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On the interpretation of the measurement equation

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ABSTRACT

An appropriate understanding of the process of measurement and its results rests on the acknowledgment of the fundamental role played by the models of the object under measurement and of the relevant quantities. In this paper, we first introduce two strategies – stemming from a classical and a representational tradition respectively – to understand the information produced by measurement in the form of an equation stating that a certain value is to be assigned to certain measurand. Then we compare such strategies by discussing their different ability to take a model-based interpretation of measurement into account. The conclusion is that, when models enter the picture, the interpretation provided within the classical tradition is more adequate than the rival interpretation.

1. Introduction

A *basic equation* in measurement is a statement reporting a measurement result, that – by focusing on quantities having a unit and until measurement uncertainty can be neglected – is of the form

$$Q(a) = \{Q\}[Q] \quad (1)$$

In this statement

1. $Q(a)$ is the *measurand*, i.e., the quantity we intend to measure of a given object a;[2]
2. $\{Q\}[Q]$ is a *measured value*, constituted by a numerical value $\{Q\}$ and a unit $[Q]$ for the relevant kind of quantity Q.

The equation is commonly interpreted as stating that the measurand is in a certain relation with a measured value, typically up to some uncertainty. So, if a is a certain object, for example a given rod, and $L(a)$ is its length, then a basic equation like $L(a) = 1.25\,\mathrm{m}$ states that the length of a is $1.25\,\mathrm{m}$, where 1.25 is a numerical value in a certain range and m is the meter, that is a unit of length.

The key questions we address in the present paper are:

KQ1: what kind of information does a basic equation convey?
KQ2: how should we define its truth conditions?

The second question is about the conditions that have to obtain in the world for the equation to be true. Therefore, in order to provide an answer to these questions, we should be able to clarify the following preliminary issues:

- what are entities like $Q(a)$? that is, what are measurands?
- what are entities like $\{Q\}[Q]$? that is, what are measured values and, more generally, values of quantities?
- how can measurands and measured values be related to each other?

We propose that these problems can be answered in the context of two conceptually competing, though operationally largely compatible, frameworks, each corresponding to a different tradition.

In the first framework, developed in what we call the *classical tradition*, properties, and then in particular quantities, of objects are assumed to exist. As a consequence, basic equations are interpreted as actual equations, whose truth is based on the fact that the same quantity is identified alternatively both as the property of a given object and as a quantity identified as a function of a unit. We will refer to this interpretation as *equational*. In the other framework, developed in what we call the *representational tradition*, properties are (or, as we will discuss, at least can be) dispensed with. As a consequence,

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[1] The authors contributed equally to the whole manuscript.
[2] We use the term "object" to refer to anything that has properties. We assume that quantities are specific properties, characterized by a structure whose features are not important to discuss here (see [1] for a general characterization). Further, we assume the distinction between *kind of quantity*, or *general quantity* (like length or temperature) and *individual quantities* of that kind (like the length and the temperature of a given object). In the following, the term "quantity" is often used as a shorthand to designate an individual quantity when no risk of confusion exists. We use "Q" for denoting a generic kind of quantity and "$Q(\cdot)$" for denoting the function that returns an individual quantity $q = Q(a)$ of kind Q when applied to an object a that has that quantity.

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basic equations are interpreted as convenient statements reporting that, under appropriate conditions, objects are represented by means of mathematical entities, usually but not necessarily numbers. We will refer to this interpretation as *representational*. The aim of this paper is to provide evidence supporting the classical interpretation in light of the fact that it enables us to better account for the role of models in measurement. This contributes to a foundational issue of measurement science, and as such can be assumed as preliminary to any theory of measurement that on the understanding of the basic equation may be grounded and built.

The structure of the paper is the following. Sections 2 and 3 discuss the main tenets of the classical and the representational traditions respectively, and the way they interpret basic equations. Section 4 explores how the models about the entities involved in a measurement can be understood in each of these traditions. This leads to a comparison of the interpretations of the basic equation in Section 5, which highlights some of the benefits of the equational interpretation over the representational one.

The focus of the analysis is on ratio quantities, either physical or psychosocial ones, but we suggest that it can also be adapted, with possible modifications, to other types of physical properties, such as interval or ordinal quantities. It is also worth noting that this analysis does take neither definitional nor measurement uncertainty into account. This has no effect on the dialectic of the paper, since a discussion of these notions would be irrelevant to the main argument we are going to propose. We acknowledge that the characterization of uncertainty from a classical and a representational point of view constitutes an important topic in itself, to be left for some possible further works.

2. The classical tradition

The classical tradition supports a realist interpretation of the notions of quantity and ratio between quantities. In general, up to a slight idealization, this tradition followed four main steps in its development: (i) the characterization of the notion of ratio of mathematical entities; (ii) the extension of this notion from mathematical to empirical entities; (iii) the generalization of the notion of number to include ratios as numbers in themselves; (iv) the identification of the empirical conditions that allow for the use of ratios to characterize empirical entities. Let us review these steps and discuss the kind of realism that is endorsed by this tradition.[3]

(i) the characterization of the notion of ratio of mathematical entities This key step was accomplished in the ancient mathematical tradition and codified in Book V of Euclid's *Elements* [9]. In the Greek tradition, a quantity is characterized as being either a plurality or a magnitude,[4] where numbers, i.e., positive integers, are viewed as pluralities, i.e., quantities composed by indivisible entities, while magnitudes are viewed as quantities that are divisible in further entities without limit. Euclid posits three relations between magnitudes:

1. the relation of *part*, according to which a is said to be part of b provided that there is a number n such that $b = na$. If a is part of b, then a is said *to measure* b;
2. the relation of *parts*, according to which a is said to be parts of b provided that there is a magnitude x and there are numbers m and n such that $a = mx$ and $b = nx$. If a is parts of b, then a is said *to have a common measure with* b, i.e. *to be commensurable with* b;
3. the relation of *having a ratio*, according to which a is said to have a ratio with respect to b provided that each can exceed the other if multiplied by appropriate numbers. The relation of having a ratio is an equivalence relation by definition and a property of positive integers. It is also evident that if a is part or parts of b, then a and b have a ratio with respect to each other.

