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RESEARCH ARTICLE

Minimal generation of finite simple groups of Lie type by regular unipotent elements

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Dedicated to the memory of Bhama Srinivasan

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Abstract

We prove that every finite simple group of Lie type G can be generated by three regular unipotent elements. In certain cases we show that two regular unipotents are sufficient to generate G.

Keywords Simple groups of Lie type · Regular unipotent element · Generation

Mathematics Subject Classification 20F05 · 20D06

1 Introduction

Problems of the generation of finite simple groups by certain their elements attract a significant attention of group theorists and have many applications. For many occasions one requires a group to be generated by specific elements. This kind of generation problems forms a large area of research in group theory. One of the aspects of this research is obtaining a good upper bound for the minimal number of conjugates of a given group element to generate the group in question.

The most universal results have been obtained by Guralnick and Saxl [24]. For a finite nonabelian simple group G and $g \in G$, let $\mathrm{gn}_G(g)$ be the minimal number of conjugates of g that generate G, and let $\mathrm{gn}(G) = \max_{1 \neq g \in G} \mathrm{gn}_G(g)$. In [24] the authors obtained a rather sharp lower bound for $\mathrm{gn}(G)$. However, for a specific $g \in G$

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the actual minimal number of conjugates of g needed to generate G dramatically depends on the choice of g. The problem of understanding the dependence of $\mathrm{gn}_G(g)$ on the choice of $g \in G$ is of great interest. Some results in this direction are proved in the literature (for instance, see [51] that provides optimal bounds for almost all sporadic groups), but in general the problem is far from a satisfactory solution. A natural and important question is that of determining $g \in G$ with $\mathrm{gn}_G(g) = 2$. Examples of such elements can be found in various publications.

Lübeck and Malle proved in [38] that every finite simple group of exceptional Lie type (apart from the Suzuki groups) is generated by two elements of respective order 2 and 3. Similar results for classical groups are obtained in [46–49]. It is shown in [55] that, for $1 \neq u \in G$ unipotent, $G = \langle u, h \rangle$ for a suitable semisimple element $h \in G$. In [61] the authors prove (modulo some open conjectures) that all but two finite nonabelian simple groups are generated by three conjugate involutions.

Our interest in this paper concerns regular unipotent elements of finite simple groups of Lie type. We conjecture that if $G \neq \mathrm{PSL}_2(q)$ with q even, then G is generated by two regular unipotent elements that can be chosen conjugate in G provided that $G \neq \mathrm{PSL}_2(9)$, see below for details.

There is a certain connection of these problems with those on presentations of an arbitrary group element as a product of two elements of special kind. An important contribution was done by Gow [20] who proved that every semisimple non-identity element of a finite simple group G of Lie type is a product of two elements from an arbitrary conjugacy class of regular semisimple elements. This leads to the conclusion that G is generated by three elements from any conjugacy class of regular semisimple elements [10, Lemma 2.8]. Ellers, Gordeev and Herzog [12, Theorem H, p. 344] proved that every non-identity element of G is a product of two unipotent elements (take there $h = 1$). The argument [10, Lemma 2.8 (2)] implies that G is generated by three unipotent elements, one can be chosen arbitrarily.

Conjecture 1.1 *Let G be a finite quasisimple group of Lie type such that $G \neq \mathrm{SL}_2(q)$ with q even. Then G is generated by two regular unipotent elements.*

Groups $\mathrm{SL}_2(q)$ with $q > 2$ even are not generated by two unipotent elements as these are involutions, and two involutions generate either an abelian group or a dihedral one.

Recall that the regular unipotent elements of G are not always conjugate. A stronger version of Conjecture 1.1 is that G is generated by two conjugate regular unipotents if, additionally, $G \neq \mathrm{PSL}_2(9)$. If $G = \mathrm{PSL}_2(q)$ with $q \geqslant 5$ odd, then the validity of the stronger version of Conjecture 1.1 is contained in [24, Lemma 3.1]; alternatively it can be deduced form a well-known result by L.E. Dickson (e.g., see [18]). If p is a good prime for $\mathbf{G}$ then the regular unipotents are conjugate in $\mathrm{Aut}(G)$, see for instance [60, Lemma 5.2]. (Here $\mathbf{G}$ is a simple algebraic group used to define G, see below. Recall that a prime p is good unless $p = 2$ and $\mathbf{G}$ is not of type A_n, $p = 3$ for $\mathbf{G}$ of the exceptional Lie type, and $p \leqslant 5$ for $\mathbf{G}$ of type E_8. In addition, if G is of adjoint type then $Z(G) = 1$.) This implies (for p good) that, if G is generated by k conjugate regular unipotents, then for every regular unipotent element $u \in G$ there are k conjugates of u that generate G.

Our main results are the following.

Theorem 1.2 *Every finite quasisimple group of Lie type is generated by three regular unipotent elements.*

Theorem 1.3 *Conjecture 1.1 holds for the following groups of Lie type:*

(1) $\mathrm{SL}_2(q)$ *with* $q \geqslant 4$, *and* $\mathrm{SL}_3(q)$*;*
(2) $\mathrm{SU}_3(q)$ *with* $q \geqslant 3$*;*
(3) $\mathrm{Sp}_4(q)$ *and* $G_2(q)$ *with* q *odd;*
(4) ${}^2B_2(2^{2m+1})$, ${}^2G_2(3^{2m+1})$ *and* ${}^2F_4(2^{2m+1})$, *with* $m \geqslant 1$*;*
(5) ${}^3D_4(q)$*;*
(6) $\mathrm{SU}_4(2)$, $\mathrm{SU}_4(3)$, $\mathrm{SU}_5(2)$, $\mathrm{SU}_5(3)$, $\mathrm{Sp}_6(9)$, $G_2(4)$, $F_4(2)$, ${}^2F_4(2)'$ *and* ${}^2E_6(2)$*;*
(7) $\mathrm{SL}_n(q)$ *with* $n \geqslant 4$.

Furthermore, in cases (1)–(6)*, and in* (7) *for* q *a prime, the stronger version of Conjecture 1.1 holds.*

The problem on whether the unipotents in Theorem 1.2 can be chosen conjugate to each other is more delicate. For this we obtain the following result.

Theorem 1.4 *The following finite quasisimple groups of Lie type are generated by three conjugate regular unipotent elements:*

(1) $\mathrm{SL}_2(q)$ *with* $q \geqslant 4$ *even,* $\mathrm{SL}_2(9)$ *and* $\mathrm{SL}_n(q)$ *with* $n \geqslant 3$*;*
(2) $\mathrm{SU}_n(q)$ *with* $n \geqslant 3$ *odd;*
(3) $\mathrm{Sp}_{2n}(q)$ *with* $n \geqslant 2$*;*
(4) $\Omega_{2n+1}(q)$ *with* $n \geqslant 3$ *and* q *odd;*
(5) $\Omega^+_{2n}(q)$ *with* $n \geqslant 4$ *and* q *even;*
(6) $\Omega^-_{2n}(q)$ *with* $n \geqslant 4$*;*
(7) $\mathrm{SU}_4(q)$ *with* q *even;*
(8) $G_2(q)$, $F_4(q)$ *and* ${}^2E_6(q)$.

For a subgroup X of a finite group G of Lie type, let X_{u} be the subgroup generated by the unipotent elements of X. Our strategy in proving Theorem 1.3 is in showing that there is a maximal subgroup X of G such that X_{u} is generated by two regular unipotents. In many occasions for this a maximal parabolic subgroup can be taken. So we state our result in this direction as follows.

Theorem 1.5 *Let* G *be one of the following quasisimple groups:* $\mathrm{SL}_n(q)$ *where* $n \geqslant 3$, $\mathrm{SU}_n(q)$ *where* $n \geqslant 6$ *is even,* $\mathrm{Sp}_{2n}(q)$ *where* $n \geqslant 3$, $\Omega^{\pm}_{2n}(q)$ *where* $n \geqslant 4$, $\Omega_{2n+1}(q)$ *where* $n \geqslant 3$ *and* q *is odd,* ${}^2E_6(q)$ *where* q *is odd,* $E_6(q)$, $E_7(q)$, $E_8(q)$*. Then* G *contains a maximal parabolic subgroup* P *such that* P_{u} *is generated by two regular unipotent elements.*

Notation

We denote by $\mathbb{F}_q$ the finite field of q elements. In the following, J_n denotes an upper triangular unipotent Jordan block of size n. We write $\mathrm{diag}(d_1, \ldots, d_k)$ for a block-diagonal matrix with diagonal blocks $d_1, \ldots, d_k$. We denote by ${}^tA$ the transpose of a matrix A. For a group G we write $Z(G)$ for the center of G, G' for the derived subgroup and $O_p(G)$ for the maximal normal p-subgroup of G (where p is a prime). For subgroups $X, Y \leqslant G$ we write $[X, Y]$ for the group generated by $xyx^{-1}y^{-1}$ with $x \in X$, $y \in Y$. For $g \in G$, $|g|$ denotes the order of g.

Let $\mathbf{G}$ be a connected reductive algebraic group in characteristic p. We view it as $\mathbf{G}(F)$, where F is an arbitrary algebraically closed field of characteristic p, since our reasoning does not depend on the choice of F. Let σ be a Frobenius endomorphism of $\mathbf{G}$, and let $G = \mathbf{G}^\sigma$ be a finite group of Lie type. Then σ stabilizes some Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ and a maximal torus $\mathbf{T}$ of $\mathbf{B}$. These determine a root system of $\mathbf{G}$ and, for every root α, the root subgroups x_α, each is isomorphic to the additive group of the ground field F. By $\Phi(G)$ we denote the set of positive roots and S the subset of simple roots. Every unipotent element of $\mathbf{B}$ can be written as the product $\prod_{\alpha \in \Phi} x_\alpha(t_\alpha)$ for some $t_\alpha \in F$. In these terms, a regular unipotent element of $\mathbf{B}$ can be defined as such that $t_\alpha \neq 0$ for every $\alpha \in S$, and a regular unipotent element of $\mathbf{G}$ can be defined as a conjugate (in $\mathbf{G}$) of a regular unipotent element of $\mathbf{B}$ (see [6, Proposition 5.1.3]). We keep this term for elements of $G = \mathbf{G}^\sigma$ that are regular unipotents in $\mathbf{G}$, and also apply this to the group ${}^2F_4(2)'$ which is not of such a form.

Recall that parabolic subgroups $\mathbf{P}$ of $\mathbf{G}$ are those containing a Borel subgroup of $\mathbf{G}$, and those containing our fixed subgroup $\mathbf{B}$ are referred here as standard parabolics. Every parabolic subgroup $\mathbf{P}$ is conjugate to a standard one, which, in turn, is a semidirect product $\mathbf{P} = \mathbf{UL}$, where $\mathbf{L}$ is a connected reductive group containing $\mathbf{T}$ and $\mathbf{U}$ is the unipotent radical of $\mathbf{P}$. As $\mathbf{T} < \mathbf{L}$, the $\mathbf{T}$-roots of $\mathbf{L}$ are $\mathbf{T}$-roots of $\mathbf{G}$. We call $\mathbf{L}$ standard if the simple roots of $\mathbf{L}$ are simple roots of $\mathbf{G}$.

By general theory, we can choose $\mathbf{B}$ and $\mathbf{T}$ σ-stable. So $\mathbf{T}^\sigma < \mathbf{B}^\sigma < G$. If $\mathbf{P}$ is a standard parabolic and $\sigma(\mathbf{P}) = \mathbf{P}$, the group $P := \mathbf{P}^\sigma$ is called a standard parabolic subgroup of G. (For our purposes we can always assume that P is standard.) In addition, if $\sigma(\mathbf{P}) = \mathbf{P}$ then there exists a Levi subgroup $\mathbf{L}$ of $\mathbf{P}$ such that $\sigma(\mathbf{L}) = \mathbf{L}$. If $\mathbf{P}$ is standard then there exists a standard Levi subgroup of $\mathbf{P}$.

Let $N = N_{\mathbf{G}}(G)$. The automorphisms of G induced by inner automorphisms of N are called diagonal.

For an algebraic group $\mathbf{G}$, or a finite group G of Lie type, we denote by $\mathbf{G}_{\mathsf{u}}$ and G_{u} the subgroups generated by the unipotent elements. Every connected reductive algebraic group $\mathbf{G}$ is a central product $Z(\mathbf{G}) \cdot \mathbf{G}_{\mathsf{u}}$, where $\mathbf{G}_{\mathsf{u}}$ is a semisimple algebraic group. In particular, $\mathbf{L} = Z(\mathbf{L}) \cdot \mathbf{L}_{\mathsf{u}} = \mathbf{T} \cdot \mathbf{L}_{\mathsf{u}}$, and hence the automorphisms of $\mathbf{L}_{\mathsf{u}}^\sigma$ arising from the action of $\mathbf{T}^\sigma$ on $\mathbf{L}_{\mathsf{u}}^\sigma$ are diagonal.

2 Preliminaries

Lemma 2.1 ([58, Lemma 2.6]) *Let* $\mathbf{G}$ *be a reductive algebraic group, and* $\mathbf{P}$ *a parabolic subgroup of* $\mathbf{G}$ *properly containing a Borel subgroup of* $\mathbf{G}$. *Let* $\mathbf{L} < \mathbf{P}$ *be a Levi subgroup of* $\mathbf{P}$. *Let* $u \in \mathbf{P}$ *be a regular unipotent. Then* $u \notin \mathbf{L}$*; in addition, the projection of* u *into* $\mathbf{L}$ *is a regular unipotent in* $\mathbf{L}$. *Consequently, every regular unipotent* $u \in \mathbf{L}$ *is the projection of some regular unipotent element of* $\mathbf{P}$.

The last claim is not stated in [58, Lemma 2.6], but it follows from the previous claim as the regular unipotent elements of an algebraic group are conjugate.

Now, let σ be a Frobenius endomorphism of $\mathbf{G}$, a simple algebraic group, and let $G = \mathbf{G}^\sigma$ be a finite group of Lie type. Then there exist σ-stable subgroups $\mathbf{T} < \mathbf{L} < \mathbf{P}$, where $\mathbf{T}$ is a maximal torus, $\mathbf{P}$ is a standard proper parabolic and $\mathbf{L}$ a standard Levi of $\mathbf{P}$. Let $P = \mathbf{P}^\sigma$ and $L = \mathbf{L}^\sigma$ is called a standard subgroup of P. Whenever we prove that the group P_{u} is generated by two regular unipotents we start from a Levi subgroup L of P such that L_{u} is generated by two regular unipotents of L; therefore, we may assume that L is standard. In this notation we have:

Lemma 2.2 *The projection of a regular unipotent element $u \in P$ is a regular unipotent of L. Moreover, every regular unipotent element of L is a projection of a regular unipotent of P into L.*

Proof The first assertion follows from Lemma 2.1 as the notion of a regular unipotent element of G and L is described in terms of algebraic groups $\mathbf{G}$ and $\mathbf{L}$ [6, Proposition 5.1.3].

Suppose first that the groups $B < L < P$ are standard. Recall that every unipotent element u in a standard Borel subgroup is expressed as a product $\prod_\alpha (x_\alpha(t_\alpha))$ with the product ranging over positive roots of G, and u is regular if and only if $t_\alpha \neq 0$ for all simple roots α.

