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A Truthmaker-based Epistemic Logic

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Abstract
The aim of this work is to investigate the problem of Logical Omniscience in epistemic logic by means of truthmaker semantics. We will present a semantic framework based on $\boldsymbol{W}$-models extended with a partial function, which selects the *body of knowledge* of the agents, namely the set of verifiers of the agent's *total knowledge*. The semantic clause for knowledge follows the intuition that an agent knows some information $\boldsymbol{\phi}$, when the *propositional content* that $\boldsymbol{\phi}$ is *contained* in her total knowledge. We will argue that this idea mirrors the philosophical conception of *immanent closure* by Yablo (2014), giving to our proposal a strong philosophical motivation. We will discuss the philosophical implications of the semantics and we will introduce its axiomatization.

Keywords Epistemic logic · Truthmaker semantics · Logical omniscience · Subject matter · Total knowledge

1 Introduction

The aim of this work is to employ truthmaker semantics to address the problem of Logical Omniscience in epistemic logic.

The traditional epistemic modal logics are due to Jaakko Hintikka [2, 3], which are *normal* modal logics based on the relational possible worlds semantics, where the necessity operator is generally taken to represent knowledge and belief: ϕ is known at a world w if and only if it is true at every possible world that is epistemically *indistinguishable* from w. An immediate consequence of this approach is that whenever an agent knows all of the formulas in a set Γ, and Γ logically entails the formula ϕ, then the agent also knows ϕ. We call this closure principle *Full Logical Omniscience* [4].

This approach works well for modeling the knowledge of ideal agents, namely agents with unconstrained cognitive resources. However, our aim is to provide a theory for actual non-ideal reasoners, whose performances might be inhibited by their limited

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memory, computational capacity, faulty reasoning etc. That is to say that real life agents are fallible and resource-bounded.[1]

Therefore the predictions of Hintikkian approaches are not accurate for our purposes; in fact the brightest mathematician might know all the axioms of set theory without thereby knowing all their consequences. Accordingly, we will investigate how to model a notion of knowledge lacking this implausibly strong feature by adopting truthmaker semantics.

Truthmaker semantics is a novel mathematical and philosophical framework, which sheds new light on traditional questions about meaning and content, such as what a *proposition* expresses and what is its *subject matter*. Fine [5] defines a notion of *truthmaker content*, which is more fine-grained than the possible-world-based cognate, so that it has proved useful in the reconstruction of the semantics and the logic of different hyperintensional operators.

The application of truthmaker semantics to modal operators is still at its infant stage.[2] Yet, it has already proved to be beneficial for the philosophical analysis of metaphysical modalities such as necessity and possibility [7] as well as deontic modalities, i.e. obligation and permission [8, 9]. This paper will significantly contribute to provide a viable alternative to possible worlds semantics also for the epistemic modalities. In particular, Fine in [10] formalizes a relation of *containment* between propositions which is arguably analogous to Yablo's parthood: P is part of Q if and only if the inference P, *therefore* Q is both truth-preserving and subject matter preserving [1]. Accordingly, we introduce an epistemic logic where the knowledge operator is not closed under classical logic, but it is closed under the logic of *containment*. The result is then a hyperintensional framework where Logical Omniscience fails, but the agents are nevertheless logically competent. We call this account *Total Knowledge*.

The paper is structured as follows. Section 2 is dedicated to an overview of the issue of Logical Omniscience: on the one hand, we will identify a list of principles which we take to be inadequate for a non-idealized conception of knowledge, on the other hand, we will argue for a suitable theory of closure of knowledge based on the idea of subject matter sensitivity. In Section 3, we introduce the truthmaker semantics. In Section 4 we present our account of Total Knowledge and in the following two sections we analyze its philosophical and technical features; the rest of the paper is dedicated to

[1] We do not focus on modeling normative epistemic constraints: for example, given that an agent knows P, what is she thereby *permitted* to know, or *obliged* to know? Also, our research is not based on empirical studies and we do not aim to test it with experiments. As we will see, our goal is to identify and model a philosophical diagnosis of the sources of non-omniscience, namely the *subject matter* sensitivity of knowledge.

[2] Hawke and Özgün [6] explore the application of Fine's truthmaker semantics to the logic of *conditional* knowledge. They provide six accounts of conditional knowledge that behave differently with respect to the principles of Logical Omniscience. One of the main differences with our approach is that the epistemic modalities are modeled as global properties of the model: they are not verified (falsified) by a particular state or world but by every state of the model. On the contrary, we are interested in elaborating an account of *unconditional knowledge* which is locally verified or falsified, which represents a fundamental property of knowledge [4, p. 339]: we want to assume that every agent in the system is in some local state at any point in time. This means that an agent decides whether a certain formula follows from the information in the agent's local state.

the introduction of the proof-theory and the development of the completeness results, developed in the Appendixes A and B.

2 Epistemic Logic and Logical Omniscience

In this section we will identify a list of principles that represent instances of Logical Omniscience and that, therefore, we want to invalidate in our account of epistemic logic. To introduce them, we work with the language $\mathcal{L}_K$ of knowledge, defined recursively as

$$\phi := p \mid \phi \wedge \phi \mid \neg\phi \mid K\phi$$

where $p \in Prop$, a countable set of propositional variables, and $K\phi$ is read as 'the agent knows ϕ'; disjunction and implication are defined as usual.

We assume the reader's knowledge of standard modal logic, in particular of the standard notions of Kripke *models*, denoted by $\mathcal{M}$, and the usual notion of *truth* in a model and a world, denoted by $\mathcal{M}, w \models \phi$. Moreover, we say ϕ is a *logical consequence* of Γ, denoted by $\Gamma \models \phi$, if and only if for all Kripke models, for all worlds w and for all $\psi \in \Gamma$, if $\mathcal{M}, w \models \psi$, then $\mathcal{M}, w \models \phi$. We say that a formula ϕ is *valid*, denoted with $\models \phi$, if and only if for all $\mathcal{M}$ and all worlds w, $\mathcal{M}, w \models \phi$. Sometimes we write $\phi \dashv \Gamma$ to indicate that ϕ is a logical consequence of Γ.

The label 'Logical Omniscience' indicates a broad group of closure conditions on knowledge. The most relevant instances of Logical Omniscience are the following.

- *Closure under (Classical) Consequence*: If $\phi \models \psi$, then $K\phi \models K\psi$.
- *Closure under (Classical) Validity*: If $\models \phi$, then $\models K\phi$;
- *Closure under (Classical) Equivalence*: If $\phi \dashv\models \psi$ then $K\psi \dashv\models K\psi$.

Closure under (Classical) Validity is a special case of *Closure under (Classical) Consequence* as validity boils down to logical consequence from the empty set. The discrepancy between the standard treatment and real agents is again apparent: we do not know whether Goldbach's conjecture is true or false, although it is one of the two. Also, arguably it is perfectly rational for an agent to know basic arithmetic truths, without thereby knowing that – say – Bézout's identity is true (against *Closure under (Classical) Equivalence*).

There are, in addition, other closure principles which sounds inappropriate for modeling the logic of real epistemic agents:

- *Closure under Disjunction*: $K\phi \models K(\phi \vee \psi)$

Closure under Disjunction is a specific case of *Closure under Consequence* and it represents an unreasonably strict constraint on non-idealized knowledge, as the agents might lack the awareness of some formula ψ: suppose Alma knows that Jorge is Colombian, it might well be the case that she does not know that Jorge is Colombian or the cardinality of real numbers is $2^{\aleph_0}$.

On the other hand, there is a conflict between the representation of the non-ideal component of agents' knowledge and its *rational* component. Since ordinary agents are non-ideal, they do not know all the consequences of what they know. However, since they are rational, we cannot model an account of knowledge in which knowing something does not imply knowing anything else in particular. This is what Jago [11] calls the problem of bounded rationality: the conflict between normative principles of rationality and our limited cognitive resources.

Accordingly, the research for an adequate logic of knowledge should find a fair balance between these two components. This means that some idealization is inevitable in the development of an epistemic logic which preserves agent's rationality and logical competence. We need then to identify a restricted form of closure of knowledge which carries an explanation of why, on the one hand, some epistemic principles can be preserved (the defensible core), while, on the other hand, there are some logical consequences more epistemically precarious.

Yablo's theory of *immanent closure* offers a solution to this issue: knowledge is closed under containment, that is "to know that snow is cold and white, you should know it is cold *already*, whereas there is no requirement of first knowing that snow is cold *or* white before you count as knowing that snow is cold" [1, p. 116]. Yablo's understanding of containment (PA) between propositions can be defined as follows:

(PA) ψ is part of ϕ iff the inference ϕ, *therefore* ψ is

- truth-preserving – whenever ϕ is true, ψ is true, and
- subject matter preserving – whatever ψ si about, ϕ is about.

We can think of subject matter as a comprehensive set of *ways things can be*. For example, the sentence 'the snow is white or cold', $\phi \vee \psi$, is true both when the snow is white and when it is cold, thus the transition from 'the snow is white', ϕ, to the sentence $\phi \vee \psi$ extends the set of the ways for the disjunction to be true, by introducing the whole set of ψ-ways to be true and, similarly, new ways to be false. New ways for a sentence to be true are new opportunities to believe the consequence for the wrong reasons. Hence, this belief is more epistemically vulnerable, because, even if we are right in believing that $\phi \vee \psi$, a confusion on *how* it is true undermines my justification in believing it, and, hence, my knowledge – as Gettier's cases have taught us. Similarly, new ways for a sentence to be false are more counterpossibilities for us against what to be on guard [1, p. 118-119]. Yablo calls this form of closure under parts *immanent* (IC):

(IC) If S knows that ϕ, and ψ is part of ϕ, then S knows that ψ.

The distinction between immanent closure and full logical omniscience mirrors the distinction between *pure* and *deductive* principles of closure.[3] Pure principles are those that hold independently from the agent's capacity to perform inferences. Similarly, the idea behind immanent closure is that knowledge is preserved necessarily only with respect to those principles that do *not* add new informational content to the pieces of knowledge already owned by the agent. On the other hand, deductive principles are those that, to move from premises to conclusions, require the agent to do something,

[3] This distinction can be found in Holliday [12]. Thanks to an anonymous reviewer for suggesting this reference.

namely to perform some inferential act. This distinction helps us to clarify the type of idealization we will model with our account: we represent agents with limited computational abilities, so that their knowledge might fail with respect to their capacity of performing deductions.

Closure under Disjunction can be seen as an instance of a deductive principle, but we could add to this family also the following principles, that we will comment further in Section 5.1:

- *Closure under Known Material Implication*: $K(\phi \to \psi) \wedge K\phi \models K\psi$.
- *Closure under Disjunctive Syllogism*: $K(\neg\phi) \wedge K(\phi \vee \psi) \models K\psi$.

Paradigmatic examples of pure principles are Conjunctive Distribution and Weak Simplification, which hence represent virtuous instances of closure of knowledge:

- *Conjunctive Distribution*: $K(\phi \wedge \psi) \models K\phi \wedge K\psi$
- *Weak Simplification*: $K(\phi \wedge \psi) \models K(\phi \vee \psi)$

Conjunctive Distribution is often considered a paradigmatic principle of ordinary knowledge (compare [13, 14]), as it is an explicit form of closure under parts. Holliday [15, p. 280] argues also that *Weak Simplification* is very intuitive and difficult to deny. The reason is that when the agent knows $\phi \wedge \psi$, she already possesses the whole information required to know $\phi \vee \psi$, and, in a sense, more. The proposition that $\phi \vee \psi$ is weaker than the one already known by the agent, in the sense that it *says less* about the same topics. In other words, the inference from $\phi \wedge \psi$ to $\phi \vee \psi$ does not require to learn anything new, where we mean 'learning something new' as an informal expression which is connected to the topic of the information that a proposition expresses and its subject matter.[4]

It is almost unanimously accepted that knowledge if *factive*, namely knowledge implies truth:

- *Factivity*: $K\phi \models \phi$.

Connected to *Factivity* is the principle that knowledge is consistent:

- *Consistency*: $\models \neg K(\phi \wedge \neg\phi)$.

Consistency states that an agent cannot know contradictory information. This principle is an immediate consequence of the assumption of the *Factivity* of knowledge and of the fact that there are no true contradictions.

[4] It might be argued that all classical inferences express *trivial* information and we do not learn anything new by performing them. Following the previous reasoning, then, we would have Logical Omniscience with respect to classical consequence over again. However, we do not accept this quick argument and we follow Jago [16] in claiming that at least some results in logic are informative. Just to hint at a possible explanation, Jago links the informativeness of the logical inferences to their computational complexity: roughly, the more inferential steps an agent needs to infer some conclusion, the more informative this conclusion is; accordingly, an agent is committed to know only those inferences which are 'simple' enough to be considered trivial.

3 Truthmaker Semantics

Fine's truthmaker semantics is a systematic procedure to assign to each formula in a propositional language a set of *states* which count as its *trutmakers* and a set of states which count as its *falsemakers*. In what follows we present the standard framework of exact verification in the *inclusive* version and of Analytic Entailment [10] as well as the standard notions of modalized state spaces and W-models.

We use letters $\phi, \psi, \gamma ...$ to denote formulas and we stick to a language, call it $\mathcal{L}$, consisting of propositional variables $p, q, r...$, logical constants "$\neg, \wedge, \vee$" and auxiliary symbols "(,)"; a well-formed formula in the language $\mathcal{L}$ is defined as:

$$\phi := p \mid \top \mid \neg\phi \mid \phi \vee \phi \mid \phi \wedge \phi$$

where p is a propositional variable; let $Prop$ indicate the set of propositional variables.

Definition 1 A state space is a tuple $\mathcal{S} = (S, \sqsubseteq)$ where

- S non-empty set of states;
- $\sqsubseteq$ (*relation*) is a partial order over S, namely a reflexive, transitive and anti-symmetric relation, such that:
  - S is complete, namely every $T \subseteq S$ has a least upper bound $\bigsqcup T \in S$ ($s \sqcup t$ denotes the *fusion* of s and t, namely $\bigsqcup\{s, t\}$);
  - we use $\square$ to denote the least upper bound of the empty set, $\square := \bigsqcup \varnothing$, and we call it "null state"; observe that it is such that $\square \sqsubseteq s$ for any $s \in S$;
  - we use $\blacksquare$ to denote the least upper bound of the set S, $\blacksquare := \bigsqcup S$, and we call it *full state*.

Note that, given completeness, state spaces always contain the the null state and the full state. A state space is extended to a *state model*, defined as follows.

Definition 2 A state model is a tuple $\mathcal{M} = (S, \sqsubseteq, |.|^+, |.|^-)$ such that:

- $(S, \sqsubseteq)$ is a state space;
- $|.|^+, |.|^- : Prop \rightarrow \mathcal{P}(S)$ are valuation functions such that
  - $|p|^+ \subseteq S$ is a *non-empty* set of exact truthmakers of p;
  - $|p|^- \subseteq S$ is a *non-empty* set of exact falsemakers of p;
  - for every non-empty $T \subseteq |p|^+(|p|^-)$, $\bigsqcup T \in |p|^+(|p|^-)$ (complete closure).

We distinguish (at least) two different relations of verification(and falsification): exact verification ($\Vdash$) is meant to capture the idea of the complete relevance of a state responsible for the truth (falsity) of a formula; inexact verification ($\Vvdash$), on the contrary, is defined in terms of the previous relation, and it admits an element of irrelevance among the parts of a state responsible for the truth (falsity) of a formula.