Let (a_1, b_1) and (a_2, b_2) be pairs of magnitudes and suppose that both (a_1, b_1) and (a_2, b_2) have a ratio with respect to each other. Then, we can ask whether (a_1, b_1) and (a_2, b_2) have the same ratio with respect to each other. In order to answer this question the notion of proportionality, i.e., sameness in ratio, has to be defined. The general definition of proportionality constitutes a crucial achievement of Book V, and runs as follows:

> (a_1, b_1) and (a_2, b_2) have the same ratio if and only if for every numbers m and n
>
> – (i) $ma_1 < nb_1$ if and only if $ma_2 < nb_2$
> – (ii) $ma_1 = nb_1$ if and only if $ma_2 = nb_2$
> – (iii) $ma_1 > nb_1$ if and only if $ma_2 > nb_2$

In light of this definition, it is evident that, if a_1 measures b_1, so that $b_1 = na_1$ for a certain n, then (a_1, b_1) and (a_2, b_2) have the same ratio precisely when $b_2 = na_2$ for the same n. Similarly, if a_1 has a common measure with b_1, so that $a_1 = mx_1$ and $b_1 = nx_1$ for certain m, n and a certain magnitude x_1, then (a_1, b_1) and (a_2, b_2) have the same ratio precisely when $a_2 = mx_2$ and $b_2 = nx_2$ for the same m, n and a corresponding magnitude x_2. Still, two pairs of magnitudes can be characterized by the relation of having the same ratio even if they are not such that the first element of a pair is part or parts of the second element. In particular, when a measures b the ratio between a and b is the same as the ratio between 1 and n, where n is the number such that $b = na$, and therefore the ratio between a and b is the same as a numerical ratio. Similarly, when a has a common measure with b the ratio between a and b is the same as the ratio between m and n, where m is the number such that $a = mx$ and n is the number such that $b = nx$ for a certain x, and therefore the ratio between a and b is again the same as a numerical ratio. Still, it is possible for two pairs of magnitudes to be characterized by the relation of having the same ratio even if their ratio corresponds to no numerical ratio. This implies the possibility of distinguishing two kinds of ratios:

- *rational* ratios, that characterize commensurable magnitudes and correspond to numerical ratios (what we now call *rational numbers*);
- *irrational* ratios, that characterize incommensurable magnitudes and do not correspond to numerical ratios (what we now call *irrational numbers*).

Hence, any two magnitudes that can exceed each other if multiplied by appropriate numbers have a ratio, but not all such pairs have a rational number as their ratio. The definition of the notion of having the same ratio is crucial for fully appreciating the ontological framework of the classical tradition. As we will see, this definition allows for introducing real numbers as ratios, so that real numbers are available in this tradition as soon as the existence of individual quantities is acknowledged.

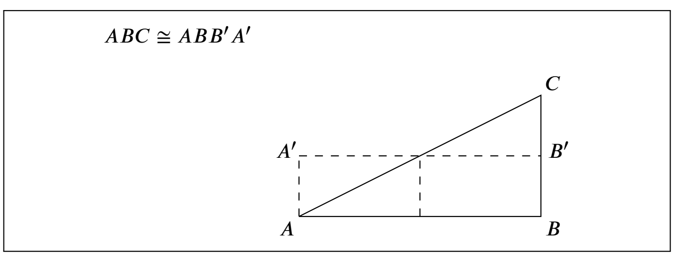
[3] See [2–5] for a historical introduction to the main ancient and medieval figures in this tradition and [6–8] for an insightful presentation of the key ideas.

[4] According to Aristotle, Metaphysics, book Δ, 13, "We call a quantity that which is divisible into two or more constituent parts of which each is by nature a one and a 'this'. A quantity is a plurality if it is numerable, a magnitude if it is measurable. We call a plurality that which is divisible potentially into non-continuous parts, a magnitude that which is divisible into continuous parts; in magnitude, that which is continuous in one dimension is length, in two breadth, in three depth. Of these, limited plurality is number, limited length is a line, breadth a surface, depth a solid." It is worth noting here that the ancient notion of quantity can be applied not only to the quantitative properties, but also to the objects having such properties. A trace of this general usage is also present nowadays, for example when the term "diameter of a circle" is referred both to segments (a circle has an uncountable number of diameters) and to their length (the diameter of a circle is in a ratio of π to its circumference).

(ii) the extension of the notion of ratio to empirical entities This key step was accomplished in the medieval mathematical tradition and codified in Oresme's Treatise [3]. In the initial part of the book, he introduces a general model for studying qualities, i.e., what we now know as *kinds of empirical quantities*, having intensities, i.e., what we now know as *individual quantities*.

> Every measurable thing except numbers is imagined in the manner of continuous quantity. Therefore, for the mensuration of such a thing, it is necessary that points, lines, and surfaces, or their properties, be imagined. For in them (i.e., the geometrical entities), as the Philosopher has it, measure or ratio is initially found, while in other things it is recognized by similarity as they are being referred by the intellect to them (i.e., to geometrical entities). Although indivisible points, or lines, are nonexistent, still it is necessary to feign them mathematically for the measures of things and for the understanding of their ratios. [3, p. 165]

The idea proposed by Oresme is that, if we want to study qualities having intensities in a systematic way, we can model them in terms of mathematical quantities, like segments, and then study the relations between such qualities in light of the corresponding relations between the mathematical entities we use to model them. For example, to study a uniformly accelerated motion we draw the intensity of the motion, i.e., its velocity, as vertical lines on an axis representing time and we get a configuration whose area represents the distance traveled during the motion. It can then be shown that the distance traveled during the motion is the same as the distance traveled during a uniform motion of half intensity.



The notion of knowledge by similarity is then specified by Oresme as follows:

> For whatever ratio is found to exist between intensity and intensity, in relating intensities of the same kind, a similar ratio is found to exist between line and line, and vice versa. For just as one line is commensurable to another line and incommensurable to still another, so similarly in regard to intensities certain ones are mutually commensurable and others incommensurable in any way because of their continuity. Therefore, the measure of intensities can be fittingly imagined as the measure of lines, since an intensity could be imagined as being infinitely decreased or infinitely increased in the same way as a line. [3, p. 167]

Hence, qualities having intensities are studied in terms of mathematical quantities by modeling such intensities as having ratios, thus modeling them as what we now know as *ratio quantities*. In summary, in this step the notion of ratio is extended from mathematical to empirical entities.

(iii) the generalization of the notion of number This key step was accomplished in the modern mathematical tradition and codified in Newton's *Arithmetic* [10].

> By Number we understand not so much a multitude of unities, as the abstracted ratio of any Quantity to another Quantity of the same Kind, which we take for Unity. [10, p. 2]

The difference between the ancient notion of ratio and the notion introduced by Newton is that ratios between quantities are acknowledged as entities that exist in themselves. In view of this new characterization two entities having ratio with each other *because* they have quantities which stand with respect to each other in a specific ratio. Accordingly, we are allowed to say not only that the ratio in length between a and b is the same as the ratio between 5 and 4, but also that:

1. a has a length $L(a)$ and b has a length $L(b)$;
2. the ratio $L(a) : L(b)$ between $L(a)$ and $L(b)$ is 1.25.

The twofold assumption is that entities having quantities are what they are, i.e., they are entities comparable with respect to a certain kind of quantity, because they have given quantities, and that quantities are what they are, i.e., entities comparable with each other, because they have specific ratios with respect to each other. Hence, this ontology is rich: besides entities having quantities, it includes both quantities and ratios. In addition, numbers, as positive integers, can now be viewed as a kind of ratios in virtue of the fact that any number n has a ratio $n : 1$ with the number 1.