If G is not of twisted type then the second claim follows by exactly the same argument as in the proof of [58, Lemma 2.6]. Suppose that G is twisted. Then the argument requires some adjustment. Note that standard σ-stable parabolic subgroups of G are determined by some σ-orbits on the set of simple roots of $\mathbf{G}$ and on the respective sets of root subgroups $x_\alpha(t_\alpha)$, see [41, Proposition 26.1]. Unipotent elements in a standard Levi subgroup are products $\prod_\alpha (x_\alpha(t_\alpha))$ over the positive roots of $\mathbf{L}$, which are σ-stable in the sense that each term $x_\alpha(t_\alpha)$ occurs together with $x_{\sigma(\alpha)}(\sigma(t_\alpha))$. Note that $t_\alpha \neq 0$ implies $\sigma(t_\alpha) \neq 0$. So, $u \in L$ is regular unipotent if all $t_\alpha \neq 0$. If S' is the set of simple roots of G that are not simple roots of $\mathbf{L}$, then $\sigma(S') = S'$. Then we can extend the above product by the multiple $\prod_{\alpha \in S'} x_\alpha(1)$ to obtain a regular unipotent element $u' \in P$ that is regular in G and whose projection into $\mathbf{L}$ coincides with u.

For the general case let $\mathbf{L}_1$ be an arbitrary σ-stable Levi subgroup of $\mathbf{P}$ and let $\mathbf{L} < \mathbf{P}$ be a standard Levi subgroup. Then $\mathbf{L}_1 = x\mathbf{L}x^{-1}$ for some $x \in \mathbf{P}$. Let $\mathbf{T}_1 = x\mathbf{T}x^{-1}$ and $\mathbf{B}_1 = x\mathbf{B}x^{-1}$. Then $\mathbf{L}_1$ is standard with respect to $\mathbf{T}_1 < \mathbf{B}_1$, so the result follows by the above reasoning as the notion of a regular unipotent element does not depend on the choice of a maximal torus in $\mathbf{G}$. ◻

Note that Levi subgroups of P are not always conjugate but it suffices for our purpose to deal with the standard Levi subgroup. In the following we will use Lemma 2.2 with no explicit reference.

The following two lemmas can be deduced from [44], but for reader's sake we give here short proofs. The first part of the statement of Lemma 2.3 follows also from [52, (1.6)].

Lemma 2.3 *Let G be a simple group of Lie type in defining characteristic $p > 0$ and U be a Sylow p-subgroup of G. Suppose that H is a maximal subgroup of G containing U. Then H is a maximal parabolic subgroup of G. In addition, H is the only maximal subgroup of G containing H_{u}.*

Proof It is well known that G is a group with BN-pair [41, Theorem 24.10]. These groups satisfy the assumption of [3, Chapter IV, Section 2.7, Theorem 5], which states that H is either parabolic or normal in G, whence the result. For the additional claim, if H is a maximal parabolic, then $H = N_G(O_p(H))$. Suppose that Q is a maximal subgroup of G containing H_{u}. Then $H \leqslant Q$ by a theorem of Borel and Tits, see [19, 13.1]. As H is maximal, we have $H = Q$. □

Lemma 2.4 *Let G be a quasisimple group of Lie type, P a maximal parabolic of G and let $u \notin P$ be a unipotent element. Then $\langle u, P_{\mathsf{u}}\rangle = G$.*

Proof Set $X = \langle u, P_{\mathsf{u}}\rangle$. By Lemma 2.3, P is the only maximal subgroup of G containing P_{u}. Since $u \notin P$, X cannot be contained in P and hence $X = G$. □

Lemma 2.5 *The groups* $\mathrm{SL}_4(2)$ *and* $\mathrm{Sp}_4(2)$ *are generated by two conjugate elements with Jordan form J_4.*

Proof Observe that $\mathrm{SL}_4(2) \cong \mathrm{Alt}(8)$ and $\mathrm{Sp}_4(2) \cong \mathrm{Sym}(6)$. Let $G = \mathrm{SL}_4(2)$. Then $C_G(J_4)$ is an abelian 2-group of order 8. So, J_4 lies in class $4B$ in the notation of [7] and is realized as the permutation $(1, 2, 3, 4)(5, 6)$ in $\mathrm{Alt}(8)$. Let $t = (2, 5, 8, 4, 6, 3, 7)$. Then $h := t^{-1}gt = (1, 7, 6, 8)(2, 4)$ and $g^2h = (1, 7, 6, 8, 3)$ is of order 5. The group $\langle g, h\rangle$ is primitive and contained in $\mathrm{Alt}(8)$. By [11, Theorem 3.3E], $\langle g, h\rangle = \mathrm{Alt}(8)$.

Let $G = \mathrm{Sp}_4(2) \cong \mathrm{Sym}(6)$. The isomorphism $\mathrm{Sym}(6) \xrightarrow{\cong} \mathrm{Sp}_4(2)$ can be chosen from taking the non-trivial irreducible constituent of the natural permutational module of degree 6 over $\mathbb{F}_2$. Then we choose for g the 4-cycle $(1, 2, 3, 4)$ in $\mathrm{Sym}(6)$, by viewing $\mathrm{Sym}(6)$ as the group of permutations on $\{1, 2, 3, 4, 5, 6\}$. Let $t = (2, 6, 4, 5, 3)$. Then $h := t^{-1}gt = (1, 3, 5, 6)$ and $(g^2h)^2 = (3, 6, 5)$ is of order 3. The group $\langle g, h\rangle$ is primitive and not contained in $\mathrm{Alt}(6)$. By [11, Theorem 3.3E], $\langle g, h\rangle = \mathrm{Sym}(6)$. □

3 Groups $\mathrm{PSL}_n(q)$

Lemma 3.1 *If $G = \mathrm{SL}_n(q)$ with $n > 2$, then Conjecture 1.1 is true.*

Proof If q is a prime then the result is contained in [21], unless $(n, q) = (4, 2)$. For $G = \mathrm{SL}_4(2)$ see Lemma 2.5. So we can assume that $q > 3$ is not a prime.

By [32, Theorem 6], if $q > 3$ then every non-central element of G is a product of two $\mathrm{GL}_n(q)$-conjugates of J_n; these are of course regular unipotents in G. In particular, an irreducible element t of order $(q^n-1)/(q-1)$ can be written as a product $t = gh$, where g, h are regular unipotents in G. So $\langle g, h\rangle$ is an irreducible subgroup of G containing $T = \langle t\rangle$. It is well known that T acts transitively on the lines of V, the underlying space of G, and, consequently, irreducibly on V. Groups H acting transitively on the lines of V are determined in [25], see also [33, pp. 512–513]. We use the list provided in [33] to exclude those containing no element of order $(q^n-1)/(q-1)$ and no regular unipotent. Let H' be the derived subgroup of H (the case $H = H'$ is not excluded).

Suppose that $H \neq \mathrm{SL}_n(q)$. By [33, pp. 512–513], one of the following holds:

(A) $H' \cong \mathrm{SL}_{n/k}(q^k)$;

(B) $H' = \mathrm{Sp}_n(q)$, $n \geqslant 2$ even, or $G_2(q)'$ for $n = 6$ and q even;
(C) H normalizes an irreducible extraspecial 2-group E, and either $E \cong Q_8$, $n = 2$ and $q \in \{5, 7, 11, 23\}$ or $n = 4$, $q = 3$ and $E \cong Q_8 \circ D_8$ and $H'/E \leqslant S_5$;
(D) either H' normalizes $\mathrm{SL}_2(5)$, $n = 2$ and $q \in \{9, 11, 19, 29, 59\}$ or $H' = A_6, A_7 < \mathrm{SL}_4(2)$, or $H' = \mathrm{SL}_2(13) < \mathrm{Sp}_6(3)$.

Suppose that $g, h \in H$. Then H is not a group from (A), as such a group contains no element with Jordan form J_n.

If (B) holds then $n > 2$ is even, and the order of an irreducible cyclic subgroup of H' divides $q^{n/2} - 1$ or $q^{n/2} + 1$. The maximum order of an element of $\mathrm{PCSp}_n(q)$ is less than $q^{(n+2)/2}/(q-1)$ (see [22, Lemma 2.10]). Since $q^{(n+2)/2} < q^n - 1$, we obtain that $t \notin H$. If $n = 6$ and $H' = G_2(q)'$ then $H' < \mathrm{Sp}_6(q)$, so this case is ruled out.

As $n > 2$ and q is not a prime, cases (C) and (D) cannot occur. This completes the proof. □

Corollary 3.2 *Let $G = \mathrm{SL}_n(q)$, where $n > 2$ and $\gcd(n, q-1) = 1$. Then G is generated by two conjugate regular unipotent elements.*

Proof The result follows from Lemma 3.1, as in this case the regular unipotent elements are conjugate in G. Indeed, let d be the number of conjugacy classes of regular unipotents in G. Recall that d equals $|Z(G)|$ (for $G = \mathrm{SL}_n(q)$), see [60, Lemma 5.2 (i)]. □

Remark 3.3 The case $n = 2$ is treated in [24, Lemma 3.1], assuming $q \geqslant 4$. In particular, $\mathrm{SL}_2(q)$ for $q \neq 9$ odd is generated by two conjugate regular unipotents, while $\mathrm{SL}_2(9)$ is generated by three conjugate regular unipotents. One can also verify that $\mathrm{SL}_2(9)$ is generated by the regular unipotents $a = J_2$ and $b = \left(\begin{smallmatrix} 1 & 0 \\ \beta & 1 \end{smallmatrix}\right)$, where $\beta \in \mathbb{F}_9$ is such that $\beta^2 = \beta + 1$. In fact, ab^2 and $[a, b]$ have respective order 8 and 10, but no maximal subgroup of $\mathrm{SL}_2(9)$ contains elements of both such orders.

Lemma 3.4 *The matrix ${}^tJ_n$ is conjugate in $\mathrm{SL}_n(q)$ to J_n or to J_n^{-1}.*

Proof Consider the matrix M with 1 at the second diagonal and 0 elsewhere. Then $MJ_nM^{-1} = {}^tJ_n$. If $q = 2$ then $\det M = 1$ and the result follows. If q is odd then $\det M = 1$ if and only if $n \equiv 0, 1 \pmod 4$. Otherwise, $\det M = -1$. If $n \equiv 3 \pmod 4$ then $-M \in \mathrm{SL}_n(q)$, and the result follows.

Let $n \equiv 2 \pmod 4$. If $-1 = \lambda^2$ for some $\lambda \in \mathbb{F}_q$ then $\det(\lambda\,\mathrm{Id}) = -1$ and $\lambda M \in \mathrm{SL}_n(q)$. Note that J_n is not conjugate to ${}^tJ_n$. Indeed, if $xJ_nx^{-1} = {}^tJ_n$ then $Mx \in C_{\mathrm{GL}_n(q)}(J_n)$. As $C_{\mathrm{GL}_n(q)}(J_n)$ consists of the upper triangular matrices with scalar diagonal, $\det(Mx) = 1$ implies the existence of a scalar matrix with determinant -1, which is false.

We show that ${}^tJ_n$ is conjugate in $\mathrm{SL}_n(q)$ to J_n^{-1}. Set $N = \mathrm{Id} - (J_n - \mathrm{Id})$. Then N is a regular unipotent with -1 above the diagonal. Let M_1 be the antidiagonal matrix $\mathrm{antidiag}(+1, -1, \ldots, +1, -1)$. Then $M_1{}^tJ_nM_1^{-1} = N$ and $\det M_1 = 1$. Note that J_n^{-1} is another matrix with -1 above the diagonal. By [60, Lemma 5.1], J_n^{-1} and N are conjugate in B, the group of upper triangular matrices in G. □

Corollary 3.5 *Let $G = \mathrm{SL}_n(q)$, where $n \geqslant 2$ and q is a prime. Then G is generated by two conjugate regular unipotent elements.*

Proof If $(n, q) = (4, 2)$ then the result follows from Lemma 2.5. Otherwise, by [21], $G = \langle J_n, {}^tJ_n \rangle$. By Lemma 3.4, the matrix ${}^tJ_n$ is conjugate in $\mathrm{SL}_n(q)$ to J_n or to J_n^{-1}. Since $\langle J_n, {}^tJ_n \rangle = \langle J_n^{-1}, {}^tJ_n \rangle$, the result follows. □

From the previous corollary, it follows in particular that the group $\mathrm{SL}_2(3)$ is generated by two conjugate regular unipotents. This fact will be used in Lemmas 5.21 and 5.22.

Remark 3.6 If a group G is generated by k conjugate cyclic groups $\langle g_1 \rangle, \dots, \langle g_k \rangle$ then, obviously, G is generated by k conjugate elements. In particular, the case of our interest is where G is a finite group of Lie type and g_i are regular unipotent elements.

Proposition 3.7 *Let $G = \mathrm{SL}_n(q)$, where $n \equiv 2 \pmod 4$ and $\gcd(q-1, n) = 2$. Then G is generated by two conjugate regular unipotent elements.*

Proof By [60, Lemma 5.2 (i)], the number of conjugacy classes of regular unipotents in G is $|Z(G)| = 2$. Let $u \in G$ be a regular unipotent. By [60, Theorem 1.7 (ii)], u is not rational if q is odd, $n = 2m$ is even and $n/\gcd(n, q-1) = 2m/|Z(G)| = m$ is odd. It follows that each conjugacy class of regular unipotents in G meets the cyclic group $\langle u \rangle$. Therefore, if $G = \langle u, v \rangle$ for some regular unipotent $v \in G$, by replacing v by a suitable power of v, we obtain two conjugates of u that generate G. □

Let d be the number of conjugacy classes of regular unipotents in a finite group $G = \mathbf{G}^\sigma$ of Lie type. If $\mathbf{G}$ is simple then $d \geqslant |Z(G)|$, where the equality holds if and only if p is a good prime for $\mathbf{G}$, see for instance [60, Lemma 5.2]. (Good primes are defined prior to Theorem 1.2 above.) So the regular unipotent elements of G are conjugate if and only if $|Z(G)| = 1$ and p is a good prime for $\mathbf{G}$. If these conditions hold then either $G = \mathrm{SL}_n(q)$ with $\gcd(q, n-1) = 1$ or $G = F_4(q), G_2(q)$ with $p > 3$ or $E_8(q)$ with $p > 5$. If $G = \mathrm{Spin}_{2n+1}(q)$ with q odd then $|Z(G)| = 2$, so regular unipotents form in two conjugacy classes. If $u \in G$ is a regular unipotent then the generators of $\langle u \rangle$ form two conjugacy classes unless $2n + 1 \not\equiv \pm 1 \pmod 8$ ([60, Theorem 1.7]). So the cyclic groups $\langle u \rangle$ are conjugate to each other for all regular unipotents of G. Hence the above reasoning works and in these cases G is generated by three conjugate regular unipotents.

If every regular unipotent is rational then k coincides with the number of cyclic subgroups generated by regular unipotents. These cases are listed in [60, Theorem 1.8] for $G \neq E_8(q)$. In particular, if q is even or a square then this is the case, so Remark 3.6 is not useful if we wish to improve the result of Lemma 3.1 by stating that G is generated by two conjugate unipotents.

