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BOUNDED RATIONALITY AND HETEROGENEOUS EXPECTATIONS IN ECONOMIC DYNAMICS

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Aristoteles

One may say: "the eternal mystery of the world is its comprehensibility".

Albert Einstein, 1936

Someday girl/ I don't know when/ we're gonna get to that place
where we really wanna go/ and we'll walk in the sun/ but till then tramps like us
baby we were born to run.

Bruce Springsteen, 1975

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Questa tesi è dedicata a voi.

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Chapter 1

General introduction and theoretical background

Economic systems are characterized by a mutual interaction between the actions of economic actors, e.g. individual consumers or firms, and the economic environment in which these actions are evaluated and realized. This kind of dependance is due to the existence of a kind of aggregation of all individual choices and, at the same time, individual choices are affected by the overall economic environment.

Financial markets are an example of this interaction between actions and economic environment: the demand of investors for an asset is driven by expected future returns whereas, at the same time, asset returns are determined by investors' demand through realized asset prices. Additionally, in a market of a perishable consumption good, firms have to establish today how much to produce for tomorrow market supply. In this way firms base their production decisions upon tomorrow's expected profits and, conversely, expected profits depend on the total amount of firms production. Considering these simple examples, the mutual dependance that links individual decisions and economic environment is related through a feedback relation between expectations and realizations of economic variables. This is the reason why we can call these systems *expectational feedback systems.*

Starting from this assumption, we can state that expectations play a central role in financial markets and in every segments of macroeconomics. Indeed individual economic decisions today depend upon expectations about the future state of the global economy. Through these decisions, expectations feed back into the actual realizations of economic variables and therefore any dynamic model depends crucially on its underlying expectations hypothesis.

Traditional economic analysis has circumvented the troubles arising through the type of interaction of many individual actions, by assuming that all agents are *rational.* Rational expectations (RE) have become the leading paradigm on modeling expectations in economics, starting from the seminal works of Muth (1961), Lucas (1972) and Sargent (1993). The idea of rational expectations lies on two components: the first considers the behavior of each individual that can be represented as the outcome of maximizing an objective function subject to perceived constraints; the second is the mutual consistency of the constraints perceived by all the individuals that populate the economic environment. In such a framework, all agents are the same, expectations are model consistent and coincide on average with realizations, without systematic forecasting errors. This would lead directly to an equilibrium point where choices need not to be revised, unless unanticipated changes of the exogenous parameters characterizing the environment or the decision makers, take place. Therefore, if agents were rational, observed changes in economic variables should come from a response to unexpected changes in some exogenous characteristics, or fundamentals, of the economy.

However RE models rest on the unrealistic assumption of perfect knowledge of the economy. Sargent, for example, argues that not only the agents have to be endowed with a substantial amount of information in order to form RE, but even if perfect knowledge of the market was available, RE requires too strong computing abilities of the agents to make decisions such that all predictions and beliefs are consistent with the outcome of all agents' choices.

Several tests and empirical analysis have shown that agents' predictions and beliefs are often at odds with the requirements of RE, and enlighten that economic variables fluctuate, even when changes in the fundamentals of the economy do not occur. Direct evidence against rationality consists, for example, in showing that individual responses to simple economic decisions typically present systematic errors and psychological biases. Indirect evidence against rationality has been gathered from empirical tests of the predictions of economic models built under the assumptions of RE, including households consumption data, survey data on expectations of inflation and other variables, which commonly reject the unbiased hypothesis of RE and their efficiency in predictions.

1.1 Bounded rationality and heterogeneous expectations

In this thesis we investigate the possibility that economic fluctuations can be explained through the interaction of *boundedly rational agents*, that is, agents are not assumed to be rational and are not necessarily able to solve the mutual interaction implied by the expectational feedback.

Generally speaking, a boundedly rational agent is modeled as being able to select what he perceives as the best alternative in a decision making process, but he does not know the exact structure of the economic environment. Simon (1955) argues that individuals are limited in their knowledge and in their computing abilities, they face search costs to obtain sophisticated information in order to pursue optimal decision rules. These limitations make agents able to use simple behavioral rules and adapt they behavior over time, switching from time to time to better performing rules. Hence bounded rationality with agents using simple satisfying rules of thumb for their decisions under uncertainty, could be a more accurate and realistic description of human behavior than perfect rationality with fully optimal decision rules.

