginning* of the time period for the production process to take place — and therefore to be *replaced* during the production process itself:

$$\mathbf{s} = \sum_{i=1}^{m} \mathbf{s}^{(i)} = \mathbf{Aq} = \mathbf{A}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{y} = \mathbf{Hy} \tag{4.5}$$

with

$$\mathbf{s}^{(i)} = \mathbf{Aq}^{(i)} = \mathbf{A}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{y}^{(i)} = \mathbf{Hy}^{(i)} = \mathbf{h}_i y_i \tag{4.6}$$

where $\mathbf{h}_i$ is the i-th column of matrix $\mathbf{H}$.

In the same way, we can express the total amount of labour required for the production of the net output $\mathbf{y}$ as:

$$x_n = \sum_{i=1}^{m} x_n^{(i)} = \mathbf{a}_{ni}^T \mathbf{q} = \mathbf{a}_{ni}^T (\mathbf{I} - \mathbf{A})^{-1} \mathbf{y} = \mathbf{v}^T \mathbf{y} \tag{4.7}$$

with

$$x_n^{(i)} = \mathbf{a}_{ni}^T \mathbf{q}^{(i)} = \mathbf{a}_{ni}^T (\mathbf{I} - \mathbf{A})^{-1} \mathbf{y}^{(i)} = \mathbf{v}^T \mathbf{y}^{(i)} = v_i y_i \tag{4.8}$$

where v_i is the i-th element of row vector $\mathbf{v}^T$.

Given these definitions, system (3.2) can be equivalently written as:

$$\mathbf{q} = \mathbf{A}(\mathbf{I} - \mathbf{A})^{-1} \mathbf{y} + \mathbf{y} = \mathbf{H}\mathbf{y} + \mathbf{y} = (\mathbf{I} + \mathbf{H})\mathbf{y} \tag{4.9}$$

Comparing expressions (3.2) and (4.9), we notice that:[10]

$$(\mathbf{I} - \mathbf{A})^{-1} \equiv (\mathbf{I} + \mathbf{H}) \tag{4.10}$$

Expressions (4.5) and (4.7) can thus be written, respectively, as:

$$\mathbf{s} = \mathbf{A}(\mathbf{I} + \mathbf{H})\mathbf{y} \equiv \mathbf{A}\mathbf{y} + \mathbf{A}\mathbf{H}\mathbf{y} \tag{4.11}$$

i.e. *direct* plus *indirect* capital requirements for the production of net output $\mathbf{y}$, and

$$x_n = \mathbf{a}_{ni}^T (\mathbf{I} + \mathbf{H})\mathbf{y} \equiv \mathbf{a}_{ni}^T \mathbf{y} + \mathbf{a}_{ni}^T \mathbf{H}\mathbf{y} \tag{4.12}$$

i.e. *direct* plus *indirect* labour.

4.2 Vertically hyper-integrated sectors — Pasinetti (1988)

In his 1988 paper, Pasinetti adopts a different approach, *generalising* the concept of vertically integrated sectors to that of vertically *hyper*-integrated sectors.

As already hinted at above, the key difference between the two is the way in which *new investment* is treated.

[10] Clearly, this also follows from the series expansion of matrix $(\mathbf{I} - \mathbf{A})^{-1}$:

$$(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \ldots = \mathbf{I} + \mathbf{A}(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \ldots) = \mathbf{I} + \mathbf{A}(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{H}$$

In Pasinetti (1973), the net product of each vertically integrated sector i is given by $x_i + j_i$, i.e. the quantity of commodity i demanded *both* as a consumption good *and* as a net investment good: new investments are taken as exogenous with respect to technology.

As a consequence, each vertically integrated sector i produces the quantity of commodity i needed by the *whole economic system* as an investment good — and should get from the other sectors the quantities of commodities $j \neq i$ it needs to increase *its own* productive capacity.

On the contrary, Pasinetti (1988) provides a re-definition of the concept of net output, by separating what re-enters the circular flow, namely new investment, from what does not, namely consumption. As a consequence, the net output of a vertically *hyper* integrated sector i is given only by $\mathbf{x}_i$, i.e. the quantity of commodity i demanded as a *consumption* good. New investment is no more considered as exogenous with respect to technology, but as part of it, being determined, in each vertically hyper-integrated sector i $(i = 1, 2, \ldots, m)$, by technology itself, once the *growth requirements*, i.e. the rate of growth of final demand for the corresponding consumption commodity, are known. This means that the new investments are determined by the evolution of *both* technological progress *and* final demand.

The gross quantities produced during the time period by each vertically hyper-integrated sector i are therefore given by a quantity x_i of commodity i demanded for consumption purposes, and by a *batch* of intermediate commodities produced both to replace those used up during the production process and to provide the *additional productive capacity* which will be needed at the beginning of the following period in order to satisfy the increased demand for commodity i as a consumption good.

This approach provides us with a *dynamic* generalisation of Sraffa's subsystems: a subsystem sector is defined as "a system of industries [...] which is in a self-replacing state" (Sraffa 1960, p. 89). It should now be clear, however, that a vertically integrated sector is self-replacing only in a single period of time, within a *static* framework. As soon as we introduce growth, the m vertically integrated sectors conforming the economic system as a whole fail to be independent of each other, having to exchange part of their net output — that devoted to new investments — with the others.

On the contrary, vertically hyper-integrated sectors continue to be self-replacing systems through time when growth is introduced, since they produce *all* the intermediate commodities they need not only to replace what has to be used up in the current period to carry on the production process, but also to *expand* their productive capacity in line with the expansion of demand for the corresponding consumption good.

Analytically, the consequences are straightforward. Each vertically hyper-integrated sector grows at its own rate $g+r_i = c_i$ — the rate of change of demand for the consumption good it produces. Following Pasinetti (1988) and Pasinetti (1989), the total quantities to be produced by each 'hyper-subsystem' — or *growing subsystem* — i are given by:

$$\mathbf{q}^{(i)} = \mathbf{A}\mathbf{q}^{(i)} + \mathbf{A}c_i\mathbf{q}^{(i)} + \mathbf{x}^{(i)} \tag{4.13}$$

i.e.:

$$\mathbf{q}^{(i)} = (\mathbf{I} - \mathbf{H}c_i)^{-1}(\mathbf{I} + \mathbf{H})\mathbf{x}^{(i)} \tag{4.14}$$

At the aggregate level, total quantities $\mathbf{q}$ are given by the sum of the sectoral quantities $\mathbf{q}^{(i)}$, i.e:

$$\mathbf{q} = \sum_{i=1}^{m} \mathbf{q}^{(i)} = \sum_{i=1}^{m} (\mathbf{I} - \mathbf{H}c_i)^{-1}(\mathbf{I} + \mathbf{H})\mathbf{x}^{(i)} \tag{4.15}$$

As shown below in appendix A, expression (4.15) can equivalently be written, under certain conditions, as:

$$\mathbf{q} = (\mathbf{I} + \mathbf{H})(\mathbf{I} - \mathbf{H}\widehat{\mathbf{c}})^{-1}\mathbf{x} \tag{4.16}$$

Using these definitions, we can derive the expressions for sectoral and aggregate capital stocks and labour employment.

The aggregate and sectoral capital stocks are given by:

$$\mathbf{s} = \sum_{i=1}^{m} \mathbf{s}^{(i)} = \mathbf{A}\mathbf{q} = \mathbf{H}(\mathbf{I} - \mathbf{H}\widehat{\mathbf{c}})^{-1}\mathbf{x} = \mathbf{M}\mathbf{x} \tag{4.17}$$

with

$$\mathbf{s}^{(i)} = \mathbf{A}\mathbf{q}^{(i)} = \mathbf{H}(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} = \mathbf{M}^{(i)}\mathbf{x}^{(i)} \tag{4.18}$$

or, since $(\mathbf{I} - \mathbf{H}c_i)^{-1} = \mathbf{I} + \mathbf{H}c_i(\mathbf{I} - \mathbf{H}c_i)^{-1}$:

$$\begin{aligned}\mathbf{s}^{(i)} &= \mathbf{A}(\mathbf{I} + \mathbf{H})(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} = \left(\mathbf{A}(\mathbf{I} - \mathbf{H}c_i)^{-1} + \mathbf{A}\mathbf{H}(\mathbf{I} - \mathbf{H}c_i)^{-1}\right)\mathbf{x}^{(i)} = \\ &= \mathbf{A}(\mathbf{I} + c_i\mathbf{M}^{(i)})\mathbf{x}^{(i)} + \mathbf{A}\mathbf{M}^{(i)}\mathbf{x}^{(i)} = \mathbf{A}\mathbf{x}^{(i)} + \mathbf{A}\mathbf{M}^{(i)}\mathbf{x}^{(i)} + c_i\mathbf{A}\mathbf{M}^{(i)}\mathbf{x}^{(i)}\end{aligned} \tag{4.19}$$

At the beginning of the time period, therefore, each vertically hyper-integrated sector i needs to be provided with a productive capacity which is the sum of three components:

- Intermediate commodities directly required for the production of commodity i as a consumption good — *direct productive capacity* $\mathbf{A}\mathbf{x}^{(i)}$;
- Intermediate commodities directly required for the replacement of those intermediate commodities which will be used up, in the whole vertically hyper-integrated sector, during the production process — *indirect productive capacity* $\mathbf{A}\mathbf{M}^{(i)}\mathbf{x}^{(i)}$;
- Intermediate commodities directly required for the expansion of productive capacity according to the over-all increase in the demand for commodity i as a consumption good — *hyper-indirect productive capacity* $c_i\mathbf{A}\mathbf{M}^{(i)}\mathbf{x}^{(i)}$.

Thus, $\mathbf{M}$ is the matrix of direct, indirect and hyper-indirect *aggregate* productive capacity for the production of one unit of each commodity entering final demand for consumption goods $\mathbf{x}$. Matrices $\mathbf{M}^{(i)}$ are the matrices of vertically hyper-integrated productive capacity. More specifically, $\mathbf{m}_i^*$, i.e. the i-th column of $\mathbf{M}^{(i)}$, is a unit of vertically hyper-integrated productive capacity for the corresponding vertically hyper-integrated sector i.

Symmetrically, the aggregate and sectoral quantities of employed labour are given by:

$$x_n = \sum_{i=1}^{m} x_n^{(i)} = \mathbf{a}_{ni}^T\mathbf{q} = \mathbf{v}^T(\mathbf{I} - \mathbf{H}\hat{\mathbf{c}})^{-1}\mathbf{x} = \mathbf{z}^T\mathbf{x} \tag{4.20}$$

with

$$\begin{aligned} x_n^{(i)} &= \mathbf{a}_{ni}^T\mathbf{q}^{(i)} = \mathbf{v}^T(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} = \mathbf{z}^{(i)T}\mathbf{x}^{(i)} \\ &= \mathbf{a}_{ni}^T\mathbf{x}^{(i)} + \mathbf{a}_{ni}^T\mathbf{M}^{(i)}\mathbf{x}^{(i)} + c_i\mathbf{a}_{ni}^T\mathbf{M}^{(i)}\mathbf{x}^{(i)} \end{aligned} \tag{4.21}$$

where $\mathbf{z}^T$ is the vector of aggregate direct, indirect and hyper-indirect labour, and z_i^*, i.e. the i-th component of each vector $\mathbf{z}^{(i)T}$, is the vertically hyper-integrated labour coefficient for sector i.

5 Structural change and economic growth

In *Structural Change and Economic Growth* Pasinetti himself states that "all production processes will be considered as vertically integrated" (Pasinetti 1981, p. 29), and that "the notion of 'vertically integrated sectors, which is here used, has been generalised in my article 'Vertical Integration in Economic Analysis', *Metroeconomica*, 1973" (Pasinetti 1981, p. 29n). All sectors are split up into two parts, i.e. a final industry producing the net output — consisting of the *consumption* good — and a 'vertically integrated' industry producing the capital goods directly, indirectly and hyper-indirectly needed by the former.

But now that the difference between vertically integrated and hyper-integrated sectors has been made clear, it should be straightforward to conclude that Pasinetti (1981) framework is actually formulated in vertically *hyper*-integrated terms. In fact, the net output is made up *only* by consumption goods; new investments commodities are produced together with the intermediate ones used up during the production process and therefore to be replaced. Thus, new investments are treated here as in Pasinetti (1988): they are *all* the capital goods — in this case one homogeneous commodity due to the particular simplifying assumptions made on the technique in use — needed by the final industry to expand its productive capacity in order to produce, period after period, the quantity of commodity i demanded as a *consumption good*; their production takes place at the vertically (hyper-)integrated level, i.e. in the capital goods industry, not in the final one, and each subsystem is independent of all the others, producing all intermediate commodities it needs, without buying anything from or selling anything to the others.[11]

In what follows we will first give a synthetic exposition of Pasinetti's (1981) original formulation,[12] and then try to re-state both Pasinetti (1973) and Pasinetti (1988) in the same analytical terms, in order to show that Pasinetti's (1981) approach is a vertically *hyper*-integrated one.

5.1 Pasinetti's formulation

In *Structural Change and Economic Growth*, Pasinetti adopts a step-by-step approach: he first presents a pure labour model, in which all production activities are carried out with labour alone — the system produces consumption goods only. Then, he extends the framework by adding capital goods, which are used together with labour for the production of consumption goods, but whose production again requires labour alone — we shall refer to this 'version' of the model as the *intermediate case*. Finally, he presents what he defines the more general version of the

[11]As argued elsewhere (Garbellini & Wirkierman 2010, section 6), even if a complete and explicit recognition of the notion of vertical hyper-integration has been reached and exposed only in Pasinetti (1988), the idea had already emerged in 1977. The way in which new investments are treated clearly shows that, though not always explicitly stated, Pasinetti's (1981) sectors actually are vertically *hyper*-integrated.

[12]Though with the simplification of considering only stocks of *circulating* capital, in order to avoid further complications and keep the analysis as simple as possible. See footnote 3.

framework, in which both consumption and capital goods are produced by means of both labour and capital goods.

This last version of the model, anyway, has been left aside by Pasinetti (1981) himself — the most important results are developed also for this case, but mainly in footnotes, and the focus is entirely on the intermediate step.

I will follow here exactly the same procedure, by briefly exposing the intermediate version of Pasinetti's (1981) framework.[13] In this case, however, the reason for doing so is a very specific one. As will be shown later on,[14] in all Pasinetti's (1981) formulations, productive capacity is measured in terms of units of *direct* productive capacity, that is to say, the amount of intermediate commodities *directly* required for the production of one unit of a certain commodity. But, due to the particular simplifying assumptions adopted, there is no *analytical* — even if a fundamental and deep *conceptual* — difference, in the intermediate case, between direct, indirect, and hyper-indirect productive capacity, since capital goods are produced by means of labour alone. Therefore, it is particularly convenient to adopt this formulation, since it is straightforward to interpret the units of productive capacity as vertically hyper-integrated ones, and therefore to read the main results in these terms.

It is my contention that this reading key is useful first of all to fully understand how far reaching Pasinetti's (1981) work is. Many implications have not been fully grasped before due to the failure in understanding its vertically hyper-integrated character. In the second place, it provides a link between Pasinetti (1981) and Pasinetti (1988), allowing to use the more complete analytical formulation of the latter to generalise and extend the conclusions of the former.

Pasinetti's (1981) quantity system, in this intermediate case and in matrix terms, is given by:

$$\begin{bmatrix} \mathbf{I} & \mathbf{O} & -\mathbf{a}_{in} \\ -\mathbf{I} & \mathbf{I} & -\mathbf{a}_{k_i n} \\ -\mathbf{a}_{ni}^T & -\mathbf{a}_{nk_i}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x_k} \\ x_n \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ 0 \end{bmatrix} \tag{5.1}$$

where:

(i) $\mathbf{x}$ is the vector of physical quantities of final consumption commodities $i = 1, 2, \ldots, m$;

(ii) $\mathbf{x_k}$ is the vector of physical quantities of intermediate (capital) commodities $k_i = k_1, k_2, \ldots, k_m$ (measured in units of productive capacity). Here, the simplifying assumption is made that each intermediate commodity k_i is specific for the production of the corresponding consumption commodity i

[13] A very concise exposition of the more complex case is given in appendix A.4.

[14] And also briefly exposed in Garbellini & Wirkierman (2010).

— and that intermediate commodities themselves are produced by means of labour alone. As I have already said, with respect to Pasinetti's (1981) original formulation, an additional simplifying assumption is made, i.e. that there is circulating capital only;[15]

(iii) $\mathbf{a}_{in}$ is the vector of demand coefficients for final consumption commodities $i = 1, 2, \ldots, m$;

(iv) $\mathbf{a}_{k_i n}$ is the vector of demand coefficients of intermediate commodities $k_i = k_1, k_2, \ldots, k_m$ for new investment, i.e. of per-capita demand for the *units* of (vertically hyper-integrated) productive capacity;

(v) $\mathbf{a}_{ni}^T$ is the vector of (direct) labour requirements for the production of final consumption commodities $i = 1, 2, \ldots, m$;

(vi) $\mathbf{a}_{nk_i}^T$ is the vector of (direct) labour requirements for the production of intermediate commodities $k_i = k_1, k_2, \ldots, k_m$.

It must be further stressed that intermediate commodities are measured by means of a particular unit of measurement, i.e. *units of vertically hyper-integrated productive capacity*: direct, indirect and hyper-indirect requirements for the production of one unit of commodity i as a consumption good.

System (5.1) is made up by three series of equations.

The first one concerns consumption goods, the quantities of which are determined by consumers' effective demand.

The second one concerns capital goods. The quantity to be produced of each capital good i must be enough to replace worn-out productive capacity and provide for the new investment commodities demanded by the final sector.

The last equation is the full-labour-employment one.

The price system is given by:

$$\begin{bmatrix} \mathbf{p}^T & \mathbf{p_k}^T & w \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{O} & -\mathbf{a}_{in} \\ -(\mathbf{I} + \widehat{\boldsymbol{\pi}}) & \mathbf{I} & \widehat{\boldsymbol{\pi}}\mathbf{a}_{in} - \mathbf{a}_{k_i n} \\ -\mathbf{a}_{ni}^T & -\mathbf{a}_{nk_i}^T & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{0}^T & \mathbf{0}^T & 0 \end{bmatrix} \qquad (5.3)$$

where $\mathbf{p}^T$ is the vector of consumption commodities prices, $\mathbf{p_k}^T$ is the vector of intermediate commodities prices and $\widehat{\boldsymbol{\pi}}$ is a diagonal matrix with the sectoral rates of profit on the main diagonal.