Definition 3 (Exact Verification) Given a *state model* $\mathcal{M} = (S, \sqsubseteq, |.|^+, |.|^-)$, the conditions for a formula to be *exactly verified* ($\Vdash$) or *exactly falsified* ($\dashv\mid$) by a state $s \in S$ are defined recursively:

$$\begin{array}{ll}
s \Vdash p & \Leftrightarrow s \in |p|^+ \\
s \dashv\mid p & \Leftrightarrow s \in |p|^- \\
s \Vdash \neg\phi & \Leftrightarrow s \dashv\mid \phi \\
s \dashv\mid \neg\phi & \Leftrightarrow s \Vdash \phi \\
s \Vdash \phi \wedge \psi & \Leftrightarrow \text{for some } t, u\ (t \Vdash \phi, u \Vdash \psi \text{ and } s = t \sqcup u) \\
s \dashv\mid \phi \wedge \psi & \Leftrightarrow s \dashv\mid \phi \text{ or } s \dashv\mid \psi \text{ or for some } t, u\ (t \dashv\mid \phi, u \dashv\mid \psi \text{ and } s = t \sqcup u) \\
s \Vdash \phi \vee \psi & \Leftrightarrow s \Vdash \phi \text{ or } s \Vdash \psi \text{ or for some } t, u\ (t \Vdash \phi, u \Vdash \psi \text{ and } s = t \sqcup u) \\
s \dashv\mid \phi \vee \psi & \Leftrightarrow \text{for some } t, u\ (t \dashv\mid \phi,\ u \dashv\mid \psi \text{ and } s = t \sqcup u)
\end{array}$$

Definition 4 (Inexact Verification) Given a state model $\mathcal{M} = (S, \sqsubseteq, |.|^+, |.|^-)$, for any $s \in S$, we say that s *inexactly verifies* a formula ϕ if s contains an exact verifier of ϕ; more formally $s \Vvdash \phi$ iff for some $t \sqsubseteq s$, $t \Vdash \phi$.

Definition 5 (Modalized State Space) A modalized state space is a tuple $(S, S^\Diamond, \sqsubseteq)$ with:

- $(S, \sqsubseteq)$ a state space;
- $S^\Diamond \subseteq S$ is a non-empty set of *possible states* such that for any $t \in S$ and $s \in S^\Diamond$, $t \sqsubseteq s$ implies $t \in S^\Diamond$ (closure under parts).

Definition 6 (Compatibility) A set of states $T \subseteq S$ is compatible when $\bigsqcup T \in S^\Diamond$ and incompatible otherwise. We say that s ans t are (in)compatible when $\{s, t\}$ is.

From the notion of possible states we can define a possible world as a maximal extension of a possible state, namely as a maximal member of $S^\Diamond$.

Definition 7 (Possible world) A possible world is a possible state which contains as a part all states compatible with it: for a world w and any state s, $s \sqsubseteq w$ if $w \sqcup s \in S^\Diamond$. We call W the set of possible worlds and $W \subseteq S^\Diamond$.

Definition 8 (W-space) A W-space is a *modalized state space* in which all possible states are part of a possible world: for all $s \in S^\Diamond$, there is a $w \in W$ such that $s \sqsubseteq w$.

Definition 9 (W-model) A W-model is a tuple $(S, S^\Diamond, \sqsubseteq, |.|^+, |.|^-)$, such that $(S, S^\Diamond, \sqsubseteq)$ is a W-space, and $|.|^+, |.|^-$ are valuation functions defined as in *state model*, and

- for all $s, t \in S$, if for some $p \in Prop$, $s \in |p|^+$ and $t \in |p|^-$, then $s \sqcup t \notin S^\Diamond$ (exclusivity);
- for all $w \in W$, either some part of w is a member of $|p|^+$ or some part of w is a member of $|p|^-$ for all $p \in Prop$ (exhaustivity).

We can then show that the properties of non-emptiness of the valuation functions, their exclusivity and exhaustivity can be extended from the propositional letters to all sentences in the language by a simple induction.

Proposition 1 *Let* $(S, S^{\Diamond}, \sqsubseteq, |.|^{+}, |.|^{-})$ *be a W- model, then the following propositions hold:*

1. *for all* $\phi \in \mathcal{L}$, $|\phi|^{+} \neq \varnothing$ *and* $|\phi|^{-} \neq \varnothing$*;*
2. *for all* s *and* $t \in S$*, for all* $\phi \in \mathcal{L}$*, if* $s \in |\phi|^{+}$ *and* $t \in |\phi|^{-}$*, then* $s \sqcup t \notin S^{\Diamond}$.
3. *for all* $w \in W$ *and for all* $\phi \in \mathcal{L}$*, either some part of* w *is a member of* $|\phi|^{+}$ *or some part of* w *is a member of* $|\phi|^{-}$.

We are now in the position to define new modal notions of verification and logical consequence which corresponds to classical logic. Note that we will use the symbol $\models$ to refer to loose consequence, even though we used the same for the notion of truth in a Kripke model. No confusion will occur, because from now on we will need to refer only to loose verification and loose consequence.

Definition 10 (Loose Verification) Given a sentence ϕ of $\mathcal{L}$, a W-model $\mathcal{M}$ and world $w \in \mathcal{M}$, ϕ is *loosely verified* in w, denoted with $\mathcal{M}, w \models \phi$, if and only if $s \in |\phi|^{+}$ for some $s \sqsubseteq w$.

Definition 11 (Loose Logical Consequence) For $\Gamma \cup \{\phi\} \subseteq \mathcal{L}$, ϕ is a *loose consequence* of Γ, denoted with $\Gamma \models \phi$, if and only if , for every W-model $\mathcal{M}$ and world $w \in \mathcal{M}$, if $\mathcal{M}, w \models \psi$ for all $\psi \in \Gamma$ entails $\mathcal{M}, w \models \phi$.

Definition 12 (Loose Validity) For all $\phi \in \mathcal{L}$, ϕ is *loosely valid* if and only if for every W-model $\mathcal{M}$ and world $w \in \mathcal{M}$, $\mathcal{M}, w \models \phi$.

Theorem 2 *For all W-models* $\mathcal{M}$ *and for all* $\Gamma \cup \{\phi\} \subseteq \mathcal{L}$, $\Gamma \models \phi$ *if and only if* $\Gamma \models_{CL} \phi$*, where* CL *stands for classical logic.*

For further considerations on the relation between loose consequence and the classical one see [5], in particular the section titled *Classical Truth-Conditions*.

3.1 Propositions and Containment

A proposition is a set of states. The exact content of a proposition is denoted as $|\phi|^{+} = \{s \in S \mid s \Vdash \phi\}$, $|\phi|^{-} = \{s \in S \mid s \dashv\mid \phi\}$. The inexact content is denoted as $||\phi||^{+}$ and $||\phi||^{-}$, namely $||\phi||^{+} = \{s \in S \mid s \Vvdash \phi\}$ and $||\phi||^{-} = \{s \in S \mid s \dashv\mid\mid \phi\}$.

Sometimes we might want to enrich propositions with the following closure properties.

Definition 13 (Convex closure) X is convex when, if $s \in X$, $u \in X$, and $s \sqsubseteq t \sqsubseteq u$, then $t \in X$ too. We write X^{c} for the smallest convex set containing X.

Definition 14 (Complete closure) X is completely closed (or complete) when,for any nonempty subset $Y \subseteq X$, its fusion $\bigsqcup Y \in X$. We write X^{f} for the smallest complete set containing X.

Definition 15 (Regular closure) X is regular when it is both (completely) closed and convex. We write X^{r} for the smallest regular set containing X.

The relation of containment between two propositions is defined as follows.

Definition 16 (Containment) For any set X and $Y \subseteq S$, Y is a partial content of X, i.e. $Y \preceq X$, iff the following two conditions hold:

- X **subsumes** Y: for all $s \in X$, there is a $t \in Y$ and $t \sqsubseteq s$.
- Y **subserves** X: for all $t \in Y$, there is an $s \in X$ and $t \sqsubseteq s$.

The notation $Y \succeq X$ denotes the fact that X is a partial content of Y.

As an example of containment between propositions, Fine [5] argues that in saying that he is an American philosopher, he is saying that he is a philosopher. However, in saying that he is a philosopher, he is not saying that he is a philosopher or American. The truthmaker-based definition of containment correctly predicts that, in the former case, the second content is part of the first while, in the latter, it is not.

We can look at this relation of containment as an entailment relation, which is called *Analytic Entailment*. Following the literature we refer to this relation with the notation **AC**, from Angellic Content.

Definition 17 (Analytic Entailment) For all formulas ϕ and $\psi \in \mathcal{L}$, ϕ analytically entails ψ, $\phi >_{AC} \psi$, iff $|\phi|^+_f \succeq |\psi|^+_f$ in every non-empty model $\mathcal{M}$. And ϕ is analytic equivalent to ψ – $\phi \approx_{AC} \psi$ – when $|\phi|^+_r = |\psi|^+_r$.

The logical system with respect to which this consequence relation is complete corresponds to Angell's system of analytic implication [17, 18], which is meant to represent the notion of containment of meaning or synonymousness.

4 Semantics of Total Knowledge

The concept of containment and Analytic Entailment will play a key role in building our epistemic semantics. More specifically, different entailment relations are involved in two of the issues that concern us. One is involved in the question: *what follows from a statement of the form $K\phi$?* The other is involved in the question: *what consequence relation (if any) is the modality 'K' closed under?* In other words, what is the *internal* logic of knowledge?

To avoid omniscience problems convincingly, when we address the first question, we need to consider a classical modal consequence: indeed, we want to say that it is *metaphysically* possible that an ordinary agent knows ϕ and does not know ψ, even if ψ is a classical logical consequence of ϕ. Hence, in order to model a framework that answers to the first question, we will adopt *loose consequence*, which, as we have seen, is equivalent to the classical consequence (Definition 11).

On the other hand, in order to answer the second question, we will appeal to a non-classical logical consequence. In particular, the consequence under which our K-operator is closed is Analytic Entailment. As a result, loose consequence and Analytic Entailment will interact in a logical system which delivers good results with respect to the philosophical problems we are concerned with: it avoids Logical Omniscience with respect to classical logic, but it preserves the agents' logical competence, as we will argue in the following sections.

Our Truthmaker-based epistemic logic extends the framework provided by Fine in a epistemic modal sense. Recall the language $\mathcal{L}$:

$$\phi ::= p \mid \neg\phi \mid \phi \vee \phi \mid \phi \wedge \phi$$

where $p \in Prop = \{p, q, ...\}$. An atomic epistemic formula is of the form $K\phi$ with ϕ a formula in $\mathcal{L}$, which is read as 'the agent knows ϕ'.[5] We take an *epistemic formula* (or K-formula) to be a truth-functional compound of formulas of the form $K\phi$ – which we call *K-atoms*. Note, therefore, that the epistemic language does not admit embedded modalities such as $KK\phi$ and $K\neg K\phi$, because $K\phi$ itself is not a formula in $\mathcal{L}$. We then call $\mathcal{L}_e$ the epistemic language, which is the closure under the truth-functional connectives of $\mathcal{L}$ and all the K-formulas, and we define a well formed formula α in $\mathcal{L}_e$ as follows:

$$\alpha ::= p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid K\phi$$

where p is a propositional letter and $\phi \in \mathcal{L}$. Accordingly, in what follows, we will use Greek letters $\phi, \psi, \chi, \ldots$ to denote specifically non-modal formulas in $\mathcal{L}$, and we use $\alpha, \beta, \gamma, \ldots$ to refer to arbitrary formulas in $\mathcal{L}_e$.

An epistemic state space extends a W-space with a partial function mapping (some) members of S into subsets of S. Let us call $\mathsf{dom}(f)$ the domain of the partial function f.

The idea behind this epistemic function is to select for some states and agents their *body of knowledge*, namely the body of knowledge is the set of verifiers of the agent's total knowledge. However, before analyzing deeper the nature of the epistemic function, we will introduce the formal definition of epistemic models and exact verification.

Definition 18 (Epistemic space) An epistemic space is a tuple $\mathcal{S} = (S, S^{\Diamond}, \sqsubseteq, f)$, where

- $(S, S^{\Diamond}, \sqsubseteq)$ is a W-space;
- f is a partial function: $f : S \to \mathcal{P}(S)$.

Definition 19 (Epistemic model) An epistemic model $\mathcal{M}$ is a tuple $(\mathcal{S}, |.|^{+}, |.|^{-})$ such that $\mathcal{S}$ is an epistemic space, and $|.|^{+}$ and $|.|^{-}$ are functions mapping the atomic sentence letters of $\mathcal{L}$ into subsets of S such that for every sentence letter $p \in Prop$:

- $|p|^{+}$ and $|p|^{-}$ are nonempty;
- for every non-empty $T \subseteq |p|^{+}(|p|^{-})$, $\bigsqcup T \in |p|^{+}(|p|^{-})$ (complete closure);
- if $s \in |p|^{+}$ and $t \in |p|^{-}$ then $s \sqcup t \notin S^{\Diamond}$ (exclusivity);
- if $w \in W$, then either some part of w is a member of $|p|^{+}$ or some part of w is a member of $|p|^{-}$ (exhaustivity).

[5] Since we will work on the single-agent formalism, we will not indicate the agent as subscript on the modality.

Definition 20 (Epistemic semantics) Given an epistemic model $\mathcal{M} = (S, S^{\Diamond}, \sqsubseteq, f, |.|^+, |.|^-)$ and a state $s \in S$, exact verification is defined for propositional variables and Boolean operation as in the inclusive truthmaker semantics (Def. 3) and if $s \in \mathsf{dom}(f)$, then

$$s \Vdash K\phi \Leftrightarrow |\phi|^+ \preceq f(s)$$
$$s \dashv\!| \, K\phi \Leftrightarrow |\phi|^+ \not\preceq f(s)$$

Informally, an agents knows ϕ at s, if and only if ϕ is contained in her total knowledge compatible with s. For example, if her total knowledge includes the proposition 'it is raining', it does not necessarily include the proposition 'is it raining or the atomic bomb exploded', because the latter propositional content is not necessarily contained in the former. Similarly, if her total knowledge contains the conjunction of all the Peano's axioms, it does not necessarily contain complex propositional content related to Bézout's identity. Note that when $s \notin \mathsf{dom}(f)$, then there is no K-formula that is verified or falsified in s. This means that the metatheory of the semantic is non-classical as it admits gaps.

In the following sections we will see how these intuitive ideas follow from our semantic clauses, but we will first explore further the interpretation of the epistemic function as the agents' total knowledge.

4.1 Epistemic Function and Total Knowledge

We have already described the epistemic function as the agent's body of knowledge, i.e. the set of the verifiers of the agent's total knowledge. In other words, if s is a state in the domain of the epistemic function f, then s is the state that says that the agent knows exactly the proposition $f(s)$ and nothing more. Hence, we understand total knowledge as a maximal epistemic state. In addition, the knowledge operator K tells us that what follows from the agents' total knowledge is what is analytically entailed by it, namely what it is contained in it.