4 Generation by two conjugate regular unipotent elements

Corollary 4.1 *The group $\Omega_6^+(q)$ is generated by two regular unipotent elements.*

Proof Note that $\Omega_6^+(q)$ is isomorphic to $\mathrm{PSL}_4(q)$ and the isomorphism in question arises from an irreducible representation $\phi\colon \mathrm{SL}_4(q) \to \mathrm{GL}_6(q)$. Then the regular unipotent $u \in \mathrm{SL}_4(q)$ remains a regular unipotent in $\Omega_6^+(q)$. The Jordan form of it is $\mathrm{diag}(J_2, J_4)$ if q is even, and it is $\mathrm{diag}(J_5, J_1)$ if q is odd. □

Lemma 4.2 *For every regular unipotent element u of $G = Sz(2^{2m+1})$, $m \geqslant 1$, there exists $g \in G$ such that $G = \langle u, u^g \rangle$.*

Proof Let u be a regular unipotent element of $G = Sz(2^{2m+1})$. Since $G \leqslant \mathrm{Sp}_4(q)$, the Jordan form of u is J_4, whence $|u| = 4$. As shown in [57], G admits a unique class of involutions and exactly two classes of elements of order 4, respectively those of u^2, u and u^{-1}. Always by [57], G can be generated by an element x of order 2 and an element y of order 4. Then y is regular unipotent and $H = \langle y, y^x \rangle$ is a normal subgroup of $\langle x, y \rangle = G$. Since G is simple, it follows that $H = G$. □

The knowledge of the character table of a group G (and of the list of its maximal subgroups) can be very useful to our purposes. Namely, given a finite group G, let c_1, c_2, c_3 be (not-necessarily distinct) conjugacy classes of G. Denote by $\Delta_G(c_1, c_2, c_3)$ the number of distinct triples (g_1, g_2, g_3), where g_3 is a fixed element of the class c_3 and $g_1 \in c_1$, $g_2 \in c_2$ are such that $g_1 g_2 = g_3$. This structure constant can be computed using the (complex) character table of G. Namely, it is given by the formula

$$\Delta_G(c_1, c_2, c_3) = \frac{|c_1| \cdot |c_2|}{|G|} \cdot \sum_{i=1}^{r} \frac{\chi_i(g_1)\chi_i(g_2)\overline{\chi_i(g_3)}}{\chi_i(1)}, \tag{4.1}$$

where $\chi_1, \ldots, \chi_r$ are the irreducible complex characters of G.

Lemma 4.3 *For every regular unipotent u of $G = \mathrm{SU}_5(2)$, there exists $g \in G$ such that $G = \langle u, u^g \rangle$.*

Proof The regular unipotent elements of $G = \mathrm{SU}_5(2)$ have Jordan form J_5 and belong to the class $8a$. Using the character table of G, one can compute $\Delta_G(8a, 8a, 11a) = 53416$. So, given a regular unipotent element u, there exists $g \in G$ such that uu^g has order 11. Now, the only maximal subgroup of G whose order is divisible by 11 is $\mathrm{PSL}_2(11)$, but this subgroup does not contain elements of order 8. Hence, $\langle u, u^g \rangle = G$. □

Lemma 4.4 *For every regular unipotent u of $G = {}^2E_6(2)$, there exists $g \in G$ such that $G = \langle u, u^g \rangle$.*

Proof The group $G = {}^2E_6(2)$ has three classes of regular unipotent elements. In fact, G has nine classes of elements of order 16, three of them ($16a$, $16b$ and $16c$) with centralizer of size 256, the other six with centralizer of size 128.

Let c be any conjugacy class of regular unipotents and let c_3 be the class $19a$. Since $\Delta_G(c, c, c_3) > 0$, for every regular unipotent element $u \in G$ there exists $g \in G$ such that the product uu^g has order 19. The list of the maximal subgroups of G is provided in [7, 8]. It is easy to see that the only maximal subgroup of G whose order is divisible by 19 is $\mathrm{PSU}_3(8):3_1$. However, this subgroup does not contain elements of order 16, proving that $G = \langle u, u^g \rangle$. □

Lemma 4.5 *For every regular unipotent u of $G = {}^2G_2(3^{2m+1})$, $m \geqslant 1$, there exists $g \in G$ such that $G = \langle u, u^g \rangle$.*

Proof The maximal subgroups of $G = {}^2G_2(3^{2m+1})$ are described in [4, 27]. In particular, as shown in [62], the only maximal proper overgroup of the cyclic torus T of order $t = 3^{2m+1} + 3^{m+1} + 1$ is its normalizer $H = T:6$. Note that the unipotent regular elements of G have order 9, as $G \leqslant \Omega_7(3^{2m+1})$.

Our aim is to show that there exist two conjugate regular unipotents whose product has order t. This goal can be achieved by computing the structure constants for G thanks to (4.1). In particular, we make use of [16], that contains the character table of G and all the relevant information we need. So, following the notation of [16], let c be a conjugacy class consisting of regular unipotents (so, $j = 5, 6, 7$) and let c_{14} be a class of elements of order t (with the parameter $\mathtt{iI} = 1$). Computing $\Delta_G(c_j, c_j, c_{14})$ we obtain the value

$$3^{10m+3} - 3^{8m+2} + 4 \cdot 3^{6m+1} + 2 \cdot 3^{7m+2} + 2 \cdot 3^{5m+1} + 2 \cdot 3^{4m+1} + 3^{3m+1}$$

for $j = 5$, and

$$3^{10m+3} - 3^{8m+2} - 3^{7m+2} - 2 \cdot 3^{6m+1} - 3^{5m+1}$$

for $j = 6, 7$. Note that $\Delta_G(c_j, c_j, c_{14}) > 0$ for all $j = 5, 6, 7$. This implies that, given a regular unipotent element u, there exists $g \in G$ such that uu^g has order t. We conclude that $\langle u, u^g \rangle$ contains T, whence the result. □

Lemma 4.6 *For every regular unipotent u of $G = {}^3D_4(q)$, there exists $g \in G$ such that $G = \langle u, u^g \rangle$.*

Proof The maximal subgroups of $G = {}^3D_4(q)$ are described in [4, 26]. In particular, as shown in [62], the only proper maximal overgroup of the cyclic torus T of order $t = q^4 - q^2 + 1$ is its normalizer $H = T:4$. Note that the regular unipotent elements of $G \leqslant \Omega_8^+(q^3)$ have Jordan form $\mathrm{diag}(J_2, J_6)$, so they have order 8 when q is even.

We proceed, as done in proving Lemma 4.5, with using the character table of G in order to compute the structure constants of G. Following the notation of [16], let c_j be a conjugacy class consisting of regular unipotents and let c_k be a class of elements of order t (with the parameter $\mathtt{kK} = 1$). So, $j = 7, 8$ and $k = 28$ if q is even, $j = 7$ and $k = 32$ if q is odd. Computing $\Delta_G(c_j, c_j, c_k)$ we obtain the value

$$\frac{1}{4}q^{20} - \frac{1}{4}q^{18} + \frac{1}{4}q^{16} - \frac{1}{2}q^{14} + q^{12} - q^{10} + \frac{1}{2}q^8$$

if q is even, and

$$q^{20} - q^{18} + q^{16} - 2q^{14} + 2q^{12} - 2q^{10}$$

if q is odd. Note that $\Delta_G(c_j, c_j, c_k) > 0$ for all values of j and k. This implies that, given a regular unipotent element u, there exists $g \in G$ such that uu^g has order t. We conclude that $\langle u, u^g \rangle$ contains T, whence the result. □

Lemma 4.7 *For every regular unipotent element u of $G = {}^2F_4(2^{2m+1})$, $m \geqslant 1$, there exists $g \in G$ such that $G = \langle u, u^g \rangle$.*

Proof Set $q^2 = 2^{2m+1}$. The maximal subgroups of $G = {}^2F_4(2^{2m+1})$ are described in [39]. In particular, the only proper maximal overgroup of the cyclic torus T of order $t = q^4 + \sqrt{2}q^3 + q^2 + \sqrt{2}q + 1$ is its normalizer $H = T:12$. Note that the regular unipotent elements of G have order 16. So, to prove the result it suffices to show that, given a regular unipotent element $u \in G$, there exists $g \in G$ such that the product uu^g has order t.

We proceed as done in proving Lemma 4.5, with using the character table of G in order to compute the structure constants of G. Unfortunately, [16] only contains a partial character table, missing 10 character families. However, we can show that these irreducible characters take value 0 on $T \setminus \{1\}$. In fact, looking at [37], one can verify that the missing characters have the following degrees:

$$\begin{aligned}
\eta_1 &= tt_2(q^4 - q^2 + 1)(q^2 - \sqrt{2}q + 1)(q + 1)(q - 1)(q^2 + 1)^2,\\
\eta_2 &= tt_2(q^4 - q^2 + 1)(q^2 + \sqrt{2}q + 1)(q + 1)(q - 1)(q^2 + 1)^2,\\
\eta_3 &= \frac{\sqrt{2}q}{2}(q + 1)(q - 1)\eta_1,\\
\eta_4 &= \frac{\sqrt{2}q}{2}(q + 1)(q - 1)\eta_2,\\
\eta_5 &= (q^2 - \sqrt{2}q + 1)(q + 1)(q - 1)\eta_1,\\
\eta_6 &= (q^2 + \sqrt{2}q + 1)(q + 1)(q - 1)\eta_2,\\
\eta_7 &= q^4\eta_1,\\
\eta_8 &= q^4\eta_2
\end{aligned}$$

where $t_2 = q^4 - \sqrt{2}q^3 + q^2 - \sqrt{2}q + 1$. Note that $|G:T| = q^{24}t_2(q^8 - 1)(q^6 + 1)(q^4 + 1)(q^2 - 1)$ and that $\gcd(|T|, |G:T|) = 1$. Now, let r be a prime dividing t. By the above, r does not divide $\frac{|G|}{\eta_k}$ and so all the missing characters are of r-defect 0, and hence vanish at the r-singular elements. So, we can still use [16] for our computations.

Let c be a conjugacy class consisting of regular unipotents (4 classes) and let c_3 be a class of elements of order t (with the parameter $\mathtt{aA} = 1$). Computing $\Delta_G(c, c, c_3)$ we obtain the value $\frac{1}{16}q^{17}tb$, where

$$\begin{aligned}
b = q^{23} - \sqrt{2}q^{22} + q^{19} - \sqrt{2}q^{16} - q^{15} + \sqrt{2}q^{14}\\
+ 2q^{13} - 2q^{11} - 2\sqrt{2}q^{10} + 2q^9 + 3\sqrt{2}q^8 - q^7\\
+ 6\sqrt{2}q^6 + 14q^5 + 18\sqrt{2}q^4 + 33q^3 + 16\sqrt{2}q^2 + 14q + 8\sqrt{2}.
\end{aligned}$$

Since $\Delta_G(c, c, c_3) > 0$, we have the desired result. □

Lemma 4.8 *For every regular unipotent element u of $G = \mathrm{Sp}_4(q)$, q odd, there exists $g \in G$ such that $G = \langle u, u^g \rangle$.*

Proof As before, we can compute the structure constants using the character table of $G = \mathrm{Sp}_4(q)$ (see [2, 50, 54]). We follow the notation of [54]. So, let c be any of the two conjugacy classes of the regular unipotent elements $A_{4,1}$, $A_{4,2}$, and let c_3 be the conjugacy class of $B_1(1)$, a semisimple element of order $q^2 + 1$. The only irreducible characters taking nonzero values on both classes c and c_3 are $\chi_1(j)$, θ_7 and θ_8. So,

$$\Delta_G(c, c, c_3) = \frac{q^5 - q}{4}\left(q + (-1)^{(q-1)/2}\right),$$

which is positive for all $q \geqslant 3$. This implies that, given any regular unipotent element u, there exists $g \in G$ such that uu^g has order $q^2 + 1$. Note that the Jordan form of u is J_4.

If $q = 3$, there is no maximal subgroup containing elements of both orders 9 and 10. So, the result easily follows. Then suppose $q > 3$. By [40] and [4], the only maximal subgroup containing elements of order $q^2 + 1$ is $M = H : 2$, where $H \cong \mathrm{Sp}_2(q^2)$. So, assume that $X = \langle u, u^g \rangle$ is contained in M. Since the unipotent elements of $\mathrm{SL}_2(q^2)$ have Jordan form J_2, we easily get a contradiction, proving that $X = \mathrm{Sp}_4(q)$. □

For a fixed $g_3 \in c_3$, denote by $\Delta^*_G(c_1, c_2, c_3)$ the number of distinct triples (g_1, g_2, g_3) such that $g_1 \in c_1$, $g_2 \in c_2$, $g_1 g_2 = g_3$, and $G = \langle g_1, g_2 \rangle$. We aim to show that $\Delta^*_G(c_1, c_2, c_3)$ is positive for certain classes c_1, c_2, c_3. To this end, let H be a maximal subgroup of G containing a fixed element $g_3 \in c_3$, and denote by $\Sigma_H(c_1, c_2, c_3)$ the number of distinct pairs $(g_1, g_2) \in c_1 \times c_2$ such that $g_1 g_2 = g_3$ and $\langle g_1, g_2 \rangle \leqslant H$. The value of $\Sigma_H(c_1, c_2, c_3)$ can be obtained as the sum of the structure constants $\Delta_H(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3)$ of H for all the H-conjugacy classes $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$ such that $\tilde{c}_i \subseteq H \cap c_i$.

The following holds (e.g., see [23]):

Lemma 4.9 *Let G be a finite group and let H a subgroup of G containing a fixed element x. Denote by $h(x, H)$ the number of the distinct conjugates of H containing x. If $\gcd(|x|, |N_G(H) : H|) = 1$, then*

$$h(x, H) = \sum_{i=1}^{s} \frac{|C_G(x)|}{|C_{N_G(H)}(x_i)|},$$

where $x_1, \ldots, x_s$ are representatives of the $N_G(H)$-conjugacy classes fused to the G-class of x.

As a consequence, we obtain a useful lower bound for $\Delta^*_G(c_1, c_2, c_3)$. Namely:

$$\Delta^*_G(c_1, c_2, c_3) \geqslant \Theta_G(c_1, c_2, c_3),$$

where

$$\Theta_G(c_1, c_2, c_3) = \Delta_G(c_1, c_2, c_3) - \sum h(g_3, H)\Sigma_H(c_1, c_2, c_3),$$

g_3 is a representative of the class c_3, and the sum is taken over the representatives H of the G-classes of maximal subgroups of G containing elements of all the classes c_1, c_2, c_3. For groups of small order, $\Theta_G(c_1, c_2, c_3)$ can be computed using [59].

Lemma 4.10 *For every regular unipotent element u of $G = \mathrm{SU}_4(2)$, there exists $g \in G$ such that $G = \langle u, u^g \rangle$.*

Proof As explained before, the result for $G = \mathrm{SU}_4(2)$ can be obtained by using the character table, showing that $\Theta^*_G(c, c, c_3) > 0$ for a class c consisting of regular unipotents and a suitable class c_3.