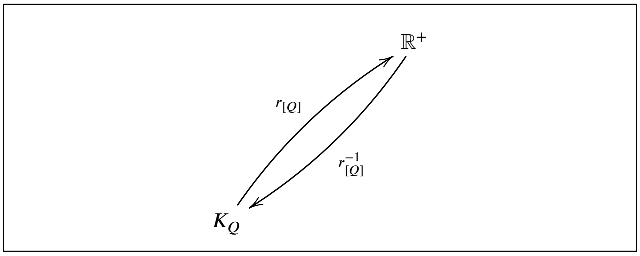
(iv) the identification of the conditions of comparison by ratio The last key step in the classical tradition was the identification of the conditions for the notion of ratio to be used with respect to empirical quantities. In step *(ii)* it was assumed that empirical objects can be modeled in terms of mathematical objects. In step *(iii)* it was assumed that the comparability of mathematical objects is based on their being characterized by quantities having ratios with respect of each other. So, combining the two steps, we conclude that *empirical objects that are comparable with respect to a certain kind of quantity are so comparable in virtue of the fact that they are characterized by quantities having a certain ratio with respect to each other*. In this sense, for example, two rods are comparable with respect to length in virtue of the fact that they are characterized by certain lengths having a certain ratio with respect of each other.

Still, this conclusion, and its underlying assumption that empirical entities are like mathematical entities with respect to certain kinds of quantities, has to be justified. In particular, we have to justify that rods, like segments, are comparable with respect to length, namely that it is possible to describe a rod as having *one* length, like a segment has. This assumption of similarity can be substantiated in two steps, by first proving that ratio quantities can be completely specified in terms of some abstract relations and operations, and then showing that corresponding empirical relations and operations can be found in relation to the system of empirical quantities we want to study. The first step was accomplished at the beginning of the last century [11,12]. The second one is left to the specific empirical disciplines that study the relevant kinds of quantities.

2.1. The emergence of the equational interpretation

Within the classical tradition, a simple interpretation of the basic equation suggests itself. First, quantities are characterized by the fact that they are related to each other via well-defined ratios. Second, once a quantity is chosen as the unit, all other quantities are at least in principle determined as ratios with respect to it. Indeed, quantities that are commensurable to the unit are such that their ratio with respect to the unit is rational, i.e., a ratio between integer numbers. By contrast, quantities that are incommensurable to the unit are such that their ratio with respect to the unit is irrational, that can be modeled as a sequence of ratios between integer numbers.

Once a unit $[Q]$ is chosen for a given kind of quantity Q, this can be represented as:



where:

- K_Q is a set of quantities of kind Q;
- $\mathbb{R}^+$ is the set of non-negative real numbers;
- $r_{[Q]} : K_Q \to \mathbb{R}^+$ is the function that takes a quantity $\mathrm{q} \in K_Q$ and returns its ratio $\{Q\} \in \mathbb{R}^+$ with the unit $[Q]$;
- $r^{-1}_{[Q]} : \mathbb{R}^+ \to K_Q$, the inverse of $r_{[Q]}$, takes a ratio $r \in \mathbb{R}^+$ and returns the corresponding quantity $\mathrm{q} \in K_Q$ as identified with respect to $[Q]$.

In this framework, for any $\mathrm{q} \in K_Q$ it is true that

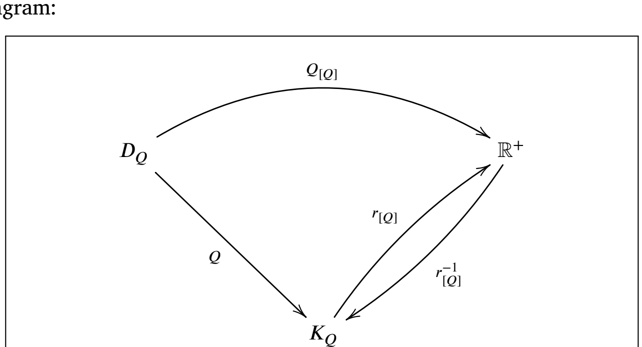
1. $r_{[Q]}(\mathrm{q}) = r \in \mathbb{R}^+$ for some r;
2. $r_{[Q]}(\mathrm{q}) = \{Q\}$, since $\{Q\}$ is the ratio between q and $[Q]$;
3. $\mathrm{q} = \{Q\}[Q]$, since $\mathrm{q} = r^{-1}_{[Q]}(\{Q\})$ is the quantity whose ratio with $[Q]$ is $\{Q\}$.

Therefore, any quantity q of kind Q can be presented as a ratio with any other, somehow chosen, quantity $[Q]$ of the same kind, and this is precisely the view we find at the end of the classical tradition. To be sure, in his 1870 lecture notes Clifford, following Newton, says

> Every quantity is therefore measured by the ratio which it bears to some fixed quantity, called the unit. But between any two ratios is an infinite number of ratios; it is therefore impossible to tabulate all ratios, or to give them names. A ratio then can only be described approximately, as being very near to the ratio of two numbers, that is, of two quantities which have a common measure. [13, p. 525]

As mentioned, this means that in the equation $\mathrm{q} = \{Q\}[Q]$ it is possible for $\{Q\}$ to be a rational number, when q and $[Q]$ are commensurable, or to be modeled in terms of sequences of rational numbers, when q and $[Q]$ are not commensurable.

The final move that allows us to get an interpretation of the basic equation is the idea that the same quantity q can be presented both as the quantity characterizing a certain object (a presentation *by address*) and as the quantity determined by a certain ratio with a unit (a presentation *by value*). So, once a unit $[Q]$ is chosen, we get the following diagram:



where

- D_Q is a set of objects characterized by quantities in K_Q, i.e., of kind Q
(for example, D_Q could be a set of rods);
- $Q : D_Q \to K_Q$ is the function that takes an object $a \in D_Q$ and returns its quantity $Q(a) = \mathrm{q} \in K_Q$
(for example, Q could map each rod of the given set to a length that is presented as the length of that rod);
- $Q_{[Q]} : D_Q \to \mathbb{R}^+$ is the function that takes an object $a \in D_Q$ and returns the ratio $\{Q\}$ between $Q(a)$ and the unit $[Q]$.

Hence, supposing that the length unit $[Q]$ is the meter, $Q_{[Q]}$ maps each rod of the given set to its length in meters. So, evidently, $Q_{[Q]} = r_{[Q]} \circ Q$, so that $\mathrm{q} = r^{-1}_{[Q]} \circ Q_{[Q]}$, and we get

1. $Q(a) = \mathrm{q}$
(the length of any rod is a length);
2. $\mathrm{q} = r^{-1}_{[Q]}\{Q\} = \{Q\}[Q]$
(a length can be presented as a number times a unit: a value of length);
3. and so $Q(a) = r^{-1}_{[Q]}\{Q\} = \{Q\}[Q]$
(the length of any object can be presented as a value of length).

Finally, we recover the classical idea:

> Every expression of a Quantity consists of two factors or components. One of these is the name of a certain known quantity of the same kind as the quantity expressed, which is taken as a standard of reference. The other component is the number of times the standard is to be taken in order to make up the required quantity. [14, p. 1]

Thus, a basic equation $Q(a) = \{Q\}[Q]$ is actually an identity, where the same individual quantity q is presented both by address, as the quantity $Q(a)$ characterizing a certain object a, and by value, as the quantity determined by a certain ratio $\{Q\}$ with the chosen unit $[Q]$: if the equation is true, $Q(a)$ and $\{Q\}[Q]$ are the same individual quantity presented in two different ways.