When predicting future variables, bounded rationality implies that individuals do not know the true equilibrium distribution of aggregate variables and, as a matter of fact, ex-ante predictions and ex-post outcomes need not to coincide on average. Moreover a boundedly rational agent keeps on updating his strategies as he learns about the economic environment through feedbacks about his past decisions.

The decision whether to model agents as rational or boundedly rational is part of the assumptions of an economic model. Researchers adopting RE models often argue that rationality is a useful assumption to describe the equilibrium outcome of the trial-and-error processes that agents employ. According to this view, the repeated interaction of boundedly rational agents leads to the same outcome *as if* agents were perfectly rational. The general underlying idea is that agents who are not rational would learn to be rational over time since incentives to behave rationally, such as higher profits or utility, are constantly at work. In summary, assuming rationality is often based on the presumption that this approach offers the equilibrium outcome of repeated interaction.

Convergence to rational behavior has been the topic of investigation of many theoretical papers on bounded rationality. In macroeconomics, much work has been done on *adaptive learning*, see e.g. Sargent (1993) and Evans and Honkapohja (2001) for detailed overview. Boundedly rational agents do not know the true law of motions of the economy but in-

stead use time series observations to form expectations based on their own perceived law of motion, trying to learn the model parameters as more and new observations become available. Much of these works focused on the stability of RE equilibria and addressed the possibility of agents learning to form rational expectations. In fact, adaptive learning may enforce convergence to RE equilibria but it may also lead to non-RE equilibria.
Learnability of RE equilibria depends on the structure of the interaction mechanism between individual expectations and the economic environment, as the signals received from the market might be deceptive to agents trying to obtain rational expectations through learning.
As Bullard (1994) synthesized, *some rational expectations equilibria are learnable while others not. Furthermore, convergence will in general depend on all the economic parameters of the system, including policy parameters.*
Although adaptive learning has become increasingly popular as an alternative paradigm to model private-sector expectations, most models still assume a representative agent who is learning about the economy.
In deviating from rationality and modeling agents as boundedly rational, it is often assumed that agents are *heterogeneous*. There are several arguments in support of *heterogeneous expectations*, as summarized by Kirman (1992, 2010) and Hommes (2006). One commonly referred is the "no trade" argument, which states that in a world where all agents are rational and it is common knowledge that everyone is rational, there will be no trade. However, no trade theorems are in contrast with the high daily trading observed in real markets, and this reinforces the idea of heterogeneous expectations. Moreover heterogeneity in individual expectations has been widely documented empirically. For example, Frankel and Froot (1990), Allen and Taylor (1990), Taylor and Allen (1992) find that financial experts use different forecasting strategies to predict exchange rates. More recently Carrol (2003), Mankiw, Reis and Wolfers (2003), Branch (2004) and Pfajfar and Santoro (2010) provided supporting evidence for heterogeneous beliefs using survey data on inflation expectations, while Hommes, Sonnemans, Tuinstra and van de Velden (2005, 2007), Adam (2007), Pfajar and Zakelj (2010) and Hommes (2011) find evidence for heterogeneity in learning to forecast laboratory experiments.
Bounded rationality and learning in a complex environment naturally fit with *heterogeneous expectations*, with the economy viewed as complex evolving system composed of many different, boundedly rational, interacting agents, using different decision strategies. Hence in this thesis we assume that agents are heterogeneous, in the sense that they choose

different simple decision rules to address the same decision problem. In general, rules can differ in terms of sophistication, where the most sophisticated rule correspond to rationality. We also assume that the higher the sophistication of a rule, the higher the deliberation cost an agent pays in order to use it. Rules can also differ in terms of information they use, where information can also be costly. Moreover, instead of considering fixed fractions of agents adopting each rule, we will let them evolve over time as a function of their "fitness". We employ an evolutionary approach: a rule that has performed better according to some measure, to be defined case by case, is used by a higher fraction of agents. Hence, in general, we assume that agents have only knowledge of their objectives and of the constraints that they face, but they do not have a full economic model of determination of aggregate variables. Individual decisions are taken optimally on the basis of subjective expectations of future evolution of endogenous variables. Different agents will generally make different choices when facing the same economic problem.

The *wilderness* of bounded rationality in agent-based models leaves many degrees of freedom in economic modeling, and it seems far from clear which rules are the most reasonable out of an infinite class of potential behavioral rules. In model with RE *perfect consistency* between beliefs and realizations is assumed. Alternatively, in a bounded rationality research program it is required a reasonable and conceivable form of consistency between beliefs and realization.