[15]Pasinetti, on the contrary, considers fixed capital also. Therefore, his physical quantity coefficient matrix would be:

$$\begin{bmatrix} \mathbf{I} & \mathbf{O} & -\mathbf{a}_{in} \\ -\widehat{\mathbf{T}}^{-1} & \mathbf{I} & -\mathbf{a}_{k_i n} \\ -\mathbf{a}_{ni}^T & -\mathbf{a}_{nk_i}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x_k} \\ x_n \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ 0 \end{bmatrix} \qquad (5.2)$$

where T_i^{-1} is the depreciation rate for (vertically hyper-integrated) sector i.

Both the quantity and the price system are linear and homogeneous systems of equations, and can be written as eigenproblems:[16]

$$\begin{cases} (\mathbf{A_x} - \lambda_x \mathbf{I})\mathbf{x} = \mathbf{0} \\ \lambda_x^* = 1 \\ \lambda_x^* = \lambda_x^{max} \end{cases} \tag{5.6}$$

for the physical quantity system; and:

$$\begin{cases} \mathbf{p}^T(\mathbf{A_p} - \lambda_p \mathbf{I}) = \mathbf{0}^T \\ \lambda_p^* = 1 \\ \lambda_p^* = \lambda_p^{max} \end{cases} \tag{5.7}$$

for the commodity price system.

As to the quantity system, the characteristic equation associated to expression (5.6) is:

$$\begin{aligned} &\begin{vmatrix} -\lambda_x \mathbf{I} & \mathbf{O} \\ \mathbf{I} & -\lambda_x \mathbf{I} \end{vmatrix} \left(-\lambda - \begin{bmatrix} \mathbf{a}_{ni}^T & \mathbf{a}_{nk_i}^T \end{bmatrix} \begin{bmatrix} -\lambda_x \mathbf{I} & \mathbf{O} \\ \mathbf{I} & -\lambda_x \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{a}_{in} \\ \mathbf{a}_{k_i n} \end{bmatrix} \right) = \\ &= \lambda_x^{2m} \left(-\lambda_x + \frac{1}{\lambda_x} \mathbf{a}_{ni}^T \mathbf{a}_{in} + \frac{1}{\lambda_x^2} \mathbf{a}_{nk_i}^T \mathbf{a}_{in} + \frac{1}{\lambda_x} \mathbf{a}_{nk_i}^T \mathbf{a}_{k_i n} \right) = \\ &= \lambda_x^{2m-2} \left(-\lambda_x^3 + \lambda_x (\mathbf{a}_{ni}^T \mathbf{a}_{in} + \mathbf{a}_{nk_i}^T \mathbf{a}_{k_i n}) + \mathbf{a}_{nk_i}^T \mathbf{a}_{in} \right) = 0 \end{aligned}$$

Therefore, the first $2m - 2$ eigenvalues are repeated eigenvalues equal to zero. The remaining three eigenvalues are the solution of the equation obtained by setting the third degree polynomial in brackets in expression (5.8) equal to zero:

$$(-\lambda_x^3 + \lambda_x (\mathbf{a}_{ni}^T \mathbf{a}_{in} + \mathbf{a}_{nk_i}^T \mathbf{a}_{k_i n}) + \mathbf{a}_{nk_i}^T \mathbf{a}_{in}) = 0 \tag{5.8}$$

What we are left to do now is to find out the conditions for λ_x^* to be one root of this equation, and then to show that, once such condition is satisfied, the two other solutions are smaller than λ_x^* itself.

[16] Where

$$\mathbf{A_x} = \begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{a}_{in} \\ \mathbf{I} & \mathbf{O} & \mathbf{a}_{k_i n} \\ \mathbf{a}_{ni}^T & \mathbf{a}_{nk_i}^T & 0 \end{bmatrix} \tag{5.4}$$

and:

$$\mathbf{A_p} = \begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{a}_{in} \\ \mathbf{I} + \widehat{\boldsymbol{\pi}} & \mathbf{O} & \mathbf{a}_{k_i n} - \widehat{\boldsymbol{\pi}} \mathbf{a}_{in} \\ \mathbf{a}_{ni}^T & \mathbf{a}_{nk_i}^T & 0 \end{bmatrix} \tag{5.5}$$

Finding the condition for $\lambda_x^* = 1$ to be a solution of equation (5.8), means finding the condition that, if satisfied, allows us to find three scalars a, b and c such that equation (5.8) can be written as:

$$(\lambda_x - 1)(a\lambda_x^2 + b\lambda_x + c)$$

By using Ruffini's rule, the condition for being able to decompose the third degree polynomial in equation (5.8) in this way emerges as:

$$\mathbf{a}_{ni}^T \mathbf{a}_{in} + \mathbf{a}_{nk_i}^T \mathbf{a}_{in} + \mathbf{a}_{nk_i}^T \mathbf{a}_{k_i n} = 1 \tag{5.9}$$

This condition being satisfied, we can finally write:

$$(\lambda_x - 1)\left(-\lambda_x^2 - \lambda_x - \mathbf{a}_{nk_i}^T \mathbf{a}_{in}\right) \tag{5.10}$$

or, equivalently:

$$(\lambda_x - 1)\left(-\lambda_x^2 - \lambda_x - 1 + \mathbf{a}_{ni}^T \mathbf{a}_{in} + \mathbf{a}_{nk_i}^T \mathbf{a}_{k_i n}\right) \tag{5.11}$$

The last thing that we have to show is thus that both solutions of the second degree polynomial in brackets are smaller than one. In principle, in computing such solutions, we should consider all the possible cases as to the discriminant of the second degree polynomial, i.e.:

1. $\Delta < 0$: we have two complex solutions;

2. $\Delta > 0$: we have two real, distinct solutions;

3. $\Delta = 0$: we have two real, repeated solutions.

However, we are looking for *real* eigenvalues, and therefore we can rule out case 1 and focus attention on cases 2 and 3.

In the simplest case, i.e. when $\Delta = 0$,[17] in which two repeated eigenvalues are equal to $-1/2 < 1$.

Finally, consider the case in which $\Delta > 0$; here we have two distinct solutions, i.e.:

$$\lambda_q^{1,2} = \frac{-1 \pm \sqrt{\Delta}}{2} = \frac{-1 \pm \sqrt{1 - 4\mathbf{a}_{nk_i}^T \mathbf{a}_{in}}}{2}$$

Clearly, we are interested in the greater one only, which is not greater than 1 when:

$$\mathbf{a}_{ni}^T \mathbf{a}_{in} + \mathbf{a}_{nk_i}^T \mathbf{a}_{k_i n} \leq 3 \tag{5.12}$$

[17] For this to be true, the demand and direct labour coefficients must be such that $\mathbf{a}_{nk_i}^T \mathbf{a}_{in} = 0.25$. Clearly, this is a very special case.

or, equivalently, when:

$$\mathbf{a}_{nk_i}^T \mathbf{a}_{in} \geq -2 \tag{5.13}$$

i.e. in all economically meaningful cases.

Hence, when condition (5.9) is satisfied, $\lambda_x^* = 1 = \lambda_x^M$. Such a condition is the *macroeconomic condition* for full employment of the labour force, and it is the sum of three addenda:

- $\mathbf{a}_{ni}^T \mathbf{a}_{in}$: direct labour required for the production of consumption commodities — *direct labour*;
- $\mathbf{a}_{nk_i}^T \mathbf{a}_{in}$: direct labour required for replacing the units of productive capacity used up during the production process — *indirect labour*;
- $\mathbf{a}_{nk_i}^T \mathbf{a}_{k_i n}$: direct labour required for the production of the units of productive capacity demanded as new investment commodities, i.e. in order to expand productive capacity — *hyper-indirect labour*.

The vector of physical quantities for consumption and intermediate commodities, therefore, is the right-hand-side eigenvector associated to $\lambda_x^* = 1$, which is completely determined once we fix one component — in this case, following Pasinetti (1981), once we set $x_n = \overline{x}_n$:

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{x_k} \\ x_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_{in}\overline{x}_n \\ \mathbf{x} + \mathbf{a}_{k_i n}\overline{x}_n \\ \overline{x}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_{in}\overline{x}_n \\ (\mathbf{a}_{in} + \mathbf{a}_{k_i n})\overline{x}_n \\ \overline{x}_n \end{bmatrix} \tag{5.14}$$

As to the price system, we have first of all to notice that, in order for matrix $\mathbf{A_p}$ to be non-negative, the following condition should hold:

$$\widehat{\boldsymbol{\pi}} \leq (\widehat{\mathbf{a}_{k_i n}^{-1}})\mathbf{a}_{in}$$

or:

$$\pi_i \leq \frac{a_{in}}{a_{k_i n}}, \qquad i = 1, 2, \ldots, m \tag{5.15}$$

which means that the number of units of productive capacity that, evaluated at current prices, give us the profit component of prices is smaller than or at most equal to the number of units of final consumption commodities to be produced during the production process. Since the profit component of prices defines the amount of value created in excess with respect to replacements (and wages), this condition being satisfied would imply that the realised profits could allow, *at most*, to produce *in each sector* a number of units of productive capacity exactly equal to that required by the expansion of productive capacity in line with the evolution of final demand for consumption commodities.[18]

This is not necessarily so. Therefore, we will follow here the same procedure followed in section 3 to solve the industry-level price system, i.e. that of looking for the condition(s) guaranteeing that $\lambda_p^* = 1$ be an eigenvalue of matrix $\overline{\mathbf{A}}_p$, then computing the associated left-hand eigenvector, and then again finding out the conditions for this vector to be non-negative.

The characteristic equation associated to expression (5.7) is:

$$\begin{vmatrix} -\lambda_p \mathbf{I} & \mathbf{O} \\ \mathbf{I} + \widehat{\boldsymbol{\pi}} & -\lambda_p \mathbf{I} \end{vmatrix} \left(-\lambda_p - \begin{bmatrix} \mathbf{a}_{ni}^T & \mathbf{a}_{nk_i}^T \end{bmatrix} \begin{bmatrix} -\lambda_p \mathbf{I} & \mathbf{O} \\ \mathbf{I} + \widehat{\boldsymbol{\pi}} & -\lambda_p \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{a}_{in} \\ \mathbf{a}_{k_i n} \end{bmatrix} \right) =$$
$$= \lambda_p^{2m} \left(-\lambda_p + \frac{1}{\lambda_p} \mathbf{a}_{ni}^T \mathbf{a}_{in} + \frac{1}{\lambda_p^2} \mathbf{a}_{nk_i}^T \mathbf{a}_{in} + \frac{1}{\lambda_p^2} \mathbf{a}_{nk_i}^T \mathbf{a}_{in} - \frac{1}{\lambda_p} \mathbf{a}_{nk_i}^T \mathbf{a}_{in} + \frac{1}{\lambda_p} \mathbf{a}_{nk_i}^T \mathbf{a}_{k_i n} \right) =$$
$$= \lambda_p^{2m-2} \left(-\lambda_p^3 + \lambda_p \mathbf{a}_{ni}^T \mathbf{a}_{in} + \mathbf{a}_{nk_i}^T \mathbf{a}_{in} + \mathbf{a}_{nk_i}^T \widehat{\boldsymbol{\pi}} \mathbf{a}_{in} - \lambda_p \mathbf{a}_{nk_i}^T \widehat{\boldsymbol{\pi}} \mathbf{a}_{in} + \lambda_p \mathbf{a}_{nk_i}^T \mathbf{a}_{k_i n} \right) = 0 \tag{5.16}$$

What we want to do now is finding the condition that makes $\lambda^* = 1$ an eigenvalue of matrix $\overline{\mathbf{A}}_p$, i.e. to be a solution of equation (5.16). For this to be true, by substituting $\lambda_p = \lambda_p^*$ into (5.16), the expression in brackets must be equal to zero.

Therefore, the condition for $\lambda_p^* = 1$ to be an eigenvalue of matrix $\overline{\mathbf{A}}_p$ is:

$$\mathbf{a}_{ni}^T \mathbf{a}_{in} + \mathbf{a}_{nk_i}^T \mathbf{a}_{in} + \mathbf{a}_{nk_i}^T \mathbf{a}_{k_i n} = 1 \tag{5.17}$$

which is exactly the same condition as (5.9), obtained above for $\lambda_x^* = 1$ to be an eigenvalue of matrix $\overline{\mathbf{A}}_x$.

Hence, expression (5.9) is a necessary condition for the price system also to have non-trivial solutions, and moreover, it is a *macroeconomic condition* for full expenditure of income and — as we know from the quantity system — for full employment of the labour force.

The vector of commodity prices is the left-hand-side eigenvector of matrix $\mathbf{A}_p$ associated to $\lambda_p^* = 1$, and is completely determined once one component is arbitrarily fixed. In this case, this amounts to choosing a *numéraire* for the price system;

[18]As will become clear later on, after having provided the single-period equilibrium conditions, and assuming that all sectoral profits are invested, this requirement is satisfied in stock-equilibrium. It is therefore also satisfied in a condition of stock disequilibrium characterised by a lack of productive capacity. It would not be satisfied though in a situation of stock disequilibrium with non-utilised productive capacity, which is anyway a perfectly possible situation.

again following Pasinetti (1981), we chose labour as the *numéraire* commodity, therefore setting $w = \overline{w}$, and obtaining:

$$\left[\begin{array}{c} \mathbf{p}^T \\ \mathbf{p_k}^T \\ w \end{array}\right]^T = \left[\begin{array}{l} \overline{w}\mathbf{a}_{ni}^T + \mathbf{p_k}^T(\mathbf{I} + \widehat{\boldsymbol{\pi}}) \\ \overline{w}\mathbf{a}_{nk_i}^T \\ \overline{w} \end{array}\right]^T = \left[\begin{array}{l} \overline{w}(\mathbf{a}_{ni}^T + \mathbf{a}_{nk_i}^T(\mathbf{I} + \widehat{\boldsymbol{\pi}})) \\ \overline{w}\mathbf{a}_{nk_i}^T \\ \overline{w} \end{array}\right]^T \tag{5.18}$$

which is always non-negative provided that

$$\pi_i \geq -\frac{a_{ni} + a_{nk_i}}{a_{nk_i}}, \quad \forall i = 1, 2, \ldots, m$$

Now that we have the solution vectors for physical quantities and commodity prices, we can analyse in more details the (single-period) *equilibrium conditions*.

We have already said what the word 'equilibrium' means within Pasinetti's (1981) framework: it is a situation in which labour force is fully employed, income is fully spent and *productive capacity is fully utilised*. Macroeconomic condition (5.9) concerns the *flows* of the economic system, and guarantees to comply with the first two equilibrium requirements.

The third equilibrium requirement, on the contrary, concerns the *stocks* of the economic system: each vertically hyper-integrated sector must be provided, at the beginning of the time period, with the number of units of productive capacity allowing it to carry on the production process in line with final demand requirements. Hence, we do not have a single condition, but rather a *series* of *sectoral conditions*. Before stating them, we must accordingly introduce a new series of *sectoral* magnitudes:

$$\mathbf{k} = [k_i], \qquad i = 1, 2, \ldots, m$$

where k_i is the number of units of (vertically hyper-integrated) productive capacity necessary *at the beginning of the production process* for it to be carried out.

Therefore, in order for productive capacity to be fully utilised, the following series of sectoral conditions must be satisfied:

$$\mathbf{k} = \mathbf{x} \tag{5.19}$$

The statement of macroeconomic condition (5.9) and of sectoral conditions (5.19) closes the exposition of Pasinetti's (1981) framework analysing production in the short run.

For the purposes of the present paper, this is all we need to know about Pasinetti's (1981) analytical formulation in order to compare it with Pasinetti (1973) and Pasinetti (1988).

5.2 Vertically integrated sectors

We now want to re-state Pasinetti's (1973) framework in terms formally analogous to Pasinetti's (1981) formulation. In doing so, we have to take into account the already mentioned major differences between the two as to the description of the technique.

In Pasinetti (1981), a specific commodity is *either* a consumption good *or* an intermediate commodity; moreover, each specific capital good k_i is only devoted to the production of the corresponding consumption commodity i. The only *inter-industry* flows are therefore those going from industry k_i to industry i $(i = 1, 2, \ldots, m)$, and each sector is conformed by two industries.

In Pasinetti (1973) — and also in Pasinetti (1988) — on the contrary, any of the m commodities produced in the economic system can be used *both* as a consumption good *and* as an intermediate commodity. Therefore, there is no neat distinction, in general, between consumption and intermediate commodities. Such a distinction arises only within each vertically integrated sector i, where only commodity i is produced as net output, while *all* commodities (included commodity i itself) are produced as intermediate commodities. In a few words, each commodity i appears as a final commodity only in the corresponding sector i, while it appears as an intermediate commodity in *all* sectors.

It is still possible to think of particular intermediate commodities specific to each sector, but of course in this case they will be *composite* commodities, whose constituent elements are the same in all sectors, though entering them in *sector-specific proportions*.

Moreover, as already mentioned above, in Pasinetti (1973) the net product of each sector is made up by the sum of two components: x_i, i.e. the quantity of commodity i demanded as a *consumption good*, and j_i, i.e. the quantity of commodity i demanded as an *investment commodity*. In this way, vertically integrated sector i produces a part of *its own* new productive capacity, i.e. the i-th component, *and* a part of that of *all other sectors*. Therefore, the batch of commodities to be devoted to new investment are not produced together with the capital goods, but together with, and 'in the same way as', the consumption goods.

In terms of inter-industry and inter-sectoral flows, all inter-industry relations are reintroduced, and there are also some inter-sectoral flows, all sectors selling to the others part of their net product, and buying from all the others part of their net product, in order to build up new productive capacity.