2.2. The ontology underlying the equational interpretation

The ontology behind this picture is such that

- there are objects;
- there are quantities of objects;
- quantities are classified in kinds by their comparability;
- there are numbers, possibly interpreted as ratios of quantities of the same kind;
- some objects have a quantity of a given kind, so that some quantities of that kind can be presented as quantities of given objects;
- a quantity of that kind can be singled out as the unit of the kind, so that all the quantities of that kind can be presented as ratios with respect to the unit.

Accordingly, as we saw, a basic equation is interpreted as stating that *the quantity presented in terms of an object* (left hand side) *is the same quantity presented in terms of the unit* (right hand side).

3. The representational tradition

The representational tradition stems from the idea that measurement with respect to a domain of empirical objects is possible provided that such a domain can be characterized in terms of some specific relations and operations, where a measurement is conceived of as a process producing information that properly represents such relations and operations, a condition that leads to model such a process as a morphism between the empirical system having that domain as support and a given numerical system.[5] In a nutshell, the representational tradition aims at analyzing and generalizing the notion of similarity in regard to intensities introduced in the ancient and medieval reflection about the relationship between quantities, at the same time reducing the emphasis on the role of quantities, or entirely doing without quantities. In fact, this tradition is typically associated with an antirealist position about quantities, even if its main results are consistent with a realist interpretation, and for this reason in the analysis that follows we discuss representationalism as implying antirealism about quantities, and more generally properties.[6]

3.1. The emergence of the representational interpretation

The key points of the representational interpretation can be identified from the basic problem that a measurement theory is expected to solve according to representationalism: understanding the conditions that justify the practice of associating numbers with empirical objects, so that the assignment reflects the structure of the relations that in some sense are observed among the objects under measurement. This leads to positing two fundamental problems (see [19, Chapter 1] for a general, and classical, introduction to the topic).

The first fundamental problem: possibility of representation The first problem is known as the *problem of representation*, and concerns the discovery of the conditions that an empirical system has to satisfy in order to be represented in terms of a numerical system.[7]

> The first basic problem of measurement theory is the *representation problem*: Given a particular numerical relational system $\mathfrak{B}$, find conditions on an observed relational system $\mathfrak{A}$ necessary and sufficient for the existence of a homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$. The emphasis is on finding sufficient conditions. If all the conditions in a collection of sufficient conditions are necessary as well, that is all the better. [22, p. 54]

The fact that the assignment of numbers to the empirical objects in a system is constrained by a set of conditions on the empirical system is crucial, since it prevents the assignment from being purely conventional.[8] The identification of suitable conditions and the proof that they are sufficient for the existence of a homomorphism is the content of the *representation theorem* in a measurement theory. The main result of a representation theorem is then the possibility of defining a scale for measuring objects in empirical systems, since a scale is nothing more than the homomorphism whose existence is to be proved.

The second fundamental problem: uniqueness of representation The second problem is known as the *problem of uniqueness*, and concerns the discovery of the group of transformations that allow us to pass from a representation to a different but still admissible representation for the object of a given empirical system.

> The second basic problem of measurement theory is the *uniqueness problem*: How unique is the homomorphism f? [22, p. 55]

This problem is related to the fact observed above, about ratio quantities, that the numerical representation of a quantity depends on the choice of a unit. Thus, different numbers can be used to represent the same quantity in terms of different units. Still, the possible representations of a ratio quantity are related by a suitable group of transformations that allows us to pass from a representation to another. For example, in the case of ratio quantities the transformations are similarities, of the form $f(x) = cx$, where c is the ratio between the initial unit and the target unit. Hence, if the length of a given object is represented by 1.25 in meters, then the same length is represented by 125 in centimeters, since the ratio between the meter and the centimeter is $c = 100$. The identification of suitable group of transformations and the proof that this group completely characterizes the set of scales for a certain empirical system is the content of the *uniqueness theorem* in a measurement theory. The main result of a uniqueness theorem is then the possibility of defining a set of scales for measuring empirical systems, where any scale in the set can be obtained from any other given the group of transformations.

[5] See [15, Part II], [16,17], [18, Chapter 5], for an overview of the way in which the representational tradition developed. See [19–25] for a throughout introduction to the general framework and a presentation of the main ideas and tools employed in this tradition. Antirealism about properties is well-documented in these references. Some sort of moderate representationalism is also sometimes endorsed, that allows for the existence of quantities of objects while denying that values of quantities are themselves quantities, thus assuming that quantities of objects are represented by numbers and still denying that numbers are ratios of quantities. However, this position does not deserve our consideration here for its lack of basic consistency. Indeed, on the one hand it should accept that any measurement unit $[Q]$ is a quantity of a given kind and therefore that numbers, $\{Q\}$, can be assigned to other quantities of the same kind by comparing these quantities, $Q(a)$, and the unit by ratio, $\{Q\} = Q(a) : [Q]$; but, on the other hand, it appears to reject the then obvious consequence that by inverting such a relation an actual equation is obtained. In other words, the compromise implied by this moderate representationalism makes it unable to provide a consistent explanation of a position according to which, say, the meter is acknowledged to be a length but the concatenation of two meters, i.e., the value 2 m, is not a length in turn, but only a "symbol" – whatever this means – that represents a length.

[6] It is important to note that, according to the representational interpretation, properly speaking we do not measure the length of an object, but to an object we apply a procedure whose outcome is a length-related representation of the object. The contrast with the classical tradition on this point is evident: where for example Campbell characterizes measurement as an assignment of numbers to represent *qualities of objects or events* (see [15, p. 267]), in the representational tradition it is widely assumed that measurement is an assignment of numbers to *objects or events* themselves (see for example Stevens' position, [26, p. 667]. So, in what is arguably one of the cornerstone of this tradition, it is said that "In measuring length ordinally, we confine our observations to comparisons between simple, unconcatenated rods, and we are concerned only with assigning numbers $\phi(a), \phi(b)$, etc. to rods a, b, etc. so as to reflect the results of these comparisons". (see [19, section 1.1]). Similarly, Roberts says that measurement has something to do with assigning numbers that represent observed relations, so that, for instance, if a is less long than b then $\phi(a) < \phi(b)$, making evident that the numbers $\phi(a)$ and $\phi(b)$ are assigned to objects, not to their quantities (see [22, section 2.1]). However, we will maintain here the less cumbersome and more usual phrasing that we measure quantities of objects, that a representationalist might accept as a linguistic shorthand.

[7] In the following formulation of the problem, a relation of homomorphism between an empirical system and a numerical system is exploited. A slightly different formulation can be obtained in terms of a relation of isomorphism by taking the quotient of the empirical system (see, for example [21, p. 4]: "The most fundamental problem for a theory of representational measurement is to construct the following representation: Given an empirical structure satisfying certain properties, to which numerical structures, if any, is it isomorphic? These numerical structures, thus, represent the empirical one. It is the existence of such isomorphisms that constitutes the representational claim that measurement of a fundamental kind has taken place").