The regular unipotent elements of G have Jordan form J_4 and belong to the class $4b$. We have $\Delta_G(4b, 4b, 9a) = 486$. The unique maximal subgroup of G containing elements of both classes $4b$ and $9a$ is $H = 3^3 : S_4$. Furthermore, $h(g_3, H) = 1$ and $\Sigma_H(4b, 4b, 9a) = 81$, where g_3 is a fixed elements of $9a$. So, $\Theta_G(4b, 4b, 9a) = 405$, which implies the result. □

Lemma 4.11 *For every regular unipotent element u of $G = F_4(2)$, there exists $g \in G$ such that $G = \langle u, u^g \rangle$.*

Proof The regular unipotent elements of $G = F_4(2)$ have order 16 and belong to the classes $16a, 16b, 16c, 16d$. Using the character table of G, we have $\Delta_G(16a, 16a, 13a) = \Delta_G(16b, 16b, 13a) = 808763850752$ and $\Delta_G(16c, 16c, 13a) = \Delta_G(16d, 16d, 13a) = 808582807552$. The list of the maximal subgroups of G were determined in [43]. The maximal subgroups of G of order divisible by 13 do not contain elements from the classes $16a$ or $16b$. On the other hand, the unique maximal subgroup of G containing elements of classes $13a$ and $16X$ ($X \in \{c, d\}$) is $H = {}^2F_4(2)$. Furthermore, $h(g_3, H) = 1$ and $\Sigma_H(16c, 16c, 13a) = \Sigma_H(16d, 16d, 13a) = 519168$, where g_3 is a fixed elements of the class $13a$. So, $\Theta_G(16c, 16c, 13a) = \Theta_G(16d, 16d, 13a) = 808582288384$, which implies the result. □

Remark 4.12 For $G = {}^2F_4(2)'$ the elements of order 16 are regular unipotent in the simple group of type F_4. We have $\Theta_G(c, c, 13a) = \Delta_G(c, c, 13a) = 64896$ for $c \in \{16a, 16b, 16c, 16d\}$. So, the simple group ${}^2F_4(2)'$ is generated by two conjugate regular unipotents.

Lemma 4.13 *Let G be one of the groups $\mathrm{SL}_3(q)$ with $q \geqslant 2$, or $\mathrm{SU}_3(q)$ with $q \geqslant 3$. For every regular unipotent element u of G, there exists $g \in G$ such that $G = \langle u, u^g \rangle$.*

Proof If $q \leqslant 5$, we can compute with [59] the values of $\Theta_G(c, c, c_3)$ for every conjugacy class c of regular unipotent elements and a suitabile class c_3:

G	c	c_3	$\Theta_G(c, c, c_3)$	G	c	c_3	$\Theta_G(c, c, c_3)$
$\mathrm{SL}_3(2)$	$4a$	$7a$	7				
$\mathrm{SL}_3(3)$	$3b$	$13a$	39	$\mathrm{SU}_3(3)$	$3b$	$7a$	56
$\mathrm{SL}_3(4)$	$4a, 4b, 4c$	$7a$	56	$\mathrm{SU}_3(4)$	$4a$	$13a$	247
$\mathrm{SL}_3(5)$	$5b$	$31a$	589	$\mathrm{SU}_3(5)$	$5b, 5c, 5d$	$7a$	7

Since $\Theta_G(c, c, c_3) > 0$, the group G is generated by two conjugate regular unipotents.

Now, suppose $q \geqslant 7$ and set $\delta = +1$ if $G = \mathrm{SL}_3(q)$ and $\delta = -1$ if $G = \mathrm{SU}_3(q)$. Also, set $d = \gcd(q - \delta, 3)$. In the notation of [53] (see also [15, 45]), the regular unipotent elements belong to the d classes $C_3^{(0,\ell)}$, where $0 \leqslant \ell < d$. So, let c be any of these classes, and let c_3 be the conjugacy class of $C_8^{(1)}$. Then c_3 consists of elements of order $q^2 + \delta q + 1$. Using the character table of G, one can easily compute that $\Delta_G(c, c, c_3) = \frac{(q^2+\delta q+1)(q^2-d\delta q-1)}{d^2}$. Since $\Delta_G(c, c, c_3) > 0$, there exists $g \in G$ such that the product uu^g has order $q^2 + \delta q + 1$.

Consider the subgroup $H = \langle u, u^g \rangle$. Since H contains an element of prime order p (where $p \mid q$) and an element of order $q^2 + \delta q + 1$, looking at the list of the maximal subgroups of G, we obtain that either $H = G$ or $p = 3$ and H is conjugate to a maximal subgroup $A:3$ in $\mathcal{C}_3$ (a unique class of subgroups), where A is a cyclic group of order $q^2 + \delta q + 1$. Hence, if $p \neq 3$ we are done.

Finally, assume $p = 3$. There are a unique class c of regular unipotent elements and $(q^2 + \delta q)/3$ conjugacy classes of elements of order $q^2 + \delta q + 1$. By Ito's theorem, the irreducible characters of H are linear (three characters) or have degree 3 ($(q^2 + \delta q)/3$ characters). The irreducible characters of degree 3 have 3-defect zero, so they take non-zero values only on A. Hence, it is quite easy to compute the structure constants. Let $\tilde{c}_1, \tilde{c}_2$ be the two conjugacy classes of H consisting of elements of order 3 and let $\tilde{c}_3$ be a conjugacy class consisting of elements of order $q^2 + \delta q + 1$. For $i, j = 1, 2$, we have $\Delta_H(\tilde{c}_i, \tilde{c}_j, \tilde{c}_3) = (q^2 + \delta q + 1)(1 - \delta_{i,j})$. Given $g_3 \in H$ of order $q^2 + \delta q + 1$, we have $h(g_3, H) = 1$ and then $\Theta_G(c, c, c_3) \geqslant (q^2 + \delta q + 1)(q^2 - \delta q - 1) - 2(q^2 + \delta q + 1) = (q^2 + \delta q + 1)(q^2 - \delta q - 3) > 0$. □

Lemma 4.14 *The group* $\mathrm{SU}_3(2)$ *is generated by two regular unipotents.*

Proof The group $G = \mathrm{SU}_3(2)$ admits three conjugacy classes of regular unipotent elements. We can compute with [59] the character table of G and the structure constant $\Delta_G(4a, 4b, 4c) = 10$. Looking at the maximal subgroups of G, only the parabolic subgroups have at least three conjugacy classes of elements of order 4. Hence, we can also easily compute $\Theta_G(4a, 4b, 4c) = 8$. This implies that there exist two regular unipotent elements u, v such that $G = \langle u, v \rangle$. □

Lemma 4.15 *Let* $G = G_2(q)$ *with* $q \in \{3, 4, 5\}$. *For every regular unipotent element* u *of* G, *there exists* $g \in G$ *such that* $G = \langle u, u^g \rangle$.

Proof We can compute with [59] the values of $\Theta_G(c, c, c_3)$ for every conjugacy class c of regular unipotent elements and a suitable class c_3:

G	c	c_3	$\Theta_G(c, c, c_3)$
$G_2(3)$	$9a$	$13a$	7293
$G_2(3)$	$9b, 9c$	$13a$	7410
$G_2(4)$	$8a$	$13a$	245440
$G_2(4)$	$8b$	$13a$	241540
$G_2(5)$	$25a$	$31a$	9373625

As $\Theta_G(c, c, c_3) > 0$, the group G is generated by two conjugate regular unipotents. □

Lemma 4.16 *Let $G = G_2(q)$ with $q \geqslant 7$ odd. For every regular unipotent element u of G, there exists $g \in G$ such that $G = \langle u, u^g \rangle$.*

Proof The maximal subgroups of $G = G_2(q)$ are described in [4]. In particular, as shown in [62], the only proper maximal subgroups of G containing a cyclic torus T of order $t = q^2 - q + 1$ are of type $\mathrm{SU}_3(q).2$.

Firstly we show that, given a unipotent element $u \in G$, there exists $x \in G$ such that uu^x has order t. This can be achieved by computing the structure constants of G with [16]. We follow the notation of [16], distinguishing two cases.

Suppose $q \equiv 0 \pmod 3$. There are three conjugacy classes of regular unipotent elements (labeled by c_7, c_8 and c_9, whose centralizers have order $3q^2$). The class c_{28} (with the parameter `iI=1`) consists of elements of order t. We have

$$\Delta_G(c_j, c_j, c_{28}) = \begin{cases} \frac{1}{9}q^3(q^7 - q^5 - 2q^4 + 8q^3 - 10q^2 + 6q - 1) & \text{if } j = 7, \\ \frac{1}{9}q^3(q^7 - q^5 - 5q^4 + 2q^3 - 7q^2 - 4) & \text{if } j = 8, 9. \end{cases}$$

Next, suppose that q is odd and such that $q \equiv 1, 2 \pmod 3$. There is only one conjugacy class of regular unipotent elements (labeled by c_7 whose centralizer has order q^2). The class c_{31} (with the parameter `iI=1`) consists of elements of order t. We have

$$\Delta_G(c_7, c_7, c_{31}) = \begin{cases} q^3(q^7 - q^5 - 2q + 1) & \text{if } q \equiv 1 \pmod 3, \\ q^5(q^5 - q^3 + q^2 - 1) & \text{if } q \equiv 2 \pmod 3. \end{cases}$$

Since all these values are positive, we obtain that there exists $x \in G$ such that uu^x has order t. This implies that the subgroup $H = \langle u, u^x \rangle$ is contained in a maximal subgroup $X = \mathrm{SU}_3(q).2$.

As q is odd, u lies in X_{u}. By [42, Theorem 1.1], if $q \neq 9$ then X_{u} is completely reducible. As $u \in G \leqslant \Omega_7(q)$ is similar to J_7, X_{u} must be irreducible. However, by [56, Theorem 1.9] X_{u} has no irreducible representation of degree greater than 3 containing J_7. It follows that $H = G$. The same conclusion can be obtained also for $q = 9$ since the group $\mathrm{SU}_3(9)$ does not contain elements of order $|u| = 9$. □

Lemma 4.17 *The group $\mathrm{Sp}_6(9)$ is generated by two conjugate regular unipotents.*

Proof Let $G = \mathrm{Sp}_6(9)$ be defined by the Gram matrix $a = \begin{pmatrix} 0 & \mathrm{Id}_3 \\ -\mathrm{Id}_3 & 0 \end{pmatrix}$. Take $x = \mathrm{diag}(J_3, {}^tJ_3^{-1})$ and

$$y = \begin{pmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&\beta \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix},$$

where $\beta \in \mathbb{F}_9$ is such that $\beta^2 = \beta + 1$. Then x, y are elements of G and $u = xy$ is regular unipotent (a general construction will be described in Lemma 5.14). Let $v = h^{-1}uh$, where $h = (ax)^2a$. Then, $|uv| = 120$, $|uvu| = 164$, $|uvu^2| = 146$ and

$|uv^2u| = 728$. This implies that the order of $H = \langle u, v\rangle$ is divisible by same primes dividing the order of G. By [34, Corollary 5] it follows that $G = H$. □

Lemma 4.18 *The group* $\mathrm{SU}_4(3)$ *is generated by two conjugate regular unipotents.*

Proof Let $G = \mathrm{SU}_4(3)$ be defined by the antidiagonal matrix $a = \mathrm{antidiag}(1, 1, 1, 1)$. Take

$$u = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \beta^2 \\ \beta^2 & \beta^6 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and $x = \mathrm{diag}(\beta^7, \beta^5, \beta, \beta^3)$, where $\beta \in \mathbb{F}_9$ is such that $\beta^2 = \beta + 1$. Then u, x are elements of G with u regular unipotent. Let $v = x^{-1}ux$. Then, $|uv^6| = 28$ and $|u^2v^6| = 36$. Looking at the list of the maximal subgroups of G, it follows that $\langle u, v\rangle = G$. □

Lemma 4.19 *The group* $\mathrm{SU}_5(3)$ *is generated by two conjugate regular unipotents.*

Proof Let $G = \mathrm{SU}_5(3)$ be defined by the antidiagonal matrix $a = \mathrm{antidiag}(1, 1, 1, 1, 1)$. Take

$$u = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$x = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \beta^2 & 0 \\ 1 & 0 & 0 & 0 & \beta^2 \end{pmatrix},$$

where $\beta \in \mathbb{F}_9$ is such that $\beta^2 = \beta + 1$. Then u, x are elements of G with u regular unipotent. Let $v = x^{-1}ux$. Then, $|uv| = 80$, $|uv^3| = 61$ and $|uv^5| = 28$. This implies that the order of $H = \langle u, v\rangle$ is divisible by same primes dividing the order of G. By [34, Corollary 5] it follows that $G = H$. □

We combine the results of this section in a uniform statement as follows.

Proposition 4.20 *The following finite quasisimple groups of Lie type are generated by two conjugate regular unipotents:*

$$\mathrm{SL}_3(q),\ \mathrm{SU}_3(q),\ \mathrm{Sp}_4(q)\ (q \text{ odd}),\ \mathrm{Sz}(2^{2m+1}),\ {}^2G_2(3^{2m+1}),$$
$${}^2F_4(2^{2m+1}),\ {}^3D_4(q),\ G_2(q)\ (q \text{ odd}),\ \mathrm{SU}_4(2),\ \mathrm{SU}_4(3),\ \mathrm{SU}_5(2),$$
$$\mathrm{SU}_5(3),\ \mathrm{Sp}_6(9),\ G_2(4),\ F_4(2),\ {}^2F_4(2)',\ {}^2E_6(2).$$

Theorem 1.3 now follows from Lemma 3.1, Corollary 3.5 and Proposition 4.20.

5 Generation by three regular unipotents

In this section we prove Theorem 1.2, i.e., that every finite quasisimple group of Lie type is generated by three regular unipotent elements. In several cases, we show that these three elements can be chosen to be conjugate, proving Theorem 1.4.

By [24, Lemma 3.1] the groups $\mathrm{SL}_2(q)$, $q > 2$ even, and $\mathrm{SL}_2(9)$ are generated by three conjugate regular unipotents. In addition, we have:

Lemma 5.1 *The group* $\mathrm{SL}_n(q)$*,* $n > 2$*, is generated by three conjugate regular unipotents.*

Proof The group $\mathrm{SL}_3(q)$ is generated by two regular unipotents by Lemma 4.13. Suppose that $n > 3$. Let P be the stabilizer of a line and $P_1 \leqslant P$ the stabilizer of a vector. Then $P_1 = O_p(P)\cdot L_1$, where $L_1 \cong \mathrm{SL}_{n-1}(q)$. Then L_1 is generated by two of its regular unipotents. As $P = O_p(P)\cdot L$, where $L \cong \mathrm{GL}_{n-1}(q)$, these are conjugate in L, and hence P_1 is generated by two conjugate regular unipotents u, u' of G. Let $v \notin P$ be a conjugate of u. Then $\langle v, u, u'\rangle = G$ by Lemma 2.4. □

Lemma 5.2 *Let* $G = \mathrm{GL}_n(q)$ *and* $S = \mathrm{SL}_n(q)$*,* $n > 1$*. Then every irreducible* $\mathbb{F}_q G$*-module is irreducible on* S*.*

Proof Suppose the contrary. Then, by Clifford's theorem, $V = V_1 + \cdots + V_k$, where $V_1, \ldots, V_k$ are irreducible $\mathbb{F}_q S$-modules, and $V_i = V_1^{g_i}$ for some $g_i \in G$. As $\mathbb{F}_q$ is a splitting field for G (see [28, Proposition 5.4.4] or [14, Lemma 8.5]), the modules $V_1, \ldots, V_k$ are absolutely irreducible. By Steinberg's theorem, V_1 extends to $\mathrm{SL}_n(\overline{\mathbb{F}}_q)$, and we can assume that $V_i = V_1^{g_i}$ as $\mathrm{SL}_n(\overline{\mathbb{F}}_q)$-modules. These are still non-equivalent to each other. As $\mathrm{GL}_n(\overline{\mathbb{F}}_q) = Z\cdot\mathrm{SL}_n(\overline{\mathbb{F}}_q)$, where Z the group of non-zero scalar matrices in G, we have $g_i \in Z\cdot\mathrm{SL}_n(\overline{\mathbb{F}}_q)$, and hence the conjugation by g_i yields an inner automorphism of $\mathrm{SL}_n(\overline{\mathbb{F}}_q)$. Then V_1, V_i are equivalent $\mathrm{SL}_n(\overline{\mathbb{F}}_q)$-modules. This is a contradiction. □

Lemma 5.3 ([24, Lemma 2.6]) *Let* G *be a finite group and* $\phi\colon G \to \mathrm{GL}_m(q)$ *be a non-trivial absolutely irreducible representation of* G*. Suppose that* q *is minimal in the sense that* ϕ *is not equivalent to a representation into* $\mathrm{GL}_m(q_0)$ *for any* $q_0 \mid q$*. Let* V *be the underlying* $\mathbb{F}_q G$*-module. Then* V *is irreducible as* $\mathbb{F}_p G$*-module (where* $p \mid q$ *is a prime).*

Lemma 5.4 *Let* $u \in \mathrm{GL}_n(F) = \mathrm{GL}(V)$ *be a unipotent element. Let* $V_1 \subset V_2$ *be* u*-stable subspaces of* V*. Suppose that* $[u, V_2] \subset V_1$ *and* $\dim V_2/V_1 = d$*. Then the Jordan form of* u *has at least* d *Jordan blocks, equivalently,* $\dim V/[V, u] = \dim C_V(u) \geqslant d$*.*

The following lemma is obvious.