Hence, the physical quantity system is:

$$\begin{bmatrix} \mathbf{I} & \mathbf{O} & -(\mathbf{a}_{in} + \mathbf{a}_{k_i n}) \\ -\mathbf{I} & \mathbf{I} & \mathbf{O} \\ -\mathbf{a}_{ni}^T & -\mathbf{a}_{ni}^T \mathbf{H} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{x}_k \\ x_n \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ 0 \end{bmatrix} \tag{5.20}$$

where $\mathbf{a}_{ni}^T\mathbf{H} = \mathbf{a}_{nk_i}^T$ and a_{k_in} is per-capita demand for commodity i as an investment good.

The system stated above can also be written as an eigenproblem as follows:[19]

$$\begin{cases} (\lambda_x\overline{\mathbf{I}} - \overline{\mathbf{A}}_x)\overline{\mathbf{q}} = \overline{\mathbf{0}} \\ \lambda_x^* = 1 \\ \lambda_x^* = \lambda_x^{max} \end{cases} \tag{5.22}$$

the characteristic equation being:

$$\left| \begin{array}{cc} -\lambda_x\mathbf{I} & \mathbf{O} \\ \mathbf{I} & -\lambda_x\mathbf{I} \end{array} \right| \left(-\lambda_x - \left[\begin{array}{cc} \mathbf{a}_{ni}^T & \mathbf{a}_{ni}^T\mathbf{H} \end{array} \right] \left[\begin{array}{cc} -\lambda_x\mathbf{I} & \mathbf{O} \\ \mathbf{I} & -\lambda_x\mathbf{I} \end{array} \right]^{-1} \left[\begin{array}{c} \mathbf{a}_{in} + \mathbf{a}_{k_in} \\ \mathbf{0} \end{array} \right] \right) = 0 \tag{5.23}$$

Solving and rearranging we get:

$$(\lambda_x)^{2m} \left(-\lambda_x + \frac{1}{\lambda_x}\mathbf{a}_{ni}^T(\mathbf{a}_{in} + \mathbf{a}_{k_in}) + \frac{1}{\lambda_x^2}\mathbf{a}_{ni}^T\mathbf{H}(\mathbf{a}_{in} + \mathbf{a}_{k_in}) \right) = 0 \tag{5.24}$$

or:

$$(\lambda_x)^{2m-2} \left(-\lambda_x^3 + \lambda_x\mathbf{a}_{ni}^T(\mathbf{a}_{in} + \mathbf{a}_{k_in}) + \mathbf{a}_{ni}^T\mathbf{H}(\mathbf{a}_{in} + \mathbf{a}_{k_in}) \right) = 0 \tag{5.25}$$

The characteristic equation associated to this eigenproblem has $2m-2$ repeated roots equal to zero. We are left with three other possibly *real* eigenvalues, the solutions to the polynomial in the second brackets.

If

$$\mathbf{a}_{ni}^T(\mathbf{a}_{in} + \mathbf{a}_{k_in}) + \mathbf{a}_{ni}^T\mathbf{H}(\mathbf{a}_{in} + \mathbf{a}_{k_in}) = 1$$

i.e.:

$$\mathbf{a}_{ni}^T(\mathbf{I} + \mathbf{H})(\mathbf{a}_{in} + \mathbf{a}_{k_in}) \equiv \mathbf{v}^T(\mathbf{a}_{in} + \mathbf{a}_{k_in}) = 1 \tag{5.26}$$

then expression (5.25) can be re-written as:

$$\lambda_x^{2m-1}(\lambda_x^* - 1)(\lambda_x^2 + \lambda_x + 1 - \mathbf{a}_{ni}^T(\mathbf{a}_{in} + \mathbf{a}_{k_in})) = 0 \tag{5.27}$$

The solution resulting from the first expression in brackets is precisely $\lambda_x^* = 1$, while the last two are the solutions of the second degree equation in the second brackets. If real, i.e. if:

$$\mathbf{a}_{ni}^T\mathbf{H}(\mathbf{a}_{in} + \mathbf{a}_{k_in}) < \frac{3}{4}$$

[19] Where:

$$\overline{\mathbf{A}}_x = \left[\begin{array}{ccc} \mathbf{O} & \mathbf{O} & \mathbf{a}_{in} + \mathbf{a}_{k_in} \\ \mathbf{I} & \mathbf{O} & \mathbf{O} \\ \mathbf{a}_{ni}^T & \mathbf{a}_{ni}^T\mathbf{H} & 0 \end{array} \right] \tag{5.21}$$

we want them not to be greater than one; this would imply:

$$\mathbf{a}_{ni}^T \mathbf{H}(\mathbf{a}_{in} + \mathbf{a}_{k_i n}) \geq -2$$

which of course is true in all economically meaningful cases.

Expression (5.26) is the *macroeconomic condition* for full-employment of the labour force, analogous to expression (5.9) found out in the previous section for Pasinetti's (1981) original framework. In this case, however, we can see that it is the sum of *two* components:

- $\mathbf{a}_{ni}^T(\mathbf{a}_{in} + \mathbf{a}_{k_i n})$: labour directly needed for the production of consumption *and* new investment commodities: *direct labour*;

- $\mathbf{a}_{ni}^T \mathbf{H}(\mathbf{a}_{in} + \mathbf{a}_{k_i n})$: labour directly needed for the replacement of the intermediate commodities used up during the production process: *indirect labour*.

Once we set $x_n = \overline{x}_n$, the right-hand eigenvector associated to $\lambda_x^* = \lambda_x^{max} = 1$ gives us the solutions for physical quantities, i.e.:

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{x}_k \\ x_n \end{bmatrix} = \begin{bmatrix} (\mathbf{a}_{in} + \mathbf{a}_{k_i n})\overline{x}_n \\ (\mathbf{a}_{in} + \mathbf{a}_{k_i n})\overline{x}_n \\ \overline{x}_n \end{bmatrix} \tag{5.28}$$

As to the price system, it can be written as:

$$\begin{bmatrix} \mathbf{p}^T & \mathbf{p}_k^T & w \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{O} & -(\mathbf{a}_{in} + \mathbf{a}_{k_i n}) \\ -\pi \mathbf{I} & \mathbf{I} - \mathbf{H}\pi & \pi(\mathbf{a}_{in} + \mathbf{a}_{k_i n}) \\ -\mathbf{v}^T & -\mathbf{v}^T\mathbf{H} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{0}^T & \mathbf{0}^T & 0 \end{bmatrix} \tag{5.29}$$

or, in eigen form, as:[20]

$$\begin{cases} \overline{\mathbf{p}}^T(\lambda_p^* \overline{\mathbf{I}} - \overline{\mathbf{A}}_p) = \overline{\mathbf{0}}^T \\ \lambda_p^* = 1 \end{cases} \tag{5.31}$$

[20]Where:

$$\overline{\mathbf{A}}_p = \begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{a}_{in} + \mathbf{a}_{k_i n} \\ \pi \mathbf{I} & \mathbf{H}\pi & -\pi(\mathbf{a}_{in} + \mathbf{a}_{k_i n}) \\ \mathbf{v}^T & \mathbf{v}^T\mathbf{H} & 0 \end{bmatrix} \tag{5.30}$$

Here we do not want $\lambda_p^* = 1$ to be the maximum eigenvalue, since the matrix has at least some negative element, and therefore Perron Frobenius theorems do not apply. We will instead follow the same procedure already followed above, namely that of stating the condition for it to be an eigenvalue, in order to compute the associated eigenvector and therefore finding out the conditions for it to be real and non-negative.

The characteristic equation is:

$$\begin{vmatrix} -\lambda_p \mathbf{I} & \mathbf{O} \\ \pi \mathbf{I} & \mathbf{H}\pi - \lambda_p \mathbf{I} \end{vmatrix} \times$$
$$\times \left(-\lambda_p - \begin{bmatrix} \mathbf{v}^T & \mathbf{v}^T\mathbf{H} \end{bmatrix} \begin{bmatrix} -\lambda_p \mathbf{I} & \mathbf{O} \\ \pi \mathbf{I} & \mathbf{H}\pi - \lambda_p \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{a}_{in} + \mathbf{a}_{k_i n} \\ -\pi(\mathbf{a}_{in} + \mathbf{a}_{k_i n}) \end{bmatrix} \right) = 0 \tag{5.32}$$

Since the determinant appearing as the first factor of expression (5.32) is:

$$(-1)^m \lambda_p^m d_{\mathbf{H}\pi}$$

where $d_{\mathbf{H}\pi}$ is the determinant of matrix $(\mathbf{H}\pi - \lambda_p^* \mathbf{I})$, and the inverse of such matrix can be written as:

$$\frac{1}{d_{\mathbf{H}\pi}} (\mathbf{H}\pi - \lambda_p \mathbf{I})^{(+)}$$

where $(\mathbf{H}\pi - \lambda_p^* \mathbf{I})^{(+)}$ is the adjoint matrix, the inverse matrix in expression (5.32) can be written as:

$$\begin{bmatrix} -\lambda_p^{m-1} d_{\mathbf{H}\pi} \mathbf{I} & \mathbf{O} \\ \pi \lambda_p^{m-1} (\mathbf{H}\pi - \lambda_p^* \mathbf{I})^{(+)} & \lambda_p^m (\mathbf{H}\pi - \lambda_p \mathbf{I})^{(+)} \end{bmatrix} \tag{5.33}$$

Hence, expression (5.32) becomes, after some manipulations:

$$(-1)^m \Big(-\lambda_p^{m+1} d_{\mathbf{H}\pi} - \lambda_p^{m-1} d_{\mathbf{H}\pi} \mathbf{v}^T (\mathbf{a}_{in} + \mathbf{a}_{k_i n}) - \pi \lambda_p^{m-1} \mathbf{v}^T \mathbf{H} (\mathbf{H}\pi - \lambda_p^* \mathbf{I})^{(+)} (\mathbf{a}_{in} + \mathbf{a}_{k_i n}) +$$
$$+ \pi \lambda_p^m \mathbf{v}^T \mathbf{H} (\mathbf{H}\pi - \lambda_p^* \mathbf{I})^{(+)} (\mathbf{a}_{in} + \mathbf{a}_{k_i n}) = 0 \Big) \tag{5.34}$$

With $\lambda_p^* = 1$ it reduces to:

$$(-1)^m d_{\mathbf{H}\pi} \left(-1 + \mathbf{v}^T (\mathbf{a}_{in} + \mathbf{a}_{k_i n}) \right) = 0 \tag{5.35}$$

since up to this point $d_{\mathbf{H}\pi}$, with $\lambda_p^* = 1$, is a scalar which does not depend any more on λ_p itself.

Therefore, the condition for $\lambda_p^* = 1$ to be an eigenvalue of matrix $\mathbf{A_p}$ is:

$$\mathbf{v}^T (\mathbf{a}_{in} + \mathbf{a}_{k_i n}) = 1 \tag{5.36}$$

which is precisely the same condition as the one found above for the quantity system, guaranteeing full expenditure of income as well as full employment of the labour force.

What is left to do is to check whether the associated eigenvector is real and non-negative. Such eigenvector, once we set $w = \overline{w}$, is:

$$\left[\begin{array}{ccc} \mathbf{p}^T & \mathbf{p}_k^T & w \end{array} \right] = \left[\begin{array}{c} \overline{w}\mathbf{v}^T(\mathbf{I} - \mathbf{H}\pi)^{-1} \\ \overline{w}\mathbf{v}^T\mathbf{H}(\mathbf{I} - \mathbf{H}\pi)^{-1} \\ \overline{w} \end{array} \right]^T \tag{5.37}$$

Using Perron-Frobenius theorems (see also Pasinetti 1977, p. 89) we can conclude that it is non-negative when:

$$\pi^{max} < \frac{1}{\lambda_H^{max}} \tag{5.38}$$

i.e. when (uniform) rate of profit is smaller than the maximum eigenvalue of matrix $\mathbf{H}$.

5.3 Vertically hyper-integrated sectors

Before going to the reformulation of Pasinetti's (1988) framework, it is worth spending a few words on the particular unit of measurement used for intermediate commodities. In Pasinetti (1981), *direct* productive capacity is used as a unit of measurement; however, not only such a choice cannot be made here, but it is also possible to read Pasinetti's (1981) framework in vertically hyper-integrated terms.

Going back to section 4.2, the physical quantity system for the i-th vertically integrated sector can be written as:

$$\mathbf{q}^{(i)} = (\mathbf{I} + \mathbf{H})(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} \tag{5.39}$$

or

$$\mathbf{q}^{(i)} = \mathbf{x}^{(i)} + \mathbf{H}\mathbf{x}^{(i)} + (\mathbf{I} + \mathbf{H})\mathbf{H}c_i(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} = \mathbf{x}^{(i)} + \mathbf{q_k}^{(i)} \tag{5.40}$$

where $\mathbf{q_k}^{(i)}$ is the batch of commodities entering vertically hyper-integrated sector i's *gross* investment.

As we have already said, in Pasinetti (1981) the capital goods in each vertically integrated sector are measured in units of direct productive capacity for the corresponding final (i.e. consumption) commodity. This is possible thanks to the assumption according to which such productive capacity consists of a homogeneous capital good. In the more general formulation, this not the case anymore, since productive capacities are *composite* commodities. This entails no *conceptual* difficulty, since the productive capacity of each vertically integrated sector can be seen as a particular composite commodity, in which intermediate goods enter in particular proportions.

The analytical problem arises because the proportions in which the various commodities enter direct, indirect, and hyper-indirect productive capacity are different, so that it is not possible to say that the batch of commodities necessary, for example, for the production of one unit of direct productive capacity for a certain final commodity is a scalar multiple of such productive capacity itself. This could happen only in the very particular case in which the eigenvectors of matrix $\mathbf{A}$ were its own columns.

More specifically, the demand for capital goods in 'traditional' units, using a formulation analogous to Pasinetti's (1981) one, is:

$$\mathbf{q_k}^{(i)} = \mathbf{A}\mathbf{x}^{(i)} + \mathbf{A}\mathbf{q_k}^{(i)} + \mathbf{H}c_i(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} \tag{5.41}$$

i.e. as the sum of: the intermediate commodities directly necessary for the production of the final consumption commodities ($\mathbf{A}\mathbf{x}^{(i)}$); the intermediate commodities directly necessary for the production of the whole set of capital goods ($\mathbf{A}\mathbf{q_k}^{(i)}$); and new investment ($\mathbf{H}c_i(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)}$).[21]

In order to express it in units of *direct* productive capacity for consumption good i, such capacity being the i-th column of matrix $\mathbf{A}$, we should be able to write the above expression as:

$$\beta_k \mathbf{a}_i x_i = \beta_x \mathbf{a}_i x_i + \beta_{Ak} \mathbf{a}_i x_i + \beta_{Mc_i} \mathbf{a}_i x_n \tag{5.42}$$

with β_k, β_x, β_{Ak}, and β_{Mc_i} being all scalars. Direct comparison of the left-hand side of equations (5.41) and (5.42) reveals that this is not possible. Since

$$\mathbf{q_k}^{(i)} = \mathbf{H}(1 + c_i)(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} = (\mathbf{I} + \mathbf{H})(1 + c_i)(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{a}_i x_i \tag{5.43}$$

such comparison would imply that:

$$\mathbf{H}(1 + c_i)(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} = (1 + c_i)(\mathbf{I} + \mathbf{H})(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{a}_i x_i = \beta_k \mathbf{a}_i x_i \tag{5.44}$$

which could be possible only if $\mathbf{a}_i$ were an eigenvector of matrix $(\mathbf{I}+\mathbf{H})(\mathbf{I}-\mathbf{H}c_i)^{-1}$ and β_k the associated eigenvalue. But this matrix, by definition, has exactly the same eigenvectors as matrix $\mathbf{A}$. Therefore, unless in very special cases (e.g. if $\mathbf{A}$ is diagonal, i.e. the very special case considered by Pasinetti (1981)), equivalence (5.44) shall not hold, and direct productive capacity cannot be used as a unit of measurement for capital goods in the vertically hyper-integrated sector.

What we must actually do, therefore, is choosing another unit of measurement for intermediate commodities. While in the previous section, dealing with vertical integration, we solved the problem by using a unit of vertically integrated

[21] See appendix A.2 for details.

productive capacity as the unit of measurement, here the most obvious choice is represented by the units of vertically *hyper*-integrated productive capacity for consumption good i.

As to Pasinetti's (1981) formulation, if we stick to the *intermediate case* — i.e. with consumption commodities produced by means of labour and intermediate commodities, and capital goods produced by means of labour alone — we can actually read all the analytical formulations as if they were written in vertically hyper-integrated terms. The reason is very simple: since no capital goods are required for the production of capital goods themselves, there is no difference, in physical terms, between direct, vertically integrated and vertically hyper-integrated productive capacity. In fact, both indirect and hyper-indirect requirements are those intermediate commodities required for the production of capital goods directly used up in the production of the final consumption commodity, either in the current or in future periods, and therefore, in this simplified case, they are simply equal to zero. All productive capacity needed by this simplified economic system therefore reduces to the direct one.

Going back to the general case, the expression for $\mathbf{q_k}^{(i)}$ can be equivalently written as:[22]

$$\mathbf{q_k}^{(i)} = \mathbf{M}^{(i)}\mathbf{x}^{(i)} + c_i\mathbf{M}^{(i)}\mathbf{x}^{(i)} = \mathbf{m}_i^* x_i + c_i \mathbf{m}_i^* a_{in} x_n \tag{5.45}$$

where $\mathbf{m}^*$ is the i-th — i.e. the *relevant* — column of matrix $\mathbf{M}^{(i)}$. Therefore, using $\mathbf{m}_i^*$ as the measurement unit, we can write the above expression as:

$$x_{k_i} = x_i + c_i a_{in} x_n \tag{5.46}$$

This means that the number of units of vertically hyper-integrated productive capacity to be produced during the production process (x_{k_i}) in sector i ($i = 1, 2, \ldots, m$) is given by the number of units of final consumption commodity i necessary to satisfy final demand (x_i) plus the number of units of such a consumption commodity which will be additionally demanded in the following period ($c_i a_{in} x_n$), for which additional productive capacity must be set up.