Here we focus on the role of *behavioral rationality* and *heterogeneous expectations* within stylized different models. Behavioral rationality emphasizes the use of simple decision rules - *heuristics* - which are not perfect and need not to be optimal. The *endogenous evolutionary selection* or *reinforcement learning* among heterogeneous decisions is the form of learning that we employ to discipline the class of decision heuristics, according to the switching framework of Brock and Hommes (1997). The main idea is that agents tendo to switch to rules that have performed better, according to some suitable economic performance measure, in the recent past.

Behavioral rationality and heterogeneous expectations lead to highly *nonlinear* dynamical systems, because the fractions attached to the different rules are changing over time. Often, the evolutionary system does not necessary lead to a rational expectation equilibrium but it can exhibit complicated dynamics or perpetual fluctuations. As stated by Hommes (2013), when some rules act as "far from the steady state stabilizing forces" and other rules act as "close to the steady state destabilizing forces", evolutionary selection of expectation rules may generate potential instability and chaos in a complex adaptive system with behaviorally rational agents.

1.2 Thesis outline

This thesis is built around three main economic frameworks, which are developed in separate chapters. Each chapter is self-contained, with its own introduction, conclusion, notes and appendices as needed. Thus each chapter can be read independently from the others. This section briefly discusses the main contents of each chapter.

1.2.1 Booms and Busts in a Housing Market with Heterogeneous Agents

In chapter 2 we study the housing market using a partial equilibrium model in which the rational expectations hypothesis is relaxed in favor of chartist-fundamentalist mechanism to allow for the endogenous development of bubbles.

Although boom and bust home price cycles have occurred for centuries, the recent boom-bust development seems to dwarf anything seen before. Since the late 1990s, dramatic home price rallies have been observed in cities in countries such as Australia, Canada, China, France, India, Ireland, Italy, Korea, Russia, Spain, the United Kingdom, and the United States. Some of these price movements can be called spectacular.

It seems impossible to explain this phenomena merely on a rational point of view because fundamentals such as real rents or construction costs do not match up with this incredible price boom. The speculative thinking and the use of non rational expectations deriving from market psychology are elements that play an important role in determining house prices. In particular Shiller (2005, 2008) was the first who emphasized the role played by speculative thinking in particular in recent spectacular price movements. He suggested that the same forces of human psychology in the form of optimism and pessimism, herd behavior and social contagion of new ideas, and positive feedback dynamics are elements that play an important role in determining housing prices.

Furthermore, the recent fluctuations in housing market have increased the interest of researchers in this field but it is still difficult to explain the large and rapid rise and fall in housing prices using a purely rational model. Some recent papers use models of learning to explain the observed phenomena. Adam, Marcet and Kuang (2010) developed a model which can replicate quantitatively the house price dynamics from 2001 to 2008 in the G7 economies as well as the associated current account, relaxing the rational expectations hypothesis and allowing households to be uncertain about how house prices are related to the economic fundamentals. They use the idea of *internal rationality*, previously developed by Adam and Marcet (2010, 2011), where utility maximizer agents do not fully understand how price are formed, so that their subjective probability distribution about prices may not exactly be equal to the true equilibrium distribution.

Recurrent boom-bust house price cycles have raised the need to incorporate bounded rationality into housing market models, (e.g. Schiller 2007a-b, Piazzesi and Schneider 2009, Disci and Westerhoff 2012-2013, Tomura 2012, Bolt at al. 2011) and to provide endogenous explanations for such phenomena.

The goal of the present work is to develop a simple model of a stylized housing market to account for these observations. Our approach is inspired by recent work on agent-based financial market models (see Hommes 2006 and LeBaron 2006 for comprehensive surveys). In these models, the dynamics of financial markets depends on the expectation formation of boundedly rational heterogeneous interacting agents. As indicated by a number of empirical papers (summarized in Menkhoff and Taylor 2007), financial market participants rely on technical and fundamental trading rules when they determine their orders.

The structure of our model reflects the one of Adam, Marcet and Kuang (2010) but it moves away from it in the type of expectations we adopt. Houses are seen as assets that can be driven by fundamentals and by animal spirits. Starting from this point, the possibility to predict future changes in house prices and the deviations between housing prices and fundamentals create opportunity of large gains.