[8] To illustrate, see [18, pp. 53–55]. Here Carnap shows how to define the relations E, of being equal in weight, and L, of being lighter or less than in weight, in terms an empirical procedure consisting of taking any pair of objects and determine how they compare by using a balance scale. Two rules are introduced: (1) if the two objects balance each other on the scale, they are of equal weight; (2) if the two objects do not balance, the object on the pan that goes down is heavier than the object on the pan that goes up. This procedure is not completely conventional: as E is conceived of as an equivalence relation, the relation of balancing on a balance scale has to be reflexive, symmetric and transitive; similarly, as L is conceived of as a strict partial order, the relation of being on the pan that goes down has to be irreflexive, asymmetric and transitive. Still, nothing ensures us that the empirical relations we use to define E and L actually fulfill these properties: hence, this is an empirical condition of the possibility of the definition of E and L.

The interpretation of the basic equation The interpretation of the basic equation is now straightforward: since objects are related to numbers by a scale which is unique up to a specific group of transformations, measuring on a given scale the quantity of an object coincides with assigning a number to the object by identifying the kind-related empirical relations that hold for the object and mapping the result to numbers through the scale. So, once a scale based on unit $[Q]$ is chosen, we get the following diagram:

$$D_Q \xrightarrow{Q_{[Q]}} \mathbb{R}^+$$

where

- D_Q is a domain of objects to which a set of relevant empirical relations can be applied
  (for example, D_Q could be again a set of rods);
- $Q_{[Q]} : D_Q \to \mathbb{R}^+$ is the function that takes an object $a \in D_Q$ and returns a number in the $[Q]$-related scale.

Hence, we get

- $Q_{[Q]}(a) = r$
  (the length in the $[Q]$-related scale of any rod is a number)

where a basic equation $Q(a) = \{Q\}[Q]$ is interpreted in this case only as an idiomatic form of $Q_{[Q]}(a) = \{Q\}$, so avoiding any reference to quantities as independent entities.

3.2. *The ontology underlying the representational interpretation*

The ontology behind this picture is such that

- there are objects;
- some objects are comparable by some empirical relations;
- there are numbers;
- some objects can be represented by numbers via a morphism that preserves the empirical relations among objects to relations among numbers.

Accordingly, a basic equation is interpreted as stating that an object in the empirical system determined by a set of relations that in some (usually unspecified) sense defines a kind of quantity (left hand side) is represented by a number in a scale that in some (usually unspecified) sense is characterized by a unit (right hand side).

4. The role of models in measurement

While referred to an empirical property, measurement is an information production process, not a purely empirical one. Since the outcome of the transduction performed by the measuring instrument needs to be interpreted in terms of information entities – for example the voltage induced on a thermocouple in terms of values of the applied temperature – a model of the object under measurement is unavoidably, though sometimes only implicitly, exploited in the production of the measurement result. Accordingly, model-based accounts of measurement have been developed since the last two decades by studying, in particular, measurement practices in the sciences.[9] If quantities are part of the framework, as in the classical tradition, three mutually related models are actually present, about (*i*) the kind of quantity that is measured (e.g., temperature), (*ii*) the object under measurement (the body whose temperature we are interested in), and (*iii*) the individual quantity that is intended to be measured (the temperature of the body).[10] Let us shortly discuss about these models before turning our attention to the way the basic equation can be interpreted in the light of them.

Models of kinds of quantities Kinds of quantities, and more generally kinds of properties, are modeled in two interdependent ways: with respect to their type, thus along the line of Stevens' theory of scales [26], and with respect to their relations to other kinds of quantities. Taking temperature as an example, different models can be – and historically have been – adopted as to its type, sometimes as only ordinal, when only the warmer-than and colder-than relations are assessed, but then as an interval quantity with thermometric scales, and finally as a ratio quantity when equipped with absolute zero in the context of thermodynamics. Depending on the adopted type, and then particularly when interpreted as a ratio quantity, temperature can be modeled as functionally related to other kinds of quantities, like volume and pressure in the case of gases, and voltage in the case of the Seebeck effect. In both cases, hypotheses on what physical laws hold about the concerned kind of quantity play a crucial role: in terms of the internal structure of the kind, i.e., what relations can be empirically observed among quantities of that kind, and the possible structure of such laws, as depending on the type of the quantity; and in terms of the laws according to which we interpret what the kind is as related to other kinds. It is typically basing on such models that measuring instruments are designed, and their behavior is understood, as in the case of the Seebeck effect that explains the transduction performed by a thermocouple. Thus, the first role of models in measurement is that kinds of quantities are modeled based on sets of laws that state connections, typically functional relations, among different kinds.

Models of objects under measurement What we measure are quantities of objects, and objects under measurement are entities that we usually interpret according to a model. Taking again temperature as an example, when we decide to measure the temperature of a given body, we are modeling the body as something that has one temperature, even if we know that different parts of the body might have different temperatures and that attributing a temperature to sufficiently small parts of the body is pointless. Hence, in order to attribute one temperature to the body as a whole, it is necessary to model it as thermally homogeneous. In view of this model we are then free to measure the temperature of any part of the body and assign the measured value the body as such. Thus, the second role of models in measurement is that objects under measurement are modeled as entities that have the quantity intended to be measured.

Models of individual quantities The quantities we measure are individual quantities, that we usually model in view of the purposes of the measurement and the circumstances in which it is performed. This assumes the critical distinction between the quantity we intend to measure and to which the measured value is attributed, i.e., the measurand, and the quantity that by interacting with the measuring instrument produces an effect on its state, and therefore that may be called the effective quantity. Again in the case of the measurement of the temperature of a body, while the effective quantity is the temperature of the part of body with which the thermometer interacts in the, typically only partially known, conditions of the interaction, we could be interested in measuring the temperature of the body in specified conditions, and this would require us to model the measurand by identifying the quantities by which it is affected, and then either to intervene to control them when measurement is performed or to correct the obtained measured value.

[9] For a general presentation of this emergent approach to the study of measurement, see [27–29]. For a more detailed analysis of the different levels at which models are exploited in measurement, see [30,31].

[10] Of course, other models need to be introduced in a more encompassing treatment of measurement, and in particular a model of the behavior of the measuring instrument, as provided by the information obtained from its calibration and the characterization of the conditions of the environment when the measurement is performed. See [32] for a detailed presentation of such a model.

Thus, the third role of models in measurement is that measurands are modeled based on sets of laws that state the connections, typically some functional relations, between what we want to measure and the other quantities that are to be taken into account when the measurement is performed.

In summary,

- any measuring instrument is designed so as to be sensitive to quantities of a given kind, and this is an empirical feature of the instrument, still independent of its calibration and the way the results of the interaction will be represented, and
- the object under measurement is modeled as having an individual quantity of that kind, and this involves both modeling the object and the individual quantity.

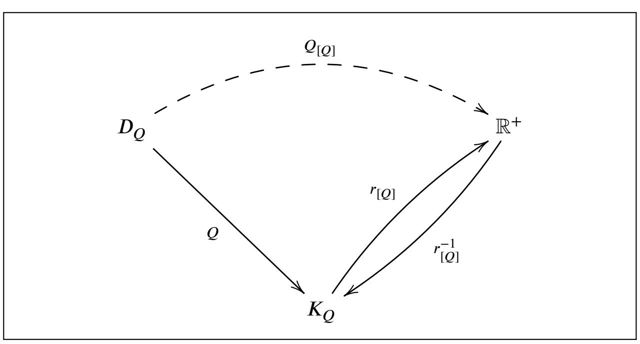
Thus, when for example the length of a given rod is measured

- *the rod is modeled as something having a length*, say as a cylindrical object;
- *this model allows us to design a suitable measurement procedure*, say by stipulating that measuring the distance between two appropriate points on the opposite sides of the rod using a meter stick suffices to get what we want.