Lemma 5.5 *Let* G *be a quasisimple group of Lie type,* P *a parabolic subgroup of* G*, and* U *the unipotent radical of* P*. Let* $V := Q_1 < Q_2 \leqslant U$ *be a composition factor of* P *on* U*. Then* V *is an elementary abelian* p*-group.*

In the notation of previous lemma, assume $G \neq {}^2B_2(q^2), {}^2G_2(q^2), {}^2F_4(q^2)$. Since V is an elementary abelian p-group, V is an irreducible $\mathbb{F}_p L$-module for every Levi subgroup L of P. The fact going back to [1] is that V is completely reducible, in fact, an irreducible $\mathbb{F}_q L$-module in the cases arising below.

Proposition 5.6 ([1, Theorem 1]) *Suppose that G is a split Chevalley group over a field $\mathbb{K}$, except those arising from a simple algebraic group of type B_n, C_n, F_4, G_2 in characteristic 2 and G_2 in characteristic 3. Let $P = QL$ be a parabolic subgroup of G with U the unipotent radical of P and L a Levi factor.*

(1) ([1, Theorem 1]) *The quotients of subsequent terms of the central series of U have the structure of completely reducible $\mathbb{K}L$-modules with the simple constituents M being highest weight modules. The highest weights are independent of $\mathbb{K}$ and each simple module is a chief factor of P.*
(2) ([1, Theorem 2(c)]) *Let L_{u} be the subgroup of L generated by the unipotent elements of L. If L_{u} acts on M non-trivially, then M is irreducible as an $\mathbb{F}_q L_{\mathrm{u}}$-module.*
(3) ([1, Lemma 7(b)]) *If G is non-twisted and L_{u} acts on M non-trivially, then M is irreducible as an $\mathbb{F}_p L_{\mathrm{u}}$-module.*

In the following lemma we exclude the groups $\mathrm{SU}_4(3)$ and $\mathrm{SU}_5(3)$ as these are generated by two conjugate regular unipotents by Lemma 4.18 and 4.19.

Lemma 5.7 *Let $G = \mathrm{SU}_n(q)$, where either $n > 5$, or $n = 4, 5$ and $q \neq 3$ is odd. Then G is generated by three regular unipotents. If n is odd then these can be chosen conjugate to each other.*

Proof Let V be the underlying $\mathbb{F}_{q^2}G$-module, W be a maximal totally isotropic subspace of V and $m = \dim W$. Let P be the stabilizer of W in G and U be the unipotent radical of P. Then $P = UL$, where L is a Levi subgroup of P. Then $L_{\mathrm{u}} \cong \mathrm{SL}_m(q^2)$. Let $g \in P$ be a regular unipotent and $X = \langle L_{\mathrm{u}}, g\rangle$. By Lemma 2.1, $g \notin L$, and hence $U_1 = X \cap U \neq 1$.

Suppose first that n is even. Then $m = n/2$ and U is an elementary abelian subgroup. By [13, Lemma 4.6(1)], U is irreducible as an $\mathbb{F}_p L_{\mathrm{u}}$-module. This means that U has no proper non-trivial L_{u}-invariant subgroup. As U_1 is L_{u}-invariant, we have $U_1 = U$ and $X = UL_{\mathrm{u}}$. Lemma 3.1 implies that $X = \langle g, h\rangle$ for a regular unipotent element $h \in P$, so X is generated by two regular unipotents.

Next, let y be a conjugate of g that is not in P, and let $Y = \langle g, h, y\rangle$. Then $UL_{\mathrm{u}} < Y$ and hence Y contains a Sylow p-subgroup of G for p dividing q. By Lemma 2.4, $Y = G$.

Suppose now that $n = 2m + 1$ is odd. Then $m = (n-1)/2$. Let $R = C_P(W)$. Choose a basis $\{b_1, \dots, b_n\}$ of V such that $(b_{m+1+j}, b_i) = \delta_{i,j}$ for $1 \leqslant i, j \leqslant m$, $(b_{m+1}, b_{m+1}) = 1$ and $(b_{m+1}, b_i) = (b_{m+1}, b_{m+1+i}) = 0$. Then the elements of R are of shape $\begin{pmatrix} \mathrm{Id} & {}^tA & B \\ 0 & 1 & -\gamma(A) \\ 0 & 0 & \mathrm{Id}\end{pmatrix}$, where γ is the Galois automorphism of $\mathbb{F}_{q^2}/\mathbb{F}_q$, B is an $(m \times m)$-matrix and A is a $(1 \times m)$-matrix. So, R is a unipotent normal subgroup of P, in fact, $R = U$. Let Q be a subgroup of U consisting of the above matrices with $A = 0$. Then the action of P on U by conjugation leaves Q invariant and U/Q is an $\mathbb{F}_{q^2}L$-module. Note that $L = \mathrm{diag}(h^{-1}, \det(h\,{}^t\gamma(h)^{-1}), {}^t\gamma(h))$, where h ranges over $\mathrm{GL}_m(q^2)$. Then $L \cong \mathrm{GL}_m(q^2)$ and U/Q can be viewed as an $\mathbb{F}_{q^2}\mathrm{GL}_m(q^2)$-module with natural action of $\mathrm{SL}_m(q^2)$. The action of $L \cong \mathrm{GL}_m(q^2)$ on Q is as above.

Observe that U is not abelian. If $U_1 \cap Q = 1$ then the projection $\overline{U}_1$ of U_1 into Q is non-trivial. As U/Q is irreducible as an $\mathbb{F}_p\mathrm{SL}_m(q^2)$-module, we conclude that $\overline{U}_1 = U/Q$. As $[U, U] \neq 1$, we conclude that $U_1 \cap Q \neq 1$. As above, this implies $Q \leqslant U_1$.

We wish to show that $U_1 \cap Q \neq Q$. For this it suffices to observe that $g \notin LQ$. With respect to the above basis, Q consists of matrices $\begin{pmatrix} \mathrm{Id} & 0 & B \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix}$ and $L = \mathrm{diag}(h^{-1}, \det(h\ {}^t\gamma(h)^{-1}), {}^t\gamma(h))$. It follows that $Qb_{m+1} \subseteq \langle b_{m+1}\rangle$ and $Lb_{m+1} = b_{m+1}$. Therefore, $\langle b_{m+1}\rangle$ is stable under LQ. In addition, if $x \in LQ$ is unipotent then $LQW = W$ implies that x stabilizes a line of W. Therefore, $g \in LQ$ implies that g stabilizes two distinct lines on V, which is false as g is similar to J_n.

In addition, if n is odd then $L \cong \mathrm{GL}_{(n-1)/2}(q^2)$. Indeed, in this case $V = W + W_1 + W'$, where W, W' are totally isotropic of dimension $(n-1)/2$ and $W_1 = (W + W')^{\perp}$. Then L can be chosen to be the stabilizer of W and W' in G. There is a basis with Gram matrix $\begin{pmatrix} 0 & 0 & \mathrm{Id}_m \\ 0 & 1 & 0 \\ \mathrm{Id}_m & 0 & 0 \end{pmatrix}$ with $m = (n-1)/2$, such that L consists of all matrices $\mathrm{diag}(M^{-1}, \det(M\ {}^t\gamma(M)^{-1}), {}^t\gamma(M))$, where $M \in \mathrm{GL}_m(q^2)$. It follows that the regular unipotent elements of $P/O_p(P) \cong \mathrm{GL}_m(q^2)$ are conjugate. So the above reasoning shows that we can choose the elements in question to be conjugate in G. □

The previous argument does not work for q even and $n = 4, 5$. In Lemmas 5.8 and 5.9 below we assume $q > 2$, as for $q = 2$ we have a stronger conclusion, see Lemmas 4.3 and 4.10.

Lemma 5.8 *Let $G = \mathrm{SU}_5(q) = \mathrm{SU}(V)$, $q > 2$ even. Then G is generated by three conjugate regular unipotents.*

Proof Choose a basis $\{b_1, \ldots, b_5\}$ of V such that the associate Gram matrix is the antidiagonal matrix $\mathrm{antidiag}(1, 1, 1, 1, 1)$. Let $W = \langle b_1\rangle$ be a totally isotropic subspace of V and P the stabilizer of W in G. Then P is a parabolic with Levi L, and $L_{\mathsf{u}} \cong \mathrm{SU}_3(q)$. Let $U = O_p(P)$. Then U is of extraspecial type and the only L_{u}-invariant subgroup of U is $Q = Z(U)$. By Lemma 4.13, L_{u} is generated by two conjugate regular unipotents. It follows that $L_{\mathsf{u}}U$ is generated by two conjugate regular unipotents.

Indeed, let X be a subgroup of P generated by two conjugate regular unipotents such that $X/(X \cap U) = L_{\mathsf{u}}$. We know that $U_1 := X \cap U \neq 1$. Then $U_1 \cap Q \neq 1$ as U/Q is an irreducible $\mathbb{F}_p L_{\mathsf{u}}$-module and $[U, U] = Q$. Then $Q < U_1$.

Then the upper unitriangular matrices of P forms a Sylow 2-subgroup S of G and P. In addition, L_{u} consists of matrices of the form $\mathrm{diag}(1, D, 1)$, with $D \in \mathrm{SU}_3(q)$, and Q consists of matrices $\mathrm{Id} + x$, where x is a (5×5)-matrix with $(1, 5)$-entry in $\mathbb{F}_q$ and 0 elsewhere. Then $P_{\mathsf{u}} = L_{\mathsf{u}}Q$ consist of matrices of the form $\begin{pmatrix} 1 & 0 & b \\ 0 & D & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with $b \in \mathbb{F}_q$; in particular the $(1, 2)$-entry of a matrix in $L_{\mathsf{u}}Q$ equals 0. In contrast, the $(1, 2)$-entry of any regular unipotent in S is non-zero. This is a contradiction.

It follows that that $X = P_{\mathsf{u}}$, so P_{u} is generated by two conjugate regular unipotents u, v, say. There is a conjugate u' of u that is not in P. Then $\langle X, u'\rangle = \langle u, v, u'\rangle = G$ by Lemma 2.4. □

Lemma 5.9 *Let $G = \mathrm{SU}_4(q) = \mathrm{SU}(V)$, $q > 2$ even. Then G is generated by three conjugate regular unipotents.*

Proof Choose a basis $\mathcal{B} = \{b_1, b_2, b_3, b_4\}$ of V with Gram matrix Id_4. Let $h = \mathrm{diag}(a, a, a, a^{-3}) \in G = \mathrm{SU}_4(q)$, where a is a primitive $(q+1)$-root of unity and let $x, y \in G$ be permutational matrices that act on the basis $\mathcal{B}$, respectively, as the 4-cycle (b_1, b_2, b_3, b_4) and the 3-cycle (b_2, b_3, b_4). Then $x^4 = \mathrm{Id}_4 \neq x^2$, so x is similar to J_4 and hence x is a regular unipotent.

Let $x_1 = (yh)x(yh)^{-1}$. Set $H = \langle x, x_1\rangle$, so H is generated by two conjugate regular unipotents. Since $x^2((xx_1)^3)x^{-2} = h^{-4}$ and $\gcd(q+1, 4) = 1$, we observe that H contains h and also $x^i h x^{-i}$ for $i = 2, 3, 4$. So $H > D$, where D is the group of diagonal matrices in G under the basis $\mathcal{B}$. Note that $|D| = (q+1)^3$.

In particular, H contains the element $d = \mathrm{diag}(a^{-8}, a^4, 1, a^4) \in D$ and then it contains $d(x_1x)^2 = y$. Note that $\langle x, y\rangle \cong S_4$. It follows that $H/D \cong S_4$ and $H = N_G(D)$. By [4, Table 8.10], H is a maximal subgroup of G. Therefore, G is generated by x, x_1 and any conjugate x_2 of x such that $x_2 \notin H$. □

Lemma 5.10 *Let $G = \Omega^+_{2n}(q) = \Omega(V)$, $n > 3$, and let P be the stabilizer of a totally singular subspace of dimension n. Then P_{u} is generated by two regular unipotents and G is generated by three regular unipotents. If q is even or n is odd then these can be chosen conjugate.*

Proof Let U be the unipotent radical of P, and L a Levi. Then L is isomorphic to a subgroup of index $\gcd(2, q-1)$ of $\mathrm{GL}_n(q)$. It follows that the regular unipotent elements of L partition in at most 2 conjugacy classes; if q is even then they are conjugate in L. In n is odd then

$$|\mathrm{GL}_n(q) : Z(\mathrm{GL}_n(q)) \cdot \mathrm{SL}_n(q)| = \gcd(n, q-1)$$

is odd. Therefore, if we view L as a subgroup of $\mathrm{GL}_n(q)$, we observe that $|\mathrm{GL}_n(q) : L \cdot Z(\mathrm{GL}_n(q))|$ is odd, whence $\mathrm{GL}_n(q) = L \cdot Z(\mathrm{GL}_n(q))$. So the regular unipotent elements of L are conjugate in L.

Under a certain basis of V we can write $U = \begin{pmatrix} \mathrm{Id}_n & M \\ 0 & \mathrm{Id}_n \end{pmatrix}$, where M is the set of skew-symmetric matrices if q odd, and symmetric matrices with zero diagonal if q is even. Note that $P_{\mathsf{u}} = L_{\mathsf{u}}U$ and $L_{\mathsf{u}} \cong \mathrm{SL}_n(q)$. We know that M is an irreducible $\mathbb{F}_p L_{\mathsf{u}}$-module (e.g., see [13, Lemma 4.6]). As in Lemma 5.7, we observe that there are regular unipotent elements $g, h \in P_{\mathsf{u}}$ that generate a subgroup X such that $X/(X \cap U) \cong L_{\mathsf{u}}$. In addition, $X \cap U \neq 1$. Therefore, $U < X$, and hence $X = P_{\mathsf{u}}$. Now, choose a conjugate g' of g with $g' \notin P$. As P_{u} contains a Sylow p-subgroup of G, we have $\langle P_{\mathsf{u}}, g'\rangle = G$ by Lemma 2.4. □

Lemma 5.11 *Let $G = \Omega_{2n+1}(q)$ with $n \geqslant 3$ and q odd, or $\Omega^-_{2n+2}(q)$ with $n \geqslant 3$. Let V be the underlying space of G and let P be the stabilizer of a maximal totally singular subspace. Then P_{u} is generated by two conjugate regular unipotents and G is generated by three conjugate regular unipotents.*

Proof Let W be a maximal totally singular subspace of V. Then $\dim W = n$ and $V = W + V_1 + W'$, where V_1 is complement of W in $W^\perp$ and W' is a complement of $W^\perp$ in V and is totally singular. Let U be the unipotent radical of P. Under a certain basis of V the group $Z(U)$ has form $\begin{pmatrix} \mathrm{Id}_n & 0 & M \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id}_n \end{pmatrix}$, where M is the set of skew-symmetric matrices if q odd, and symmetric matrices with zero diagonal if q is even (see, for instance, [13, Lemma 4.14]). Let $L = \{h \in P : hV_1 = V_1, hW' = W'\}$ be the stabilizer of the above decomposition. Then L consists of matrices $\mathrm{diag}(x, y, {}^t x^{-1})$, where $x \in \mathrm{GL}_n(q)$ and $y \in \mathrm{SO}(V_1)$, see [13, Lemma 4.8] for details. As $\dim V_1 \leqslant 2$ and $\mathrm{SO}(V_1)$ is abelian, we have $L_{\mathsf{u}} \cong \mathrm{SL}_n(q)$.