We assume that housing prices adjust with respect to excess demand in the usual way by an Agent-Based mechanism of chartism and fundamentalism, where agents use adaptive learning rules and the continuous evaluation of those strategies according to past performance: this leads to changes in the size of the different groups and finally to the price dynamics.

The reason for this choice is that we want to analyze how households beliefs and psychological variables can influence the housing boom and bust dynamic. As stressed in Piazzesi and Schneider (2009), who present evidences from the Michigan Survey of Consumers, the percentage of the households, believing it was a good time to buy a house because price would be raised further, increased towards the end of the boom. The mechanism of chartism and fundamentalism is one of the simplest method to take into account two different strategies but it is also sufficient to create endogenous movement in house price due to the different weight that each strategy plays. In particular the interaction between heterogeneous agents allows for the behavioral foundation of the expectations, the endogenous

development of bubbles and contributes to replicate the recent house price dynamics.
Adam, Marcet and Kuang (2010) also discuss the role of the interest rate during this crisis: the house price boom would be caused by the persistent reductions in the interest rate. They suggest that for the U.S. economy the boom would have been largely avoided if the interest rate had fallen by less at the beginning of the 2000's. Gelain, Lansing and Mendicino (2013) evaluate various policy actions that might be used to dampen the excess volatility in a DSGE model where the introduction of simple moving-average forecasting rules for a subset of agents can significantly magnify the volatility and persistence of house prices and household debt relative to an otherwise similar model with fully rational expectations. They find that a debt-to-income type constraint is the most effective tool for dampening overall excess volatility in the model economy.
We also analyze the effect of a policy that takes into account the deviation and the volatility of the house price supporting the idea that the Central Bank is able to reduce the price volatility by connecting the interest rate to the house price dynamic.

1.2.2 Heuristics selection and heterogeneity

In chapter 3 we present evidence that evolutionary selection among different forecasting heterogeneous heuristics can explain coordination on individual behavior. The model we develop is able to exhibit either convergence to an equilibrium price or persistent deviations from that, with the appearance of strange dynamics, similar to what it is possible to observe in reality: indeed asset price fluctuations are characterized by high volatility with large price changes irregularly interchanged by episodes of low volatility with small price changes.
Asset markets, involving an extremely large number on investors of different characteristics, are a suitable context for modeling the interaction of heterogenous boundedly rational agents.The failure of the representative rational agent framework in replicating properties of asset returns, persistent deviations from fundamental values, explains why most of the research in the area of bounded rationality and heterogeneity has been pursued in the context of financial markets. As a matter of facts, the evolution of economic variables, is affected by expectations of agents operating in the financial and real markets: for this reason it is possible to think to the market as an expectations feedback system: market history shapes individual expectations which, in turn, determine current aggregate market variables and so on.
In this work we present a simple model with evolutionary selection among different simple

forecasting strategies where the economic environment is seen as a complex evolutionary system between competing boundedly rational trading strategies. In this work we develop a simple nonlinear model which is able to exhibit path dependance explaining how both stable steady states and attracting curves can arise endogenously in the model.
In this multi-agent model, endogenous fluctuations are caused by a generic phenomenon, that is coexistence of two attractors (a steady state and a periodic or quasi-periodic orbit). The economic intuition behind these different outcomes (persistent oscillations and convergence) could be explained by the interaction and the evolutionary switching between trend extrapolation and stabilizing fundamental analysis that may lead to coexistence of locally stable fundamental steady state and a locally attracting closed curve far from the steady state. In particular the model presented here is an attempt to generalize the idea of *adapted belief system* (ABS) introduced by Brock and Hommes (1997, 1998) and then developed by Anufriev and Hommes (2012).
Laboratory experiments with human subjects have shown that individuals do not behave in a full rational way but follow simple *heuristics* which can account for persistent biases in taking decisions. This occurrence explains why prices may persistently deviate from fundamentals in laboratory markets, similarly to what can be observed in real stock markets. Moreover heterogeneity is crucial to the aim of expounding a number of evocative findings of the recent learning to forecast experiments. In a typical session, as described by Hommes et al. (2005, 2007), a limited number of human subjects have to make forecasts about the price of an asset for 50 periods, having knowledge of the fundamental parameters and previous price realization. The data coming from these experiments can be used as a benchmark for different expectations hypotheses, such as rational expectations or adaptive learning models. Many sessions of these kind of experiments have been conducted: some of them exhibited price convergence, others showed that prices can persistently fluctuate and temporary bubbles emerge. Three different patterns in aggregate price dynamics have been observed in recent *learning to forecast* experiments: slow monotonic convergence, permanent oscillations and dampened fluctuations.
We present a simple model with evolutionary selection among different simple forecasting heuristics and the economic environment is seen as a complex evolutionary system between competing boundedly rational trading strategies. The choice of heuristics will be governed by an evolutionary selection mechanism, based on the principle that more successful strategies will attract more followers. Furthermore this work tries to give an explanation to the outcomes observed in the learning to forecast experiments.