In order to complete our reformulation of Pasinetti's (1988) quantity system, what is left is the last equation. In particular, we need to specify the meaning of the coefficients a_{nk_i} in the present context. In Pasinetti (1981), such coefficients were the direct labour necessary for the production of one unit of *direct* productive capacity for the consumption good i. Since the definition of productive capacity adopted here is that of vertically hyper-integrated productive capacity, these coefficients come to have a different meaning — namely, that of direct labour necessary

[22]See appendix A.2 for details.

for the production of one unit of vertically hyper-integrated productive capacity — and we can accordingly write:

$$a_{nk_i} = \mathbf{a}_{ni}^T \mathbf{m}^{(i)} \tag{5.47}$$

Now we have all the elements we need to write down the physical quantity system which, in matrix form, is:

$$\begin{bmatrix} \mathbf{I} & \mathbf{O} & -\mathbf{a}_{in} \\ -\mathbf{I} & \mathbf{I} & -\widehat{\mathbf{c}}\mathbf{a}_{in} \\ -\mathbf{a}_{ni}^T & -\mathbf{a}_{ni}^T \overline{\mathbf{M}} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x_k} \\ x_n \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ 0 \end{bmatrix} \tag{5.48}$$

where $\overline{\mathbf{M}}$ is a matrix made up by the relevant columns of matrices $\mathbf{M}^{(i)}$, $\forall i = 1, 2, \ldots, m$, i.e. a matrix whose i-th column is $\mathbf{m}^{(i)}$.[23]

As usual, we can state expression (5.48) as an eigenproblem:

$$\begin{cases} (\lambda_x \mathbf{I} - \overline{\mathbf{A}}_x)\overline{\mathbf{x}} = \overline{\mathbf{0}} \\ \lambda_x^* = 1 \\ \lambda_x^* = \lambda_x^{max} \end{cases} \tag{5.49}$$

In order for $\overline{\mathbf{x}}$ to be a real and positive eigenvector of the *non-negative* matrix $\overline{\mathbf{A}}_x$, $\lambda_x^* = 1$ must be a solution of the characteristic equation associated to this eigenproblem; more specifically, the *maximum* solution.

The characteristic equation of this eigenproblem is:

$$\lambda_x^{2m} \left(-\lambda_x - \begin{bmatrix} \mathbf{a}_{ni}^T & \mathbf{a}_{ni}^T \overline{\mathbf{M}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\lambda_x}\mathbf{I} & \mathbf{O} \\ -\frac{1}{\lambda_x^2}\mathbf{I} & -\frac{1}{\lambda_x}\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{a}_{in} \\ \widehat{\mathbf{c}}\mathbf{a}_{in} \end{bmatrix} \right) = 0 \tag{5.50}$$

i.e.:

$$\lambda^{2m-2} \left(-\lambda^3 + \lambda(\mathbf{a}_{ni}^T \mathbf{a}_{in} + \mathbf{a}_{ni}^T \overline{\mathbf{M}} \widehat{\mathbf{c}} \mathbf{a}_{in}) + \mathbf{a}_{ni}^T \overline{\mathbf{M}} \mathbf{a}_{in} \right) = 0 \tag{5.51}$$

Thus, the first $2m-2$ solutions are all zeros. We are left with the second factor, which is a third degree equation in λ_x. If

$$\mathbf{a}_{ni}^T \mathbf{a}_{in} + \mathbf{a}_{ni}^T \overline{\mathbf{M}} \mathbf{a}_{in} + \mathbf{a}_{ni}^T \overline{\mathbf{M}} \widehat{\mathbf{c}} \mathbf{a}_{in} = 1 \tag{5.52}$$

[23] As it is shown in appendix A.1, under certain conditions — the same as the ones making (4.16) equivalent to (4.15) — $\overline{\mathbf{M}}$ is *approximately* equal to $\mathbf{M}$. However, I have preferred to develop all the following elaborations in terms of $\overline{\mathbf{M}}$ rather than $\mathbf{M}$. In view of the possibility of empirical applications of this framework, the fact that they are not precisely the same means that the less disaggregated the data available are, the bigger the difference between the two matrices is. The same arguments will hold for the two vectors $\overline{\mathbf{z}}^T$ and $\mathbf{z}^T$.

such equation can be decomposed in the following way:

$$(\lambda_x^* - 1)(-\lambda_x^2 - \lambda_x - 1 + \mathbf{a}_{ni}^T\mathbf{a}_{in} + \mathbf{a}_{ni}^T\overline{\mathbf{M}}\widehat{\mathbf{c}}\mathbf{a}_{in}) = 0 \tag{5.53}$$

i.e. we have one solution equal to 1, which is the one we are looking for, and then two other solutions, resulting from the second degree equation in (5.53).

These solutions are real when:

$$\mathbf{a}_{ni}^T\mathbf{a}_{in} + \mathbf{a}_{ni}^T\overline{\mathbf{M}}\widehat{\mathbf{c}}\mathbf{a}_{in} \geq \frac{3}{4}, \quad \text{or} \quad \mathbf{a}_{ni}^T\overline{\mathbf{M}}\mathbf{a}_{in} \leq \frac{1}{4}$$

and they are smaller than (or equal to) unity when:

$$\mathbf{a}_{ni}^T\mathbf{a}_{in} + \mathbf{a}_{ni}^T\overline{\mathbf{M}}\widehat{\mathbf{c}}\mathbf{a}_{in} \leq 3, \quad \text{or} \quad \mathbf{a}_{ni}^T\overline{\mathbf{M}}\mathbf{a}_{in} \geq -2$$

i.e. in all economically relevant cases.

Expression (5.52) is the *macroeconomic condition* for full employment of the labour force, analogous to expressions (5.9) — for Pasinetti (1981) case — and (5.26) — for Pasinetti (1973) case. Anyway, if there was an asymmetry between (5.9) and (5.26), we can see that such asymmetry has disappeared with respect to (5.52), since we again have the sum of *three* components:

- $\mathbf{a}_{ni}^T\mathbf{a}_{in}$: direct labour for the production of consumption commodities — *direct labour*;
- $\mathbf{a}_{ni}^T\overline{\mathbf{M}}\mathbf{a}_{in}$: direct labour for the replacement of the units of productive capacity used up during the production process — *indirect labour*;
- $\mathbf{a}_{ni}^T\overline{\mathbf{M}}\widehat{\mathbf{c}}\mathbf{a}_{in}$: direct labour for the production of the units of productive capacity needed to expand productive capacity in line with the evolution of demand for consumption commodities — *hyper-indirect labour*.

Thus, once condition (5.52) is satisfied, and once we set $x_n = \overline{x}_n$, the right-hand eigenvector associated to $\lambda_x^* = 1$ is the solution vector for physical quantities, i.e.:

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{x}_k \\ x_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_{in}\overline{x}_n \\ (\mathbf{I} + \widehat{\mathbf{c}})\mathbf{a}_{in}\overline{x}_n \\ \overline{x}_n \end{bmatrix} \tag{5.54}$$

As to the price system, by following the procedure suggested by Pasinetti (1988, section 4), we may notice that we have m *equivalent* ways of expressing the price system:

$$\begin{gathered}\mathbf{p}^T = w\mathbf{a}_{ni}^T + \mathbf{p}^T\mathbf{A} + \mathbf{p}^T\mathbf{A}\pi \equiv w\mathbf{a}_{ni}^T + \mathbf{p}^T\mathbf{A}(1 + c_i) + \mathbf{p}^T\mathbf{A}(\pi - c_i) \\ \forall i = 1, 2, \ldots, m\end{gathered} \tag{5.55}$$

and hence:

$$\mathbf{p}^T = w\mathbf{z}^{(i)T} + \mathbf{p}^T\mathbf{M}^{(i)}(\pi - c_i), \qquad \forall i = 1, 2, \ldots, m \tag{5.56}$$

Since the i-th element of vector $\mathbf{p}^T$, i.e. the price of commodity i, can be written as:

$$p_i = \overline{w}z_i^* + \mathbf{p}^T\mathbf{m}_i^*(\pi - c_i) \tag{5.57}$$

and thus expression (5.56) can be equivalently written also as:

$$\mathbf{p}^T \equiv \overline{w}\overline{\mathbf{z}}^T + \mathbf{p}^T\overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}}) \equiv \overline{w}\mathbf{z}^{(i)T} + \mathbf{p}^T\mathbf{M}^{(i)}(\pi - c_i) \tag{5.58}$$

Now, vertically hyper-integrated sector i does not produce only commodity i, but also all the intermediate commodities that utilises as inputs for producing commodity i itself as a consumption good. This means that the sector also produces the corresponding units of vertically hyper-integrated productive capacity, i.e. $\mathbf{m}_i^*$, *and* all the commodities directly, indirectly and hyper-indirectly necessary for producing these units of productive capacity, and so on.

Therefore, in principle we should derive, for each vertically hyper-integrated sector i ($i = 1, 2, \ldots$), not a single price for composite commodity $\mathbf{m}_i^*$, i.e. p_{k_i}, but rather a whole vector of prices $\mathbf{p}_k^{(i)T}$, given by

$$\mathbf{p}_k^{(i)T} = \mathbf{p}^T\mathbf{M}^{(i)}, \qquad i = 1, 2, \ldots, m \tag{5.59}$$

The i-th element of such vector is the price of one unit of productive capacity for consumption commodity i. But then, all the elements, included the i-th one, can be used to compute the price of a unit of productive capacity for productive capacity itself, and for productive capacity for it, and so on.

Consider one unit of vertically hyper-integrated productive capacity for sector i:

$$\mathbf{m}_i^* = [m_{ji}^{(i)}], \qquad i = 1, 2, \ldots, m$$

Which is the composite commodity we need to produce *this* composite commodity? For each element $m_{ji}^{(i)}$, we need a quantity $m_{bj}^{(i)}m_{ji}^{(i)}$ of each commodity b, with $b = 1, 2, \ldots, m$, used as an intermediate commodity. We thus have a vector of commodities for each element of column vector $\mathbf{m}_i^*$, and hence a matrix; more precisely, matrix $(\mathbf{M}^{(i)})^2$, which must be evaluated at current prices, that is to say, by using expression (5.59):

$$\mathbf{p}^T(\mathbf{M}^{(i)})^2 = \mathbf{p}_k^{(i)}\mathbf{M}^{(i)}$$

The price of one unit of vertically hyper-integrated productive capacity for consumption commodity i therefore is the i-th component of vector $\mathbf{p}_k^{(i)T}$:

$$\mathbf{p}_k^{(i)T} = \overline{w}\mathbf{z}^{(i)T}\mathbf{M}^{(i)} + \mathbf{p}_k^{(i)T}\mathbf{M}^{(i)}(\pi - c_i) \tag{5.60}$$

which, defining $\mathbf{z}_k^{(i)T} \equiv \mathbf{z}^{(i)T}\mathbf{M}^{(i)}$ becomes:

$$\mathbf{p}_k^{(i)T} = \overline{w}\mathbf{z}_k^{(i)T} + \mathbf{p}_k^{(i)T}\mathbf{M}^{(i)}(\pi - c_i) \tag{5.61}$$

The price of a unit of vertically hyper-integrated for sector i is the i-th element of this vector, i.e.:

$$p_{k_i} = \overline{w}z_{k_i}^* + \mathbf{p}_k^{(i)T}\mathbf{m}_i^*(\pi - c_i) \tag{5.62}$$

where $z_{k_i}^*$ is the i-th element of vector $\mathbf{z}_k^{(i)T}$.

Since, however, expression (5.59) holds for all $i = 1, 2, \ldots, m$, we can also write:

$$p_{k_i} = \mathbf{p}^T\mathbf{m}_i^*, \qquad i = 1, 2, \ldots, m \tag{5.63}$$

and therefore

$$\mathbf{p}_k^T \equiv \mathbf{p}^T\overline{\mathbf{M}} \tag{5.64}$$

In this way, equation (5.58) becomes:

$$\mathbf{p}^T = \overline{w}\mathbf{z}^T + \mathbf{p}_k^T(\pi\mathbf{I} - \widehat{\mathbf{c}}) \tag{5.65}$$

Moreover, by substituting expression (5.65) into (5.64), the latter can be written as:

$$\mathbf{p}_k^T = \overline{w}\overline{\mathbf{z}}^T\overline{\mathbf{M}} + \mathbf{p}_k^T(\pi\mathbf{I} - \widehat{\mathbf{c}})\overline{\mathbf{M}} = \overline{w}\overline{\mathbf{z}}_k^T + \mathbf{p}_k^T(\pi\mathbf{I} - \widehat{\mathbf{c}})\overline{\mathbf{M}} \tag{5.66}$$

which means that the i-th element of vector $\mathbf{p}_k^T$, i.e. the price of one unit of vertically hyper-integrated productive capacity for consumption commodity i, can be written as:

$$p_{k_i} = \overline{w}\overline{\mathbf{z}}^T\mathbf{m}_i^* + \mathbf{p}_k^T(\pi\mathbf{I} - \widehat{\mathbf{c}})\mathbf{m}_i^* = \overline{w}z_{k_i} + \mathbf{p}_k^T(\pi\mathbf{I} - \widehat{\mathbf{c}})\mathbf{m}_i^* \tag{5.67}$$

where z_{k_i} is the i-th element of vector $\overline{\mathbf{z}}_k^T$. Expressions (5.67) and (5.62) are therefore equivalent.[24]

[24]For a comparison between these two expressions, see section 6 below.

By using the expressions for consumption commodity and productive capacity prices, i.e. expressions (5.65) and (5.66), we can now formulate the price system, in matrix form and analogously to Pasinetti's (1981) one, as:

$$\begin{bmatrix} \mathbf{p}^T & \mathbf{p}_k^T & w \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{O} & -\mathbf{a}_{in} \\ -(\pi\mathbf{I} - \widehat{\mathbf{c}}) & \mathbf{I} - (\pi\mathbf{I} - \widehat{\mathbf{c}})\overline{\mathbf{M}} & (\pi\mathbf{I} - \widehat{\mathbf{c}})\mathbf{a}_{in} \\ -\overline{\mathbf{z}}^T & -\overline{\mathbf{z}}^T\overline{\mathbf{M}} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{0}^T & \mathbf{0}^T & 0 \end{bmatrix} \tag{5.68}$$

or, as an eigenproblem, as:[25]

$$\begin{cases} \overline{\mathbf{p}}^T(\lambda_p\overline{\mathbf{I}} - \overline{\mathbf{A}}_p) = \overline{\mathbf{0}}^T \\ \lambda_p^* = 1 \end{cases} \tag{5.70}$$

the characteristic equation being:

$$\begin{vmatrix} -\lambda_p\mathbf{I} & \mathbf{O} \\ \pi\mathbf{I} - \widehat{\boldsymbol{c}} & (\pi\mathbf{I} - \widehat{\boldsymbol{c}})\overline{\mathbf{M}} - \lambda_p\mathbf{I} \end{vmatrix} \times$$
$$\times \left(-\lambda_p - \begin{bmatrix} \mathbf{a}_{ni}^T & \mathbf{a}_{ni}^T\overline{\mathbf{M}} \end{bmatrix} \begin{bmatrix} -\lambda_p\mathbf{I} & \mathbf{O} \\ \pi\mathbf{I} - \widehat{\boldsymbol{c}} & (\pi\mathbf{I} - \widehat{\mathbf{c}})\overline{\mathbf{M}} - \lambda_p\mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{a}_{in} \\ (\widehat{\mathbf{c}} - \pi\mathbf{I})\mathbf{a}_{in} \end{bmatrix} \right) = 0 \tag{5.71}$$

By defining $d_{\overline{\mathbf{M}}\pi} = |(\pi\mathbf{I} - \widehat{\boldsymbol{c}})\overline{\mathbf{M}} - \lambda_p\mathbf{I}|$, the first factor reduces to $(-1)^m\lambda_p^m d_{\overline{\mathbf{M}}\pi}$ and the inverse matrix in the second factor can be written as:

$$\begin{bmatrix} -\frac{1}{\lambda_p} & \mathbf{O} \\ -\frac{1}{\lambda_p}\frac{1}{d_{\overline{\mathbf{M}}\pi}}((\pi\mathbf{I} - \widehat{\mathbf{c}})\overline{\mathbf{M}} - \lambda_p\mathbf{I})^{(+)}(\pi\mathbf{I} - \widehat{\mathbf{c}}) & \frac{1}{d_{\overline{\mathbf{M}}\pi}}((\pi\mathbf{I} - \widehat{\mathbf{c}})\overline{\mathbf{M}} - \lambda_p\mathbf{I})^{(+)} \end{bmatrix}$$

Hence, by substituting these last two results into the characteristic equation, the latter can be written as:

$$(-1)^m \Big(-\lambda^{m+1}d_{\overline{\mathbf{M}}\pi} + \lambda_p^{m-1}\mathbf{z}^T\mathbf{a}_{in} + \lambda^{m-1}\mathbf{z}^T\overline{\mathbf{M}}((\pi\mathbf{I} - \widehat{\mathbf{c}})\overline{\mathbf{M}} - \lambda_p\mathbf{I})^{(+)}(\pi\mathbf{I} - \widehat{\mathbf{c}})\mathbf{a}_{in} +$$
$$-\lambda_p^m\mathbf{z}^T\overline{\mathbf{M}}((\pi\mathbf{I} - \widehat{\mathbf{c}})\overline{\mathbf{M}} - \lambda_p\mathbf{I})^{(+)}(\pi\mathbf{I} - \widehat{\mathbf{c}})\mathbf{a}_{in} \Big) = 0 \tag{5.72}$$

[25]Where:

$$\overline{\mathbf{A}}_p^T = \begin{bmatrix} \mathbf{O} & \overline{\mathbf{O}} & \mathbf{a}_{in} \\ \pi\mathbf{I} - \widehat{\mathbf{c}} & (\pi\mathbf{I} - \widehat{\mathbf{c}})\overline{\mathbf{M}} & -(\pi\mathbf{I} - \widehat{\mathbf{c}})\mathbf{a}_{in} \\ \overline{\mathbf{z}}^T & \overline{\mathbf{z}}^T\overline{\mathbf{M}} & 0 \end{bmatrix} \tag{5.69}$$

Since the matrix has at least one negative element, we cannot use the Perron Frobenius theorems. We will therefore use the same procedure as in the previous section.