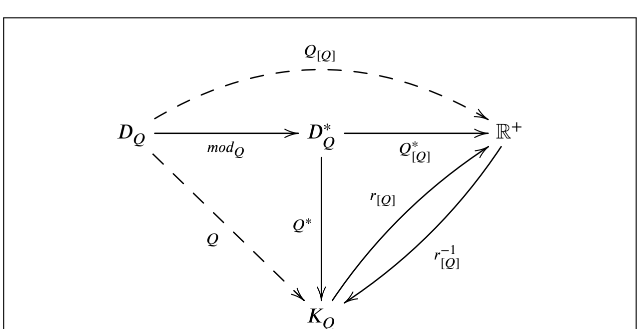
Still, while the process is made possible due to the model, we know that the rod with which the measuring instrument interacts is not a cylinder, as sufficiently precise measurements would show.

4.1. The role and significance of models on the equational interpretation

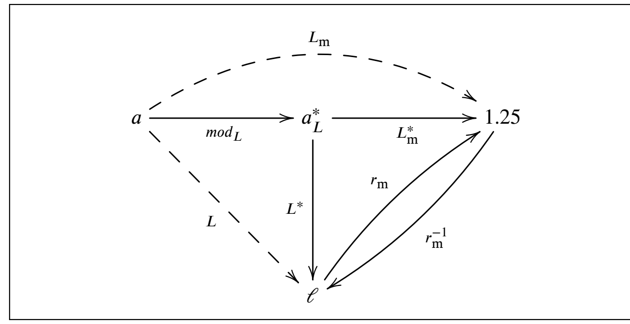
Let us go back to the equational interpretation of the basic equation. In accordance with the diagram



objects $a \in D_Q$ are characterized by quantities $Q(a) = \mathrm{q} \in K_Q$ having ratios with a unit $[Q]$. The previous analysis about the role of models leads us to introduce, as a new element in the diagram, the set D^*_Q of the mathematical models of the objects in the domain D_Q. The resulting picture is



an instance of which, about the measurement of the length of a rod, is



This diagram summarizes the following ideas.

1. For some given purposes, a given rod a is considered with respect to length L.
2. The rod a is modeled as a cylinder $a^*_L = mod_L(a)$, that by definition has a unique length $\ell = L^*(a^*_L)$.
3. The model is assumed to be adequate for the given purposes, and this justifies attributing that unique length $\ell = L(a) = L^*(a^*_L)$ to the rod a. This corresponds to assume that the lower left triangle $\langle a, a^*_L, \ell \rangle$ in the diagram above commutes, i.e., $L(.) = L^* \circ mod_L(.)$.
4. The meter, $\mathrm{m} \in K_L$, is chosen as the unit of length, and $L^*(a^*_L)$ has a unique ratio $L^*(a^*_L) : \mathrm{m} = r_{\mathrm{m}} \circ L^*(a^*_L)$ with the meter. This corresponds to assume that the lower right triangle $\langle a, 1.25, \ell \rangle$ in the diagram above commutes, i.e., $L^*_{\mathrm{m}}(.) = r^{-1}_{\mathrm{m}} \circ L^*(.)$.
5. The definition of the meter is realized by a measurement standard $s_{\mathrm{m}} \in D_L$, and a is compared with s_{m} with respect to length according to a suitable procedure. The process of comparison produces a numerical value 1.25.
6. The hypothesis of adequacy of the model justifies attributing the numerical value 1.25 to the ratio $L^*(a^*_L) : \mathrm{m}$, and therefore the value 1.25 m to the length $L^*(a^*_L)$ of the cylinder a^*_L. In this case the hypothesis bridges the empirical side to the mathematical side, since a numerical value that results from an empirical comparison is associated to a mathematical entity, namely the ratio $L^*(a^*_L) : \mathrm{m}$.
7. The hypothesis of adequacy of the model also justifies attributing the numerical value 1.25 to the ratio $L(a) : \mathrm{m}$, and therefore the value 1.25 m to the length $L(a)$ of the rod a. In this case the same hypothesis bridges the mathematical side to the empirical side, since the ratio $L^*(a^*_L) : \mathrm{m}$ is identified with the ratio $L(a) : L(s_{\mathrm{m}})$, thus justifying the attribution of the numerical value to this ratio.

Such a process is then a back-and-forth sequence between empirical objects and their quantities on the one hand and mathematical models and their quantities on the other hand, with the preliminary condition that a unit of length is chosen and its definition is realized:

- a numerical value can be attributed to $L(a)$ only if a is adequately modeled as a^*_L, i.e., only if the equation $a^*_L = mod_L(a)$ is adequate;
- the numerical value attributed to $L(a)$ in the chosen unit is determined by means of an empirical process involving a but interpreted through the model, so that what the process produces is $L_{\mathrm{m}}(a) = L^*_{\mathrm{m}}(a^*_L)$.

In summary, in order to measure a rod with respect to length and relatively to a chosen unit, we model it as having a cylindrical shape, and so as having a definite unique length. This assumption justifies us comparing by length the rod with a realization of the definition of the unit and producing a numerical value as the result. Hence, for a rod a, a length-related model of which is a^*_L, $L(a) = L^*(a^*_L) = r^{-1}_m \circ L^*_{\mathrm{m}}(a^*_L)$: through the model, the length of a is taken to be the same as the length that is 1.25 times the meter, and therefore the same as the ratio between $L^*(a^*_L)$ and m.

The introduction of the new element a^*_L in this framework deserves some comments.

1. The length-related model a^*_L of the empirical object a is an ideal entity, typically a mathematical entity, that has a definite length by definition: any cylinder has one length.
2. The rod a itself is considered to have a definite length $L(a)$ only because it is modeled so: no rod is perfectly cylindrical and at a certain scale the notion of length of an empirical macroscopic object is not well-defined.
3. The procedure that specifies how to measure $L(a)$ is based on the model of the rod: we measure one distance between two opposite vertices precisely because the rod is modeled as a cylinder (where a different model could require for example to measure the distance between several pairs of points and average the measured values).

Introducing an element like a^*_L, i.e., a measurand-related mathematical model of the object under measurement, is well and consistently understood in the equational interpretation, in which the existence of mathematical entities characterized by quantities is no surprise. So, the assumption that empirical objects become measurable provided that they are modeled by means of mathematical objects can be accepted with no modifications of the basic ontology sketched in Section 2.2.

In conclusion, we get a response to the key questions proposed in the introduction.

KQ1: How should we interpret a basic equation?

A basic equation states that the quantity identified as a property of an object on the basis of an assumed model (left hand side) and the quantity identified by its ratio with the unit (right hand side) are the same quantity.

KQ2: How should we define its truth conditions?