To give a different intuitive image of what we mean for body of knowledge and total knowledge, think of $f(s)$ as the set of epistemic possibilities which are *in some relevant sense* compatible with s: $t \in f(s)$ if it is compatible with the agent?s evidence in s that t obtains. However, compatibility must not be understood in a technical sense, namely as the possibility of the fusion of every state in $f(s)$ and s itself: impossible states can be part of bodies of knowledge as well, while *technically* they cannot be compatible with any states.[6] In fact, we take that a body of knowledge might be subject to misrepresentations, so that an agent might consider epistemically possible something that cannot consistently obtain. To say it with Hintikka: "we should allow for options which only look possible but which contain hidden contradictions" [3,

[6] The fact that impossible states cannot be compatible with any state is a direct consequence of the definition of compatibility and also a poor feature of the framework. An idea to work around it is to introduce a primitive relation of incompatibility. However, this topic goes behind the scope of the present work. For more discussion on this issue, see [19].

p. 476]. In other words, any state can be in principle compatible in this loose sense with the agent's evidence and therefore it can be a member of the relevant body of knowledge.

In principle, an agent can have different total knowledge at different states. However, there is also an intuitive requisite of totality. We expect f to determine the *maximal* amount of information that is compatible with the starting state. This conception of total knowledge, in fact, comes with some constraints.

Partiality of f

Since the proposition $f(s)$ represents the agent total knowledge, we want to allow f to be undefined for some states. The idea is that some states are too *small* to determine any knowledge of an agent and they must be silent with respect to any knowledge ascription, in the sense that they leave open what the agent knows. For example, the state of snow being white leaves open the color of blood, as well as whether I know that the sun is shining. Not only because this state contains little information in general, but also because it has no specific information about the agent's epistemic state.

The case in which the function is undefined has a different meaning than the function that assigns to an agent the empty set. When $f(s)$ is empty, the agent?s body of knowledge is the unverifiable proposition, i.e. the impossible proposition, which is different from saying that a *thin* state does not determine any knowledge. In the latter case, agents might or might not know certain information, while in the former, agents have access only to the trivial falsity and nothing else, which entails that she does not know anything.

Proposition 3 *For all* $s \in S$*, if* $f(s) = \varnothing$*, then for all* $\phi \in \mathcal{L}$*,* $s \Vdash \neg K\phi$*.*

Compatibility

There might be more than one state that determines the agent?s total knowledge, but if they are compatible, they must agree on its content. The idea is that an agent cannot have different total knowledge states in the same possible world, otherwise they would not be maximal epistemic states. On the contrary, there might be two incompatible states which determine different bodies of knowledge of an agent, hence the agent may have different total knowledge in different possible worlds. Therefore, we assume that if two compatible states are both in the domain of f they assign to the agent the same total knowledge. We state a *Compatibility* condition as follows.

Condition 3.1 (Compatibility) *For all* $s, t \in S$*, if* $s \sqcup t \in S^{\Diamond}$ *and* $s, t \in \mathsf{dom}(f)$*, then* $f(s) = f(t)$*.*

This condition is plausible because $f(s)$ is the agent's *total* knowledge and not just – say – the strongest proposition that is known by the agent at s. We understand the total knowledge as the maximal information that an agent has compatibly to the state where she is located. Hence, if $s \in \mathsf{dom}(f)$, s is the state that says that the agent's total knowledge is $f(s)$. If there is a $t \in \mathsf{dom}(f)$ which is compatible with s, then $f(s)$ and $f(t)$ must coincide, as both $f(s)$ and $f(t)$ must individuate the maximal epistemic state of the agent, which includes the whole compatible information.

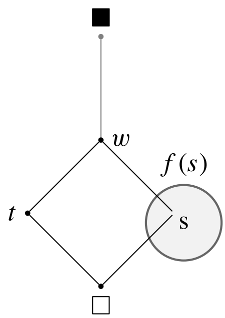
This idea of total knowledge affects also the relation between verification and falsification of the epistemic formulas. The exact verification for knowledge depends on whether f is defined in a state or not. If the function is not defined then K-formulas are neither true nor false in that state. This means that our metatheory admits gaps with respect to knowledge ascription. On the other hand, the verification and falsification clauses are defined in a mutually exclusive way. This means that each state for which the epistemic function is defined must decide about both truth and falsity of knowledge. There is no difference between the verifiers and the falsifiers of K-formulas.

This feature of our semantics might seem at odds with the bilateral spirit of truthmaker semantics. However, it is justified by the interpretation of the epistemic function just presented. As already mentioned, in principle we expect that f is defined only on states *robust* enough to determine the total knowledge of an agent, hence it is not surprising that those states express information concerning both verification and falsification of the attribution of knowledge.

4.2 Features of Knowledge

Negation and Falsification

The relation between negation and falsification is very important in an epistemic context, especially in our framework, as the clauses for knowledge do not really distinguish between truthmakers and falsemakers of knowledge. Yet, our approach delivers the right results: the falsification of a K-atom, i.e. the failure of knowing some formula, is not equivalent to the verification of the knowledge of the negated formula: $\neg K\phi$ is not equivalent to $K\neg\phi$. The idea is that one can fail to know – say – that it is raining, without thereby knowing that it is not raining.[7] See for instance the following model.



Let $Prop = \{p\}$ and the valuation functions be as follows: $|p|^+ = \{t\}$ and $|p|^- = \{\blacksquare\}$. Moreover, the epistemic function is defined only with respect to s and $f(s) = \{s\}$. Hence, we have by construction that $|p|^+ \not\preceq f(s)$, which means that $s \dashv\!\mid Kp$, i.e. $s \Vdash \neg K\ p$. Moreover, $|p|^- \not\preceq f(s)$, which means that $s \nVdash K\neg p$. In words, an agent in the state s, with $f(s)$ as total knowledge, fails to know that p, and it does not know $\neg p$.

Exclusivity

Proposition 1 guarantees that the condition of exclusivity is preserved for each formula

[7] Even if this aspect is obvious in Hintikkian epistemic logics, it is worth checking it in our setting, for testing our definitions.

$\phi \in \mathcal{L}$. We need to check that we can extend this result to each formula in $\mathcal{L}_e$ as well, namely also to the K-formulas. In order to do so, the fact that verification and falsification are defined in a mutual exclusive way is not a sufficient condition to obtain the desired result, as the Compatibility Condition 3.1 is required.

Proposition 4 *If Condition 3.1 holds, Proposition 1.2 holds for all $\alpha \in \mathcal{L}_e$, namely for all s and $t \in S$, if $s \in |\alpha|^+$ and $t \in |\alpha|^-$, then $s \sqcup t \notin S^\Diamond$.*

Proof It suffices to check the case of $\alpha := K\beta$. Suppose $s \in |K\beta|^+$ and $s' \in |K\beta|^-$, namely $s \Vdash K\beta$ and $s' \dashv\mid K\beta$, which means that $|\beta|^+ \preceq f(s)$ and $|\beta|^+ \not\preceq f(s')$. Hence, $f(s) \neq f(s')$ and by Condition 3.1, we can conclude that $s \sqcup s' \notin S^\Diamond$. □

Exhaustivity

In the previous section, we have extensively discussed the interpretation of the epistemic function and the reasons why it is partial, namely the fact that not every state determines the total knowledge of every agent and, as a consequence, that the clauses of verification and falsifications are not defined in a mutual exhaustive way. If the function is not defined on a state, then no knowledge ascription is verified or falsified in that state. Hence, our framework leaves room for gaps with respect to knowledge: it is indeed perfectly conceivable that an agent finds herself *undecided* between assenting or dissenting to some information. These considerations are particularly compelling in our framework, as we are considering the relation of exact verification, namely the relation between a formula and a state wholly relevant and responsible for its verification (or falsification).

On the other hand, unlike states, possible worlds are maximal entities, namely they are exhaustive with respect to the truth of every formula. Hence, given that the function f selects the total knowledge of an agent, we expect that there is some proposition P that is the total knowledge of the agent in (at least) a part of each possible world. Also, there should be exactly one such P, which is guaranteed by the Compatibility condition. Therefore, we impose an additional constraint on the epistemic function which guarantees that it is always defined with respect to some part of every possible world:

Condition 3.2 (Definability) *For all $s \in S^\Diamond$, there is some $t \in \mathsf{dom}(f)$, such that $s \sqcup t \in S^\Diamond$.*

Proposition 5 *If Condition 3.2 holds, then for all $w \in W$, there is a $s \sqsubseteq w$ for which $s \in \mathsf{dom}(f)$.*

Proof Consider an arbitrary $w \in W$. By Condition 3.2, it follow that there is a $t \in S$ such that $t \sqcup w \in S^\Diamond$ for which f is defined. By definition of possible world, we know that $t \sqsubseteq w$. □

We have seen that Proposition 1.3 guarantees that possible worlds are exhaustive with respect to every formula $\phi \in \mathcal{L}$, with the Definability condition we can prove that this property is preserved also with respect to the K-formulas.

Proposition 6 *For all $w \in W$ and for all $\alpha \in \mathcal{L}_e$, either there is a $s \sqsubseteq w$ such that $s \in |\alpha|^+$ or there is some $s' \sqsubseteq w$ such that $s' \in |\alpha|^-$.*

Proof It suffices to check the case of $\alpha := K\beta$. Consider an arbitrary $w \in W$. Suppose that, for all $s \sqsubseteq w$ such that $s \in \mathsf{dom}(f)$, $s \nVdash K\beta$ for some β, i.e. $|\beta|^+ \npreceq f(s)$. In particular, by Condition 3.2, we know that there is at least one $t \sqsubseteq w$ such that $t \in \mathsf{dom}(f)$. Hence, it follows by verification clauses that $s \dashv\mid K\beta$. □

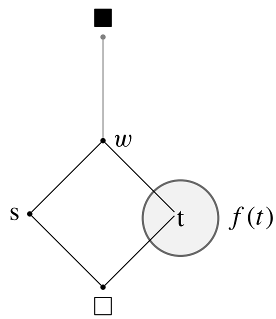
In conclusion, this proposition tells us that, even if there are states silent with respect to knowledge, possible worlds always contain a state which determines the total knowledge of the agents.

The property of non-emptiness on the valuation functions cannot be extended to every K-formula: the fact that for all $\phi \in \mathcal{L}$, $|\phi|^+ \neq \varnothing$ and $|\phi|^- \neq \varnothing$ does not imply that, for all $\phi \in \mathcal{L}$, $|\phi|^+ \preceq f(s)$ for some $s \in S$ and $|\phi|^+ \npreceq f(s')$, for some $s' \in S$.

Proposition 7 *The Total Knowledge account invalidates the following principles:*

- *Closure under (Classical) Consequence: If $\phi \models \psi$, then $K\phi \models K\psi$;*
- *Closure under (Classical) Validity: If $\models \phi$, then $\models K\phi$;*
- *Closure under (Classical) Equivalence: If $\phi \dashv\models \psi$ then $K\psi \dashv\models K\psi$;*
- *Closure under Disjunction: $K\phi \models K(\phi \vee \psi)$.*

Proof Consider the following counter-examples. Let $Prop = \{p\}$ and $\phi := p \vee \neg p$, which is a classical validity. Consider the model $\mathcal{M}$, structured as in the figure below:



Moreover we let:

- $S^{\Diamond} = \{w, s, t, \square\}$,
- $|p|^+ = \{s\}$, and $|p|^- = \{\blacksquare\}$,
- $\mathsf{dom}(f) = \{t\}$, $f(t) = \{t\}$.

$\mathcal{M}$ is an epistemic model, where f is defined only with respect to t. Hence, Condition 3.1 is vacuously satisfied and so is Condition 3.2, because every element in $S^{\Diamond}$ is compatible with t. Moreover, both the valuations functions are non-empty for every propositional letter and trivially closed under fusion. Note that w is a possible world and, since $s \Vdash p \vee \neg p$, it follows that $w \models p \vee \neg p$ by definition of loose verification. However, since $f(t) = \{t\}$ and $s \not\sqsubseteq t$, $|p \vee \neg p|^+ \npreceq f(t)$. Hence, $t \nVdash K(p \vee \neg p)$, and thus $w \not\models K(p \vee \neg p)$. This shows that *Closure under (Classical) Validity* does not hold.

The same frame can be adopted to construct a counter-model to the principle of *Closure under (Classical) Consequence*. One classical logical consequence of interest in this context is the introduction of the disjunction, i.e. $\phi \models \phi \vee \psi$, for all ϕ and some $\psi \in \mathcal{L}$.

Let $\phi := q$ and $\psi := p$. Moreover, we set $|p|^+ = \{s\}$, $|q|^+ = \{t\}$ and $|p|^- = |q|^- = \{\blacksquare\}$ and $f(t) = \{t\}$. Hence $|q|^+ \preceq f(t)$ and so $w \models Kq$. However, we have that $|p \vee q|^+ = \{s, t, w\}$. Since $w \not\sqsubseteq t$, $|p \vee q|^+ \not\preceq f(t)$. Therefore, we can conclude that $w \not\models K(p \vee q)$. Which shows also that *Closure under Disjunction* is invalid.

Consider the previous model and the following formulas: $\phi := q$ and $\psi := q \wedge (q \vee p)$. The two formulas are a classical equivalence, i.e. $\phi \dashv\models \psi$, which corresponds to one of the law of absorption. In fact, in our model, $|q|^+ = \{t\}$ and $|q \wedge (q \vee p)|^+ = \{t, w\}$, because $t \Vdash q \vee p$ and $t \Vdash q$, thus $t = t \sqcup t \Vdash q \wedge (q \vee p)$. Moreover, $s \Vdash q \vee p$ and $w = s \sqcup t$, hence $w \Vdash q \wedge (q \vee p)$. Hence, $w \models q$ and $w \models q \wedge (q \vee p)$. However $f(t) = \{t\}$, hence it follows that $|q|^+ \preceq f(t)$, i.e. $w \models Kq$. On the other hand, $w \not\sqsubseteq t$, which means that $|q \wedge (q \vee p)|^+ \not\preceq f(t)$, i.e. $w \not\models K(q \wedge (q \vee p))$. In other words, an agent located in the world w, who happens to know some information q, does not necessarily know everything which is logically equivalent to q, against *Closure under (Classical) Equivalence*. □

5 Closure of Knowledge

After discussing what does *not* follow from a K-formula, we look at the logical structure of the framework. The first feature to underline is that a knowledge ascription is always consistent: we can prove that no possible world makes both true and false the same knowledge ascription:

Proposition 8 *For all $\phi \in \mathcal{L}$, for all epistemic model $\mathcal{M}$, and for all $w \in W$, $\mathcal{M}, w \not\models K\phi \wedge \neg K\phi$.*

Proof Assume by contradiction that there is a $w \in W$ and a $\phi \in \mathcal{L}$ such that, $\mathcal{M}, w \models K\phi \wedge \neg K\phi$. This means that there are $s, s' \sqsubseteq w$, such that $|\phi|^+ \preceq f(s)$ and $|\phi|^+ \not\preceq f(s')$, but this is impossible because, by Compatibility Condition 3.1, $f(s) = f(s')$. □

Moreover, we can prove that in our account there is no egregious violation of closure and both *Conjunction Distribution* and *Weak Simplification* are valid principles:

Proposition 9 *For all ϕ and $\psi \in \mathcal{L}$ and for all epistemic models $\mathcal{M}$, Conjunction Distribution and Weak Simplification are valid principles.*

This proposition shows that our epistemic semantics delivers already very good results for the knowledge operator. However, the flexibility of our approach allows us to go further and consider other principles that we may want to validate. We can therefore consider different classes of epistemic frames, varying with respect to the constraints on the epistemic function.