When $\lambda_p^* = 1$ the above expression reduces to:

$$(-1)^m \Big(-\overline{d}_{\overline{\mathbf{M}}\pi} + \overline{d}_{\overline{\mathbf{M}}\pi} \mathbf{z}^T \mathbf{a}_{in} + \mathbf{z}^T \overline{\mathbf{M}}((\pi \mathbf{I} - \widehat{\mathbf{c}})\overline{\mathbf{M}} - \lambda_p \mathbf{I})^{(+)} (\pi \mathbf{I} - \widehat{\mathbf{c}}) \mathbf{a}_{in} + \\ - \mathbf{z}^T \overline{\mathbf{M}}((\pi \mathbf{I} - \widehat{\mathbf{c}})\overline{\mathbf{M}} - \lambda_p \mathbf{I})^{(+)} (\pi \mathbf{I} - \widehat{\mathbf{c}}) \mathbf{a}_{in} \Big) = 0 \tag{5.73}$$

i.e:

$$(-1)^m \overline{d}_{\overline{\mathbf{M}}\pi} \left(-1 + \mathbf{z}^T \mathbf{a}_{in} \right) = 0 \tag{5.74}$$

and therefore:

$$\overline{\mathbf{z}}^T \mathbf{a}_{in} = 1 \tag{5.75}$$

i.e. the same condition derived from the eigenproblem associated to the quantity system, now also guaranteeing full expenditure of income.

The left-hand eigenvector associated to this eigenvalue is the solution for commodity prices, i.e., with $w = \overline{w}$:

$$\begin{bmatrix} \mathbf{p}^T & \mathbf{p}_k^T & w \end{bmatrix} = \begin{bmatrix} \overline{w} \left[\overline{\mathbf{z}}^T \left(\mathbf{I} - \overline{\mathbf{M}}(\pi \mathbf{I} - \widehat{\mathbf{c}}) \right)^{-1} \right] \\ \overline{w} \left[\overline{\mathbf{z}}^T \overline{\mathbf{M}} \left(\mathbf{I} - (\pi \mathbf{I} - \widehat{\mathbf{c}}) \overline{\mathbf{M}} \right)^{-1} \right] \\ \overline{w} \end{bmatrix}^T = \begin{bmatrix} \overline{w} \overline{\mathbf{z}}^T \mathbf{\Phi}(\pi) \\ \overline{w} \overline{\mathbf{z}}_k^T \mathbf{\Phi}_k(\pi) \\ \overline{w} \end{bmatrix}^T \tag{5.76}$$

where $\overline{\mathbf{z}}_k^T = \overline{\mathbf{z}}^T \overline{\mathbf{M}}$, $\mathbf{\Phi}(\pi) = \left(\mathbf{I} - \overline{\mathbf{M}}(\pi \mathbf{I} - \widehat{\mathbf{c}}) \right)^{-1}$ and $\mathbf{\Phi}_k(\pi) = \left(\mathbf{I} - (\pi \mathbf{I} - \widehat{\mathbf{c}}) \overline{\mathbf{M}} \right)^{-1}$.

Notice that since $\overline{\mathbf{p}}_k^T$ can be written either as $\mathbf{p}^T \overline{\mathbf{M}}$ or as $\overline{\mathbf{z}}_k^T \mathbf{\Phi}_k(\pi)$, it follows that

$$\overline{\mathbf{z}}^T \mathbf{\Phi}(\pi) \overline{\mathbf{M}} = \overline{\mathbf{z}}^T \overline{\mathbf{M}} \mathbf{\Phi}_k(\pi) \tag{5.77}$$

and therefore that

$$\mathbf{\Phi}(\pi) = \overline{\mathbf{M}} \mathbf{\Phi}_k(\pi) \overline{\mathbf{M}}^{-1} \tag{5.78}$$

i.e. $\mathbf{\Phi}(\pi)$ and $\mathbf{\Phi}_k(\pi)$ *similar matrices*. Thus they have the same eigenvalues, and their eigenvectors, call them $\boldsymbol{\theta}$ and $\boldsymbol{\theta}_k$, respectively, are such that $\boldsymbol{\theta} = \overline{\mathbf{M}} \boldsymbol{\theta}_k \overline{\mathbf{M}}^{-1}$.

Since we want these solutions to be non-negative, we first have to check the condition guaranteeing non-negativity of matrix $\overline{\mathbf{M}}$.[26]

[26] As we can see from appendix A.1, such condition is:

$$c_i \leq \frac{1}{\lambda_H^{max}}, \qquad \forall i = 1, 2, \ldots, m \tag{5.79}$$

Then, in order for $(\pi\mathbf{I} - \widehat{\mathbf{c}})$ to be non-negative, all c_i's must be smaller than, or equal to, the rate of profit. This ensures that $\overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}})$ is non-negative. We can now use the Perron Frobenius theorems to conclude that for prices to be non-negative the maximum eigenvalue of matrix $\overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}})$ must be smaller than or equal to 1.

The results for all cases considered so far are summarised in table 1.

6 The price system

It is worth devoting some time to the analysis of prices (5.76) in comparison to those derived by Pasinetti (1981, pp. 41-3) for the intermediate case — or, exploiting the result obtained in Garbellini & Wirkierman (2010, p. 15), in comparison to:

$$p_i = \overline{w}\left(\ell_i + a_{nk_i}(\pi_i - (g + r_i))\right) \tag{6.1}$$

As Pasinetti points out:

> [O]ur approach has made it possible to express *all* price components in such a way as to allow the wage rate to be factored out. This means that what appears in the square brackets, by being multiplied by the wage rate, must obviously be either a physical quantity of labour or something which is made to be equivalent to a physical quantity of labour. [...] [L]et us notice that — whatever the way in which the rates of profit are determined, the [(5.76)] imply a theory of value which is based on quantities of physical labour and on quantities which are made to be equivalent to physical labour. The prices thereby express a theory of value which is indeed no longer in terms of pure labour, but in terms of what we may call *labour equivalents*.
>
> (Pasinetti 1981, pp. 42-3)

Pasinetti thus characterises the theory of value coming from formulation (5.76) as a *labour-equivalents theory of value*, as opposed to the pure labour one. With Pasinetti's (1981) simplified description of the technique in use, this is reflected by the fact that indirect (and hyper-indirect) labour is weighted more than direct one when a positive rate of profit is present.

Once inter-industry relations are considered, the adoption of 'labour-equivalents' values is slightly more complicated. The pure labour value of commodity i ($i = 1, 2, \ldots, m$) would be given by $\overline{w}z_i^*$ — i.e. by the wage rate multiplied by the quantity of vertically hyper-integrated labour necessary for the production of one unit of commodity i as a consumption good — while its labour equivalent value is given by the wage rate multiplied by a *linear combination* of *all* the labour coefficients,

	Physical quantities	**Commodity prices**	**Macroeconomic condition**	**Non-negativity condition (prices)**
Industry-level	$\mathbf{q} = (\mathbf{I}-\mathbf{A})^{-1}(\mathbf{a}_{in}+\mathbf{a}_{k_in})\overline{x}_n$ $x_n = \overline{x}_n$	$\mathbf{p}^T = \overline{w}\mathbf{a}_{ni}^T(\mathbf{I}-\mathbf{A}(1+\pi))^{-1}$ $w = \overline{w}$	$\mathbf{a}^T(\mathbf{I}-\mathbf{A})^{-1}(\mathbf{a}_{in}+\mathbf{a}_{k_in}) = 1$	$\pi^{max} \leq \frac{1-\lambda_A^{max}}{\lambda_A^{max}}$
Pasinetti (1981)	$\mathbf{x} = \mathbf{a}_{in}\overline{x}_n$ $\mathbf{x}_k = (\mathbf{a}_{in}+\mathbf{a}_{k_in})\overline{x}_n$ $x_n = \overline{x}_n$	$\mathbf{p}^T = \overline{w}(\mathbf{a}_{ni}^T + \mathbf{a}_{nk_i}^T(\mathbf{I}+\widehat{\boldsymbol{\pi}}))$ $\mathbf{p}_k^T = \overline{w}\mathbf{a}_{nk_i}^T$ $w = \overline{w}$	$\mathbf{a}_{ni}^T\mathbf{a}_{in} + \mathbf{a}_{nk_i}^T\mathbf{a}_{in} + \mathbf{a}_{nk_i}^T\mathbf{a}_{k_in} = 1$	$\pi_i \geq -\frac{a_{ni}+a_{nk_i}}{a_{nk_i}}$
Pasinetti (1973)	$\mathbf{y} = (\mathbf{a}_{in}+\mathbf{a}_{k_in})\overline{x}_n$ $\mathbf{x}_k = (\mathbf{a}_{in}+\mathbf{a}_{k_in})\overline{x}_n$ $x_n = \overline{x}_n$	$\mathbf{p}^T = \overline{w}\mathbf{v}^T(\mathbf{I}-\mathbf{H}\pi)^{-1}$ $\mathbf{p}^T = \overline{w}\mathbf{v}^T\mathbf{H}(\mathbf{I}-\mathbf{H}\pi)^{-1}$ $w = \overline{w}$	$\mathbf{a}_{ni}^T(\mathbf{a}_{in}+\mathbf{a}_{k_in})+\mathbf{a}_{ni}^T\mathbf{H}(\mathbf{a}_{in}+\mathbf{a}_{k_in}) = 1$ or $\mathbf{v}^T(\mathbf{a}_{in}+\mathbf{a}_{k_in}) = 1$	$\pi^{max} = \frac{1}{\lambda_H^{max}}$
Pasinetti (1988)	$\mathbf{x} = \mathbf{a}_{in}\overline{x}_n$ $\mathbf{x}_k = (\mathbf{I}+\widehat{\mathbf{c}})\mathbf{a}_{in}\overline{x}_n$ $x_n = \overline{x}_n$	$w = \overline{w}\mathbf{z}^T(\mathbf{I}-\overline{\mathbf{M}}(\pi\mathbf{I}-\widehat{\mathbf{c}}))^{-1}$ $w = \overline{w}\mathbf{z}^T\overline{\mathbf{M}}(\mathbf{I}-(\pi\mathbf{I}-\widehat{\mathbf{c}})\overline{\mathbf{M}})^{-1}$ $w = \overline{w}$	$\mathbf{a}_{ni}^T\mathbf{a}_{in} + \mathbf{a}_{ni}^T\overline{\mathbf{M}}\mathbf{a}_{in} + \mathbf{a}_{ni}^T\overline{\mathbf{M}}\widehat{\mathbf{c}}\mathbf{a}_{in} = 1$ or $\mathbf{z}^T\mathbf{a}_{in} = 1$	$c_i^{max} \leq \frac{1}{\lambda_H^{max}}$

Table 1: Price and quantity systems, their solutions, macroeconomic conditions, and non-negativity conditions for commodity prices, in the four cases considered.

the multipliers being the elements of the corresponding column of matrix $\mathbf{\Phi}(\pi)$ — call it $\boldsymbol{\phi}_i(\pi)$, $(i = 1, 2, \ldots, m)$.

When considering the price of the units of vertically hyper-integrated productive capacity, things become more complicated. As stated in section 5.3 (page 34) above, there are two equivalent ways to express such prices, given by expressions (5.67) and (5.62):

$$p_{k_i} = \overline{w}\mathbf{z}^{(i)T}\mathbf{m}_i^* + \mathbf{p}_k^{(i)T}\mathbf{m}_i^*(\pi - c_i) = \overline{w}z_{k_i}^* + \mathbf{p}_k^{(i)T}\mathbf{m}_i^*(\pi - c_i) \tag{5.62}$$

$$p_{k_i} = \overline{w}\overline{\mathbf{z}}^T\mathbf{m}_i^* + \mathbf{p}_k^T(\pi\mathbf{I} - \widehat{\mathbf{c}})\mathbf{m}_i^* = \overline{w}z_{k_i} + \mathbf{p}_k^T(\pi\mathbf{I} - \widehat{\mathbf{c}})\mathbf{m}_i^* \tag{5.67}$$

These two expressions are equivalent, but of course the two addenda constituting them are not the same.

In expression (5.62), both the (vertically hyper-integrated) labour cost and the profit components are exactly those associated to the production of composite commodity $[m_{1i}, m_{2i}, \ldots, m_{mi}]$ by vertically hyper-integrated sector i as a unit of productive capacity. On the contrary, in expression (5.67) both of them are those that we *would* have, were each of these quantities produced in the vertically hyper-integrated sector producing the corresponding commodity as a *final consumption* commodity. This is a major difference. The hyper-indirect part of each vertically hyper-integrated sector i crucially depends on the rate of growth of the sector itself, i.e. on the rate of growth of demand for the corresponding consumption good. Therefore, expression (5.67) does under- or over-estimate labour costs according to whether[27]

$$c_i \lesseqgtr \frac{\mathbf{v}^T\mathbf{D}^{(i)}\widehat{\mathbf{c}}\mathbf{m}_i^*}{\mathbf{v}^T\mathbf{D}^{(i)}\mathbf{m}_i^*}, \qquad i = 1, 2, \ldots, m \tag{6.2}$$

therefore over- or under-estimating correspondingly the profit component of price p_{k_i}.

Therefore, the labour value of a unit of vertically hyper-integrated productive capacity for sector i is not given by $\overline{w}\overline{\mathbf{z}}^T\overline{\mathbf{M}}$, but by $\overline{w}\mathbf{z}(i)_T\mathbf{M}^{(i)}$; in the solutions obtained in section 5.3, i.e. the second expression in (5.76), therefore, matrix $\mathbf{\Phi}_k(\pi)$ does not transform the true *labour values* into prices of production.

The correct formulation for the whole set of intermediate commodities prices, as hinted at in the previous section, includes the whole price *vector* for each vertically hyper-integrated sector; specifically, for each vertically hyper-integrated sector i, it is given by equation (5.60):

$$\mathbf{p}_k^{(i)T} = \overline{w}\mathbf{z}^{(i)T}\mathbf{M}^{(i)} + \mathbf{p}_k^{(i)T}\mathbf{M}^{(i)}(\pi - c_i) \tag{5.60}$$

[27]Matrix $\mathbf{D}$ is the first derivative of matrix $(\mathbf{I} - \mathbf{H}c_i)^{-1}$ with respect to c_i. For details, see appendix A.3.

i.e.

$$\mathbf{p}_k^{(i)T} = \overline{w}\mathbf{z}_k^{(i)T} + \mathbf{p}_k^{(i)T}\mathbf{M}^{(i)}(\pi - c_i) \tag{6.3}$$

and therefore

$$\mathbf{p}_k^{(i)T} = \overline{w}\mathbf{z}_k^{(i)T}(\mathbf{I} - \mathbf{M}(\pi - c_i))^{-1} = \overline{w}\mathbf{z}_k^{(i)T}\mathbf{\Phi}^{(i)}(\pi) \tag{6.4}$$

Matrix $\mathbf{\Phi}^{(i)}(\pi)$ is the matrix that, *in each vertically hyper-integrated sector* i, transforms the labour values for intermediate commodities into the respective prices of production. The i-th element of each vector $\mathbf{p}_k^{(i)T}$ clearly is equivalent to the corresponding element of vector $\mathbf{p}_k^T$.

We can therefore call matrices $\mathbf{\Phi}(\pi)$ and $\mathbf{\Phi}^{(i)}(\pi)$ the *labour transformation* matrices, the scalar $z_i^e = \overline{\mathbf{z}}^T\boldsymbol{\phi}_i(\pi)$ the *labour equivalent* for the production of consumption commodity i, and the scalar $z_{k_i}^e(\pi) = \overline{\mathbf{z}}^T\overline{\mathbf{M}}\boldsymbol{\phi}_{k_i}(\pi)$ — or equivalently $z_{k_i}^e(\pi) = \mathbf{z}^{(i)}\mathbf{M}^{(i)}\boldsymbol{\phi}_{k_i}^{(i)}(\pi)$ — the *labour equivalent* for the production of one unit of productive capacity for vertically hyper-integrated sector i. With this new notation, prices can be now written as:

$$\begin{cases} p_i = \overline{w}z_i^e(\pi) \\ p_{k_i} = \overline{w}z_{k_i}^e(\pi) \end{cases} , \quad i = 1, 2, \ldots, m \tag{6.5}$$

When the rate of change of demand is different from sector to sector, and with a uniform, exogenously given, rate of profit, prices and (labour) values are therefore diverging, due to the difference between the future rate of growth of sectoral demands for consumption goods and the rate of profit. This difference originates shifts of value among the various sectors, which aim at allowing each of them to keep satisfied its physical (i.e. quantity side) requirements for equilibrium growth *given* the distributive variables.

7 Sectoral and aggregate magnitudes through time

The last step of this discussion about production in the short run consists in explicitly stating the analytical formulation of three magnitudes, both in sectoral and aggregate terms, which are bound to acquire great importance in Pasinetti's (1981) treatment of economic dynamics: capital/output ratio(s), capital/labour ratio(s), and product per worker.

As Pasinetti (1981) explains in depth (for details, see Pasinetti 1981, chapter IX, sections 4-6),[28] it is very important to stress the conceptual difference between

[28] And as will become clear when dealing with production in the long run, i.e. with the general dynamic model (Garbellini 2010).

the capital/output ratio and the capital/labour ratio. In a few words, they both are an index of the 'roundaboutness' of a production process, but while the first ratio expresses the degree of *capital intensity* — and therefore is relevant, among the other things, for the process of *price formation* — the second one expresses the corresponding *degree of mechanisation*, and therefore is relevant for problems concerning *employment*.