The main achievements of the experiments are:

- human subjects tend to follow simple forecasting predictors and set up their decisions on past observations;
- participants are able to coordinate on a common prediction strategy even if this can be different between sessions;
- three different price patterns were observed (slow and almost monotonic convergence, persistent oscillations with almost constant amplitude, dampening fluctuations);
- realized asset are significantly different from the rational fundamental price in every sessions.

The model we are showing is able to exhibit either convergence to an equilibrium price or persistent deviations from it, with the appearance of strange dynamics, similar to what it's possible to observe in reality: indeed asset price fluctuations are characterized by high volatility with large price changes irregularly interchanged by episodes of low volatility with small price changes. In particular there's empirical evidence that many "stylized facts" observed in financial time series recall the presence of endogenous fluctuations that cannot be explained uniquely by external factors or by fundamentals. For these reasons, the purpose of this paper is to show these facts simultaneously by a simple behavioral model of individual learning. In particular the ABS here considered is a present discounted value asset pricing model with heterogeneous beliefs: there are two trader types and the fractions of these types change over time according to evolutionary fitness, as measured by utility from realized profit. The economic intuition behind these different outcomes (persistent oscillations and convergence) could be explained by the interaction and the evolutionary switching between trend extrapolation and stabilizing fundamental analysis that may lead to coexistence of locally stable fundamental steady state and an attracting closed curve far from the steady state.

1.2.3 Macroeconomic stability and heterogeneous expectations

In chapter 4 we consider a simple model made up by the standard aggregate demand function, the New Keynesian Phillips curve and a Taylor rule to deal with different issues, such as the stabilizing effect of different monetary policies in a system populated by heterogeneous agents. The response of the system depends on the ecology of forecasting rules, on agents sensitivity in evaluating the past performances of the predictors and on the reaction to inflation. In particular we investigate whether the policy makers can sharpen macroeconomic stability in the presence of heterogeneous expectations about future inflation and output gap and how this framework is able to reduce volatility and distortion in the whole system.

In this work we study the stability properties of a macroeconomic model in which agents have heterogeneous expectations. We would like to ask to the following questions: How many stable or unstable equilibria emerge if there are agents predicting future variables value using different forecasting rules? How stability conditions change in a framework with heterogeneous expectations? How monetary policy should be designed in order to guide the system to a stable equilibrium?

We address these questions using the standard three equations system composed by the IS curve, a New Keynesian Phillips curve and a Taylor rule. According to the benchmark model of Branch and McGough, our setting has the same functional form as the standard formulation except for the homogeneous expectation hypothesis which is replaced with a combination of heterogenous expectations. As a consequence, the dynamic properties of the model depend crucially on the distribution of agents. Generally most of the models introducing heterogeneous expectations consider individuals with too many cognitive skills: they do not fully understand the underlying model due to informational inertia. For this reasons we consider a parsimonious model with simple rules of thumb which is able to generate endogenous waves of optimism and pessimism; moreover the analysis of monetary policy is conducted to investigate the role of inflation and output gap in business cycle movements.

Some recent examples of macro and financial models with heterogeneous expectations include the works of Evans and Honkapohja (2003, 2006), Bullard and Mitra (2002), Hommes (2006), Ascari et al. (2012).

More recently Hommes (2011), Assenza et al. (2011) and some others studies, provided evidence in favor of heterogeneous expectations using laboratory experiments with human subjects.

However, the question how to manage expectations when forecasting rules are heterogeneous has hardly been addressed. Since agents are assumed to have cognitive limitations, they only understand small bits of the whole model and use simple rules to guide their behavior. We introduce rationality in the model through a selection mechanism in which agents evaluate the performance of the forecasting they are following and decide to change their strategy depending on how well it performs relative to other ones. In our stylized model agents form expectations about the future rate of inflation and output using different forecasting principles. We employ the heterogeneous expectations framework of Brock and Hommes (1997), where the ecology of forecasting rules is disciplined by endogenous, evolutionary selection of strategies with agents switching between forecasting rules on the basis of their past performance.