Condition 4.1 (Reflexivity) *For all $s \in \mathsf{dom}(f)$, $s \in f(s)$.*

Definition 21 A *factive* epistemic model $\mathcal{M}^F$ is an epistemic model, where f is *reflexive*, i.e. Condition 4.1 holds.

The condition of Reflexivity is necessary to validate the principle of *Factivity*, one of the distinguished features of a knowledge operator, namely the fact that we can know only true propositions.

Proposition 10 *For all* $\phi \in \mathcal{L}$ *and for all factive models* $\mathcal{M}^F$, $K\phi \models \phi$.

We can consider further constraints to impose on the epistemic function, such as its closure under fusion and convexity:[8]

Definition 22 A *complete* epistemic model $\mathcal{M}^f$ is an epistemic model, where for all $s \in \mathsf{dom}(f)$, $f(s) = f(s)^f$, i.e. it is closed under fusion.

Definition 23 A *regular* epistemic model $\mathcal{M}^r$ is an epistemic model, where for all $s \in \mathsf{dom}(f)$, $f(s) = f(s)^r$, i.e. it is closed under fusion and convexity.

Imposing convexity on the propositional content does not affect the principles validated, but from a philosophical perspective the relation of containment between convex verifiable propositions is antisymmetric, unlike what happens for arbitrary propositions and even complete propositions (Lemma 5 in [20, p. 650]). This property places our relation of containment close to a genuine notion of partial content. Insisting on convexity, then, avoids distinguishing propositions which are not distinguished by containment.

Recall that we called Analytic Entailment the relation between closed propositions ($>_{AC}$). Accordingly, we have the following closure principle for the K-operator:

Proposition 11 *For all* ϕ *and* $\psi \in \mathcal{L}$, *for all* complete *epistemic models* $\mathcal{M}^f$, *if* $\phi >_{AC} \psi$, *then* $K\phi \models K\psi$.

This is one of the most relevant desideratum for our account and a result that we have already defended in the introduction, because it represents a non-ideal logical competence which is philosophically well motivated. Closure under Analytic Entailment is in fact a truthmaker version of Yablo's *immanent closure*. We will explore further their relationship and the connection with the notion of subject matter in Section 5.2.

5.1 Fragmentation of the States of Mind

The closure under fusion of the epistemic function has the consequence of validating *Agglomeration* as well:

- *Agglomeration*: $(K\phi \wedge K\psi) \models K(\phi \wedge \psi)$

Proposition 12 *For all* ϕ *and* $\psi \in \mathcal{L}$, *and for all* complete *epistemic models* $\mathcal{M}^f$, $K\phi \wedge K\psi \models K(\phi \wedge \psi)$.

8 Note that in the counter-models above, we have intentionally designed the epistemic functions as already closed under fusion and convexity and such that reflexivity holds. Therefore, the following results do not affect the invalidities that we obtained before.

Proof Consider an arbitrary complete model $\mathcal{M}^f$ and world w such that $\mathcal{M}^f, w \models K\phi \wedge K\psi$. Then there is a $s \in S^{\Diamond}$, such that $s \sqsubseteq w$ and $s \Vdash K\phi$, and there is a $s' \in S^{\Diamond}$, such that $s' \sqsubseteq w$ and $s' \Vdash K\psi$. Given Condition 3.1, we can say that $|\phi|^+ \preceq f(s)$ and $|\psi|^+ \preceq f(s)$. Hence, it suffices to prove that $|\phi \wedge \psi|^+ \preceq f(s)$. Take an arbitrary $t \in |\phi \wedge \psi|^+$, then $t = t' \sqcup t''$ and $t' \in |\phi|^+$ and $t'' \in |\psi|^+$. There is, by assunption, a $u \in f(s)$ such that $t' \sqsubseteq u$ and there is a $u' \in f(s)$ such that $t'' \sqsubseteq u'$. Hence, $t' \sqcup t'' = t \sqsubseteq u \sqcup u''$. Since $f(s)$ is closed under fusion, $u \sqcup u' \in f(s)$, proving the first clause of the relation of containment. The second clause follows easily by the truthmaker verification clauses. □

It has been argued[9] that this principle might reasonably fail with respect to non-ideal knowledge, because ordinary agents might experience a *fragmentation* of states of mind. This means that they tend to compartmentalize information already possessed and fail to 'put them together' in a single frame of mind. The consequence is that even if an agent knows ϕ and ψ, it does not follow that she knows their conjunction, namely the two pieces of information at once. Our semantics does not force us to accept this principle.[10]

Nonetheless, the fragmentation of knowledge seems to us explicitly related to a contingent epistemic status, concerning the focus of the agent in a particular moment. The phenomenon we want to capture instead aims at a more specific picture, quite unrelated to what an agent is paying full attention in a specific moment. Rather, we focus on the general informational content of her body of knowledge even if she does not single it out in *that* very moment.

Hoek [21] argues in a similar way against the theories of fragmentation and he develops a theory of minimal rationality which has many elements in common with ours.[11] In particular, he argues that a minimally rational subject's beliefs are not perfectly integrated, but neither are they partitioned into isolated compartments, as in fragmentation theories of beliefs. A theory of fragmentation risks to lack predictive powers as the beliefs in different fragments do not constrain one another, hence 'switch fragments, and all bets are off'.

For these reasons, *Agglomeration* represents a minimal form of idealization that we are willing to accept, given that our goal is to formalize a notion of knowledge which depends on the subject-matter sensitivity, as argued by Hoek: minimally rational beliefs are linked together by their *thematic* connections rather than their entailment relations.

9 In [14], for example, the authors argue that the phenomenon of fragmentation is one of the main sources of non-omniscience.

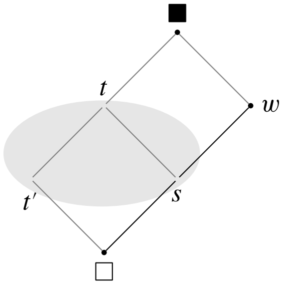
10 We will see in the second part of the paper, that we have more compelling proof-theoretical reasons to accept it, as *Agglomeration* is essential to the reduction in normal form of the K-formulas.

11 Thanks to an anonymous reviewer for suggesting this interesting reference.

Related to the phenomenon of fragmentation are the principles of *Closure under Known Material Implication* and its specific instance of *Closure under Disjunctive Syllogism*:

- *Closure under Known Material Implication*: $K(\phi \to \psi) \wedge K\phi \models K\psi$.
- *Closure under Disjunctive Syllogism*: $K(\neg\phi) \wedge K(\phi \vee \psi) \models K\psi$.

These two principles are not valid in our framework. However to find a counter-example, we need a frame more sophisticated than the previous ones. Indeed, we need to resort to the fact that the body of knowledge of some agent, even if located in a possible state, might in turn be inconsistent, in the sense that it contains impossible states.[12] Consider the following epistemic model.



Let $\phi := p$ and $\psi := q$, then consider a model $\mathcal{M}$ as in the figure, where $S^{\Diamond} = \{w\}^{\downarrow}$, $f(s) = \{s, t, t'\}$, $|p|^{-} = \{s, t, t'\}$, $|p|^{+} = \{t'\}$, $|q|^{-} = \{\blacksquare\}$, $|q|^{+} = \{s\}$. $\mathcal{M}$ is an epistemic model, where the epistemic function is defined only with respect to s; Condition 3.1 is vacuously satisfied and so is Condition 3.2, because every element in $S^{\Diamond}$ is compatible with s. In the model both the valuations functions are non empty for every propositional letter and closed under fusion, as desired. Therefore, this is an instance of an epistemic model. Note also that $f(s)$ is closed under fusion, therefore $\mathcal{M}$ is also an instance of complete epistemic model. From the valuation function, it follows that $|p \vee q|^{+} = \{t, t', s\}$. Therefore we have by construction that $|p|^{-} \preceq f(s)$, and $|p \vee q|^{+} \preceq f(s)$. However, $|q|^{+} \npreceq f(s)$, because there is an element in $f(s)$, i.e. t', such that $s \not\sqsubseteq t'$, and s is the only element in $|q|^{+}$. Accordingly, $w \models K(\neg p)$ and $w \models K(p \vee q)$, but $w \not\models Kq$.

The validity of *Closure under Disjunctive Syllogism* is controversial: on the one hand it is subject to some counter-intuitive epistemic situations, such as *Surprise Exam Paradox* [22];[13] on the other hand it is a special case of *Closure under Known Material Implication*, which can be regarded as a basic requirement of logical competence for non-ideal agents.

As argued by Rosenkranz [23, p. 35], there is general agreement on the fact that knowledge implies belief. The principle of *Closure under Known Material Implication* requires in turn that agents never fail to come to believe a conclusion of *modus*

[12] We focus only on a counter-model of *Closure under Disjunctive Syllogism*, which is in turn a counter-example to *Closure under Known Material Implication*.

[13] Compare also the 'Criterion counter-example' to Disjunctive Syllogism presented in [6, p. 8].

ponens whose premises they know, while ordinary agents cannot expect to satisfy this requirement in every occasion. To say it with Rosenkranz [23, p. 36], knowledge is a cognitive achievement, even when it comes naturally. This idea reminds of the notion of deductive principles that we mentioned before, namely those principles that require the agent to perform some cognitive act. These are distinct from the pure principles, which are those that we want to capture with our notion of Total Knowledge.

Consider, for example, the scenario in which Jones knows that Mary lives in New York, that Fred lives in Boston and that Boston is north of New York. Yet Jones fails to infer the obvious: that Mary will have to travel north to visit Fred [24]. The fact that Jones does not employ his ability to infer via *modus pones* is not necessarily a symptom of his logical incompetence. The failure of this principle has been associated with the phenomenon of fragmentation of states of mind, as famously argued by Lewis [25] (but also [26, 27] and [14]). The idea concerning the previous scenario is that Jones has two different frames of mind: he knows in one that Mary lives in New York and in a different one that Fred lives in Boston and that Boston is north of New York, but he fails to put them together, because of a lack of focus or a limited memory.

As already mentioned, this theory of fragmentation is related to the limits of human memory and focus abilities, but it cannot explain by itself in a satisfying way why agents fail to follow through the logical consequences of what they know. It seems strange to say that when agents don't follow through the logical consequences of what they believe, it is always because they have not conjoined the premises whereas, when they do, they suddenly come to believe all their infinitely many consequences [28]. Accordingly, even if it might be reasonable to say that combining one's information can constitute an important insight which brings new knowledge, performing this operation cannot guarantee knowledge of all the logical consequences of what one knows. Therefore, it is hard to look at the phenomenon of fragmentation, as much as widespread in empirical cognitive activities, as part of the *fundamental* problem of Logical Omniscience [14].

As we showed, the Total Knowledge account invalidates *Closure under Known Material Implication*, whereas *Agglomeration* holds in the class of complete epistemic models. Note that the previous counter-model of *Closure under Known Material Implication* is a complete model, which then validates *Agglomeration*. In light of this, since the failure of *Agglomeration* is a paradigmatic symptom of the fragmentation of knowledge, the Total Knowledge account does not predict that the failure of *Closure under Known Material Implication* is ascribable to the same source. The diagnosis of the Total Knowledge account is that *Closure under Known Material Implication* is not valid because $(\neg\phi \vee \psi) \wedge \phi$ does not analytically entail ψ, as the counter-model presented above witnesses.

5.2 A Subject-matter Sensitive Knowledge

In a truthmaker-based framework we formalize the notion of containment between propositions as a relation between the truth(false)makers of the propositions and their parts. In the case of *regular* propositions, this notion of containment assumes very interesting features, which display its affinity with Yablo's understanding of the same concept. Indeed, it can be argued that, for all *verifiable complete* propositions, containment expresses a form of Yablo's *immanent closure*.

To see why, we first introduce a truthmaker-based notion of subject matter. We distinguish the positive subject matter of some propositional content, its negative and its overall subject matter. The first one is the fusion of the positive content, the second one is the fusion of the negative content and the third, which amounts to the subject matter of the bilateral proposition, is the fusion of both the positive and negative contents.

Definition 24 (Positive Subject Matter) Le ϕ be a sentence and $|\phi|^+$ the respective verifiable proposition, according to the inclusive semantics, the subject-matter of $|\phi|^+$ is $\bigsqcup |\phi|^+$, i.e. the fusion of the exact verifiers of ϕ. We denote it with $\mathbf{s}_\phi^+$.

It can be shown [10, p. 212] that for regular contents, the two clauses of the definition of containment are equivalent to the two requirements for *immanent closure*:

Proposition 13 *In all modalized models $\mathcal{M}$, let $|\phi|^+$ and $|\psi|^+$ be two* verifiable *and* regular *propositions, then*

1. *$|\phi|^+$ subsumes $|\psi|^+$ if and only if $||\phi||^+ \subseteq ||\psi||^+$ – ψ is an inexact consequence of ϕ;*
2. *$|\psi|^+$subserves $|\phi|^+$ if and only if $\mathbf{s}_\psi^+ \sqsubseteq \mathbf{s}_\phi^+$ – ψ's subject matter is part of ϕ's subject matter.*

In words, the forward clause (*subsumption*) corresponds to inexact consequence, while the backward clause (*subserving*) corresponds to subject matter preservation. It is evident, at this point, the close connection with Yablo's notion of containment (PA) between two propositions, which amounts itself to the conjunction of a clause of truth-preservation and one of subject matter inclusion. For this reason, closure under Analytic Entailment is a form of *immanent closure* (as underlined in [5]): knowledge is closed under those inferences that do not change the subject.[14]

There are, however, some differences with respect to a Yablovian immanent closure.[15] First of all, the notion of truth-preservation that Yablo had in mind is classical [1, 31], unlike inexact consequence (which corresponds to FDE) that is paraconsistent and paracomplete. Secondly, we work only with the positive content of propositions and their positive subject matter, while Yablo's idea of subject matter preservation includes both the positive and the negative contents. Our choice has the drawback that the semantics does not distinguish between propositions with the same verifiers and different falsifiers.

On the other hand, it is possible to generalize our approach to include also the negative content of propositions and adapt our Total Knowledge to the notion of containment between bilateral propositions.

[14] Note that also Elgin in [29] argues that knowledge is closed under (known) analytic entailment: a relation that holds just in case the meaning of one contains the meaning of the other.

[15] Hawke [30] and Holliday [15] explore the technical details of Yablo's proposal and in particular its problems.

To implement this idea we need (i) to model the agent's total knowledge as a bilateral proposition; (ii) to adopt a semantics for knowledge that requires, in order for a sentence to be known by an agent, that the bilateral content of the sentence is contained in her Total Knowledge.

We call a *bilateral epistemic function* a partial function that selects for some state in the domain an ordered pair of unilateral propositions:

$$f : S \to \mathcal{P}(S) \times \mathcal{P}(S)$$

thus, the body of knowledge determined by a state s is a pair $\langle P, P' \rangle$, for $P, P' \subseteq S$. The notion of epistemic model can be redefined with the new bilateral epistemic function as expected. As before, we understand knowledge in direct reference to containment. Recall the definition of containment with respect to bilateral propositions:

Definition 25 (Bilateral containment) For all ϕ and $\psi \in \mathcal{L}_e$, $\langle \psi \rangle$ is contained in $\langle \phi \rangle$, denoted $\langle \psi \rangle \preceq \langle \phi \rangle$, just in case (i) $|\psi|^+ \preceq |\phi|^+$ and (ii) $|\psi|^- \subseteq |\phi|^-$, i.e. (i) the positive content of ψ is part of the one of ϕ, and (ii) every falsifier of ψ is also a falsifier of ϕ.