Let us start from the capital/output ratio. The m *sectoral* ratios are given by:

$$\gamma_i = \frac{p_{k_i} k_i}{p_i x_i} = \frac{p_{k_i} x_i}{p_i x_i} = \frac{\overline{w}\overline{\mathbf{z}}_k^T \boldsymbol{\phi}_{k_i}(\pi)}{\overline{w}\overline{\mathbf{z}}^T \boldsymbol{\phi}_i(\pi)} \equiv \frac{\overline{w}\mathbf{z}_k^{(i)T} \boldsymbol{\phi}_i^{(i)}(\pi)}{\overline{w}\overline{\mathbf{z}}^T \boldsymbol{\phi}_i(\pi)} \equiv \frac{z_{k_i}^e(\pi)}{z_i^e(\pi)} \tag{7.1}$$

where the second equality comes from the fact that here we are assuming to be in stock-equilibrium, i.e. that $k_i = x_i$ for all $i = 1, 2, \ldots, m$.[29]

Looking at expression (7.1), we can see that the *sectoral* capital/output ratios are the ratios of two quantities of labour equivalents. The wage rate appears both in the numerator and in the denominator, and therefore cancels out. Thus, such ratios only depend on technology, and on the rate of profit — or better, on the difference between it and the sectoral rates of growth:

> [...] the incidence of capital in each commodity price, i.e. that component of each price which has to be charged for the use of capital, is proportional to the capital/output ratio required in that production process, quite independently of the number or the value of machines operated by each worker. The lower the capital/output ratio, the lower the charge for capital in each price, no matter whether and how much the capital/labour ratio may be changing.
>
> (Pasinetti 1981, p. 181)

This can be seen more clearly by writing the price of consumption commodity i ($i = 1, 2, \ldots, m$) as:

$$p_i = \overline{w} z_i^* + \frac{p_{k_i}}{p_i} p_i (\pi - c_i)$$

and therefore:

$$p_i = \overline{w} z_i^* + \gamma_i p_i (\pi - c_i) \tag{7.2}$$

The second addendum in expression (7.2) is precisely the charge for capital in the price of consumption commodity i, directly proportional to the corresponding capital/output ratio.

[29]See Garbellini & Wirkierman (2010, section 3.3) for details.

The *aggregate* capital/output ratio, on the other hand, is given by:

$$\Gamma = \frac{\mathbf{p}_k^T\mathbf{k}}{\mathbf{p}^T\mathbf{x}} = \frac{\mathbf{p}_k^T\mathbf{x}}{\mathbf{p}^T\mathbf{x}} = \frac{\overline{w}\overline{\mathbf{z}}_k^T\mathbf{\Phi}_k(\pi)\mathbf{a}_{in}}{\overline{w}\overline{\mathbf{z}}^T\mathbf{\Phi}(\pi)\mathbf{a}_{in}} = \frac{\mathbf{z}_{k_i}^e(\pi)^T\mathbf{a}_{in}}{\mathbf{z}_i^e(\pi)^T\mathbf{a}_{in}} \tag{7.3}$$

As it appears clearly by looking at expression (7.3), the *aggregate* degree of capital intensity depends not only on technology and on the rate of profit, but also on the *structure* of final demand for consumption commodities. Changing the *composition* of final demand, therefore, makes the capital/output ratio of the economic system as a whole change, even if technology and income distribution are still the same.

The meaning of the capital/labour ratio is entirely a different one. The m *sectoral* ratios are given by:

$$\theta_i = \frac{p_{k_i}k_i}{z_i^* x_i} = \frac{p_{k_i}x_i}{z_i^* x_i} = \frac{\overline{w}\overline{\mathbf{z}}_k^T\boldsymbol{\phi}_{k_i}(\pi)}{z_i^*} \equiv \frac{\overline{w}\mathbf{z}_k^{(i)T}\boldsymbol{\phi}_i^{(i)}(\pi)}{z_i^*} \equiv \frac{\overline{w}z_{k_i}^e(\pi)}{z_i^*} \tag{7.4}$$

The capital/labour ratio for vertically hyper-integrated sector i ($i = 1, 2, \ldots, m$) is the ratio between a *stock* of capital, evaluated at current prices, and a *flow* of labour. In this case, as it is apparent from expression (7.4), the wage rate only appears in the numerator, so that it does not cancel out. Hence, the sectoral degree of mechanisation depends on technology, on the rate of profit, *and* on the wage rate.

The *aggregate* capital/labour ratio, on the other hand, is given by:

$$\Theta = \frac{\mathbf{p}_k^T\mathbf{k}}{\mathbf{z}^T\mathbf{x}} = \frac{\mathbf{p}_k^T\mathbf{a}_{in}}{\overline{\mathbf{z}}^T\mathbf{a}_{in}} = \frac{\overline{w}\overline{\mathbf{z}}_k^T\mathbf{\Phi}_k(\pi)\mathbf{a}_{in}}{\overline{\mathbf{z}}^T\mathbf{a}_{in}} \equiv \frac{\overline{w}\mathbf{z}_k^{e^T}(\pi)\mathbf{a}_{in}}{\overline{\mathbf{z}}^T\mathbf{a}_{in}} \tag{7.5}$$

Also in this case, as for the capital/output ratio, the aggregate expression (7.5) also depends on the *composition* of final demand for consumption goods.

We can now have a look at another quite important economic magnitude, i.e. the product per worker, sectoral and aggregate, respectively:

$$y_i = \frac{p_t x_i}{z_i^* x_i} = \frac{\overline{w}\overline{\mathbf{z}}^T\boldsymbol{\phi}_i(\pi)}{z_i^*} \equiv \frac{\overline{w}z_i^e(\pi)}{z_i^*} \tag{7.6}$$

and:

$$Y_t = \frac{\mathbf{p}^T\mathbf{a}_{in}}{\overline{\mathbf{z}}^T\mathbf{a}_{in}} = \frac{\overline{w}\overline{\mathbf{z}}^T\mathbf{\Phi}(\pi)\mathbf{a}_{in}}{\overline{\mathbf{z}}^T\mathbf{a}_{in}} \equiv \frac{\overline{w}\mathbf{z}^{e^T}(\pi)\mathbf{a}_{in}}{\overline{\mathbf{z}}^T\mathbf{a}_{in}} \tag{7.7}$$

Also in this case, the difference between expression (7.6) and expression (7.7) lies in the fact that the *sectoral* product per worker does not depend on the structure of final demand for consumption commodities, while the *aggregate* one does.

8 Conclusions

As stated in the Introduction, the aim of the present paper was first of all that of stressing the vertically hyper-integrated character of Pasinetti's (1981) framework, by underlining the conceptual and analytical analogies with Pasinetti (1988) as to the treatment of new investment and, therefore, to the definition of net output.

Secondly, I have tried to reformulate the first part of Pasinetti's (1981) book, that concerning production in the short run, by removing some simplifying assumptions on the technology in use, by using matrix algebra, and by restating both the quantity and the price system as eigenproblems. This algebraic reformulation is intended to be a first step towards a complete generalisation of the whole analysis carried out by Pasinetti (1981) on the one side — by taking full advantage of Pasinetti (1988) generalisation — and to make it possible to apply this framework for empirical and institutional analyses on the other side.

In particular, I would like to devote some lines to the discussion of three aspects which I regard as particularly relevant.

First, the deepest implications of this approach can be fully drawn when the more realistic characterisation of the technique in use is re-introduced. The complex network of inter-industry relations is in fact an aspect of primary importance of modern economic systems; disregarding it prevents us from grasping the main potentialities of this approach as to its ability of making us understand, explain, and eventually look for a way to change, reality. Thanks to vertical hyper-integration, this aspect can be re-introduced into the picture without losing the possibility of keeping the analysis itself at the most fundamental level. This task is accomplished by using vertically hyper-integrated productive capacities as the units of measurement for intermediate commodities.

In this way, capital accumulation can be studied with respect to the final consumption commodities produced in the m sectors conforming the economic system, leaving the problem of their *composition* aside, but keeping the possibility of resuming it in any moment — vertically hyper-integrated analysis simply entails a *linear* algebraic transformation of usual inter-industry matrices, which can be reverted without any problem.[30]

Second, both the physical quantity and the commodity price systems can be restated as eigenproblems, the solutions being the eigenvectors associated to a specific eigenvalue. The macroeconomic condition emerges as the condition for such a value to be actually an eigenvalue of the coefficient matrix, and therefore to get economically meaningful solutions out of these eigensystems. The restatement fol-

[30]The implications of using vertically hyper-integration for the study of capital accumulation will be explored in Garbellini (2010), where I reformule that part of Pasinetti's (1981) book dealing with production in the long run.

lows from the reformulation, using matrix algebra — and in particular partitioned matrices — of the two equation systems. It is a more compact, and therefore easier to manage, mathematical formulation with respect to that adopted by Pasinetti (1981) and Pasinetti (1988).

Matrix algebra is a powerful mathematical tool, and it should be possible to take advantage of its further utilisation within Modern Classical economic theory. It is a matter of fact that some problems which seemed unsolvable to many Classical economists — think of Ricardo and Marx — actually were so only because of the lack of proper mathematical tools. Hence, in general, trying to restate 'old' problems in 'new' ways is one of the main keys to be able to successfully re-switch back to Classical Political Economy.

Moreover, restating the quantity and price systems as eigensystems makes it easier to work out empirical applications, since there are many numerical techniques, to be performed with the main statistical software, for the computation of such measures: solving eigenproblems is a very straightforward way of managing actual data. In addition eigenvalues — though not eigenvectors — possess the property of surviving similarity transformations of matrices, which means that conclusions can be reached from their analysis even if we have inter-industry matrices in nominal, rather than in physical, terms.[31]

Third, I have tried to do a step forward with respect to the analytical formulation in Pasinetti (1988). In the latter, Pasinetti defines matrices $\mathbf{M}^{(i)}$, $(i = 1, 2, \ldots, m)$, i.e. the matrices of vertically hyper-integrated productive capacity for each sector i. But, he says, each of such matrices has only one relevant column, i.e. the i-th one. Such a column vector is that composite commodity that he calls a unit of vertically hyper-integrated productive capacity for sector i.

Then, Pasinetti (1988) uses such matrices for expressing the price system in vertically hyper-integrated terms, i.e. reformulating it in order to make the role of vertically hyper-integrated productive capacity explicit. However, this is done in order to reach the definition of natural rates of profit; we have m price systems, one for each vertically hyper-integrated sector i, each of them with its own natural rate of profit π_i^*, in each of them appearing the corresponding matrix $\mathbf{M}^{(i)}$, whose columns, in this particular case, are *all*, not only the i-th one, relevant.

The notion of natural rates of profit has not yet been introduced here. Since it is a notion closely related to capital accumulation, and therefore to dynamics, I have defined and explored it in the paper devoted to this topic, i.e. Garbellini (2010). In the present paper I have stated, in section 5.3, the price system with

[31]The lack of proper, i.e. physical, data when performing empirical analyses is an old problem. Clearly, such a restatement does not solve it. But using matrix algebra and exploring its possible further applications can be a step forward in the right direction.

a (uniform[32]) *exogenous* rate of profit, keeping its determination as a degree of freedom. I have therefore introduced a more general formulation of the price system, defining a matrix $\overline{\mathbf{M}}$ made up by the i-th column of each of matrix $\mathbf{M}^{(i)}$ ($i = 1, 2, \ldots, m$) and using it for the formulation of the price system in terms analogous to what Pasinetti does in *Structural Change and Economic Growth*.

By doing so, it emerges clearly that, for the determination of (vertically hyper-integrated) sectoral intermediate commodity prices, *all* columns of each matrix $\mathbf{M}^{(i)}$ are relevant, not only within the 'natural' economic system, but also in general. In the general case, however, there is an equivalence between the 'outcomes' of the price system formulated in terms of matrix $\overline{\mathbf{M}}$ and in terms of matrices $\mathbf{M}^{(i)}$. We can therefore rely on the former, using the latter when dealing with issues requiring particular sectoral considerations. As we will see in Garbellini (2010), this is not true anymore when non-uniform *sectoral* rates of profit — e.g. the 'natural' ones — are introduced.

Further explanations and algebraic proofs are given in the Appendix below.

A Appendix

A.1 Growing subsystems and aggregate quantities

Following Pasinetti (1989), we can write the quantity system, with non-proportional growth, as:

$$\mathbf{q}^{(i)} = \mathbf{A}\mathbf{q}^{(i)} + \mathbf{A}(g + r_i)\mathbf{q}^{(i)} + \mathbf{x}^{(i)} \tag{A.1}$$

i.e.:

$$\mathbf{q}^{(i)} = \mathbf{H}c_i\mathbf{q}^{(i)} + (\mathbf{I} + \mathbf{H})\mathbf{x}^{(i)} \tag{A.2}$$

and therefore:

$$\mathbf{q}^{(i)} = (\mathbf{I} - \mathbf{H}c_i)^{-1}(\mathbf{I} + \mathbf{H})\mathbf{x}^{(i)} \tag{A.3}$$

Quantities (A.3) are non-negative provided that matrix $(\mathbf{I} - \mathbf{H}c_i)^{-1}$ is non-negative, i.e., for Perron-Frobenius theorems, that the maximum eigenvalue of matrix $\mathbf{H}$, λ_H^{max}, is such that

$$c_i < \frac{1}{\lambda_H^{max}}, \qquad \forall i = 1, 2, \ldots, m \tag{A.4}$$

[32] Pasinetti (1981) directly formulates the price system using non uniform ones. However, this would have, in the present framework, complicated the algebraic formulation without adding anything to the main conclusions, since the introduction of a whole series of sectoral rates of profit becomes useful when introducing natural rates of profit themselves.

which is precisely the same condition as that required for matrix $\mathbf{H}c_i$, and therefore for the series expansion of matrix $(\mathbf{I}-\mathbf{H}c_i)^{-1}$ itself, to converge.

Aggregate quantities are the sum of the $\mathbf{q}^{(i)}$'s, i.e.:

$$\begin{aligned}
\sum_{i=1}^{m}\mathbf{q}^{(i)} &= \sum_{i=1}^{m}(\mathbf{I}-\mathbf{H}c_i)^{-1}(\mathbf{I}+\mathbf{H})\mathbf{x}^{(i)} = \\
&= \sum_{i=1}^{m}\left(\mathbf{I}+\mathbf{H}c_i+(\mathbf{H}c_i)^2+\ldots\right)(\mathbf{I}+\mathbf{H})\mathbf{x}^{(i)} = \\
&= (\mathbf{I}+\mathbf{H})\sum_{i=1}^{m}\mathbf{x}^{(i)} + \mathbf{H}\sum_{i=1}^{m}c_i\left(\mathbf{I}+\mathbf{H}c_i+(\mathbf{H}c_i)^2+(\mathbf{H}c_i)^3+\ldots\right)(\mathbf{I}+\mathbf{H})\mathbf{x}^{(i)} = \\
&= (\mathbf{I}+\mathbf{H})\mathbf{x} + \mathbf{H}(\mathbf{I}+\mathbf{H})\widehat{\mathbf{c}}\mathbf{x} + \mathbf{H}^2\sum_{i=1}^{m}c_i^2\left(\mathbf{I}+\mathbf{H}c_i+(\mathbf{H}c_i)^2+\ldots\right)(\mathbf{I}+\mathbf{H})\mathbf{x}^{(i)} = \\
&= (\mathbf{I}+\mathbf{H})\mathbf{x} + (\mathbf{I}+\mathbf{H})\mathbf{H}\widehat{\mathbf{c}}\mathbf{x} + (\mathbf{I}+\mathbf{H})\mathbf{H}^2\widehat{\mathbf{c}}^2\mathbf{x} + \mathbf{H}^3\sum_{i=1}^{m}c_i^3\left(\mathbf{I}+\mathbf{H}c_i+\ldots\right)(\mathbf{I}+\mathbf{H})\mathbf{x}^{(i)} = \\
&= (\mathbf{I}+\mathbf{H})(\mathbf{I}+(\mathbf{H}\widehat{\mathbf{c}})+(\mathbf{H}\widehat{\mathbf{c}})^2+(\mathbf{H}\widehat{\mathbf{c}})^3+\ldots)\mathbf{x}
\end{aligned} \tag{A.5}$$

If conditions (A.4) are satisfied, this infinite series converges, and therefore the quantity system, in the aggregate, can be written as:

$$\sum_{i=1}^{m}\mathbf{q}^{(i)} = \mathbf{q} = (\mathbf{I}+\mathbf{H})(\mathbf{I}-\mathbf{H}\widehat{\mathbf{c}})^{-1}\mathbf{x} \tag{A.6}$$

Notice now that $\mathbf{q}^{(i)}$ $(i=1,2,\ldots,m)$ can be written as:

$$\mathbf{q}^{(i)} = (\mathbf{I}-\mathbf{H}c_i)^{-1}(\mathbf{I}+\mathbf{H})\mathbf{x}^{(i)} \equiv (\mathbf{I}+\mathbf{H})(\mathbf{I}-\mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} \tag{A.7}$$

Since matrices $(\mathbf{I}-\mathbf{H}c_i)^{-1}$ and $(\mathbf{I}+\mathbf{H})$ have the same eigenvectors, we can exploit the rule according to which if two matrices $\mathbf{X}$ and $\mathbf{Y}$ have the same eigenvectors, then $\mathbf{X}=\mathbf{Y}=\mathbf{YX}$ holds.

to see that $(\mathbf{I}-\mathbf{H}c_i)^{-1}$ and $(\mathbf{I}+\mathbf{H})$ have the same eigenvectors, notice that, if

$$\mathbf{Ax} = \lambda_A\mathbf{x} \tag{A.8}$$

then

$$\mathbf{Ix} - \mathbf{Ax} = \mathbf{x} - \lambda_A\mathbf{x}$$

i.e.

$$(\mathbf{I} - \mathbf{A})\mathbf{x} = (1 - \lambda_A)\mathbf{x}$$

Since if $\mathbf{Y}\mathbf{x} = \lambda_Y \mathbf{x}$ then $\mathbf{Y}^k\mathbf{x} = \lambda_Y^k \mathbf{x}$, then the latter expression can be written as

$$(\mathbf{I} - \mathbf{A})^{-1}\mathbf{x} = \frac{1}{1 - \lambda_A}\mathbf{x} \tag{A.9}$$

and therefore

$$\mathbf{A}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{x} = \frac{1}{1 - \lambda_A}\mathbf{A}\mathbf{x}$$

which, by using expression (A.8) and the definition of $\mathbf{H}$, can be written as

$$\mathbf{H}\mathbf{x} = \lambda_H \mathbf{x}$$

where $\lambda_H = \lambda_A/(1 - \lambda_A)$. Therefore, it is shown that matrices $\mathbf{A}$ and $\mathbf{H}$ have the same eigenvectors. We can now write

$$\mathbf{H}c_i\mathbf{x} = \lambda_H c_i \mathbf{x}$$

and hence

$$(\mathbf{I} - \mathbf{H}c_i)\mathbf{x} = (1 - \lambda_H c_i)\mathbf{x}$$

and therefore

$$(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x} = \frac{1}{1 - \lambda_H c_i}\mathbf{x}$$

Moreover, since $(\mathbf{I} - \mathbf{A})^{-1} = (\mathbf{I} + \mathbf{H})$, expression (A.9) can be written as:

$$(\mathbf{I} + \mathbf{H})\mathbf{x} = \frac{1}{1 - \lambda_A}\mathbf{x}$$

We have therefore shown that matrices $(\mathbf{I} + \mathbf{H})$ and $(\mathbf{I} - \mathbf{H}c_i)^{-1}$ have the same eigenvectors, and therefore that equivalence (A.7) holds.