A basic equation is true provided that the model of the object under measurement is adequate for the measurement purposes, i.e., provided that any discrepancy between the model and the object is irrelevant given the accuracy of the measurement.[11]

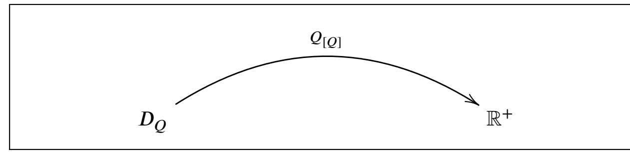
Furthermore, the derived questions are addressed in this way by the equational interpretation:

- measurands are quantities identified as properties of empirical objects, i.e., by address;
- measured values are quantities identified in terms of numerical values and units, i.e., by value;
- the relation holding between a measurand and a measured value in a basic equation is in principle an identity: a quantity identified by address and a quantity identified by value are one and the same quantity.

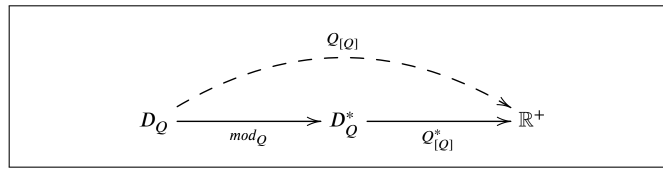
Similar conclusions can be obtained for properties other than ratio quantities.

4.2. The role and significance of models on the representational interpretation

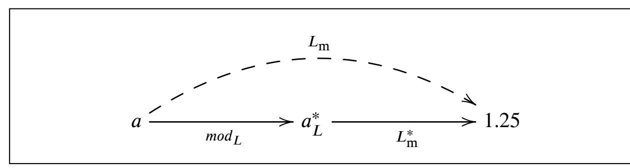
In the diagram that captures the tenets of the representational interpretation



objects $a \in D_Q$ are directly related to numbers $Q_{[Q]}(a) \in \mathbb{R}^+$. The fact that both objects under measurement and objects taken as measurement standards, i.e. objects associated with a numerical value obtained from a metrological traceability chain, are modeled entities leads us again to introduce, as a new element in the diagram, the set D^*_Q of the mathematical models of objects in D_Q. The resulting picture is



an instance of which, about the measurement of the length of a rod, is



This diagram summarizes the following ideas. In order to measure the length of a rod a in D_L, we model the rod as being an element of an empirical system that satisfies some conditions allowing us to prove a representation theorem and a uniqueness theorem with respect to a given numerical system. Since some of the conditions that are imposed on the empirical system might not be empirically checkable, this implies some degree of idealization,[12] with the (somewhat paradoxical) consequence that the empirical system is actually made of models of the empirical objects under measurement, not empirical objects themselves. Therefore, since the model $a^*_L = mod_L(a)$ of a is assumed to be an element of this system, we can assign a number to it based on a scale L^*_{m} which, in the case of ratio quantities, specifies how numbers are assigned in compliance with the condition that objects can be concatenated and compared in length for a given unit. So, since the length-related model of the rod is assumed to be comparable with the model s^*_{m} of some measurement standard that embodies the unit of length, we measure its length by comparing it with s^*_{m} and obtaining the number $L^*_{\mathrm{m}}(a^*_L)$ as a result. Finally, we attribute the measured value to the rod. Here a is length-represented by 1.25 under $L^*_{\mathrm{m}} \circ mod_L$, given the convention that $L^*_{\mathrm{m}}(s^*_{\mathrm{m}}) = 1$.

The introduction of the new element a^*_L deserves some comments.

1. The model of the rod is again an ideal entity, both because the conditions that an empirical system have to satisfy for us to be able to prove a representation theorem and a uniqueness theorem are typically ideal and because it is a cylinder.
2. The measurement procedure is again based on the model of the rod, since also in this case we decide to measure the distance between two opposite points precisely because the rod is modeled as a cylinder.

In the representational interpretation, introducing a mathematical object like a cylinder poses some problems: in fact, given the underlying antirealism, the existence of mathematical entities characterized by quantities should be suspicious. Hence, the assumption that empirical objects become measurable provided that they are modeled by means

[11] This answer is admittedly sketchy. For a more throughout discussion of this topic, see [33].

[12] It is generally assumed that empirical systems for ratio quantities include a total and transitive ordering relation and an operation of concatenation that supports infinite divisibility of objects: these are evidently ideal conditions. See [19,22,34] for a further discussion on this point.

of mathematical objects can be hardly accepted without modifying the basic ontology.

In conclusion, we get a response to the key questions proposed in the introduction.

KQ1: How should we interpret a basic equation?

A basic equation is only an idiomatic form of the actual equation $Q_{[Q]}(a) = \{Q\}$, interpreted as stating that the number to be assigned to an object to represent it in a given scale (left hand side) is a given number (right hand side).

KQ2: How should we define its truth conditions?

A basic equation is true provided that the model of the object under measurement is adequate for the measurement purposes, i.e., provided that the measurement implements the morphism between the empirical system containing the object and the measurement standard and the corresponding numerical system.[13]

Furthermore, the derived questions are addressed in this way by the representational interpretation:

- measurands are elements of a suitable empirical system, since the mapping between the empirical system and the numerical system is crucial for identifying what we intend to measure;
- measured values are elements of a suitable numerical system, since the mapping between the empirical system and the numerical system is again crucial for identifying a number as a value;
- the relation holding between measurands and measured values is in principle a representation, and mathematically a mapping between the empirical system and the numerical system: the object that we intend to measure is represented by a number given a unit.

In this case too, similar conclusions can be obtained for properties other than ratio quantities.

5. Comparing the interpretations

The advantages of the representational interpretation (RI) over the equational interpretation (EI) are evident: provided that the existence of a mapping $Q_{[Q]}$ is provable, under the assumption of the existence of objects in D_Q and the characterization of their relations, no other entities – and in particular no properties – are required to interpret a basic equation. Since the frameworks developed in the two traditions provide information whose mathematical treatment is basically the same, RI seems preferable to EI due to this ontological parsimony. Moreover, RI clearly encompasses EI, in the sense that if a value $\{Q\}[Q]$ is actually the same as a quantity $Q(a)$, as claimed by EI, then $\{Q\}[Q]$ can be also taken as a representative of $Q(a)$, as claimed by RI, whereas in general the represented object is different from the one that represents it.[14] This notwithstanding, we argue that, once an actual process of measurement is considered and the idealizations implied in introducing and using models are highlighted, the advantages of EI as a basis for a model-based interpretation of measurement become apparent.