Accordingly, we can state the clauses of knowledge as follows. For all $s \in S$ such that $s \in \mathsf{dom}(f)$,

$$s \Vdash K\phi \textit{ iff } \langle \phi \rangle \preceq f(s) \; s \dashv\!| \; K\phi \textit{ iff } \langle \phi \rangle \not\preceq f(s)$$

Note that it is possible to show ([10], Theorem 25) that bilateral containment with respect to a bilateral semantics (see also e.g. [5]) gives rise to the same logic of AC: the system AC is sound for the bilateral semantics and hence every formula valid under the unilateral semantics is also valid under the bilateral semantics.

As before, knowledge is closed under analytic entailment:

Proposition 14 *For all $\phi, \psi \in \mathcal{L}_e$ and $w \in W$ if $w \models K\phi$ and $\langle \psi \rangle \preceq \langle \phi \rangle$, then $w \models K\psi$.*

This new semantics takes in consideration both the positive and the negative content of the agents' total knowledge. This feature affects also the notion of subject matter preservation, which is a crucial part of *immanent closure*. To see why, recall that the negative subject matter of ϕ, $\mathbf{s}^-_\phi$, is the fusion of its negative content. The overall subject matter of ϕ is the fusion of its negative and positive content $\bigsqcup(|\phi|^+ \cup |\phi|^-)$, i.e.

$$\mathbf{s}_\phi = \mathbf{s}^+_\phi \sqcup \mathbf{s}^-_\phi.$$

It is possible to show that if the bilateral proposition $\langle \psi \rangle$ is analytically entailed by the bilateral proposition $\langle \phi \rangle$ then, not only ψ is an inexact consequence of ϕ, but also the overall subject matter of ψ is part of the overall subject matter of ϕ.

Proposition 15 *For all $\phi, \psi \in \mathcal{L}_e$ if $\langle \psi \rangle \preceq \langle \phi \rangle$, then $\mathbf{s}_\psi \sqsubseteq \mathbf{s}_\phi$.*

Proof We already know by Proposition 13, that $\mathbf{s}^+_\psi \sqsubseteq \mathbf{s}^+_\phi$. It is easy to see that also $\mathbf{s}^-_\psi \sqsubseteq \mathbf{s}^-_\phi$ holds: $\mathbf{s}^-_\psi \in |\psi|^-$ by closure under fusion, and since $|\psi|^- \subseteq |\phi|^-$, $\sqcup|\psi|^- \in$

$|\phi|^-$; it follows that $\sqcup|\psi|^- \sqsubseteq \sqcup|\phi|^-$, i.e. $\mathbf{s}_\psi^- \sqsubseteq \mathbf{s}_\phi^-$. Therefore we can conclude that $\mathbf{s}_\psi^+ \sqcup \mathbf{s}_\psi^- \sqsubseteq \mathbf{s}_\phi^+ \sqcup \mathbf{s}_\psi^-$, i.e. $\mathbf{s}_\psi \sqcup \mathbf{s}_\phi$. □

This bilateral approach appears to be philosophically satisfying and there are many details that deserve further investigation.

The idea that knowledge is sensitive to subject matter preservation is quite widespread in the literature of epistemic logic. Hawke et al. [14], for example, introduce a system of epistemic logic with a topic-sensitive modality of knowledge, which is very close in spirit to our idea of closure under analytic containment. On the other hand, their starting point is very different, and each account is informed by a different picture of what propositional content is.

Hawke et al. [14] follow Yablo's theory of thick content, namely they supplement truth conditions with an account of *topicality*. Topics are elements of an algebraic structure which are assigned by a topic function to every element of the logical language. Hence, the notion of topicality differs from Yablo's subject matter, but they have in common the idea that content is given by two elements which are in principle independent from each other. Accordingly, knowledge is understood as truth in all epistemic worlds and topic inclusion.

While discussing Yablo's theory, Fine [32, p. 134] remarks that "thick content is an *enhancement* of intensional content; it is intensional content plus subject matter. For me, thick content is not so much an enhancement as a modification of intensional content". This modification consists of identifying the content with the set of *states* which make a sentence true or false, unlike the traditional intensional content, which rather looks at the *possible worlds* where the sentence is true or false. Hence, the very notion of intensional content is put aside for a brand new understanding of content in terms of verification and falsification by states. The truthmaker-based notion of subject matter naturally emerges from the content of a sentence: it is not just added with mathematical idealized tools. The algebra of topics, in fact, provides the semantics with a partially syntactic element, reminiscent of the Awareness functions [26].[16] Moreover, an algebra of topics is silent about what topics are: we only know that they are non-linguistic items and that they are transparent with respect to Boolean connectives. Despite these differences, the philosophical ideas that ground both approaches are similar and so is the resulting logics in their crucial elements: knowledge is closed under conjunction elimination but not under disjunction introduction.

6 The Epistemic Logic

In this section we introduce the epistemic logic with respect to which our Total Knowledge semantics is sound and complete. We call it **EL**. As a first step, we will introduce

[16] The authors claim that "the version of awareness logic closest to our framework is the one in terms of 'awareness generated by primitive propositions', where an agent is aware of a formula ϕ just in case it is aware of all of its atomic constituents taken together. We stress the syntactic features of this approach, not shared by ours: awareness is still given by a construction based on atomic formulas, whereas our topic function assigns topics, non linguistic items in the semantics, to formulas, with a recursion on the basic non-epistemic operators (negation and conjunction, thus disjunction)" ([14], footnote 11).

the standard axiomatic system of Analytic Entailment **AC** [10]. As it will become apparent soon, the reason is that the logic **AC** is an integral part of our the system **EL**.

We shall set up a single-premise deductive system consisting of derivations of the form $\phi \vdash_{AC} \psi$, which we call *single formulas sequents* (from now on, it is convenient to abbreviate it with only *sequent*). We derive valid sequents (rather than valid sentences) from the axioms using the rules. Reasoning with multiple premises may be understood by taking their conjunction.[17] We say that two formulas ϕ and ψ are provably equivalent if $\phi \dashv\vdash_{AC} \psi$.

Definition 26 The system **AC** consists of the following axioms and rules.

(E0) $\phi \dashv\vdash_{AC} \neg\neg\phi$
(E1) $\phi \dashv\vdash_{AC} \phi \wedge \phi$
(E2) $\phi \wedge \psi \vdash_{AC} \psi \wedge \phi$
(E3) $(\phi \wedge \psi) \wedge \chi \vdash_{AC} \phi \wedge (\psi \wedge \chi)$
(E4) $\phi \dashv\vdash_{AC} \phi \vee \phi$
(E5) $\phi \vee \psi \dashv\vdash_{AC} \psi \vee \phi$
(E6) $(\phi \vee \psi) \vee \chi \dashv\vdash_{AC} \phi \vee (\psi \vee \chi)$
(E7) $\neg(\phi \wedge \psi) \dashv\vdash_{AC} \neg\phi \vee \neg\psi$
(E8) $\neg(\phi \vee \psi) \dashv\vdash_{AC} \neg\phi \wedge \neg\psi$
(E9) $\phi\wedge(\psi\vee\chi) \dashv\vdash_{AC} (\phi\wedge\psi)\vee(\phi\wedge\chi)$
(E10) $\phi\vee(\psi\wedge\chi) \dashv\vdash_{AC} (\phi\vee\psi)\wedge(\phi\vee\chi)$
(E11) $\dfrac{\phi \vdash_{AC} \psi \qquad \psi \vdash_{AC} \chi}{\phi \vdash_{AC} \chi}$
(E12) $\dfrac{\phi \vdash_{AC} \psi}{\phi \wedge \chi \vdash_{AC} \psi \wedge \chi}$
(E13) $\dfrac{\phi \vdash_{AC} \psi}{\phi \vee \chi \vdash_{AC} \psi \vee \chi}$
(E14) $\phi \wedge \psi \vdash_{AC} \phi$
(E15) $\phi \vdash_{AC} \top$
(E16) $\bot \vdash_{AC} \phi$

Fine shows [10, p. 202] a rule of positive replacement is admissible in the sense that it preserves theorem-hood. If ϕ, ψ, χ are formulas of $\mathcal{L}$, let $\chi(\phi/\psi)$ be the result of replacing the occurrences of ϕ in χ by ψ. Then the rule of positive replacement is the following.

$$(PR)\ \frac{\phi \dashv\vdash_{AC} \psi}{\chi \dashv\vdash_{AC} \chi(\phi/\psi)}$$

Lemma 16 *The following rule is derivable in* ***AC****.*

$$(E17)\ \frac{\phi_1 \vdash_{AC} \psi_1 \qquad \phi_2 \vdash_{AC} \psi_2}{\phi_1 \vee \phi_2 \vdash_{AC} \psi_1 \vee \psi_2}$$

Theorem 17 (Theorems 14 and 21 in [10]) $\phi \vdash_{AC} \psi$ *iff* $\phi >_{AC} \psi$.

Now, we introduce the language and the epistemic logic **EL**. Recall that we distinguish between the set of non-modal formulas $\mathcal{L}$ and the modal language $\mathcal{L}_e$, i.e. the union of $\mathcal{L}$ and all the K-formulas, closed under the truth-functional connectives. Recall also the following definitions:

- a *K-atom* is a formula of the form $K\phi$, for any $\phi \in \mathcal{L}$;
- a *negated K-atom* is a formula of the form $\neg K\phi$, for any $\phi \in \mathcal{L}$;
- a *K-literal* is either a K-atom or a negated K-atom;

[17] In general, this is not something we can do in truthmaker semantics, where conjunctions differ semantically from their conjuncts taken together. But here our notion of entailment is based on containment, rather than exact entailment, and so, in particular, $\phi \wedge \psi >_{AC} \phi$ is valid.

- an *epistemic formula* (or *K-formula*) is any truth-functional compound of only K-literals;
- we call simply *literal* λ any propositional letter p in $Prop$ or its negation. Let us call *Lit* the set of all such literals.

In accordance to our formulation of the proof system for **AC**, we shall set up a single-premise deductive system consisting of single formulas sequents of the form $\phi \vdash_{EL} \psi$, where **EL** stands for epistemic logic. Let us call **CL** the proof system of propositional classical logic. We do not spell out **CL** in details: the axioms and rules are those of the classical propositional sequent calculus.

Definition 27 (Epistemic Logic EL) EL extends *classical logic* with the following rule (K1) and axiom (K2):

(K0) Classical propositional axioms and rules in **CL**

(K1) $\dfrac{\phi \vdash_{AC} \psi}{K(\phi) \vdash_{EL} K(\psi)}$

(K2) $K(\phi) \wedge K(\psi) \vdash_{EL} K(\phi \wedge \psi)$

Proposition 18 *The following theorems $K3 - K6$ are deducible in **EL**:*

(K3) $K(\phi \wedge \psi) \vdash_{EL} K(\phi)$

(K4) $K(\phi) \wedge K(\psi) \vdash_{EL} K(\phi \vee \psi)$

(K5) $K(\phi \wedge \psi) \vdash_{EL} K(\phi) \wedge K(\psi)$

(K6) $\dfrac{\phi \dashv\vdash_{AC} \psi}{K(\phi) \dashv\vdash_{EL} K(\psi)}$

Proof **K3** is derivable from $E14$ $\phi \wedge \psi \vdash_{AC} \phi$ and **K1**; **K4** is derivable from **K2**, **K1** and the fact that $\phi \wedge \psi \vdash_{AC} \phi \vee \psi$ is AC-derivable; **K5** is derivable from **K3** and classical logic; **K6** is derivable from **K1**. □

Theorem 19 (Completeness of EL) ***EL** is sound and complete with respect to the class of Complete Epistemic models $\mathcal{M}^f$: for all $\phi \in \mathcal{L}_e$, $\vdash_{EL} \psi$ if and only if $\models \phi$.*

Proof The proof is developed in Appendix A. □

The class of epistemic models which validate the principle of *Factivity* is the one of *factive* epistemic models $\mathcal{M}^F$, namely the class of epistemic models where f is Reflexive (Condition 4.1). The logic of this class of models extends **EL** with the axiom of factivity:

Definition 28 The system $\mathbf{EL}^+$ consists of the axiom in **EL** plus K7: $K(\phi) \vdash_{EL^+} \phi$.

Theorem 20 (Completeness of EL^+) *$\mathbf{EL}^+$ is sound and complete with respect to the class of Factive Complete Epistemic models $\mathcal{M}^F$: for all $\phi \in \mathcal{L}_e$, $\vdash_{EL^+} \phi$ if and only if $\models \phi$.*

Proof The proof is developed in Appendix B. □

6.1 Hyperintensionality

The idea of subject matter sensitivity seems to give us good reasons to reject closure under disjunction and at the same time to include conjunction distribution within the defensible core of knowledge. Similarly, logically equivalent sentences do not necessarily *say* the same things, namely they are not necessarily *about* the same topic. Yablo suggests to give up closure under logical equivalence, which amounts to claiming that knowledge is a hyperintensional operator: equivalent propositions are those that share the same intension, because they are true in the same possible worlds.

The notion of hyperintensionality has became of prime importance in the literature of philosophical logic, to the point that Nolan [33] predicted a 'hyperintensional revolution' for the 21st century.

There are many definitions of hyperintensionality, and of hyperintensional contexts and logics in the literature [33, 34]. Leitgeb [35] developed the logic HYPE, which is presented as a 'background system for hyperintensional operators'. He defines hyperintensionality as the failure of substitutivity with respect to *classical logical* equivalents. A unary connective S is defined to be hyperintensional if prefixing sentences logically equivalent in classical logic by S can lead to sentences which differ in truth value at some state. A hyperintensional logic is a logic which can model hyperintensional operators.

In other words, from a proof-theoretic perspective, according to Leitgeb's definition, the entailment $\vdash_L$ is hyperintensional if and only if for all formulas ϕ, ψ, χ and all the propositional variables p of the language, the following claim does *not* hold:

$$\phi \dashv\vdash_{CL} \psi \text{ implies } \chi(\phi/p) \dashv\vdash_L \chi(\psi/p).$$

The Total Knowledge account creates an hyperintensional epistemic context, as the failure of closure under classical equivalence witnesses from a semantic perspective: we cannot substitute classically equivalent formulas within the scope a K-operator *salva veritate*. In other words, the knowledge operator is hyperintensional with respect to the classical equivalents, in accordance to Leitgeb's intuition of hyperintensionality.[18]

$$\phi \dashv\vdash_{CL} \psi \text{ does not imply } \chi(\phi/p) \dashv\vdash_{EL^+} \chi(\psi/p).$$

On the other hand, we can easily address the challenge of the *granularity* of the knowledge operator. The label of 'granularity problems' [28, 37] refers to a family of issues concerning the right level of fine-grainedness:

[18] [36] criticizes these definitions. If hyperintensionality is simply defined as failure of inter-substitution in L for classically equivalent formulas, then virtually every non classical logic would turn out to be hyperintensional in this sense, as classical equivalence may simply not be preserved in L. According to Odintsov and Wansing, the notion of hyperintensionality must be understood in terms of self-extensionality. A logic L with consequence relation $\vdash_L$ is self-extensional if and only if for all formulas ϕ, ψ, χ and all the propositional variables p of the language, the following claim does *not* hold: $\phi \dashv\vdash_L \psi$ implies $\chi(\phi/p) \dashv\vdash_L \chi(\psi/p)$. In the Total Knowledge account, since EL^+ is a modal extension of CL, we also meet Odintsov and Wansing's requirement for a genuine hyperintensional operator.