Now, the expression for $\mathbf{s}^{(i)}$ can be written as:

$$\mathbf{s}^{(i)} = \mathbf{A}\mathbf{q}^{(i)} = \mathbf{A}(\mathbf{I} + \mathbf{H})(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} = \mathbf{H}(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} = \mathbf{M}^{(i)}\mathbf{x}^{(i)} \tag{A.10}$$

For Perron-Frobenius theorems, matrices $\mathbf{M}^{(i)}$ $(i = 1, 2, \ldots, m)$ are non-negative provided that

$$c_i^{max} < \frac{1}{\lambda_H^{max}} \tag{A.11}$$

The aggregate vector of intermediate means of production, $\mathbf{s}$, is the sum of all the $\mathbf{s}^{(i)}$, i.e.:

$$\begin{aligned}\mathbf{s} &= \sum_{i=1}^{m} \mathbf{s}^{(i)} = \sum_{i=1}^{m} \mathbf{H}(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} = \sum_{i=1}^{m} \mathbf{H}(\mathbf{I} + \mathbf{H}c_i + (\mathbf{H}c_i)^2 + \ldots)\mathbf{x}^{(i)} = \\ &= \sum_{i=1}^{m} (\mathbf{H} + \mathbf{H}^2 c_i + \mathbf{H}^3 c_i^2 + \ldots)\mathbf{x}^{(i)} = \sum_{i=1}^{m} (\mathbf{H}\mathbf{x}^{(i)} + \mathbf{H}^2 c_i \mathbf{x}^{(i)} + \mathbf{H}^3 c_i^2 \mathbf{x}^{(i)} + \ldots) = \\ &= \mathbf{H}\sum_{i=1}^{m} \mathbf{x}^{(i)} + \mathbf{H}^2 \sum_{i=1}^{m} c_i \mathbf{x}^{(i)} + \mathbf{H}^3 \sum_{i=1}^{m} c_i^2 \mathbf{x}^{(i)} + \ldots = \mathbf{H}\mathbf{x} + \mathbf{H}^2\widehat{\mathbf{c}}\mathbf{x} + \mathbf{H}^3\widehat{\mathbf{c}}^2\mathbf{x} + \ldots = \\ &= \mathbf{H}(\mathbf{I} + \mathbf{H}\widehat{\mathbf{c}} + (\mathbf{H}\widehat{\mathbf{c}})^2 + \ldots) = \mathbf{H}(\mathbf{I} - \mathbf{H}\widehat{\mathbf{c}})^{-1}\mathbf{x} = \mathbf{M}\mathbf{x}\end{aligned} \tag{A.12}$$

Hence, $\mathbf{s} = \mathbf{M}\mathbf{x}$. But $\mathbf{s}$ can also be written as:

$$\sum_{i=1} \mathbf{s}^{(i)} = \sum_{i=1}^{m} \mathbf{M}^{(i)}\mathbf{x}^{(i)} = \sum_{i=1}^{m} \mathbf{m}_i^* x_i = \overline{\mathbf{M}}\mathbf{x} \tag{A.13}$$

Comparing expressions (A.12) and (A.13), we can thus conclude that:

$$\mathbf{M}\mathbf{x} \cong \overline{\mathbf{M}}\mathbf{x}$$

By proceeding in the same way for sectoral and aggregate vertically hyper-integrated labour requirements too, we can conclude that

$$\mathbf{z}^T\mathbf{x} \cong \overline{\mathbf{z}}^T\mathbf{x}$$

If conditions (A.11) are satisfied, i.e. if all matrices $\mathbf{M}^{(i)}$, $i = 1, 2, \ldots, m$, are non-negative, then matrix $\overline{\mathbf{M}}$ is non-negative too.

A.2 Reformulation of demand for capital goods

The total quantities of capital goods produced in one period is given by total quantities $\mathbf{q}^{(i)}$ less final demand $\mathbf{x}^{(i)}$. We can call this difference $\mathbf{q_k}^{(i)}$.

The total quantities produced in the i-th vertically hyper-integrated sector are:

$$\begin{aligned}\mathbf{q}^{(i)} &= (\mathbf{I} + \mathbf{H})(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} \\ &= (\mathbf{I} + \mathbf{H})\left(\mathbf{I} + \mathbf{H}c_i(\mathbf{I} - \mathbf{H}c_i)^{-1}\right)\mathbf{x}^{(i)} = \\ &= \mathbf{x}^{(i)} + \mathbf{H}\mathbf{x}^{(i)} + \mathbf{H}c_i(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} + \mathbf{H}(\mathbf{H}c_i)(\mathbf{I} - \mathbf{H}c_i)^{-1}\mathbf{x}^{(i)}\end{aligned} \tag{A.14}$$

i.e. the sum of final demand for consumption good i ($\mathbf{x}^{(i)}$), vertically integrated productive capacity for consumption good i ($\mathbf{Hx}^{(i)}$), new investment ($\mathbf{H}c_i(\mathbf{I}-\mathbf{H}c_i)^{-1}\mathbf{x}^{(i)}$) and vertically integrated productive capacity for new investments ($\mathbf{H}(\mathbf{H}c_i)(\mathbf{I}-\mathbf{H}c_i)^{-1}\mathbf{x}^{(i)}$).

Therefore:

$$\begin{aligned} \mathbf{q_k}^{(i)} &= \mathbf{Hx}^{(i)} + \mathbf{H}c_i(\mathbf{I}-\mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} + \mathbf{H}(\mathbf{H}c_i)(\mathbf{I}-\mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} \\ &= \mathbf{H}\left(\mathbf{I} + c_i(\mathbf{I}-\mathbf{H}c_i)^{-1} + \mathbf{H}c_i(\mathbf{I}-\mathbf{H}c_i)^{-1}\right)\mathbf{x}^{(i)} = \\ &= \mathbf{H}(1+c_i)(\mathbf{I}-\mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} \end{aligned} \tag{A.15}$$

Following Pasinetti (1981), $\mathbf{q_k}^{(i)}$ can also be written as:

$$\mathbf{q_k}^{(i)} = \mathbf{Ax}^{(i)} + \mathbf{Aq_k}^{(i)} + \mathbf{H}c_i(\mathbf{I}-\mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} \tag{A.16}$$

i.e. direct productive capacity for consumption good i plus direct productive capacity for $\mathbf{q_k}^{(i)}$ plus new investment. Using the last equality of (A.15) this expression can be written as:

$$\begin{aligned} \mathbf{q_k}^{(i)} &= \mathbf{Ax}^{(i)} + \mathbf{AH}(1+c_i)(\mathbf{I}-\mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} + \mathbf{H}c_i(\mathbf{I}-\mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} = \\ &= \mathbf{Ax}^{(i)} + \mathbf{AH}(\mathbf{I}-\mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} + \mathbf{AH}c_i(\mathbf{I}-\mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} + \mathbf{H}c_i(\mathbf{I}-\mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} = \\ &= \mathbf{Hx}^{(i)} + \mathbf{H}(\mathbf{H}c_i)(\mathbf{I}-\mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} + \mathbf{H}c_i(\mathbf{I}-\mathbf{H}c_i)^{-1}\mathbf{x}^{(i)} \end{aligned} \tag{A.17}$$

which is precisely the first line of (A.15). Hence (A.15) and (A.16) are equivalent.

A.3 The price of the units of vertically hyper-integrated productive capacity

Show that expressions (5.67) and (5.62) are equivalent, i.e. that

$$p_{k_i} = \overline{w}\overline{\mathbf{z}}^T\mathbf{m}_i^* + \mathbf{p}^T\overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}})\mathbf{m}_i^* \equiv \overline{w}\mathbf{z}^{(i)}\mathbf{m}_i^* + \mathbf{p}^T\mathbf{M}^{(i)}(\pi - c_i)\mathbf{m}_i^* \tag{A.18}$$

The first equality can be written as:

$$\begin{aligned} p_{k_i} =& \overline{w}\overline{\mathbf{z}}^T\mathbf{m}_i^* + \overline{w}\mathbf{z}^{(i)}\mathbf{m}_i^* - \overline{w}\mathbf{z}^{(i)}\mathbf{m}_i^* + \mathbf{p}^T\overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}})\mathbf{m}_i^* + \\ &+ \mathbf{p}^T\mathbf{M}^{(i)}(\pi - c_i)\mathbf{m}_i^* - \mathbf{p}^T\mathbf{M}^{(i)}(\pi - c_i)\mathbf{m}_i^* \end{aligned}$$

i.e. as:

$$\begin{aligned} p_{k_i} =& \overline{w}\mathbf{z}^{(i)}\mathbf{m}_i^* + \mathbf{p}^T\mathbf{M}^{(i)}(\pi - c_i)\mathbf{m}_i^* + \\ &+ \overline{w}(\overline{\mathbf{z}}^T - \mathbf{z}^{(i)})\mathbf{m}_i^* + \mathbf{p}^T\overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}})\mathbf{m}_i^* - \mathbf{p}^T\mathbf{M}^{(i)}(\pi - c_i)\mathbf{m}_i^* \end{aligned}$$

We therefore want to show that

$$\overline{w}(\overline{\mathbf{z}}^T - \mathbf{z}^{(i)})\mathbf{m}_i^* + \mathbf{p}^T\overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}})\mathbf{m}_i^* - \mathbf{p}^T\mathbf{M}^{(i)}(\pi - c_i)\mathbf{m}_i^* = 0 \tag{A.19}$$

By exploiting equivalence (5.58), expression (A.19) can be written as:

$$\begin{aligned}&\overline{w}(\overline{\mathbf{z}}^T - \mathbf{z}^{(i)})\mathbf{m}_i^* + \overline{w}\overline{\mathbf{z}}^T\left(\mathbf{I} - \overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}})\right)^{-1}\overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}})\mathbf{m}_i^* + \\ &- \overline{w}\mathbf{z}^{(i)T}\left(\mathbf{I} - \mathbf{M}^{(i)}(\pi - c_i)\right)^{-1}\mathbf{M}^{(i)}(\pi - c_i)\mathbf{m}_i^*\end{aligned}$$

Since, given two matrices $\mathbf{F}$ and $\mathbf{G}$, it holds that $\mathbf{F}^{-1}\mathbf{G} \equiv (\mathbf{G}^{-1}\mathbf{F})^{-1}$, the latter can be written as:

$$\begin{aligned}&\overline{w}(\overline{\mathbf{z}}^T - \mathbf{z}^{(i)})\mathbf{m}_i^* + \overline{w}\overline{\mathbf{z}}^T\left(\left(\overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}})\right)^{-1}\left(\mathbf{I} - \overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}})\right)^{-1}\right)^{-1}\mathbf{m}_i^* + \\ &- \overline{w}\mathbf{z}^{(i)T}\left(\left(\mathbf{M}^{(i)}(\pi - c_i)\right)^{-1}\left(\mathbf{I} - \mathbf{M}^{(i)}(\pi - c_i)\right)^{-1}\right)^{-1}\mathbf{m}_i^* = \\ =&\overline{w}(\overline{\mathbf{z}}^T - \mathbf{z}^{(i)})\mathbf{m}_i^* + \overline{w}\overline{\mathbf{z}}^T\left(\left(\overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}})\right)^{-1} - \mathbf{I}\right)^{-1}\mathbf{m}_i^* + \\ &- \overline{w}\mathbf{z}^{(i)T}\left(\left(\mathbf{M}^{(i)}(\pi - c_i)\right)^{-1} - \mathbf{I}\right)^{-1}\mathbf{m}_i^* = \\ =&\overline{w}(\overline{\mathbf{z}}^T - \mathbf{z}^{(i)})\mathbf{m}_i^* - \overline{w}\overline{\mathbf{z}}^T\left(\mathbf{I} - \left(\overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}})\right)^{-1}\right)^{-1}\mathbf{m}_i^* + \\ &+ \overline{w}\mathbf{z}^{(i)T}\left(\mathbf{I} - \left(\mathbf{M}^{(i)}(\pi - c_i)\right)^{-1}\right)^{-1}\mathbf{m}_i^*\end{aligned}$$

which, by defining $\left(\mathbf{M}^{(i)}(\pi - c_i)\right)^{-1} \equiv \mathbf{F}$ and $\left(\overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}})\right)^{-1} \equiv \mathbf{G}$, becomes

$$\begin{aligned}&\overline{w}(\overline{\mathbf{z}}^T - \mathbf{z}^{(i)})\mathbf{m}_i^* - \overline{w}\overline{\mathbf{z}}^T(\mathbf{I} - \mathbf{G})^{-1}\mathbf{m}_i^* + \overline{w}\mathbf{z}^{(i)T}(\mathbf{I} - \mathbf{F})^{-1}\mathbf{m}_i^* = \\ =&\overline{w}(\overline{\mathbf{z}}^T - \mathbf{z}^{(i)})\mathbf{m}_i^* - \overline{w}\overline{\mathbf{z}}^T\left(\mathbf{I} + \mathbf{G}(\mathbf{I} - \mathbf{G})^{-1}\right)\mathbf{m}_i^* + \overline{w}\mathbf{z}^{(i)T}\left(\mathbf{I} + \mathbf{F}(\mathbf{I} - \mathbf{F})^{-1}\right)\mathbf{m}_i^* = \\ =&\overline{w}\mathbf{z}^{(i)T}\mathbf{F}(\mathbf{I} - \mathbf{F})^{-1}\mathbf{m}_i^* - \overline{w}\overline{\mathbf{z}}^T\mathbf{G}(\mathbf{I} - \mathbf{G})^{-1}\mathbf{m}_i^*\end{aligned} \tag{A.20}$$

By substituting back into (A.20) the definitions of $\mathbf{F}$ and $\mathbf{G}$, we get

$$\begin{aligned}&\overline{w}\mathbf{z}^{(i)T}\left(\mathbf{M}^{(i)}(\pi - c_i)\right)^{-1}(\mathbf{I} - \left(\mathbf{M}^{(i)}(\pi - c_i)\right)^{-1})^{-1}\mathbf{m}_i^* + \\ &- \overline{w}\overline{\mathbf{z}}^T\left(\overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}})\right)^{-1}(\mathbf{I} - \left(\overline{\mathbf{M}}(\pi\mathbf{I} - \widehat{\mathbf{c}})\right)^{-1})^{-1}\mathbf{m}_i^*\end{aligned}$$

which can be written as

$$\overline{w}\mathbf{z}^T\left(\mathbf{I}-\overline{\mathbf{M}}(\pi\mathbf{I}-\widehat{\mathbf{c}})\right)^{-1}-\overline{w}\mathbf{z}^{(i)T}\left(\mathbf{I}-\mathbf{M}^{(i)}(\pi-c_i)\right)^{-1}$$

which, again using equivalence (5.58), is finally shown to be equal to zero.

Going back to what hinted at in section 6 about the reason because of which the labour costs and the profit component can differ in (5.67) with respect to (5.62), we now have to derive expression (6.2):

$$c_i \lesseqqgtr \frac{\mathbf{v}^T\mathbf{D}^{(i)}\widehat{\mathbf{c}}\mathbf{m}_i^*}{\mathbf{v}^T\mathbf{D}^{(i)}\mathbf{m}_i^*}, \qquad i=1,2,\ldots,m \tag{6.2}$$

Matrix $\mathbf{D}^{(i)}=[\mathbf{d}_j^{(i)}]$ is the first derivative of matrix $(\mathbf{I}-\mathbf{H}c_i)^{-1}$, that is to say

$$\mathbf{D}^{(i)}=\mathbf{H}(\mathbf{I}-(\mathbf{H}c_i)^2)^{-1}$$

which is non-negative provided that $c_i<1/\lambda_H^{max}$, i.e. whenever $(\mathbf{I}-\mathbf{H}c_i)^{-1}$ is itself non-negative.