Note that $Z(U)$ is an elementary abelian p-group for a prime p dividing q, so this can be seen as an $\mathbb{F}_p L_{\mathsf{u}}$-module. By [13, Lemma 4.6], this is irreducible and hence has no proper L_{u}-invariant subgroup. As in Lemma 5.7, we observe that there are two regular unipotent elements $g, h \in P_{\mathsf{u}}$ that generate a subgroup X such that $X/(X \cap U) \cong L_{\mathsf{u}}$. If $X \cap Z(U) \neq 1$ then $Z(U) < X$. Moreover, here we can assume that g, h are conjugate in G. Indeed, as $\overline{g} := g \pmod U$ is contained in a subgroup isomorphic to $\mathrm{GL}_n(q)$ and $\overline{g}$ is a regular unipotent in $\mathrm{GL}_n(q)$, where regular unipotent elements are conjugate, we can choose h to be a conjugate of g in G.

Note that U is non-abelian and $U' = Z(U)$, see [13, Lemma 4.18]. This implies $X \cap Z(U) \neq 1$. Indeed, otherwise $1 \neq X \cap U \not\leqslant Z(U)$, so $(X \cap U)' \neq 1$, a contradiction.

So, we are left with ruling out the case where $X \cap U = Z(U)$. Suppose that $X = L_{\mathsf{u}} Z(U)$. The matrices of X are

$$\begin{pmatrix} A & 0 & 0 \\ 0 & \mathrm{Id}_k & 0 \\ 0 & 0 & {}^t A^{-1} \end{pmatrix} \cdot \begin{pmatrix} \mathrm{Id}_n & 0 & M \\ 0 & \mathrm{Id}_k & 0 \\ 0 & 0 & \mathrm{Id}_n \end{pmatrix} = \begin{pmatrix} A & 0 & AM \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & {}^t A^{-1} \end{pmatrix},$$

where $A \in \mathrm{GL}_n(q)$ and $k = 1$ if $\dim V$ is odd and $k = 2$ otherwise. If $y \in L_{\mathsf{u}} Z(U)$ is unipotent then y fixes the vectors $v \in V_1$ and some vector in W. So, the fixed point space V^y of y has dimension $\dim V^y \geqslant k + 1$.

(a) If $k = 1$ then $G = \Omega_{2n+1}(q)$, q odd, and the Jordan form of a regular unipotent of G is J_{2n+1} so $\dim V^y = 1$. This is a contradiction.

(b) If $k = 2$ then $G = \Omega^-_{2n+2}(q)$ and the Jordan form of a regular unipotent of G is $\mathrm{diag}(1, J_{2n+1})$ if q is odd and $\mathrm{diag}(J_2, J_{2n})$ if q is even (see [36, Proposition 3.5] and [17, Theorem 3.1]). In both cases, we have $\dim V^y = 2$. So, this is again a contradiction.

It follows that P_{u} is generated by two conjugate regular unipotents u, v, say. Now, choose a conjugate u' of u with $u' \notin P$. As P_{u} contains a Sylow p-subgroup of G, we have $\langle u, v, u' \rangle = G$ by Lemma 2.4. □

An element $g \in \mathrm{Sp}_{2n}(q)$ is called orthogonally indecomposable if g preserves no non-generate proper subspace of the ground symplectic space.

Lemma 5.12 *Let $G = \mathrm{Sp}_{2n}(q)$, q even, and let $g \in G$ be a 2-element. Suppose that g is orthogonally indecomposable. Then* $\mathrm{Jord}(g) = J_{2n}$ *or* $\mathrm{diag}(J_n, J_n)$. *In addition, if $|g| = 2^k$ then 2^k is the minimal 2-power such that $2n \leqslant 2^k < 4n$ or $n \leqslant 2^k < 2n$, respectively.*

Proof This is well known, see for instance [36, Section 5.1 or Theorem 7.3]. Note that there are several conjugacy classes of each type of orthogonally indecomposable elements in G. The additional claim is straightforward, or see [10, Lemma 4.6]. □

Lemma 5.13 ([63, Lemma 3.12]) *Let $G = O^+_{2n}(2) = SO(W)$, $n > 2$. Suppose that $g \in G$ is indecomposable on W. Then G is generated by three conjugates of g.*

Note that J^2_{2n} is not a regular unipotent in $O^+_{2n}(2)$.

Lemma 5.14 *Let $G = \mathrm{Sp}_{2n}(q) = \mathrm{Sp}(W)$, where $n > 2$, and let P be a parabolic with Levi $\mathrm{GL}_n(q)$. Then P contains two regular unipotent elements u, v conjugate in P such that $\langle u, v\rangle = P_{\mathsf{u}}$.*

Proof Let G be defined by the Gram matrix $\begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}$. Let $b_1, \ldots, b_{2n}$ be the basis in which this matrix is written. Let $W_1 = \langle b_1, \ldots, b_n\rangle$ and $W_2 = \langle b_{n+1}, \ldots, b_{2n}\rangle$. Then W_1, W_2 are totally isotropic subspaces. Let P be the stabilizer of W_1 in G. Then P is a maximal parabolic subgroup. Let U be the unipotent radical of P. Then $U = \begin{pmatrix} \mathrm{Id}_n & M \\ 0 & \mathrm{Id}_n \end{pmatrix}$, where M is the set of symmetric matrices. The subgroup U has a unique proper normal subgroup Q of P, which has the same form $Q = \begin{pmatrix} \mathrm{Id}_n & \widetilde{M} \\ 0 & \mathrm{Id}_n \end{pmatrix}$, where $\widetilde{M}$ is the set of symmetric matrices with zero diagonal (see for instance [13, Lemma 4.6]). The set of the matrices $\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}$, $A \in \mathrm{GL}_n(q)$, is a Levi of P. Let $x = \begin{pmatrix} J_n & 0 \\ 0 & {}^tJ_n^{-1} \end{pmatrix}$, where J_n is the upper triangular Jordan block. Then $x - \mathrm{Id}_n$ is nilpotent and $(x - \mathrm{Id}_n)^i b_n = b_{n-i}$, $(x - \mathrm{Id}_n)^i b_{n+1} \in \langle b_{n+1+i}, \ldots, b_{2n}\rangle$ for all $i = 1, \ldots, n-1$. Let $y = \begin{pmatrix} \mathrm{Id}_n & C \\ 0 & \mathrm{Id}_n \end{pmatrix}$, where C is an $(n \times n)$-matrix with 1 at the (n, n)-position and 0 elsewhere (so $y = \mathrm{Id}_{2n} + C_1$ where C_1 has 1 at the $(n, 2n)$-position and 0 elsewhere). In particular, $yb_i = b_i$ for $i = 1, \ldots, 2n-1$ and $yb_{2n} = b_{2n} + b_n$. Set $u = xy$.

We claim that u is regular unipotent. Let $w \in W$ and $uw = w = \sum a_i b_i$ ($a_i \in \mathbb{F}_q$). Let $\bar{w}$ the projection of w to W/W_1. As u acts on W/W_1 as ${}^t(J_n^{-1})$, it follows that the fixed point subspace of u on W/W_1 is $\mathbb{F}_q b_{2n}$. So, $a_{n+1} = \cdots = a_{2n-1} = 0$. As $ub_{2n} = x(b_{2n} + b_n) = b_{2n} + b_n + b_{n-1}$, we conclude that $a_{2n} = 0$, and hence $w \in W_1$. As u acts on W/W_1 as J_n, it follows that $a_2 = \cdots = a_n = 0$. So, the fixed point space of u on W is one-dimensional, and the claim follows.

As $P/U \cong \mathrm{GL}_n(q)$, the regular unipotents are conjugate in P/U. So, by Lemma 3.1, P_{u}/U is generated by two conjugate regular unipotents. Therefore, $L_{\mathsf{u}} < \langle u, v\rangle$ for some P-conjugate v of u.

We claim that $\langle L_{\mathsf{u}}, u\rangle = P_{\mathsf{u}}$. Suppose first that q is odd. Then M is an irreducible $\mathbb{F}_q L$-module (see, for instance, [13, Lemma 4.6]), which remains irreducible over L_{u} by Lemma 5.3. Moreover, by Lemma 5.2, M is irreducible as an $\mathbb{F}_p L_{\mathsf{u}}$-module, and hence U has no proper non-trivial L_{u}-invariant subgroup. So $U < \langle L_{\mathsf{u}}, u\rangle$, whence the result.

Suppose that q is even. As $x \in L_{\mathsf{u}}$, we have $y \in U \cap \langle L_{\mathsf{u}}, u\rangle$, where $y \notin Q$ by the choice of y. So the projection of y into U/Q is non-trivial. As $U/Q \cong M/\widetilde{M}$ is an irreducible $\mathbb{F}_q L$-module, U/Q is irreducible also as an $\mathbb{F}_q L_{\mathsf{u}}$-module (Lemma 5.3) and as $\mathbb{F}_p L_{\mathsf{u}}$-module (Lemma 5.2). So, we conclude that $U/(U \cap \langle L, u\rangle) = U/Q$. It follows that $Q \cap \langle L, u\rangle \neq 1$, since M is an indecomposable $\mathbb{F}_q L$-module [13, Lemma 4.6]. Again, $\widetilde{M}$ is irreducible, so $Q \cap \langle L, u\rangle = Q$. This implies the result. □

Lemma 5.15 *The group* $\mathrm{Sp}_{2n}(q)$*,* $n > 2$*, is generated by three conjugate regular unipotent elements.*

Proof Let G, P, u, v be as in Lemma 5.14. Then P_{u} is generated by two conjugate regular unipotents. Let u' be a conjugate of u such that $u' \notin P$. By Lemma 2.4, we have $\langle u', P_{\mathrm{u}}\rangle = G$. □

Lemma 5.16 *Let* $G = \mathrm{Sp}_4(q)$*,* q *even. Then* G *is generated by three conjugate regular unipotents.*

Proof If $q = 2$, the result follows from Lemma 2.5. So, we may assume $q > 2$. Let $x \in G$ be of order $q^2 + 1$. Note that $N_G(\langle x\rangle)$ contains an element, a say, of order 4 (see, for instance, [4, Table 8.14], where a group $C_{q^2+1} : 4$ occurs, we do not need to decide on its maximality). The group $\langle x, a\rangle$ is of order $4(q^2+1)$ and $\langle x, a^2\rangle$ is dihedral of order $2|x|$, where $|x| = q^2 + 1$ is odd. It follows that $\langle x, a\rangle$ contains a conjugate b of a such that $\langle a, b\rangle = \langle x, a\rangle$; in particular, $\langle x, a\rangle$ is generated by two conjugate elements of order 4. Observe that the subgroups of order $q^2 + 1$ are conjugate in G.

By Lemma 2.5, the group $\mathrm{Sp}_4(2)$ can be generated by two conjugate elements with Jordan form J_4. We conclude that there are two conjugates b, c of a such that the group $X = \langle a, b, c\rangle$ contains an elements x of order $q^2 + 1$ and a subgroup $\mathrm{Sp}_4(2)$. By inspection of the maximal subgroups of G, listed in [4], the only maximal subgroups containing elements of order $q^2 + 1$ are $\mathrm{Sp}_2(q^2) : 2$ and $O_4^-(q)$. However, these subgroups do not contain $\mathrm{Sp}_4(2)$: in fact, this would imply the existence of a projective representation of $\mathrm{Sp}_4(2)' \cong \mathrm{Alt}(6)$ of degree 2 over a field of characteristic 2, in contradiction with [28, Proposition 5.3.7]. □

Lemma 5.17 *If* $q > 4$ *is even, then the group* $G_2(q)$ *is generated by three conjugate regular unipotents.*

Proof As recalled in the proof of Lemma 4.16, the only proper maximal subgroups of G containing a cyclic torus T of order $t = q^2 - q + 1$ are of type $\mathrm{SU}_3(q).2$. So it suffices to show that, given a unipotent element $u \in G$, there exist $x, y \in G$ such that uu^x has order t and uu^y has order $t_2 = q^2 + q + 1$. This can be achieved by computing the structure constants of G with [16].

There are two conjugacy classes of regular unipotent elements (labeled by c_7 and c_8, whose centralizers have order $2q^2$). The classes c_{26} and c_{27} (with the parameter `iI`=1) consist of elements of respective order t_2 and t. For all $j = 7, 8$, we have

$$\Delta_G(c_j, c_j, c_{26}) = \begin{cases} \frac{1}{4}q^3(q^7 - q^5 + 5q^3 + 7q^2 + 5q + 1) & \text{if } q \equiv 1 \pmod 3, \\ \frac{1}{4}q^4(q^6 - q^4 + q^3 + 5q^2 + 6q + 3) & \text{if } q \equiv 2 \pmod 3 \end{cases}$$

and

$$\Delta_G(c_j, c_j, c_{27}) = \begin{cases} \frac{1}{4}q^4(q^6 - q^4 - q^3 + 5q^2 - 6q + 3) & \text{if } q \equiv 1 \pmod 3, \\ \frac{1}{4}q^3(q^7 - q^5 + 5q^3 - 7q^2 + 5q - 1) & \text{if } q \equiv 2 \pmod 3. \end{cases}$$

Since all these values are positive, we are done. □

Lemma 5.18 *The group $F_4(q)$, q odd, is generated by three conjugate regular unipotents.*

Proof Let $P = P_2$ be a parabolic subgroup of $G = F_4(q)$ generated by the root subgroups $x_\alpha(t)$ with $\alpha \in \{-\alpha_1, -\alpha_3, -\alpha_4, \beta\}$, where β ranges over all positive roots and $t \in \mathbb{F}_q$. Set $L_{\mathsf{u}} = \langle x_\alpha(t) : \alpha \in \{\pm\alpha_1, \pm\alpha_3, \pm\alpha_4\}\rangle$. Then L_{u} is a subgroup of a Levi subgroup of P, in fact, $L_{\mathsf{u}} \cong \mathrm{SL}_2(q) \times \mathrm{SL}_3(q)$, see [8, Table 8]. Set $u = x_{\alpha_1}(1)x_{\alpha_2}(1)x_{\alpha_3}(1)x_{\alpha_4}(1)$. Then $u \in P$ is a regular unipotent, see [6, Proposition 5.1.3].

The information on the unipotent radicals of maximal parabolics of $F_4(q)$ for q odd is available in [30, Table 2]. In this case, $U = O_p(P)$ is described as follows. The lower central series of U has three terms: $1 < Q_1 < Q_2 < U$, and these are unique L_{u}-invariant subgroups of U. In addition, the consecutive factors are of size $|Q_1| = q^2$, $|Q_2/Q_1| = q^6$ and $|U/Q_2| = q^{12}$.