The model can show how the business cycle dynamics depends on the expectations environment and the coefficients of an interest rate rule. If the monetary policy reacts weakly to inflation, a cumulative process of rising inflation and output appears. Signals from the market lead the economy to non-fundamental steady states, reinforced by self-fulfilling expectations of high inflation. On the contrary, when the response to inflation is moderate, the heterogeneous expectations can be managed in order to correct past forecast error and to conduct the economy towards the RE equilibrium. Even with an aggressive monetary policy, the monetary authority is able to send correct signals to agents and can induce stable dynamics settling down to the fundamental steady state. It is also worth to point out that even if the Taylor principle is sufficient to guarantee convergence to the fundamental steady state, it is no longer enough to avoid multiple equilibria. Indeed the monetary policy rule must be sufficiently aggressive to guarantee a proximity between the realized inflation and the RE equilibrium.

We have also to highlight that, in the case of many beliefs types (a *continuum of beliefs*), a monetary policy rule that reacts aggressively to current inflation can fully stabilize the system. If the policy rule is not aggressive enough and the intensity of choice is large, the cumulative process of inflation and output appears again.

Finally, to get some policy outcomes, we consider two summary indexes (i.e. volatility and distortion) that link the impact of the Taylor rule coefficients to distortion and volatility of the fundamental variables. In this heterogeneous agents framework, policy makers can reduce volatility and distortion of output and inflation with a sufficient degree of reaction. If the Central Bank is keen in inflation targeting, there exists a trade-off where lower inflation variability is obtained at the cost of increased output variability. Moreover some output stabilization is good because it reduces both output and inflation variability by preventing too large switches in forecasting behavior.

Depending on the target of the monetary authority, inflation volatility and distortion can be minimized but also output stabilization can be taken into account. Indeed, if the Central Bank shifts its target from inflation to output, results suggest that there exists a trade-off between inflation and output distortion but, a strong reaction to output is more likely to stabilize the economy.

1.3 Computational tools for nonlinear dynamics

In models with bounded rationality and heterogeneity there is an emphasis on dynamics, and most of these models are nonlinear. As a consequence, the theory of nonlinear dynamical systems is an important tool of analysis. Indeed nonlinear dynamics has become a widely used instrument in recent years.

In this thesis, we model agents' interaction and decision making taking place at discrete times and we focus on discrete time dynamical systems. Typically one of our model equations is the expectational feedback map related to the equilibrium pricing condition, and the others are the updating rules for the fractions of agents using different decision strategies. The fact that these fractions are endogenously determined, yields strong nonlinearities. In general we rely on simulations of the deterministic skeletons of the models but we provide also some analytical results.

We typically proceed as follows: firstly we search for the steady state(s) of the system which typically corresponds to rational behavior. After a steady state is detected, we use local stability analysis to specify for which parameter values the interaction and adaptation of boundedly rational agents converges to it. We encompass also the occurrence of periodic or complicated chaotic fluctuations. The dynamics of nonlinear systems is richer than linear models because it can be characterized either by convergence to a stable cycle or by irregular fluctuations.

Since it is often difficult or impossible to characterize the global dynamics analytically, we have to use computational tools.

A useful numerical tool for detecting changes in the long run dynamics as one or two parameters of the model change is the bifurcation diagram in which the parameters are varied and, for a grid of parameter values, the system of equation is simulated and the resulting long run behavior plotted. In such diagrams one can see that for some parameter values the state variable converges to an equilibrium value, whereas for other parameter values the state variable oscillates along a periodic cycle or follows a more complicated path.