Since $\mathbf{z}^{(i)T}=\mathbf{v}^T(\mathbf{I}-\mathbf{H}c_i)^{-1}$ and $\overline{\mathbf{z}}^T=\mathbf{v}^T(\mathbf{I}-\mathbf{H}\widehat{\mathbf{c}})^{-1}$, variations in the rate(s) of change in final demand affect both of them only through the effect such variations have on matrices $(\mathbf{I}-\mathbf{H}c_i)^{-1}$ and $(\mathbf{I}-\mathbf{H}\widehat{\mathbf{c}})^{-1}$, respectively.[33] Moreover, we can say that:

$$\frac{\mathrm{dz_i^*}}{\mathrm{dc_i}}=\mathbf{v}^T\mathbf{d}_i^{(i)}, \quad \frac{\mathrm{dz_j^{(i)}}}{\mathrm{dc_i}}=\mathbf{v}^T\mathbf{d}_j^{(i)}, \qquad \forall i,j=1,2,\ldots,m \tag{A.21}$$

and moreover that:

$$z_j^*=z_j^{(i)}+\mathbf{v}^T\mathbf{d}_j^{(i)}(c_j-c_i), \qquad \forall i,j=1,2,\ldots,m \tag{A.22}$$

We can now look for the relation between labour cost component in the two equivalent expressions (5.67) and (5.62) and the growth differentials between the m vertically hyper-integrated sectors composing the economic system. Such a difference is given by:

$$(\mathbf{z}^{(i)T}-\overline{\mathbf{z}}^T)\mathbf{m}_i^*=\left([z_1^{(i)},\ldots,z_i^*,\ldots,z_m^{(i)}]-[z_1^*,\ldots,z_i^*,\ldots,z_m^*]\right)\mathbf{m}_i^*$$

which, by using expression (A.22) becomes:

$$(\mathbf{z}^{(i)T}-\overline{\mathbf{z}}^T)\mathbf{m}_i^*=\left(-[\mathbf{v}^T\mathbf{d}^{(i)}(c_1-c_i),\ldots,0,\ldots,\mathbf{v}^T\mathbf{d}^{(i)}(c_m-c_i)]\right)\mathbf{m}_i^*$$

[33]Recall, as a matter of notation, that $\mathbf{z}^{(i)T}=[z_1^{(i)},\ldots,z_i^*,\ldots,z_m^{(i)}]$ and $\overline{\mathbf{z}}^T=[z_1^*,\ldots,z_i^*,\ldots,z_m^*]$.

or

$$(\mathbf{z}^{(i)T} - \overline{\mathbf{z}}^T)\mathbf{m}_i^* = \mathbf{v}^T \left(\mathbf{D}^{(i)}\mathbf{m}_i^* c_i - \mathbf{D}^{(i)}\widehat{\mathbf{c}}\mathbf{m}_i^* \right)$$

Therefore, we might conclude that

$$(\mathbf{z}^{(i)T} - \overline{\mathbf{z}}^T)\mathbf{m}_i^* \lesseqgtr 0$$

according to whether

$$c_i \lesseqgtr \frac{\mathbf{v}^T \mathbf{D}^{(i)}\widehat{\mathbf{c}}\mathbf{m}_i^*}{\mathbf{v}^T \mathbf{D}^{(i)}\mathbf{m}_i^*} \tag{6.2}$$

i.e. according to whether c_i is greater than, equal to o smaller than a particular weighted average of *all* the rates of growth of demand for consumption commodities, and which is precisely what we wanted to show.

To conclude, when c_i is smaller than this weighted average, expression (5.67) does over-estimate the labour cost associated to the production of one unit of vertically hyper-integrated productive capacity for vertically hyper-integrated sector i — in terms of labour, it is less costly to produce each component in the sector producing the corresponding commodity as a consumption good than in sector i itself; this greater labour cost is however compensated by a smaller — to the same extent — profit component, since clearly the differential with respect to the various rates of growth of demand is smaller than the specific differential corresponding to production in vertically hyper-integrated sector i, whose rate of growth is smaller. The sum of these two components is anyway exactly equal to that coming from expression (5.62). As a consequence, when we simply need the total unitary price of vertically hyper-integrated units of productive capacity e can use either formulation (5.67) or (5.62); when our aim is that of analysing the two components independently, the correct formulation to be used is (5.62) and, additionally, the complete set of intermediate commodity prices (5.60), together with solutions (6.4).

A.4 The more complex formulation

We will consider here the more complex case, where both consumption goods and capital goods are produced by means of labour and capital goods (Pasinetti 1981, chapter II, section 7). Pasinetti's (1981) original physical quantity system, in this more complex case, is:[34]

$$\begin{bmatrix} \mathbf{I} & \mathbf{O} & -\mathbf{a}_{in} \\ -\mathbf{I} & \mathbf{I}-\widehat{\gamma} & -\mathbf{a}_{k_i n} \\ -\mathbf{a}_{ni}^T & -\mathbf{a}_{nk_i}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x_k} \\ x_n \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ 0 \end{bmatrix} \tag{A.23}$$

The simplifying assumption made here is that each vertically integrated sector i is made up by only two industries: one producing the final commodity i and the other producing the homogeneous capital good k_i used by both of them. Such capital goods are sector-specific — i.e. different from sector to sector — commodities, measured in units of direct productive capacity for the final commodity industry. When this particular unit of measurement is used, the production of one unit of the final commodity requires, by definition, one unit of the capital good k_i.

In order to understand the meaning of the γ_is, as described by Pasinetti (1981, p. 43), let us go back to ordinary units, calling α_i the number of units of commodity k_i necessary for the production of one unit of commodity i, and γ_i the number of units of commodity i to be used for the production of one unit of commodity k_i itself. Then, a unit of productive capacity for the productive capacity industry — i.e. the number of units of the capital goods necessary for the production of one unit of productive capacity for the consumption good — is made up by $\alpha_i\gamma_i$ ordinary units of the capital good, or by γ_i units of productive capacity for the consumption good.

If the total quantity of units of capital good available is K_i, it can be used either for producing productive capacity for the consumption good, hence obtaining $1/\alpha_i$ such units, or for producing productive capacity for productive capacity, hence obtaining $1/\gamma_i\alpha_i$ such units. The ratio of these two quantities is

$$\frac{1/\alpha_i}{1/\gamma_i\alpha_i} = \gamma_i$$

Therefore, the γ_i's express the number of ordinary units of commodity k_i necessary for its own reproduction, the number of units of productive capacity for the consumption goods necessary for the production of one such unit of productive capacity, and the ratio of the total stock of capital goods expressed in terms of units of productive capacity for the consumption good, to the stock of capital goods expressed in terms of units of productive capacity for the productive capacity itself.

From now on, for the whole section, when talking about quantity of capital goods, we will always be using units of productive capacity for the consumption good as the unit of measurement.

[34] Since we are dealing with circulating capital only, we will set here $T_i = T_{k_i} = 1$, which means that the depreciation rate is equal to 1 in all industries and hence in all sectors.

The total quantity of capital goods to be produced in each period is given by the sum of the number of units of final commodity i, the number of units of productive capacity demanded as net investment ($a_{k_i n} x_n$, where $a_{k_i n}$ is the per-capita demand of units of productive capacity as new investments) and the number of units of productive capacity that have to be used up and therefore replaced ($\gamma_i x_{k_i}$).

Vector $\mathbf{a}_{ni}^T$ is the vector of unitary direct labour requirements for the final commodities, and $\mathbf{a}_{nk_i}^T$ is the vector of direct labour requirements per unit of productive capacity.

Written as an eigenvalue problem, system (A.23) is:[35]

$$\begin{cases} (\lambda_x^* \overline{\mathbf{I}} - \overline{\mathbf{A}}_x)\overline{\mathbf{x}} = \overline{\mathbf{0}} \\ \lambda_x^* = 1 \\ \lambda_x^* = \lambda_x^{max} \end{cases} \tag{A.25}$$

$\overline{\mathbf{x}}$, i.e. the solution for physical quantities, is the right-hand eigenvector of the non-negative matrix $\overline{\mathbf{A}}$ associated to a unitary eigenvalue. Therefore, in order for the eigensystem to have a solution, $\lambda_x^* = 1$ must be an eigenvalue of matrix $\overline{\mathbf{A}}$; moreover, in order for such solution to be real and non-negative, such eigenvalue must be the maximum one.

The characteristic equation associated to system (A.25) can be written as:

$$|\mathbf{O} - \lambda_x^* \mathbf{I}||\widehat{\gamma} - \lambda_x^* \mathbf{I}| \left(-\lambda_x^* - \begin{bmatrix} \mathbf{a}_{ni}^T & \mathbf{a}_{nk_i}^T \end{bmatrix} \begin{bmatrix} -\lambda_x^* \mathbf{I} & \mathbf{O} \\ \mathbf{I} & \widehat{\gamma} - \lambda_x^* \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{a}_{in} \\ \mathbf{a}_{k_i n} \end{bmatrix} \right) = 0 \tag{A.26}$$

We notice first that the eigenvalues of matrix $\overline{\mathbf{A}}_x$ will be different from zero — those of matrix $\mathbf{O}$ — and from γ_i, $(i = 1, 2, \ldots, m)$ — those of matrix $\widehat{\gamma}$ — otherwise matrices $(\mathbf{O} - \lambda_x^* \mathbf{I})$ and $(\widehat{\gamma} - \lambda_x^* \mathbf{I})$ would not be invertible.

The $m + 1$ eigenvalues of matrix $\overline{\mathbf{A}}_x$ are obtained as the solutions to

$$\mathbf{a}_{ni}^T \frac{1}{\lambda_x^*} \mathbf{a}_{in} + \mathbf{a}_{nk_i}^T \frac{1}{\lambda_x^*} (\lambda_x^* \mathbf{I} - \widehat{\gamma})^{-1} \mathbf{a}_{in} + \mathbf{a}_{nk_i}^T (\lambda_x^* \mathbf{I} - \widehat{\gamma})^{-1} \mathbf{a}_{k_i n} = \lambda_x^* \tag{A.27}$$

which in turn tells us that the γ_is must be smaller than the maximum eigenvalue of $\overline{\mathbf{A}}_x$.

Since we want one solution to be $\lambda_x^* = 1$, the condition for this to be true is that:

$$\mathbf{a}_{ni}^T \mathbf{a}_{in} + \mathbf{a}_{nk_i}^T (\mathbf{I} - \widehat{\gamma})^{-1} \mathbf{a}_{in} + \mathbf{a}_{nk_i}^T (\mathbf{I} - \widehat{\gamma})^{-1} \mathbf{a}_{k_i n} = 1 \tag{A.28}$$

[35]Where:

$$\overline{\mathbf{A}}_x = \begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{a}_{in} \\ \mathbf{I} & \widehat{\gamma} & \mathbf{a}_{k_i n} \\ \mathbf{a}_{ni}^T & \mathbf{a}_{nk_i}^T & 0 \end{bmatrix} \quad \text{and} \quad \overline{\mathbf{x}} = \begin{bmatrix} \mathbf{x} \\ \mathbf{x_k} \\ x_n \end{bmatrix} \tag{A.24}$$

which is precisely the macroeconomic condition found by Pasinetti (1981).

If such condition holds, then $\lambda_x^* = 1$ also is the maximum solution: since all the terms in equation (A.27) are decreasing functions of λ_x^*, the presence of an eigenvalue greater than one would contradict (A.28).

Being the maximum eigenvalue of matrix $\overline{\mathbf{A}}_x$ equal to 1, the above-mentioned conditions on the value of the γ_i's reduces to $\gamma_i < 1, \forall i$, which is a *viability condition* for the physical quantity system: the production of one unit of productive capacity cannot require more than one unit of productive capacity itself. If this condition were not accomplished, the economic system would not be viable.

The eigenvector associated to $\lambda_x^* = 1$ is therefore the solution for physical quantities, completely determined once we set $x_n = \overline{x}_n$, thus obtaining:

$$\begin{cases} \mathbf{x} = \mathbf{a}_{in}\overline{x}_n \\ \mathbf{x}_k = (\mathbf{I} - \widehat{\boldsymbol{\gamma}})^{-1}(\mathbf{a}_{in} + \mathbf{a}_{k_i n})\overline{x}_n \\ w = \overline{w} \end{cases} \tag{A.29}$$

As to the price system, it is given by:

$$\left[\begin{array}{ccc} \mathbf{p}^T & \mathbf{p_k}^T & w \end{array} \right] \left[\begin{array}{ccc} \mathbf{I} & \mathbf{O} & -\mathbf{a}_{in} \\ -(\mathbf{I} + \widehat{\boldsymbol{\pi}}) & \mathbf{I} - \widehat{\mathbf{B}}_k & \widehat{\boldsymbol{\Gamma}}_{in}\mathbf{a}_{in} - \widehat{\boldsymbol{\Gamma}}_{k_i n}\mathbf{a}_{k_i n} \\ -\mathbf{a}_{ni}^T & -\mathbf{a}_{nk_i}^T & 1 \end{array} \right] = \overline{\mathbf{0}}^T \tag{A.30}$$

where:

$$\begin{aligned} \widehat{\mathbf{B}} &= \widehat{\boldsymbol{\gamma}}(\mathbf{I} + \widehat{\boldsymbol{\pi}}) \\ \widehat{\mathbf{B}}_k &= \widehat{\boldsymbol{\gamma}}(\mathbf{I} + \widehat{\boldsymbol{\pi}}_k) \\ \widehat{\boldsymbol{\Gamma}}_{in} &= (\widehat{\boldsymbol{\pi}} + \widehat{\mathbf{B}}_k - \widehat{\mathbf{B}})(\mathbf{I} - \widehat{\boldsymbol{\gamma}})^{-1} \\ \widehat{\boldsymbol{\Gamma}}_{k_i n} &= (\mathbf{I} - \widehat{\mathbf{B}}_k)(\mathbf{I} - \widehat{\boldsymbol{\gamma}})^{-1} \end{aligned}$$

As an eigenproblem, system (A.30) becomes:[36]

$$\begin{cases} \overline{\mathbf{p}}^T(\lambda_p \overline{\mathbf{I}} - \overline{\mathbf{A}}_p) = \overline{\mathbf{0}}^T \\ \lambda_p^* = 1 \end{cases} \tag{A.32}$$

[36] Where:

$$\overline{\mathbf{A}}_p = \left[\begin{array}{ccc} \mathbf{O} & \mathbf{O} & \mathbf{a}_{in} \\ \mathbf{I} + \widehat{\boldsymbol{\pi}} & \widehat{\mathbf{B}}_k & -\widehat{\boldsymbol{\Gamma}}_{in}\mathbf{a}_{in} + \widehat{\boldsymbol{\Gamma}}_{k_i n}\mathbf{a}_{k_i n} \\ \mathbf{a}_{ni}^T & \mathbf{a}_{nk_i}^T & 0 \end{array} \right] \tag{A.31}$$

with characteristic equation:

$$|\mathbf{O}-\lambda_p\mathbf{I}||\widehat{\mathbf{B}}_k-\lambda_p\mathbf{I}|\left(\begin{bmatrix} \mathbf{a}_{ni}^T & \mathbf{a}_{nk_i}^T \end{bmatrix}\begin{bmatrix} -\lambda_p^*\mathbf{I} & \mathbf{O} \\ \mathbf{I}+\widehat{\boldsymbol{\pi}} & \widehat{\mathbf{B}}_k-\lambda_p^*\mathbf{I} \end{bmatrix}^{-1}\begin{bmatrix} \mathbf{a}_{in} \\ -\widehat{\boldsymbol{\gamma}}_{in}\mathbf{a}_{in}+\widehat{\boldsymbol{\gamma}}_{k_in}\mathbf{a}_{k_in} \end{bmatrix}\right)=0 \tag{A.33}$$

Matrix $\overline{\mathbf{A}}_p$ can be either negative or non-negative, depending on the sign of the last element of the second row: it is non-negative as long as total profits do not exceed the total value of new investments ($\mathbf{p}_k^T\mathbf{a}_{k_in}x_n$). Anyway, this is not necessarily true, and therefore, in solving this eigenproblem, we are not going to use Perron Frobenius theorems. We will simply find out the condition for $\lambda_p^*=1$ to be an eigenvalue of $\overline{\mathbf{A}}_p$, compute the associated eigenvector, and set the conditions for it to be real and non-negative.

We know that the eigenvalues of matrix $\overline{\mathbf{A}}_p$ will be different from zero — the eigenvalues of matrix $\mathbf{O}$ — and from $\gamma_i(1+\pi_{k_i})$, $(i=1,2,\ldots,m)$ — the eigenvalues of matrix $\widehat{\mathbf{B}}_k$, or matrices $(-\lambda_p^*)$ and $(\lambda_p^*\mathbf{I}-\widehat{\mathbf{B}}_k)$ would not be invertible.

The $m+1$ eigenvalues of matrix $\overline{\mathbf{A}}_p$ are thus the solutions of:

$$\begin{aligned}&\frac{1}{\lambda_p^*}\mathbf{a}_{ni}^T\mathbf{a}_{in}+\mathbf{a}_{nk_i}^T(\lambda_p^*\mathbf{I}-\widehat{\mathbf{B}}_k)^{-1}\left(\frac{1}{\lambda_p^*}(\mathbf{I}+\widehat{\boldsymbol{\pi}})(\mathbf{I}-\widehat{\boldsymbol{\gamma}})^{-1}+\widehat{\mathbf{B}}-\widehat{\mathbf{B}}_k-\widehat{\boldsymbol{\pi}}\right)(\mathbf{I}-\widehat{\boldsymbol{\gamma}})^{-1}\mathbf{a}_{in}+\\&+\mathbf{a}_{nk_i}^T(\lambda_p^*\mathbf{I}-\widehat{\mathbf{B}}_k)^{-1}(\mathbf{I}-\widehat{\mathbf{B}}_k)(\mathbf{I}-\widehat{\boldsymbol{\gamma}})^{-1}\mathbf{a}_{k_in}=\lambda_p^*\end{aligned} \tag{A.34}$$

which, when $\lambda_p^*=1$, reduces to:

$$\mathbf{a}_{ni}^T\mathbf{a}_{in}+\mathbf{a}_{nk_i}^T(\mathbf{I}-\widehat{\boldsymbol{\gamma}})^{-1}\mathbf{a}_{in}+\mathbf{a}_{nk_i}^T(\mathbf{I}-\widehat{\boldsymbol{\gamma}})^{-1}\mathbf{a}_{k_in}=1 \tag{A.35}$$

which is the same condition as the one found above for the quantity system.

By fixing $w=\overline{w}$, the eigenvector associated to λ_p^* is:

$$\begin{cases}\mathbf{p}^T=\overline{w}\left(\mathbf{a}_{ni}^T+\mathbf{a}_{nk_i}^T(\mathbf{I}+\widehat{\boldsymbol{\pi}})(\mathbf{I}-\widehat{\boldsymbol{\gamma}}(\mathbf{I}+\widehat{\boldsymbol{\pi}}_k))^{-1}\right)\\ \mathbf{p}_k^T=\overline{w}\mathbf{a}_{nk_i}^T(\mathbf{I}-\widehat{\boldsymbol{\gamma}}(\mathbf{I}+\widehat{\boldsymbol{\pi}}_k))^{-1}\\ w=\overline{w}\end{cases} \tag{A.36}$$

which is real and non-negative as long as:

$$\pi_{k_i}<\frac{1-\gamma_i}{\gamma_i},\quad i=1,2,\ldots,m \tag{A.37}$$

Expression (A.37) therefore is a set of sectoral *viability conditions* for the price system, telling us the maximum rate of profit which cannot be exceeded if we want prices to be non-negative. Notice that such a viability condition involves the rates of profit of the industries producing intermediate commodities, and not producing consumption goods (see Pasinetti 1981, p. 45, with $T_{k_i}=1$).

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