Let $X = \langle L_{\mathsf{u}}, u\rangle$. By [5, Corollary 5.2.3] we have $[u, x_{\alpha_1}(1)] = x_{\alpha_1+\alpha_2}(t).y$, where $y \in Y$, the group generated by x_β with $\beta \succ \alpha_1 + \alpha_2$, as $[x_{\alpha_3}(1), x_{\alpha_1}(1)] = [x_{\alpha_4}(1), x_{\alpha_1}(1)] = 1$. It follows that $[u, x_{\alpha_1}(1)] \in U \setminus Q_2$. Therefore, $(X \cap U)/(X \cap Q_2) \neq 1$ is a non-trivial normal subgroup of $L_{\mathsf{u}}U/Q_2$. This implies $(X \cap U)/(X \cap Q_2) = U/Q_2$. (Note that $\alpha_1 + \alpha_2$ is a weight for L_{u} on U/Q_2, see [30, Table 2], strictly speaking, this is the weight of the corresponding module for algebraic group of type $A_1 \times A_2$).

Furthermore, U/Q_2 has weights α_2 and $\alpha_1+\alpha_2+\alpha_3$ [30, Table 2] and hence there are elements $y_1, y_2 \in U$ whose projection into U/Q_2 are $x_{\alpha_2}(1)$ and $x_{\alpha_1+\alpha_2+2\alpha_3}(1)$. Then $[y_1, y_2] \in Q_2$, and the projection of $[y_1, y_2]$ into Q_2/Q_1 is $x_{\alpha_1+2\alpha_2+2\alpha_3}(t)$ for $t \in \mathbb{F}_q$. As above it follows that $Q_2/Q_1 < X/Q_1$.

In addition, there is $y_3 \in Q_2$ whose projection into Q_2/Q_1 is $x_{\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4}(1)$ [30, Table 2]. Then $[y_2, y_3] = x_{2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4}(b) \in Q_1$. It follows that $Q_1 < X$.

We conclude that $U < X$, and hence $X = L_{\mathsf{u}}U$. As L_{u} is generated by two regular unipotents that can be chosen conjugate if $q \neq 9$ (see Remark 3.3 and Lemma 4.13), it follows that so is $L_{\mathsf{u}}U$. Choose a conjugate u' of u such that $u' \notin P$. Then $\langle u', X\rangle$ contains a Sylow p-subgroup of G for p dividing q. Using Lemma 2.4 we conclude that $\langle u', X\rangle = G$.

Let $q = 9$. We choose for P a maximal parabolic subgroup whose Levi satisfies $L_{\mathsf{u}} \cong \mathrm{Sp}_6(9)$. By Lemma 4.17, L_{u} is generated by 2 conjugate regular unipotents. By [9, Proposition 4.5], $U = O_3(P)$ is a group of extraspecial type of order q^{15}, $Z(U) = U'$, and L_{u} acts irreducibly on $U/Z(U)$ and trivially on $Z(U)$. In fact, $U/Z(U)$ is an irreducible $\mathbb{F}_3L_{\mathsf{u}}$-module, so $Z(U)$ is a maximal proper normal subgroup of $L_{\mathsf{u}}U$ contained in U. Let $U_1 = \langle u, v\rangle$, where u, v are conjugate regular unipotents whose projections into P/U generate L_{u}. As in the proof of Lemma 5.21 we observe that $U_1 = U$. Let v' be a conjugate of v such that $v' \notin P$. Then $\langle u, v, v'\rangle = G$ by Lemma 2.4. □

Note that the above method does not work for the case with $G = F_4(q)$, q even.

Lemma 5.19 ([9, Propositions 4.4 and 4.6]) *Let $G = E_6(q), {}^2E_6(q)$ and let Q be the root subgroup of G corresponding to the maximal root. Let $P = N_G(Q)$ and $L = P/U$, where U is the unipotent radical of P. Then $|U| = q^{21}$ and*

$L \cong \mathrm{SL}_6(q)$, $\mathrm{SU}_6(q)$, respectively. Moreover, the conjugation action of P on U defines on U/Q a structure of an $\mathbb{F}_q L$-module isomorphic to the third exterior power of the natural $\mathrm{SL}_6(q)$-, $\mathrm{SU}_6(q)$-module, respectively, and the restriction of U/Q to L is an irreducible module of dimension 20.

Lemma 5.20 *Let $G = E_n(q)$, $n = 7, 8$, and let P be the maximal parabolic subgroup of G corresponding to the maximal root α_2. Let U be the unipotent radical of P and L a Levi. Then $L_{\mathsf{u}} \cong \mathrm{SL}_n(q)$. The P-composition series of U is unique of length 2, 3 for $n = 7, 8$, respectively, the factors are elementary abelian groups. The conjugation action of P on U turns the factors to irreducible $\mathbb{F}_q L_{\mathsf{u}}$-modules of dimensions 7, 35 and 8, 28, 56, respectively. These are irreducible as $\mathbb{F}_p L_{\mathsf{u}}$-modules, and hence these are chief factors of $P_{\mathsf{u}} = L_{\mathsf{u}} U$. In addition, the P_{u}-composition series of U is unique.*

Let $Q_1 < U$, respectively, $Q_1 < Q_2 < U$ be the non-trivial proper normal subgroups of P. Then Q_1 is abelian, $[U, Q_2] = Q_1$, and $U' = Q_1, Q_2$, respectively. In addition, U and Q_2 are non-abelian.

Proof Let $p \mid q$ be a prime. By Proposition 5.6(1) and [30], the factors of the lower central series of U are non-trivial irreducible $\mathbb{F}_q P$-modules, in fact, irreducible $\mathbb{F}_q L$-modules. In [30] the author computes their dimensions as mentioned.

By Lemma 5.2 or Proposition 5.6(2), these are irreducible $\mathbb{F}_q L_{\mathsf{u}}$-modules. We can use Lemma 5.3 to conclude that these are irreducible $\mathbb{F}_p L_{\mathsf{u}}$-modules. For this we need to show that $\mathbb{F}_q$ is the minimal realization field of their realization. This is the case by [28, Proposition 5.4.6].

It follows that the lower central series terms are the only normal subgroups of P that lie in U, so there is no normal subgroup N of P such that $[Q_i, U] < N < Q_i$. □

Lemma 5.21 *Let $G = E_n(q)$, $n \in \{6, 7, 8\}$, and let P be a maximal parabolic corresponding to a simple root α_2 in notation of [3]. Then P_{u} is generated by two regular unipotent elements of G. Consequently, G_{u} is generated by three regular unipotents.*

Proof Let L be a Levi subgroup of P. By general theory, $L_{\mathsf{u}} \cong \mathrm{SL}_n(q)$. Let U be the unipotent radical of P. Let g, h be two regular unipotents in P that generate L_{u} modulo U and let $X = \langle g, h \rangle$. Observe that $X \cap U \neq 1$. Indeed, $P = UL$ is a semidirect product and $g \in UL_{\mathsf{u}} = P_{\mathsf{u}}$ as L/L_{u} is a p'-group. As $g \notin L_{\mathsf{u}}$ (Lemma 2.1), it follows that $U_1 := X \cap U \supset \langle L_{\mathsf{u}}, g \rangle \cap U \neq 1$. Note that U_1 is an L_{u}-invariant subgroup of U. If $U_1 = U$, we are done, so we assume $U_1 < U$. If $U_1 < Q_1$ then U_1 is normal in $L_{\mathsf{u}} U$. As L_{u} acts irreducibly on Q_1, it follows that $U_1 = Q_1$, unless the action of L_{u} is trivial which happens only when $n = 6$.

Suppose first that $G = E_6(q)$. By [9, Propositions 4.4], U is of extraspecial type and $Z(U)$ is a root subgroup U_α, say. Note that U_1 is not a subgroup of $Z(U)$. Indeed, as $[L_{\mathsf{u}}, Z(U)] = 1$ in fact, $U_1 < \langle L_{\mathsf{u}}, U_\alpha, U_{-\alpha} \rangle$, which is a semisimple group. This contradicts [58, Theorem 1.4]. So U_1 is not a subgroup of $Z(U)$, and hence the projection of U_1 into $U/Z(U)$ is non-trivial. By Lemma 5.19, $U/Z(U)$ is irreducible as an $\mathbb{F}_p L_{\mathsf{u}}$-module, so $U/Z(U) = U_1/(Z(U) \cap U_1)$. As $[U, U] = Z(U)$, it follows that $[U_1, U_1] = Z(U)$, and hence $U_1 = U$. This proves the result for $E_6(q)$.

Let $n > 6$. Then Q_1 is a non-trivial irreducible $\mathbb{F}_q L_{\mathsf{u}}$-module as well as an $\mathbb{F}_p L_{\mathsf{u}}$-module by Lemma 5.20. Therefore, Q_1 has no L_{u}-invariant proper non-trivial subgroup, so either $U_1 \cap Q_1 = 1$ or $Q_1 \leqslant U_1$.

Suppose that $U_1 \neq Q_1$, and let $\overline{U_1}$ be the projection of U_1 into U/U'. As above, U/U' has no L_u-invariant proper non-trivial subgroup. If $\overline{U_1} \neq 1$ then either $\overline{U_1} = U/U'$ or $\overline{U_1} < U'$. In the former case, U_1 is non-abelian and hence $1 \neq U_1' \leqslant Q_1$. So $U_1 \cap Q_1 \neq 1$, and hence $Q_1 \leqslant U_1$ as Q_1 has no proper non-trivial L_u-invariant subgroup. So, $U_1 = U$. Finally, we conclude that $U_1 = Q_1$ or Q_2.

Suppose that $U_1 = Q_1$. Then $[u, Q_2] < Q_1$. By Lemma 5.20, $\dim_{\mathbb{F}_q}(Q_2/Q_1) = 35, 28$ for $n = 7, 8$, respectively. On the other hand, by [31, Tables 8 and 9], u has at most 8 blocks on the adjoint module both of $n = 7, 8$. This contradicts Lemma 5.4. Similarly, if $G = E_8(q)$ and $U_1 = Q_2$ then $\dim U/Q_2 = 56$ by Lemma 5.20, and we conclude as above. □

Lemma 5.22 *The group ${}^2E_6(q)$, q odd, is generated by three conjugate regular unipotents.*

Proof Note that the Dynkin diagram of $G = {}^2E_6(q)$ as a group with BN-pair is of type F_4. This is obtained by gluing the nodes (1, 6), (3, 5) at the Dynkin diagram of the root system of type E_6 and the nodes (2), (4) remain unchanged. We use [29], where the nodes of the obtained F_4 diagram are denoted by $(1+6)$, $(3+5)$, (2), (4). Let P be the parabolic subgroup of G corresponding to the node (4) at the obtained F_4 diagram. Then the Levi L is described as $(\mathrm{SL}_2(q) \times \mathrm{PSL}_3(q^2)).(q-1)$, and $L_u \cong \mathrm{SL}_2(q) \times \mathrm{PSL}_3(q^2)$. The root subgroup of the multiple $\mathrm{SL}_2(q)$ is x_{α_2} of root system of G.

Let U be the unipotent radical of P. Then $|U| = q^{29}$ by [29], and then L_u is as above by [8, Table 10]. (This is to make it sure for our choice of parabolic, another choice leads to $|U| = q^{31}$.) The lower (and upper) central series of P on U has three non-trivial terms $Q_1 < Q_2 < U$, say, whose factors can be viewed as irreducible $\mathbb{F}_q L$-modules of dimensions 2, 9 and 18, respectively. By [29, Table 1], there are exactly 3 chief factors of P on U, so these are the factors of the lower (and upper) central series of U.

Let $u \in P$ be a regular unipotent. Then u is a product of positive root elements. Let $u = x_{\alpha_1}(1)x_{\alpha_6}(1)x_{\alpha_2}(1)x_{\alpha_4}(1)x_{\alpha_3}(1)x_{\alpha_5}(1)$. As $[x_{\alpha_1}(1), x_{\alpha_6}(1)] = 1$ and $[x_{\alpha_3}(1), x_{\alpha_5}(1)] = 1$, we have $u \in {}^2E_6(q)$ by the above, and $u \in P$. As $x_{\alpha_1}(1)x_{\alpha_6}(1)$, $x_{\alpha_3}(1)x_{\alpha_5}(1)$, $x_{\alpha_2}(1) \in L_u$, it follows that $\langle L_u, u\rangle$ contains an element in $U \setminus Q_2$, in fact, this is of the form $x_{\alpha_4}(1) \cdot y$, where y is a product of root elements x_β for some roots $\beta \succ \alpha_4$. (That is, $\beta - \alpha_4$ is a positive linear combination of simple roots.)

Let $X = \langle L_u, u\rangle$. By [29], there are no intermediate L_u-invariant subgroup Y, $Q_2 < Y < U$. Therefore, $(X \cap U)/(X \cap Q_2) = U/Q_2$. As $U > Q_2 > Q_1 > 1$ is a lower central series of U, it follows that $U' < Q_2$ and U' is not a subgroup of Q_1. As above, there is no intermediate L_u-invariant subgroup Y, $Q_1 < Y < Q_2$, and hence $X/(X \cap Q_1)$ contains Q_2/Q_1 and hence U/Q_1. Finally, as $|Q_1| = q^2$, in the additive notation, is not a 1-dimensional $\mathbb{F}_q L$-module, this is an irreducible $\mathbb{F}_p L_u$-module for $p \mid q$ (see also Lemma 5.3). The latter means that there is no intermediate L_u-invariant subgroup Y, $1 < Y < Q_1$. It follows that X contains U.

If $q \neq 9$ then L_u is generated by two conjugate regular unipotent elements. Therefore, $P_u = L_u U$ is generated by two conjugate regular unipotents. This in turn implies the result as in the $F_4(q)$-case.

Suppose that $q = 9$. Then we consider another parabolic subgroup P with U of order q^{31}. In this case $L_{\mathsf{u}} \cong \mathrm{SL}_3(q) \times \mathrm{SL}_2(q^2)$ [8, Table 10]. Then L_{u} is generated by two conjugate regular unipotents (see Remark 3.3 and Lemma 4.13). The composition series of L_{u} on U has four terms with factors of dimensions 3, 4, 12, 12 [29, Table 2] which are factors of the lower central series of U. Arguing as above, we conclude that P_{u} is generated by two conjugate regular unipotents, and then the result follows. □

The above results leave us with the cases where $G \in \{{}^2E_6(q), F_4(q)\}$, q even. To consider the former case let $\mathbf{G}_{ad}$ be the simple algebraic group of type E_6, $\mathbf{H}$ the simple algebraic group of type F_4, and we consider $\mathbf{H}$ as a subgroup of $\mathbf{G}$ defined by $C_{\mathbf{G}}(\gamma)$, where γ is a graph automorphism of order 2.

Lemma 5.23 ([35, Theorem 5.1 and Remark following it]) *Let* $\boldsymbol{G}$ *be a simple algebraic group in characteristic* p, σ *a Frobenius endomorphism of* $\boldsymbol{G}$ *and* $G = \boldsymbol{G}^{\sigma}$.