> This can be understood again in terms of substitution *salva veritate* for the relevant operators: X is strictly more fine-grained than Y when all substitutions that go through for X also do for Y, but X fails some, which goes through for Y [38, p. 26].

In other words, we can image a *spectrum* of fine-graininess of operators, where the lower bound is occupied by the hyperintensional operators as fine-grained as syntax and the upper bound is occupied by intensional operators. A requirement often suggested is that a hyperintensional operator should not be as fine-grained as the syntax of the language one is working with ? on pain of giving away the very point of having a semantics for it. The real challenge is to model hyperintensional operators with exactly the 'right amount' of hyperintensionality, which should be justified with independent philosophical reasons.

The knowledge operator in the Total Knowledge framework is not as fine-grained as syntax, as we can substitute within the K-operator sentences that are exactly verified (falsified) by the same states. In other words, the knowledge operator does not discriminate between propositions with the same truth(false)-maker intension. More formally, let **EX** indicate the logic of Exact Entailment [39] and **AC** the logic of Analytic Entailment, then it holds that:

$$\phi \dashv\vdash_{EX} \psi \text{ implies } \chi(\phi/p) \dashv\vdash_{EL^+} \chi(\psi/p)$$
$$\phi \dashv\vdash_{AC} \psi \text{ implies } \chi(\phi/p) \dashv\vdash_{EL^+} \chi(\psi/p)$$

We understand two sentences that are analytically equivalent as having the same meaning. Recall that when Angell [17] introduced the first original formulation of the propositional logic of analytic containment, his aim was to formalize a notion of entailment understood in terms of containment of meanings:

> The concept of entailment [...] has also been connected to the concept of containment in Kant?s sense of analytic containment: A entails B only if the meaning of B is contained in the meaning of A. [17, p. 1]

Angell claims that this concept of Analytic Entailment is also connected to the concept of synonymity: "S_1 is synonymous with S_2 if and only if S_1 entails S_2 and S_2 entails S_1".

From these considerations it emerges that our knowledge operator does not discriminate between synonyms. For example, if an ordinary agent knows that there are female foxes in the forest, then she also knows that there are vixens in the forest. We do not model situations in which the agent is not a competent user of the language and ignores that vixens are in fact female foxes. We do not deny that these situations are possible in ordinary life and even frequent. We just claim that this is not neither a semantic phenomenon nor a logical phenomenon, namely it is not related to the logical competence of ordinary agents. For these reasons, synonymity is a good lower bound of fine-graininess for non-ideal knowledge.

7 Conclusions and Further Work

In this work, we have introduced a truthmaker-based epistemic logic which aims to model a knowledge operator for non-ideal agents, namely an account where Logical Omniscience fails. In particular, our approach is based on the notion of W-models extended with a partial *epistemic function*, which associates some state to a set of states, namely a proposition. This proposition is interpreted as the agent's *body of knowledge*, namely the set of verifiers of her *total knowledge* at a certain state. The semantic clauses for knowledge, introduced and defended in Section 4, are based on the intuition that an agent knows some information ϕ, when the *propositional content* that ϕ is *contained* in her total knowledge at that state. An agent fails to know ϕ at a state, when the *propositional content* that ϕ is not *contained* in her total knowledge.

This semantics invalidates all the controversial epistemic principles we have briefly examined in Section 2. At the same time, it is able to characterize logical competent non-ideal agents, because the notion of knowledge is closed under *Analytic Entailment*, as shown in Section 5. In particular, we have seen in Section 5.2 that the truthmaker characterization of the relation of containment, under certain conditions, corresponds to Yablo's notion of *immanent closure*, which is a philosophically sound form of closure for non-ideal knowledge.

In Section 6, we developed the proof system **EL**, by showing that it is sound and complete with respect to epistemic models where the epistemic function f is closed under fusion. Moreover, we extended **EL** with the axiom corresponding to the factivity of knowledge; the result is the proof system $\mathbf{EL}^+$, which is sound and complete with respect to factive epistemic models.

Our Total Knowledge account cannot express embedded formulas, which, however, play an important role in the analysis of non-ideal agents? knowledge and, in particular, of their introspective abilities. Hintikkian epistemic semantics validates both positive and negative introspection, which, on the other hand, are principles often opposed by epistemologists (compare [40]). Accordingly, as a future work, we will develop a new version of the semantic and axiomatization which expresses embedded modalities but does not validate the introspection principles.

Moreover, it might be objected that the the clauses of verification and falsification should not be mutually exclusive, in order to properly distinguish between *exact* verification and falsification of knowledge. Even though we argued for an interpretation of the epistemic function which justifies such a choice, in the context of a bilateral semantics, it is worth exploring a different bilateral approach, which we leave to further work .

Appendix A

In this Appendix we prove the completeness theorem of the logic **EL** with respect to the class of complete epistemic models (Theorem 19).

Theorem 21 (Soundness of EL) *EL is sound with respect to the class of epistemic models: for all* ϕ *and* $\psi \in \mathcal{L}_e$*, if* $\phi \vdash_{EL} \psi$*, then* $\phi \models \psi$*.*

Proof **K0**. Classical logic is sound and complete with respect to loose consequence. **K1**. Let $\phi, \psi \in \mathcal{L}$ and assume that $\phi \vdash_{AC} \psi$. By soundness of **AC**, $\phi >_{AC} \psi$. Consider an arbitrary $w \in W$ such that $\mathcal{M}, w \models K\phi$. Hence, there is a $s \in S^{\Diamond}$, such that $s \sqsubseteq w$ and $s \Vdash K\phi$, i.e. $|\phi|^+ \preceq f(s)$. Since $\phi >_{AC} \psi$, by definition $|\psi|^+ \preceq |\phi|^+$. By transitivity of the relation of containment we can conclude $|\psi|^+ \preceq f(s)$, which proves that $\mathcal{M}, w \models K\psi$. **K2** follows from Proposition 12. □

In this section, we will develop the proof of the completeness theorem of **EL**, namely that every consistent formula in $\mathcal{L}_e$ is satisfiable.

Definition 29 For all $\alpha, \beta \in \mathcal{L}_e$, we say that α is *inconsistent*, when it classically derives a contradiction, i.e. $\alpha \vdash_{CL} \beta \wedge \neg\beta$.

We sometimes abbreviate a classical contradiction with the symbol $\bot$. Since for all $\alpha \in \mathcal{L}_e$, $\beta \wedge \neg\beta \vdash_{CL} \alpha$, then, α is inconsistent if and only if α is provably equivalent in **CL** to $\beta \wedge \neg\beta$. Otherwise it is *consistent*.

Definition 30 A formula $\alpha \in \mathcal{L}_e$ is *satisfiable* when there is an epistemic model $\mathcal{M}$ and a world $w \in \mathcal{M}$ which loosely verifies it, i.e. $\mathcal{M}, w \models \alpha$.

We can summarize the strategy of the proof as follows. We will first show that (i) every consistent K-formula – call it δ – is satisfiable, and from this it will straightforwardly follow that (ii) every consistent formula α in $\mathcal{L}_e$ is satisfiable. To prove (i), it suffices to show that δ has a model, which amounts to showing that there is an epistemic model with a possible state that exactly verifies δ. Indeed, in an epistemic model, every possible state is part of a possible world. Hence, if a possible state verifies δ, then there is a possible world that loosely verifies it. In particular, our goal is to build a *syntactic model* for δ, which we prove to be indeed an epistemic model (in Section A.2). In order to accomplish this result, we first bring δ into a specific disjunctive normal form, which we call *Maximal K-form*.

A.1 Normal Forms

As mentioned our completeness proof draws on the idea of disjunctive normal forms. We will proceed in two steps. We first shall identify a suitable class of disjunctive normal forms in $\mathcal{L}$, namely suitable for the non-epistemic formulas. Secondly, we shall identify a particular class of disjunctive normal forms for any epistemic formulas. We call the former the class of *closed disjunctive forms*, the latter the class of *maximal K-forms*.

Definition 31 (Descriptions) A *description* is a conjunction $\lambda_1 \wedge \lambda_2 \wedge ... \wedge \lambda_m$ of literals. As a limiting case a literal is also considered a description.

Definition 32 (Sub-description) Let $\phi = \lambda_1 \wedge \lambda_2 \wedge ... \wedge \lambda_m$ and ψ be descriptions, then ψ is a *sub-description* of ϕ, written $\psi \Subset \phi$, if ψ is a conjunction of some of $\lambda_1, \lambda_2, ..., \lambda_m$.

Note that the order and the repetition of the literals is not relevant for a conjunction of literals to be a sub-description of another one. Note also that, given the descriptions ϕ and ψ, $\phi \Subset \psi$ is equivalent to say that $Lit(\phi) \subseteq Lit(\psi)$.

Definition 33 (Disjunctive normal form) A *disjunctive normal form* is a disjunction $\phi_1 \vee \phi_2 \vee ... \vee \phi_m$ of descriptions $\phi_1, \phi_2, ..., \phi_m$, with $m \geq 0$.

Definition 34 (Closed disjunctive form) A disjunctive form ϕ is *closed* iff for any set Ψ of disjuncts in ϕ, ϕ includes a disjunct ψ with $Lit(\psi) = \bigcup\{Lit(\phi_i) \mid \phi_i \in \Psi\}$.

Lemma 22

1. *Any formula ϕ in $\mathcal{L}$ is provably equivalent in **AC** to a disjunctive normal form.*
1. *Any disjunctive normal form in $\mathcal{L}$ is provable equivalent in **AC** to a closed disjunctive form.*

Proof (1) is analogous to a standard result in classical logic. (2) is an adaptation of lemma 14 in [41, p. 23]. □

Now that we have a suitable normal form for the non-epistemic formulas, which are the arguments for the modality K, we shall now introduce the normal form we need for epistemic formulas.

Definition 35 (Maximal K-form) A *maximal K-form* $K^M(\phi)$ abbreviates the conjunction

$$K(\phi) \wedge \neg K(\psi_1) \wedge ... \wedge \neg K(\psi_m)$$

(with $0 \leq m$), where $\phi = \phi_1 \vee ... \vee \phi_n$ is in closed disjunctive form, and each negated atom $\neg K(\psi_k)$ (with $k \leq m$), is such that $\psi_k = \psi_1^k \vee ... \vee \psi_{m_k}^k$ is in closed disjunctive form and:

1. *either* there is ψ_i^k $(i \leq m_k)$ such that for all ϕ_j $(j \leq n)$ $\psi_i^k \not\Subset \phi_j$;
2. *or* there is ϕ_j $(j \leq n)$ such that for all ψ_i^k $(i \leq m_k)$, $\psi_i^k \not\Subset \phi_j$;

This normal form aims at spelling out the relationship between K-atoms and negated K-atoms, with respect to their arguments. K-literals are, in fact, limit cases of Maximal K-forms: $K\phi$ satisfies vacuously the disjunct (2) of the definition, because $m = 0$, namely there are no negated K-atoms; similarly, $\neg K\psi$ satisfies vacuously the disjunct (1), because $n = 0$.

The following lemma generalizes the reasoning just applied in the previous example.

Lemma 23 *Each consistent K-formula δ is provably equivalent in **EL** to a disjunction, where each disjunct is equivalent to a maximal K-form.*

Proof Let δ be an arbitrary consistent K-formula. We firstly put δ in disjunctive normal form by means of the classical rules of EL, i.e. we obtain a disjunction of conjunctions of K-literals, then we agglomerate all the K-atoms:

$$\delta \dashv\vdash_{EL} \bigvee_{i \leq l}(K(\phi_1^i) \wedge ... \wedge K(\phi_n^i) \wedge ... \wedge \neg K(\psi_1^i) \wedge ... \wedge \neg K(\psi_m^i)) \qquad \textit{by CL}$$

$$\dashv\vdash_{EL} \bigvee_{i \leq l}(K(\phi_1^i \wedge ... \wedge \phi_n^i) \wedge \neg K(\psi_1^i) \wedge ... \wedge \neg K(\psi_m^i)) \qquad \textit{by K2 and K5}$$

Since δ is consistent, then there is at least a consistent disjunct δ_i in δ, which will be of the form $\delta_i = K(\phi^i_1 \wedge ... \wedge \phi^i_n) \wedge ... \wedge \neg K(\psi^i_1) \wedge ... \wedge \neg K(\psi^i_m)$.

Let χ^i be a closed disjunctive form (Def. 34) such that $\chi^i = \chi^i_1 \vee ... \vee \chi^i_k \dashv\vdash_{AC} \phi^i_1 \wedge ... \wedge \phi^i_n$ (we obtain it by E9 and Lemma 22), and each disjunct in χ^i is a description (def. 3.2). Then $K(\chi^i) \dashv\vdash_{EL} K(\phi^i_1 \wedge ... \wedge \phi^i_n)$ by K6 and thus δ_i is provably equivalent to a conjunction of $K(\chi^i)$ and the negated K-atoms, that is $\delta_i \dashv\vdash_{EL} K(\chi^i_1 \vee .. \vee \chi^i_k) \wedge ... \wedge \neg K(\psi^i_1) \wedge ... \wedge \neg K(\psi^i_m)$.

If δ_i is equivalent to a formula with only one negated atom $\neg K(\psi^i)$, we will iterate the reasoning that follows for each negated atom. Therefore, we can suppose w.l.o.g. that δ_i is equivalent to a formula with only one negated atom $\neg K(\psi^i)$ and $\psi^i = \psi^i_1 \vee ... \vee \psi^i_j$ is a closed disjunctive form.

It suffices to prove that $K(\chi^i_1 \vee .. \vee \chi^i_k) \wedge \neg K(\psi^i)$ is a normal K-form. It amounts to prove that either (1) or (2) in Definition 35 is the case. We assume by contradiction that both (1) and (2) are false, thus:

1 for all ψ^i_h (with $h \leq j$), there is a χ^i_l (with $l \leq k$), such that $\psi^i_h \Subset \chi^i_l$.
2 for all χ^i_l (with $l \leq k$), there is a ψ^i_h (with $h \leq j$), such that $\psi^i_h \Subset \chi^i_l$.