5.1. Interpreting the role of the models

In a model-based version of EI, values are primarily attributed to quantities of the model of the empirical object under measurement, and then assigned to the quantities characterizing that object based on the modeling relation $a^*_Q = mod_Q(a)$. Thus, the length of a rod is the length of the cylinder that models the rod, and it is something comparable, for example, with the positions of the marks of a measuring tape or the product of the speed of an electromagnetic wave and the duration the wave takes to transit back and forth from a certain source. Of course, according to a different, and more refined, model, a rod is not interpreted as a cylinder anymore and therefore it does not have a definite length, but this remains consistent with the possibility of using one of the described measurement procedures, by then reporting a non-null definitional uncertainty to take into account the discrepancy between the empirical object and its model.[15]

In contrast, in a model-based version of RI the existence of a morphism to an appropriate numerical system, unique up to a certain group of transformations, is proved for D^*_Q, not D_Q, and therefore for an idealization of an empirical system of objects, not quantities [35]. This raises problems both about the connection between empirical and ideal entities and about the identification of what is measured. Indeed, empirical relations do not characterize ideal objects, such as cylinders. If ideal relations were admitted instead, either the result would be a purely mathematical framework, thus disconnected to the actual practice of measurement,[16] or we should ensure that such relations are defined with respect to the quantity on the basis of which the model of the object under measurement is constructed. However, this is not possible in a representational setting, due to its antirealism about quantities. In addition, the formally provable ideal relations should be related to something empirical, able to account for the specific interaction between what is measured and the measuring instrument, and this is again something modeled by an element of K_Q, which finds no place in RI. In sum, in the absence of quantities it is hardly conceivable how to account for the connection between the empirical objects and the ideal objects we use to model them, a connection that constitutes a necessary connection for us to develop and implement measuring instruments and measurement procedures.

Let us review our conclusions by highlighting four points that make EI more adequate than RI to explain the information provided by a basic equation when models, as discussed in Section 4, are taken into account.

1. Measurement is always measurement of properties of objects, not of objects as such, and this is consistent with the basic observation that objects have different modes of interaction with their environment, which do not depend on the way we measure or represent them. While the acknowledgment of the existence of properties is a crucial element of EI, it remains hidden in RI, in which it appears that properties are only features of representations, and not entities with causal power.[17]

[13] This answer is also admittedly sketchy. For a discussion of this topic, where a distinction between fundamental and derived measurement is at work, see [19,22].

[14] As a cogent consequence, compare how the two frameworks could describe the behavior of, say, a thermocouple, that transduces temperature to voltage. Once a value of voltage is somehow obtained, EI considers it only an intermediate, though necessary, step: under the acknowledgment that temperatures cannot be measured in volts, it requires the sensor to be calibrated so as to be able to produce a value of temperature. This condition is instead plausibly only optional in RI, according to which temperatures can be unproblematically represented also by numerical values in volts.

[15] For a discussion of how the discrepancy between models and modeled objects can be taken into account in terms of definitional uncertainty, see [33].

[16] See [7], where a strong criticism based on this point is proposed against the way measurement is being conceived in psychology, so later leading Michell to argue whether psychometrics is "pathological science".

[17] This point is stressed in [8, p. 286], where in commenting on the operationist standpoint Michell wrote that "its central principle was that the concepts investigated in science are constituted by the operations used to measure them, thereby confusing *what is measured* with *how it is measured* and denying the logical independence of what is known from the process of knowing it". On this matter Torgerson was very explicit (in his phrasing "system" stands for what we have called here "object"): "While the distinction between systems and their properties is perhaps obvious, it is nevertheless an important distinction. It is of special importance here because of the fact that

2. Measurement relates numbers to empirical objects based on empirical operations. Still, different kinds of empirical operations can be applied to the same objects, where a given operation can be applied to a given object precisely because that object has a given property. In addition, the identification of an empirical system depends on the identification of the empirical relations that characterize the objects of the system, and this identification is in turn dependent on the assumption that those objects share a given empirical kind of property. But then again, the acknowledgment of the existence of kinds of properties and of their instances, which is pivotal in EI, plays no role in RI.
3. Measurement relates numerical systems to empirical systems in virtue of the possibility of constructing a morphism between them. While EI provides a rich and consistent understanding of numbers as ratios between quantities, and this clarifies both what numbers are and why numbers can be used in measurement, the success of the representation of an empirical system by means of a numerical system remains unexplained in RI.
4. Measurement relates numbers to empirical objects in virtue of modeling operations, where models are ideal objects and the models of the empirical objects that are exploited in measurement are ideal objects having quantities. Still, the existence and suitability of such models for representing objects remain unexplained in RI. By contrast, in EI models are mathematical objects having quantities, and this clarifies both what models are and why models can be used in measurement.

In conclusion, the equational vs. representational opposition is an instance of the explanatory benefits vs. ontological costs trade-off, and as such a last word about it can be hardly said. However, an antirealist, quantity-independent, representational characterization seems to be not sufficiently specific to account for a model-based interpretation of measurement, which is instead adequately encompassed by a realist, quantity-based, equational characterization.

5.2. Conclusion

Let us summarize the basic points distinguishing EI from RI with respect to both the ontology they support and the tenets related to the entities they admit for, and then compare these in light of what we saw before.

Ontology	EI	RI
Empirical objects	✓	✓
Relations on empirical objects	✓	✓
Properties of empirical objects	✓	–
Mathematical models of objects	✓	✓
Quantities of mathematical models	✓	–
Relations on quantities of models	✓	–
Ratios between quantities	✓	–
Real numbers	✓	✓

Tenets of EI

1. empirical objects have empirical quantities and are modeled as having quantities;
2. quantities of modeled objects are associated values of quantities, and then with numbers;
3. relevant numerical relations are relations on modeled quantities;
4. values of quantities are quantities identified via units;
5. measurement is the attribution of a value to a given empirical quantity of a given object that makes the related basic equation true;
6. such an attribution is possible because
   (a) empirical objects are assumed to be characterized by empirical quantities since they are modeled as *mathematical objects having quantities*;
   (b) numerical values of quantities are real numbers, that is *quantity ratios*.

Tenets of RI

1. empirical objects are modeled as having quantitative relations;
2. modeled objects are associated with numbers;
3. relevant numerical relations represent relations on modeled objects;
4. numerical values of quantities represent modeled objects identified via scales;
5. measurement is the attribution of a numerical value to an empirical object that makes the related basic equation consistent with the available information;
6. such an attribution is possible because
   (a) empirical objects are assumed to be characterized by empirical quantities since they are modeled as *mathematical objects in a certain empirical system*;
   (b) numerical values of quantities are real numbers.

The main argument we have proposed is to the effect that point 6 is justified without problem in the equational interpretation, while it is in need of justification in the representational interpretation. To be sure, a straightforward justification for the representational interpretation of point 6 might run as follows: once empirical objects are modeled with respect to a kind of quantity, they can be compared as having individual quantities; such comparison allows for the assignment of quantity values given by numerical values and units. Still, this kind of justification is not available for the representationalists, as they do not admit for the existence of kinds of quantity and of individual quantities, and the addition of such entities to the ontology would render the representational standpoint indistinguishable from the equational one.

While reached about quantities with a unit, the core content of the analysis proposed in this paper – that the basic equation is more adequately understood in the specific interpretation of being an actual equation, instead of the generic interpretation of being a representation – does not depend on any assumption about the algebraic structure holding on properties (as in EI) or objects (as in RI). Indeed, for any kind of quantity, a unit can be simply thought of as a quantity that through its multiples and submultiples is singled out to induce a classification of the quantities of that kind. Though construed and constructed via different strategies, the idea that values of properties are basically classifiers for properties of objects applies independently of the type of the property concerned, and thus also to ordinal quantities and nominal properties. Hence, our conclusions may be framed in the broader, evolutionary picture of metrology encompassing also non-quantitative properties.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

it is always the properties that are measured and not the systems themselves. Measurement is always measurement of a property and never measurement of a system” [36, p. 14].

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