(1) *All quasisimple subgroups* X *of* $\boldsymbol{G}$ *such that* $X_{\mathsf{u}}/Z(X_{\mathsf{u}}) \cong G_{\mathsf{u}}/Z(G_{\mathsf{u}})$ *are conjugate to* G *in* $\mathrm{Aut}(\boldsymbol{G})$.
(2) *Let* $\tau = \sigma^k$, $k > 1$, *and* $H = \boldsymbol{G}^{\tau}$. *Then all quasisimple subgroups* X *of* H *such that* $X_{\mathsf{u}}/Z(X_{\mathsf{u}}) \cong G_{\mathsf{u}}/Z(G_{\mathsf{u}})$ *are conjugate to* G *in* $\mathrm{Aut}(H)$, *in fact, in* $N_{\mathrm{Aut}(\boldsymbol{G})}(H)$.
(3) ([35, Lemma 5.2]) $N_{\boldsymbol{G}}(G_{\mathsf{u}}) = G$.

Recall that the automorphism of G_{u} arising from the conjugation by elements of $N_{\mathbf{G}}(G_{\mathsf{u}})$ are called diagonal.

Note that (2) follows from (1), see [35, Lemma 5.3]. Our wording in the above statement slightly differs from that in [35, Theorem 5.1].

If $\mathbf{G}$ is of adjoint type, then $Z(G_{\mathsf{u}}) = 1$ and $N_{\mathbf{G}}(G^{\sigma}_{\mathsf{u}}) = G^{\sigma}$ [35, Lemma 5.2]. So in this case the statement of Lemma 5.23 becomes simpler. In particular, this is the case for $\mathbf{G}$ of type F_4; in this case, $G = G_{\mathsf{u}} = N_{\mathbf{G}}(G_{\mathsf{u}})$. (Note that $\mathbf{G} = F_4$ is of adjoint type.)

Lemma 5.24 *Let* $G = F_4(q)$ *and let* M *be a maximal subgroup of* G *containing* $X \cong F_4(2)$. *Then* M *is conjugate to the standard subgroup of* G *isomorphic to* $F_4(q_1)$ *with* $q = q_1^r$, r *a prime. Moreover, for every such* q_1 *there is exactly one subgroup containing* X *and isomorphic to* $F_4(q_1)$.

Proof The first claim is contained in [8]. For the second one suppose the contrary, and let M, M_1 be distinct subgroups containing X and isomorphic to $F_4(q_1)$. By Lemma 5.23 or [8], $gMg^{-1} = M_1$ for some $g \in G$. Then $gXg^{-1} < M_1$. Again by Lemma 5.23, $hgXg^{-1}h^{-1} = X$ for some $h \in M_1$. Then $hg \in N_G(X) < N_{\mathbf{G}}(X)$. By Lemma 5.23 (3), we have $N_{\mathbf{G}}(X) = X$, whence the claim. □

Lemma 5.25 *The group* $F_4(q)$, q *even, is generated by three conjugate regular unipotent elements.*

Proof By Lemma 5.24, $F_4(q_i)$, $q_i^r = q = 2^m$, where $r \mid m$ is a prime, are the only maximal subgroups of $G = F_4(q)$ containing X. As X is generated by two conjugate regular unipotents u, u', say, it suffices to show that $G \setminus \left(\bigcup_i F_4(q_i)\right)$ contains a conjugate of u. For this we show that the total number of conjugates of u in G is greater that the total number of conjugates of u in $\bigcup_i F_4(q_i)$. Note that the size the G-orbit of u is $|G|/|C_G(u)| = |G|/q^4$ and those in the above subgroups are $|F_4(q_i)|/{q_i}^4$.

We prove that $\frac{|G|}{q^4} > \mathcal{S}(q)$, where $\mathcal{S}(q) = \sum_i \frac{|F_4(q_i)|}{q_i^4}$ and the sum is taken on all 2-power q_i such that $q = 2^m = q_i^{r_i}$ with r_i a prime. Note that $2^{51m} < |F_4(q)| < 2^{52m}$. Hence,

$$\mathcal{S}(q) \leqslant \sum_i q_i^{48} \leqslant \sum_{\substack{d|m \\ d\neq m}} (2^d)^{48} \leqslant \sum_{i=0}^{\lfloor m/2 \rfloor} (2^i)^{48} = \frac{(2^{48})^{1+\lfloor m/2 \rfloor} - 1}{2^{48} - 1} < 2^{25m} < \frac{|G|}{q^4}. \quad \square$$

Let $\mathbf{G}$ be a simple algebraic group of type E_6, and let σ_j, $j = 1, 2$, be a Frobenius morphism $\mathbf{G} \to \mathbf{G}$ such that $G_k := \mathbf{G}^{\sigma_1^k} \cong E_6(2^k)$, ${}^2G_k := \mathbf{G}^{\sigma_2^k} \cong {}^2E_6(2^k)$. We choose σ_k^1 to be a standard Frobenius arising from $x \mapsto x^{2^k}$ for $x \in \overline{\mathbb{F}}_2$, and $\sigma_2^k = \gamma \cdot \sigma_k^1 = \sigma_1^k \cdot \gamma$. (Note that γ commutes with σ_1^k and σ_2^k.) We call G_k and ${}^2G_k$ standard finite subgroups of type E_6 and ${}^2E_6$, respectively, of $\mathbf{G}$.

Set $H_k^+ = C_{\mathbf{G}}(\gamma, \sigma_1^k)$ and $H_k^- = C_{\mathbf{G}}(\gamma, \sigma_2^k)$. We can apply γ to $\mathbf{G}$ and then σ_j^k to $C_{\mathbf{G}}(\gamma)$ or conversely, first σ_j^k to $\mathbf{G}$ and then γ to $C_{\mathbf{G}}(\sigma_j^k)$, to obtain H_k^+ and H_k^-. Therefore, H_k^+ is the intersection of $C_{\mathbf{G}}(\sigma_1^k) = E_6(2^k)$ and $\mathbf{H} := C_{\mathbf{G}}(\gamma)$ and H_k^- is the intersection of $C_{\mathbf{G}}(\sigma_2^k) = {}^2E_6(q^k)$ and $\mathbf{H}$. We call H_k^+ and H_k^- standard finite subgroups of type F_4 in $\mathbf{G}$.

In Lemmas 5.26, 5.27 and 5.28 we deal with groups $G = \mathbf{G}_{ad}^{\sigma}$ of adjoint type. In this case G is not always simple whereas G_{u} is a simple group of Lie type. For our purposes it is convenient to call a subgroup M of G maximal if $M \cap G_{\mathsf{u}}$ is a proper maximal subgroup of G_{u}. The conjugacy classes of maximal subgroups of G and G_{u} are determined in [8].

Lemma 5.26 ([8]) *Let $G = {}^2E_6(q)_{\mathrm{ad}}$ and let M be a maximal subgroup of G containing $X \cong F_4(2)$. Then M is conjugate to the standard subgroup of G of type $H_m \cong F_4(q)$ or of type ${}^2E_6(q_1)_{\mathrm{ad}}$ with $q = q_1^r$, r an odd prime.*

Proof Note that $X < G_{\mathsf{u}}$. By inspection of the list of maximal subgroups of G in [8], one observes that those containing a subgroup isomorphic to $F_4(2)$ are as indicated in the lemma. In addition, groups M isomorphic to each other are conjugate in G, that is, form a single conjugacy class. So one can choose the standard subgroup as a representative of each class. $\square$

Lemma 5.27 *Let X be a subgroup of $G = {}^2E_6(q)_{\mathrm{ad}}$ isomorphic to H_k^-. Then X is conjugate in G to H_k^-. In addition, $N_G(H_k^-) = H_k^-$.*

Proof Let $q = 2^m$, so $G = {}^2G_m$. Then $X < N_G(X) \leqslant M$, where M is a maximal subgroup of G. As G is of adjoint type, by Lemma 5.26, we can assume that M is the standard subgroup of G, so $M = H_m^-$ or $M = {}^2G_{m_1}$, where $q = 2^{m_1 r}$ with r an odd prime. If $M = H_m^-$ then, by Lemma 5.23, X is conjugate in M to a standard subgroup of H_m^-. As standard subgroups of H_m^- are standard in G, the result follows. In addition, this implies the result if m is a 2-power as in this case G has no subgroup of type ${}^2E_6(q_1)$ for $q_1 < q$.

Suppose that $M = {}^2G_{m_1}$. Then the result follows by induction on m; the base of induction is m_2, the 2-power part of m, case already settled.

The additional case follows from Lemma 5.23 (3) if $N_G(X) \leqslant M = H_m^-$, otherwise this follows by induction. □

Lemma 5.28 *Let $X = F_4(2) < M < G = {}^2E_6(q)_{\mathrm{ad}}$, q even, where M is a maximal subgroup of G not containing G'. Suppose that $X < gMg^{-1}$ for $g \in G$. Then $g \in M$. Consequently, if M, M' are isomorphic maximal subgroups of G containing X then $M = M'$.*

Proof Let $q = 2^m$. By Lemma 5.26, M is conjugate to a standard subgroup $H_m \cong F_4(q)$ or ${}^2G_{m_1} \cong {}^2G_{\mathrm{ad}}(q_1)$ with $q = q_1^r$, where $r = m/m_1$ is an odd prime. We can assume that M itself is standard.

Suppose the contrary, that $X < gMg^{-1} \neq M$. Then $g^{-1}Xg < M$. By Lemma 5.27, X and $g^{-1}Xg$ are conjugate in M so $X = y^{-1}g^{-1}Xgy$ for some $y \in M$. So $gy \in N_G(X)$. As $N_G(X) = X$ by Lemma 5.27, we have $gy \in M$ and $g \in M$, a contradiction.

The complementary assertion follows from this as isomorphic maximal subgroups of G (containing X) are conjugate by [8]. □

Lemma 5.29 *Let $G = {}^2E_6(q)_{\mathrm{ad}}$, q even. Distinct maximal subgroups of G_{u} containing a fixed subgroup $X \cong F_4(2)$ are not isomorphic. Consequently, if $q = 2^m$ then the number of maximal subgroups of G_{u} containing X is $d + 1$, where d is the number of odd primes dividing m.*

Proof We only need to consider the case with $G_{\mathrm{u}} \neq G$ by Lemma 5.28. This implies that $3 \mid (q + 1)$ and $|G/G_{\mathrm{u}}| = 3$.

Let M be a maximal subgroup of G, and $X < M$. If $M \cong F_4(q)$ then the reasoning in Lemma 5.28 works (in this case $M = M_{\mathrm{ad}} = M_{\mathrm{u}}$ so the subgroups of M isomorphic to X are conjugate in M).

Let $M = {}^2E_6(q_1)_{\mathrm{ad}} < G$, $q = q_1^3$. Then $3 \mid (q_1 + 1)$ and $X < M_{\mathrm{u}} < M$. As M_{u} is normal in M, M_{u} is the unique subgroup of G_{u} containing X.

The assertion on the number of maximal subgroups in question follows from [8].□

Lemma 5.30 *Let $G = {}^2E_6(q)_{\mathrm{ad}}$, q even. Then G_{u} is generated by three conjugate (in G_{u}) regular unipotent elements.*

Proof Let $X < G_{\mathrm{u}}$, $X \cong F_4(2)$ be as above. Let $q = 2^m$, and let s be the number of distinct odd prime divisors of m.

By Lemma 5.29, $F_4(q)$ and ${}^2E_6(q_i)_{\mathrm{u}}$, $q_i^r = q$, where $r \mid m$ is an odd prime, are the only maximal subgroups generated by unipotent elements and containing X. As X is generated by two conjugate regular unipotents u, u', say, it suffices to show that ${}^2E_6(q)_{\mathrm{u}} \setminus \big(F_4(q) \bigcup_i {}^2E_6(q_i)_{\mathrm{u}}\big)$ contains a conjugate of u. For this we show that the total number of conjugates of u in G_{u} is greater that the total number of conjugates of u in $F_4(q) \cup \bigcup_i {}^2E_6(q_i)_{\mathrm{u}}$.

Note that the size the G_{u}-orbit of u is $|G_{\mathrm{u}}|/|C_{G_{\mathrm{u}}}(u)| = |G_{\mathrm{u}}|/q^6$, and those in the above subgroups are $|F_4(q)|/q^4$ and $|{}^2E_6(q_i)_{\mathrm{u}}|/q_i^6$.

Table 1 Auxiliary results for Theorem 1.4

$\mathrm{SL}_2(q)$, $q \geqslant 4$ even	[24, Lemma 3.1]
$\mathrm{SL}_2(9)$	[24, Lemma 3.1]
$\mathrm{SL}_n(q)$, $n \geqslant 3$	Lemma 5.1
$\mathrm{SU}_4(q)$, $q \geqslant 4$ even	Lemma 5.9
$\mathrm{SU}_5(q)$, $q \geqslant 5$ odd	Lemma 5.7
$\mathrm{SU}_5(q)$, $q \geqslant 4$ even	Lemma 5.8
$\mathrm{SU}_n(q)$, $n \geqslant 7$ odd	Lemma 5.7
$\mathrm{Sp}_4(q)$, q even	Lemma 5.16
$\mathrm{Sp}_{2n}(q)$, $n \geqslant 3$	Lemma 5.15
$\Omega_{2n+1}(q)$, $n \geqslant 3$ and q odd	Lemma 5.11
$\Omega^+_{2n}(q)$, $n \geqslant 4$ and q even	Lemma 5.10
$\Omega^-_{2n}(q)$, $n \geqslant 4$	Lemma 5.11
$G_2(q)$, $q \geqslant 4$ even	Lemma 5.17
$F_4(q)$	Lemmas 5.18 and 5.25
${}^2E_6(q)$	Lemmas 5.22 and 5.30

We prove that

$$\frac{|G_{\mathrm{u}}|}{q^6} > \frac{|F_4(q)|}{q^4} + \sum_i \frac{|{}^2E_6(q_i)_{\mathrm{u}}|}{q_i^6}.$$

Set

$$\mathcal{S}(q) = \sum_i \frac{|{}^2E_6(q_i)_{\mathrm{u}}|}{q_i^6},$$

where the sum is taken on all 2-power q_i such that $q = 2^m = q_i^{r_i}$ with r_i odd prime.

Assume $m > 1$. First of all, note that $2^{77m} < |E_6(q)_{\mathrm{u}}| < 2^{78m}$ and $|F_4(q)| < 2^{52m}$. Now,

$$\mathcal{S}(q) \leqslant \sum_i q_i^{72} \leqslant \sum_{\substack{d|m\\ d\neq 1}} (2^{m/d})^{72}$$
$$= \sum_{\substack{d|m\\ d\neq m}} (2^d)^{72} \leqslant \sum_{i=0}^{\lfloor m/2\rfloor} (2^i)^{72} = \frac{(2^{72})^{1+\lfloor m/2\rfloor} - 1}{2^{72}-1} \leqslant 2^{37m}.$$

Hence,

$$\frac{|F_4(q)|}{q^4} + \mathcal{S}(q) < 2^{48\,m} + 2^{37\,m} < 2^{71\,m} < \frac{|{}^2E_6(q)_{\mathrm{u}}|}{q^6}.$$

If $m = 1$ (or m is a 2-power) then we only need to observe that $|G_{\mathrm{u}}| > q^2|F_4(q)|$, which is clear from the above. □

Note that the method used for $G = {}^2E_6(q)$ works for $E_6(q)$. However, we used for this group an alternative approach.

In conclusion, Theorem 1.4 follows from the previous results according to Table 1.

Theorem 1.5 follows from Lemmas 5.1, 5.10, 5.11, 5.14, 5.21, and 5.22. Finally, Theorem 1.2 follows from Theorems 1.3, 1.4 and Lemmas 5.7, 5.10, and 5.21.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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