Let f be a function such that for each $h \leq j$ of ψ^i, $\psi^i_h \Subset \chi^i_{f(h)}$. Then by (E14), and from 1*, it follows that for all $h \leq j$:

$$\begin{gathered}\chi^i_{f(1)} \vdash_{AC} \psi^i_1 \\ \vdots \\ \chi^i_{f(j)} \vdash_{AC} \psi^i_j\end{gathered}$$

Thus, by (E17), $\chi^i_{f(1)} \vee ... \vee \chi^i_{f(j)} \vdash_{AC} \psi^i_1 \vee ... \vee \psi^i_j$, which is $\chi^i_{f(1)} \vee ... \vee \chi^i_{f(j)} \vdash_{AC} \psi^i$. Moreover, let g be a function such that for each $l \leq k$ of χ^i, $\psi^i_{g(l)} \Subset \chi^i_l$. Then by (E14), and from 2, it follows that for all $l \leq k$:

$$\begin{gathered}\chi^i_1 \vdash_{AC} \psi^i_{g(1)} \\ \vdots \\ \chi^i_k \vdash_{AC} \psi^i_{g(k)}\end{gathered}$$

Thus, by (E17), $\chi^i_1 \vee ... \vee \chi^i_k \vdash_{AC} \psi^i_{g(1)} \vee ... \vee \psi^i_{g(k)}$ which is $\chi^i \vdash_{AC} \psi^i_{g(1)} \vee ... \vee \psi^i_{g(k)}$. It follows from $\chi^i_{f(1)} \vee ... \vee \chi^i_{f(j)} \vdash_{AC} \psi^i$ and $\chi^i \vdash_{AC} \psi^i_{g(1)} \vee ... \vee \psi^i_{g(k)}$ by (E17) and (E4) that $\chi^i \vdash_{AC} \psi^i$ and thus, $K(\chi^i) \vdash_{EL} K(\psi^i)$ by K1. Then $\delta^i \vdash_{EL} K(\psi^i) \wedge \neg K(\psi^i)$, against our assumption that δ^i is consistent. Hence, we can conclude that either (1) or (2) is true and so $K(\chi^i) \wedge \neg K(\psi^i)$ is a maximal K-form. □

A.2 Syntactic Models and Completeness

In the present section, we will define a 'canonical model' in which the states are taken to be sets of the literals of the language, and we can read the epistemic function off an arbitrary consistent K-formula δ. To be precise, the model we are going to built is not canonical in the sense that it satisfies in a world all the consistent formulas of our language. On the contrary, we will show that for each formula there is a model which satisfies it. It is convenient, then, to call it *syntactic* model, to distinguish it from a model which verifies all the formulas, because, as mentioned, we will built it up from the set of literals of our language.

Recall that δ is provably equivalent to a disjunction, each disjunct of which is a maximal K-form $K^M(\phi^i)$:

$$\delta_i = K(\phi_1^i \vee ... \vee \phi_n^i) \wedge \neg K(\psi_1^i) \wedge ... \wedge \neg K(\psi_m^i).$$

In particular, as it will became apparent soon, the epistemic function is based on the set of literals in such ϕ.

Definition 36 (Syntactic Epistemic Model) A *syntactic epistemic model* for δ is $\mathfrak{M}^\delta = (S, S^\Diamond, \sqsubseteq, f, |.|^+, |.|^-)$ where:

- $S = \mathcal{P}(Lit)$
- $S^\Diamond = \{s \in S \mid \{p, \neg p\} \nsubseteq s,\ \textit{for all } p \in Prop\}$
- $\sqsubseteq = \subseteq$
- $\mathsf{dom}(f) = S^\Diamond$ and $f(s) = \{\bigcup Lit(\phi_j^i)_{\phi_j^i \in \Psi} \mid$ for Ψ a set of disjuncts in $\phi^i\}$, for each $s \in S^\Diamond$.
- $|p|^+ = \{\{p\}\}, |p|^- = \{\{\neg p\}\}$, for all $p \in \mathcal{L}$.

Note that the value of f is defined in the same way for all the possible states in the domain. Moreover, it is easy to see that $f(s)$ is the closure under fusion of the sets of literals $Lit(\phi_j^i)$ for $j \leq n$, i.e. $f(s) = \{Lit(\phi_j^i) \mid \text{for any } j \leq n\}^f$.

Lemma 24 $\mathfrak{M}^\delta = (S, S^\Diamond, \sqsubseteq, f, |.|^+, |.|^-)$ *is a epistemic model.*

Proof $\mathfrak{M}^\delta = (S, S^\Diamond, \sqsubseteq,)$ is W-space, see [5, p. 647]. It suffices to show that f is an epistemic function. It is easy to see that f is a partial function, closed under union by definition. Also, since it is defined in the same way for all $s \in S^\Diamond$, Condition 3.1 holds. Moreover, for the same reason, every possible s is compatible with a state for which f is defined, namely itself, hence also Condition 3.2 holds. □

Note that in the syntactic model the valuation functions pick always a singleton for each propositional letter. On the basis of this definition, we shall prove some results regarding the valuation functions, which will be useful later on.

Lemma 25 *Let* $\mathfrak{M}^\delta$ *be a syntactic model as defined in Def. 36 and* $\lambda_1, ..., \lambda_n$ *be literals such that* $\{\lambda_1, ..., \lambda_n\} \in S$*, then* $|\lambda_1 \wedge ... \wedge \lambda_n|^+ = \{\{\lambda_1, ..., \lambda_n\}\}$.

Proof Let X be a set of literals, then:

$$\begin{aligned} X \in |\lambda_1 \wedge ... \wedge \lambda_m|^+ &\text{ iff } X = \textstyle\bigcup_{i \leq m} X_i \textit{ and } X_i \in |\lambda_i|^+ \\ &\text{ iff } X = \textstyle\bigcup_{i \leq m} X_i \textit{ and } X_i = \{\lambda_i\} \\ &\text{ iff } X = \{\lambda_1\} \cup ... \cup \{\lambda_m\} \\ &\text{ iff } X = \{\lambda_1, ..., \lambda_m\} \end{aligned}$$

It follows that $|\lambda_1 \wedge ... \wedge \lambda_m|^+ = \{\{\lambda_1, ..., \lambda_m\}\}$. □

Lemma 26 (Lemma 26 [41]) *In $\mathfrak{M}^\delta$, for any closed disjunctive form $\phi = \phi_i \vee ... \vee \phi_n$, $|\phi|^+ = |\phi|^+_f$. To unpack the definition, since in $\mathfrak{M}^\delta$, $\bigsqcup X = \bigcup X$, we have that*

$$|\phi|^+ = \{Lit(\phi_1), ..., Lit(\phi_n)\} \cup \{\bigcup_{\psi \in \Psi} Lit(\psi) \mid \Psi \text{ is a collection of disjuncts in } \phi\}.$$

Recall that we proved (Lemma 23) that a conjunction $K\phi \wedge \neg K\psi_1 \wedge \cdots \wedge \neg K\psi_m$ is inconsistent in our system if there is a ψ_i $(i \leq m)$ such that every disjunct in ϕ has a sub-description that is a disjunct in ψ_i, and every disjunct in ψ_i is a sub-description of a disjunct in ϕ. The next step is to show that the previous conjunction is satisfied by the syntactic model. The idea of the proof is that, in building the syntactic model, we picked a possible state s and let $f(s)$ be the set of verifiers of ϕ, and since the conjunction is consistent, we can see that no ψ_i has a set of verifiers analytically entailed by $|\phi|^+$. So the state s verifies $K\phi$ and falsifies each $K\psi_i$.

In the proof of following theorem, we will spell out all the details of the strategy just sketched.

Lemma 27 *Every consistent epistemic formulas δ is satisfiable with respect to the class of epistemic models.*

Proof Suppose the K-formula δ is consistent. By Lemma 23, δ is provably equivalent to a normal K-form $\delta_1 \vee \delta_2 \vee ... \vee \delta_m$, for $m \geq 1$, where each δ_i is of the form $K^M(\phi^i)$. The syntactic model for δ is $\mathfrak{M}^\delta = (S, S^\diamond, \sqsubseteq, f, |.|^+, |.|^-)$, defined as in Definition 36. It suffices to show that δ_i (for some $i \leq m$) is true in the model and consequently so is δ. Let $\delta_i = K(\phi^i_1 \vee ... \vee \phi^i_n) \wedge \neg K(\psi^i_1) \wedge ... \wedge \neg K(\psi^i_m)$.

In what follows, I will drop the superscript i for readability purposes, and I will consider as before only one negated K-atom, without loss of generality.

Consider an arbitrary state $s \in S^\diamond$. By definition of the syntactic model, that there is a possible world w, such that $s \subseteq w$.

We first prove that $\mathfrak{M}^\delta, w \models K(\phi_1 \vee ... \vee \phi_n)$, namely that $s \Vdash K(\phi_1 \vee ... \vee \phi_n)$, i.e. $|\phi_1 \vee ... \vee \phi_n|^+ \preceq f(s)$. Given Lemma 26, and the definition of $f(s)$ in $\mathfrak{M}^\delta$, we know that $|\phi_1 \vee ... \vee \phi_n|^+ = |\phi_1 \vee ... \vee \phi_n|^+_f = f(s)$, proving immediately our claim.

Secondly, we shall prove that $\mathfrak{M}^\delta, w \models \neg K(\psi)$. Recall the definition of maximal K-form, $\neg K(\psi)$ is such that $\psi = \psi_1 \vee ... \vee \psi_m$ is in closed disjunctive form and:

1. **either** there is ψ_i $(i \leq m)$ such that for all ϕ_j, $(j \leq n)$ $\psi_i \not\in \phi_j$;
2. **or** there is ϕ_j $(j \leq n)$ such that for all ψ_i $(i \leq k)$, $\psi_i \not\in \phi_j$;

Suppose (1) is the case. Since both ψ_i and ϕ_j are descriptions, and $\psi_i \not\in \phi_j$, then $Lit(\psi_i) \nsubseteq Lit(\phi_j)$, for all ϕ_j. Since ϕ is a *closed* normal form, it includes a disjunct ϕ_l such that $Lit(\phi_l) = \bigcup\{Lit(\phi_j) \mid \phi_j \in \Psi\}$ where Ψ is a set of disjuncts in ϕ. Hence, also $Lit(\psi_i) \nsubseteq \bigcup\{Lit(\phi_i) \mid \phi_i \in \Psi\}$, for any choice of Ψ. Thus, take an arbitrary $L \in f(s)$, it is of the form $\bigcup\{Lit(\phi_i) \mid \phi_i \in \Psi\}$. It follows that, $Lit(\psi_i) \nsubseteq L$. Hence, since L was arbitrary and $Lit(\psi_i) \in |\psi|^+$ (Lemma 26), we can conclude that for each $L \in f(s)$, there is $L' \in |\psi|^+$, such that $L' \nsubseteq L$, i.e. $|\psi|^+ \npreceq f(s)$.

Suppose (2) is the case. Since both ψ_i and ϕ_j are descriptions, and $\psi_i \not\in \phi_j$, then $Lit(\psi_i) \nsubseteq Lit(\phi_j)$, for all ψ_i. Take an arbitrary $L \in |\psi|^+$, which will be of the form

$L = \bigcup_{i \leq n} Lit(\psi_i)$. If $n = 1$, there is a $Lit(\phi_j) \in f(s)$, such that $Lit(\psi_i) \in |\psi|^+$ and $Lit(\psi_i) \nsubseteq Lit(\phi_j)$. Hence, since ψ_i was arbitrary, $|\psi|^+ \npreceq f(s)$. If $n > 1$, it follows from 2 that $\bigcup_{i \leq n} Lit(\psi_i) \nsubseteq Lit(\phi_j)$, and $\bigcup_{i \leq n} Lit(\psi_i) \in |\psi|^+$, by Lemma 26. Hence for the previous reasoning we can conclude that $|\psi|^+ \npreceq f(s)$. In both cases, $|\psi|^+ \npreceq f(s)$, so we can conclude that $\mathfrak{M}^\delta, w \nvDash K(\psi)$. □

Recall that we called Lit the set of all the propositional letters p and their negation. Let us now call Lit_k the set of K-literals, namely all the K-atoms. Given classical logic, we know that each α in $\mathcal{L}_e$ is provably equivalent to a formula in disjunctive normal form, i.e. $\alpha \dashv\vdash_{EL} \alpha_1 \vee \cdots \vee \alpha_n$, where each disjunct is a conjunction of literals either in Lit or in Lit_k. Moreover, we know from Lemma 23, that a conjunction of K-literal is provably equivalent to a maximal K-form $K^M(\phi)$. Let us call the conjunction of propositional literals Λ. Hence, we say w.l.o.g. that each $\alpha \in \mathcal{L}_e$ is of the form $\bigvee(\Lambda \wedge K^M(\phi))$.

Lemma 28 *Every consistent formula $\alpha \in \mathcal{L}_e$ is satisfiable with respect to the class of epistemic models.*

Proof By the previous reasoning α is provably equivalent to a disjunctive normal form. Since it is consistent, there must be at least a consistent disjunct α_i, which is of the form $\Lambda \wedge K^M(\phi)$. Since α_i is consistent then also Λ and $K^M(\phi)$ are both consistent. From Lemma 27 we know that, for each consistent $K^M(\phi)$ we can build a syntactic epistemic model that makes it true, call it $\mathfrak{M}^\delta$. In particular, we know that for all $s \in S^\Diamond$, $s \Vdash K^M(\phi)$. Moreover, there is also a $s' \in S^\Diamond$, that is the set of propositional literals in α_i, i.e. $s' = Lit(\Lambda)$. Accordingly, $s' \Vdash \Lambda \wedge K^M(\phi)$. Since $\mathfrak{M}^\delta$ is an epistemic model, there is a world w such that $s' \subseteq w$. Hence $\mathfrak{M}^\delta, w \models \Lambda \wedge K^M(\phi)$ and thereby $\mathfrak{M}^\delta, w \models \alpha$. □

Theorem 29 (Completeness) *The system **EL** is complete: for any epistemic model $\mathcal{M}$ and for any $\phi \in \mathcal{L}_e$, if $\mathcal{M} \models \phi$, then $\vdash_{EL} \phi$.*

Proof The proof is a straightforward consequence of the previous lemma. □

A.3 Convexity

As we have mentioned, the adoption of regular propositions does not entail any difference on the resulting logic of analytic containment and analytic equivalence, which is the logic **AC**. On the other hand, closure under convexity is philosophically interesting, because it makes analytic containment antisymmetric, which is arguably a desideratum for a natural characterization of analytic containment. Given these considerations, we might want to consider the introduction of convexity as a closure principle for the epistemic function as well. In this case, we would need to adapt the completeness proof to this new feature, in particular the syntactic model.

The relevant subclass of disjunctive forms in this case is the one that Fine [10] calls *maximal*. I will call it the *regular* disjunctive form, in order to avoid mix-ups with the maximal K-form of Definition 35.

Definition 37 (Regular disjunctive form) A disjunctive form ϕ is regular iff for any disjunct ϕ_i in ϕ and any literal λ occurring as a conjunct in a disjunct of ϕ, ϕ contains a disjunct ϕ_j with $Lit(\phi_j) = Lit(\phi_i) \cup \{\lambda\}$.

Lemma 30 (Lemma 17 in [10]) *Every disjunctive form is provably equivalent, within **AC**, to a regular disjunctive form.*

Lemma 31 *Every consistent K-formula δ is provably equivalent to a normal K-form $\delta_1 \vee ... \vee \delta_m$, where each δ_i is a maximal K-formula, i.e. δ_i is of the form $K^M(\phi)$, and additionally ϕ is a* regular *normal form.*

Proof By Lemmas 23 and 30. □

Definition 38 A syntactic *regular* epistemic model $\mathfrak{M}^\delta_R$ for δ is a syntactic epistemic model $\mathfrak{M}^\delta = (S, S^\Diamond, \sqsubseteq, f', |.|^+, |.|^-)$, where $f'(s) = f(s)^r$, where $f(s)$ is defined as in Definition 36, i.e. $f'(s)$ is closed under fusion and convexity.

It is easy to check that $\mathfrak{M}^\delta_R$ is an epistemic model. Moreover, it can be shown that for all regular disjunctive forms ϕ, $|\phi|^+ = |\phi|^+_r$ (see [41, p. 30]). Then, the proof of the completeness of the **EL** system is an adaptation of the previous one.

Appendix B Factivity

In this Appendix we prove the completeness theorem of the logic $\mathbf{EL}^+$ with respect to the class of factive (Theorem 20) epistemic models.

Theorem 32 (Soundness) *For all $\phi, \psi \in \mathcal{L}_e$, if $\phi \vdash_{EL^+} \psi$, then $\phi \models \psi$.*

Proof **K7**. The proof is analogous to the right-left direction of Lemma 10. **K8**. It derives from $K7$, because, suppose by contradiction that $\vdash_{EL^+} K(\phi \wedge \neg\phi)$, then by $K7$, $K(\phi \wedge \neg\phi) \vdash_{EL^+} \phi \wedge \neg\phi$, which contradicts classical logics. □

Now, we just need to adapt the previous results concerning the completeness of **EL** to the new extended logic. It turns out that a syntactic model with a reflexive epistemic function looks quite different from the previous one and we need to build a slightly more sophisticated structure.

We need to be able to talk of the negations of the literals, both the positive and negative ones. Therefore, with each literal $\lambda \in Lit$, we associate a unique *shadow* $\bar{\lambda}$, which acts as follows: if $\lambda = p$, then $\bar{\lambda} = \neg p$; if $\lambda = \neg p$, then $\bar{\lambda} = p$.

Lemma 33 *Let $K^M(\phi)$ be a consistent maximal K-form in $\mathcal{L}_e$. Since $\phi = \phi_1 \vee ... \vee \phi_n$, there is at least a disjunct ϕ^* in ϕ such that $p \wedge \neg p \notin \phi^*$, for any $p \in \mathcal{L}$.*

Proof Suppose by contradiction that for each disjunct ϕ_i $(i \leq n)$ in ϕ, there is some $p^i \in Lit$, such that $p^i \wedge \neg p^i \Subset \phi_i$. Hence $\phi_1 \vee ... \vee \phi_n \vdash_{AC} (p^i \wedge \neg p^i) \vee ... \vee (p^n \wedge \neg p^n)$, by E14 and E17. Hence, $K(\phi) \vdash_{EL^+} K((p^i \wedge \neg p^i) \vee ... \vee (p^n \wedge \neg p^n))$, by K1. Therefore,

$$\begin{aligned} K(\phi) \wedge \neg K(\psi_1) \wedge ... \wedge \neg K(\psi_m) &\vdash_{EL^+} K((p^i \wedge \neg p^i) \vee ... \vee (p^n \wedge \neg p^n)) \\ &\vdash_{EL^+} (p^i \wedge \neg p^i) \vee ... \vee (p^n \wedge \neg p^n) \\ &\vdash_{EL^+} \bot \end{aligned}$$

which contradicts the fact that $K^M(\phi)$ is consistent. Hence, there is at least a disjunct ϕ_i in ϕ which does not have any contradicting atomic letters as sub-description. □

Since ϕ^* is a description, then it follows that $Lit(\lambda \wedge \bar{\lambda}) \nsubseteq Lit(\phi^*)$, for any $\lambda \in Lit$ which is equivalent to say that $\{\lambda, \bar{\lambda}\} \nsubseteq Lit(\phi^*)$.

Our goal, now, is to read an epistemic syntactic model out of this consistent description ϕ^*. In particular, the set of *possible* states of our syntactic model will be based on the set of literals in ϕ^*.

A *syntactic state space* is a tuple $\mathfrak{S} = (S, \sqsubseteq)$, where $S = \mathcal{P}(Lit)$ and $\sqsubseteq = \subseteq$. Now, consider the state $s = Lit(\phi^*)$. We shall construct a maximally consistent state $w(s)$, namely a possible world, which contains s as a subset. In other words, we will extend s to a possible world $w(s)$ and we will adopt it as a basis for the set of possible states $S^\diamond$ in our syntactic model.

Definition 39 In a syntactic state space $\mathfrak{S} = (S, \sqsubseteq)$, let s be a state in S, s is *consistent* when for any $\lambda \in Lit$, $\{\lambda, \bar{\lambda}\} \nsubseteq s$. It is *inconsistent* otherwise.

As we have already discussed, Lemma 33 shows that $s = Lit(\phi^*)$ is consistent. From this consistent state we will build a maximal state, namely a state that, for all atoms $p \in Prop$, either contains a part which verifies p or contains a part which verifies $\neg p$. To do so, it is necessary to be able to talk of every literal of the language. Since they are countable, we can simply enumerate them, by associating them with an index $i \in \mathbb{N}$.

Let s be our base case, thus $s = s_0$ and consider the first λ_0 of our enumerated list of literals in Lit. Then, the following state s_1 will be $s_1 = s_0 \cup \{\lambda_0\}$, just in case $\bar{\lambda}_0 \notin s_0$, and $s_1 = s_0$ otherwise. Indeed, if $\bar{\lambda}_0 \in s_0$, we cannot build a new consistent state s_1 by adding λ_0 to s_0, because we would obtain $\{\lambda_0, \bar{\lambda}_0\} \subseteq s_1$, i.e. s_1 would be inconsistent.

We shall replicate the same operation, until we obtain a $w(s) = \bigcup_{n \in \mathbb{N}} s_n$, such that nothing more can be added without thereby resulting in an impossible state. In symbols:

$$s_0 = Lit(\phi^*)$$
$$s_{n+1} = \begin{cases} s_n \cup \{\lambda_n\} & \text{if } \bar{\lambda}_n \notin s_n \\ s_n, & \text{otherwise} \end{cases}$$
$$w(s) = \bigcup_{n \in \mathbb{N}} s_n.$$

Lemma 34 *$w(s)$ is a consistent state.*

Proof $w(s)$ is obviously a state, because it is a set of literals. Moreover it is consistent by construction. □

In order to check whether $w(s)$ is a possible world (and not just a consistent state), we need first to define our syntactic W-space. As before, consider δ, which is provably equivalent to a disjunction, each disjunct of which is a maximal K-form $K^M(\phi^i)$:

$$\delta_i = K(\phi_1^i \vee ... \vee \phi_n^i) \wedge \neg K(\psi_1^i) \wedge ... \wedge \neg K(\psi_m^i).$$

We fix additionally ϕ^* to be a consistent disjunct in ϕ^i. Recall that we say that $X^{\downarrow}$ is the smallest downwards closed set (w.r.t. parthood) containing X.

Definition 40 A syntactic *factive* epistemic space for δ is $\mathfrak{S}_F^{\delta} = (S, S^{\Diamond}, \sqsubseteq, f)$, where

- $(S, \sqsubseteq)$ is a syntactic state space;
- $S^{\Diamond} = \{w(s)\}^{\downarrow}$;
- $\mathsf{dom}(f) = \{s\}$, where $s = Lit(\phi^*)$ and $f(s) = \{Lit(\phi_j^i) \mid \text{for any } j \leq n\}^f$.

Lemma 35 $\mathfrak{S}_F^{\delta} = (S, S^{\Diamond}, \sqsubseteq, f)$ *is a factive epistemic space.*

Proof The first step it to prove that $(S, S^{\Diamond}, \sqsubseteq)$ is a W-space. $S^{\Diamond}$ is a non empty subset of S and it is closed under parthood by construction. Hence, it suffices to show that $w(s)$ is a possible world, namely it contains every state with which it is compatible, then, we shall check that every possible state in $S^{\Diamond}$ is part of $w(s)$. Consider an arbitrary t' such stat $t' \cup w(s) \in S^{\Diamond}$. Then, by construction of $S^{\Diamond}$, $t' \subseteq w(s)$. This shows that $w(s)$ contains all its compatible states. Now, suppose t is possible, i.e. $t \in w(s)^{\downarrow}$, then again $t \subseteq w(s)$ by construction, which completes the proof.

Secondly, we shall check that the function f is indeed an epistemic function. Since it is defined only on the state s, it vacuously satisfies the *Compatibility* Condition 3.1.

Moreover, take an arbitrary $t \in S^{\Diamond}$, it suffices to show that t is compatible with s. Since $t \in S^{\Diamond}$, then $t \subseteq w(s)$. Since also $s \subset w(s)$, then, $t \cup s \subseteq w(s)$, and so $t \cup s$ is in $S^{\Diamond}$. This shows that every possible state is compatible with a state s such that $s \in \mathsf{dom}(f)$, thus satisfying the *definability Condition* 3.2. Lastly it is easy to check that $s \in f(s)$ (*reflexivity*). □

Definition 41 A syntactic *factive* epistemic model for δ is $\mathfrak{M}_F^{\delta} = (S, S^{\Diamond}, \sqsubseteq, |.|^+, |.|^-)$ where:

- $(S, S^{\Diamond}, \sqsubseteq, f)$ is a syntactic factive epistemic space.
- $|p|^+ = \{\{p\}\}, |p|^- = \{\{\neg p\}\}$, for all $p \in \mathcal{L}$.

Lemma 36 $\mathfrak{M}_F^{\delta} = (S, S^{\Diamond}, \sqsubseteq, |.|^+, |.|^-)$ *is a W-model.*

Proof It suffices to show that the evaluation functions satisfy the conditions of exclusivity and exhaustivity and closure.

- **Exclusivity.** Consider arbitrary $t \in |p|^+$ and $t' \in |p|^-$, which amounts to saying that $t = \{p\}$ and $t' = \{\neg p\}$. It suffices to show that $t \cup t' \notin S^{\Diamond}$. If it was $\{p, \neg p\} \subseteq w(s)$, which is contrary to the fact that $w(s)$ is possible.
- **Exhaustivity.** Since $w(s)$ is the maximal element in $S^{\Diamond}$ it is also the only possible world of the model. Hence, it suffices to check that for all $p \in \mathcal{L}$, either there is a $t \subseteq w(s)$ such that $t \in |p|^+$ or there is a $t' \subseteq w(s)$ such that $t' \in |p|^-$. Suppose this is not the case, by contradiction. Then, there is a p such that, for all $t \subseteq w(s)$, $t \notin |p|^+$ and $t \notin |p|^-$, which means that $t \neq \{p\}$ and $t \neq \{\neg p\}$. However, since p is a literal there must be a number $i \in \mathbb{N}$, such that $p = \lambda_i$ and $\neg p = \bar{\lambda}_i$. Hence, there is a $s_i \subseteq w(s)$, such that either $\lambda_i \in s_{i+1}$ or $\bar{\lambda}_i \in s_{i+1}$. In the former case, $\{\lambda_i\} \subseteq s_{i+1} \subseteq w(s)$, in the latter case $\{\bar{\lambda}_i\} \subseteq s_{i+1} \subseteq w(s)$. In both cases, we contradict the assumption.
- **Closure** is trivially true.

□

Lemma 37 *Every consistent epistemic formulas δ is satisfiable with respect to the class of factive epistemic models.*

Proof Consider a factive epistemic syntactic model $\mathfrak{M}_F^{\delta}$ as in Definition 41. It suffices to check that s is a possible state, but this follows directly from Lemma 33. Thus, there is a possible world, namely $w(s)$, such that $s \subseteq w(s)$. The proof, then, is analogous to the one of Theorem 27, hence we can conclude that $\mathfrak{M}_F^{\delta}, w(s) \models K^M(\phi)$. □

As in the previous proof, we need now to make sure that every formula $\alpha \in \mathcal{L}_e$ is satisfiable with respect to the class of epistemic factive models, and not only the K-formulas. Since our syntactic factive model consist of only one possible world, it is less trivial then the previous case to show that there is a possible state for each selection of literals compatible with each K-formula. In this case we need to build a different possible world, and thus a different epistemic state space, for each α we consider. Indeed, recall that we can say w.l.o.g. that each α is of the form of $\bigvee(\Lambda \wedge K^M(\phi))$, where Λ is a certain description (of propositional literals) and $K^M(\phi)$ is an arbitrary maximal normal form.

Now, in the construction of the syntactic factive model we need to chose the one which includes $s' = Lit(\Lambda)$ as a subset. This is a legit choice as long as s' is compatible with our initial s_0, namely $\{\lambda, \bar{\lambda}\} \not\subseteq s \cup s'$ for all $\lambda \in Lit$. Recall that $s_0 = Lit(\phi^*)$, where ϕ^* is one of the disjuncts in ϕ which is consistent. Note that we can assume for simplicity, without loss of generality, that each ϕ, contains only one consistent disjunct. In fact, if there were many (and they excluded each other), we would just need to construct different possible words, each starting from the state corresponding to the set of literals of a different consistent ϕ_i in ϕ.

Lemma 38 *For all consistent disjunctive normal forms $\alpha \in \mathcal{L}_e$, there is one of its disjuncts α_i which is consistent and such that, for all $\lambda \in Lit$, $\{\lambda, \bar{\lambda}\} \not\subseteq Lit(\Lambda) \cup Lit(\phi^*)$.*

Proof That α_i exists follows by classical logic and it implies that Λ is consistent, and so is $K^M(\phi)$. As we showed in Lemma 33, since $K^M(\phi)$ is a consistent maximal K-form, then there is at least a disjunct ϕ^* in ϕ such that $\{\lambda, \bar{\lambda}\} \not\subseteq Lit(\phi^*)$, for any $\lambda \in Lit$. Hence, it suffices to check that it is not the case that there is a $\lambda \in Lit(\Lambda)$ and $\bar{\lambda} \in Lit(\phi^*)$. Suppose by contradiction that this was the case. Then, by $E14$ and classical logic, $\phi^* \wedge \Lambda \vdash_{EL^+} \lambda \wedge \bar{\lambda}$. Moreover, since we assume that ϕ^* is the only consistent disjunct in ϕ, by $K7$ we get $K^M(\phi) \vdash_{EL^+} \phi^*$. Hence,

$$K^M(\phi) \wedge \Lambda \vdash_{EL^+} \lambda \wedge \bar{\lambda} \vdash_{EL^+} \bot,$$

contradicting our assumption that α_i was consistent. Then we can conclude that for all $\lambda \in Lit$, $\{\lambda, \bar{\lambda}\} \not\subseteq Lit(\Lambda) \cup Lit(\phi^*)$. □

This lemma shows that, for each formula α there is in fact a selection of literals, i.e. a state of literals s', which is compatible with $s_0 = Lit(\phi^*)$, so that we can build a possible world up from it and including s' in a consistent way. The resulting possible world will make true Λ and $K^M(\phi)$, proving immediately the following lemma:

Lemma 39 *Every consistent formula $\alpha \in \mathcal{L}_e$ is satisfiable with respect to the class of factive epistemic models.*

Proof The proof is analogous to the one of Lemma 28, given the adequate choice of syntactic factive model, as discussed above. □

Theorem 40 (Completeness) *The system **EL** is complete: for any factive epistemic model $\mathcal{M}^F$ and for any $\phi \in \mathcal{L}_e$, if $\mathcal{M}^F \models \phi$, then $\vdash_{EL^+} \phi$.*

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