Moreover we consider also the basins of attraction as an important tool to study the dynamical properties of the models we develop. A basin of attraction is the set of points in the space of system variables such that initial conditions chosen in this set dynamically

evolve to a particular attractor. These dynamic models may have time evolutions that exhibit bounded dynamics which may be periodic, or quasi-periodic or chaotic. In such cases, a delimitation of a bounded region of the state variables space where the system dynamics are ultimately trapped, despite of the complexity of the long-run time patterns, may be an useful information for practical applications. Moreover, in the case of several attractors, the dynamic process becomes path-dependent, i.e. which kind of long run dynamics is chosen depends on the starting conditions. This naturally leads to the delimitation of the basins of attraction and their changes as the parameters of the model vary. These two problems lead to two different routes to complexity, one related to the complexity of the attracting sets which characterize the long run time evolution of the dynamic process, the other one related to the complexity of the boundaries which separate the basins when several coexisting attractors are present. Both the questions outlined above require an analysis of the global dynamical properties of the dynamical system.
Therefore nonlinear dynamics, chaos and complex systems have important consequences for the validity of the rational expectations hypothesis. In a simple, linear and stable economy with a unique steady state path, it seems natural that agents can learn to have rational expectations, at least in the long run. A representative, perfectly rational agent model fits into a linear view of a globally stable and predictable economy. But how could agents have rational expectations or perfect foresight in a complex, nonlinear world, with prices and quantities moving irregularly on a strange attractor? A boundedly rational world view with agents using simple forecasting strategies, perhaps not perfect but at least approximately right, seems more appropriate within a complex, nonlinear world. Applications of tools from nonlinear dynamics and complex systems theory have stimulated much work in heterogeneous agents models, which are almost always highly nonlinear, adaptive systems.

Chapter 2

Booms and Busts in a Housing Market with Heterogeneous Agents

2.1 Introduction

Since the late 1990's a dramatic increase in housing prices has been observed in most of the countries around the world. For example, London real house prices tripled during the period 1996-2008, and in the United States the housing prices increased by 85 percent roughly during the same period.

It seems impossible to explain these phenomena merely from a rational point of view because fundamentals such as real rents or construction costs do not match up with this large price boom. Shiller (2005, 2008) was the first to emphasize the role played by speculative thinking, extrapolative expectations, optimism/pessimism deriving from market psychology in determining the dynamics of house prices, particularly in the recent spectacular price movements. He suggested that the same forces of human psychology driving financial markets could also have the potential to affect other markets, especially the housing market. Recurrent boom-bust house price cycles generate the need for an endogenous explanation for such phenomena, possibly incorporating bounded rationality into housing market modeling.

In this paper, we try to take on this challenge. We aim to build a stylized model of the housing market with no rational expectations able to produce endogenous prolonged movements in the house prices. In this model the house is an asset that can be collateralized and whose price can be driven both by fundamentals and by *animal spirits*. More generally, we want to stress and investigate the importance of the behavioral approach and bounded rationality. Note that ample empirical evidence exists to show that agents generally act in a bounded rational way (e.g., Kahneman and Tversky, 1973, Hommes, 2011).

To do so, we build on the model of Adam, Marcet and Kuang (2011) but we modify it in two ways. First, we introduce a different timing between demand and supply to account for the fact that it takes time to build new houses. So while households take their decision daily, the supply is based on a quarterly frequency. Hence the model generates two different dynamics for the house price at daily and at quarterly frequency, respectively. Second, we change the way agents form expectations. Instead of using Bayesian learning, we employ the Agent-Based[1] framework of chartism and fundamentalism, where these two types of agents use different adaptive learning rules to forecast the future house price. Households are maximizing agents but they can be either chartists, believing the house price trend will continue, or fundamentalists, expecting mis-pricing will be corrected by the market. Moreover, they continuously evaluate these two different strategies according to past performance; this leads to endogenous shifts in the relative shares of the two groups (chartists vs. fundamentalists). These shifts have large effects on the house price dynamics. When chartists dominate the market, house price can sharply deviate from the underlying fundamental value but, if the *animal spirits* change, the market will be dominated by fundamentalists and the price will revert towards the fundamental value.

This type of framework have been used in research in financial markets.[2] Using it to model the housing market seems a logical step given the chaotic state of real-estate markets in the last decade. We thus adapt to the housing market the setup in Lengnick and Wohltmann (2010) and Westerhoff (2008) and then insert it into our model structure, derived from Adam, Marcet and Kuang (2011). The mechanism of chartism and fundamentalism is one of the simplest methods that allow taking into account households' believes and behavioral factors. As stressed in Piazzesi and Schneider (2009), the percentage of the households, believing it was a good time to buy a house because price would be raised further, increased towards the end of the boom.

We show that such a model is able to generate endogenous boom-and-bust cycles in

[1]To have a survey on the Agent-Based Computational models visit the Tesfatsion website, www.econ.iastate.edu/tesfatsi/ace.htm

[2]See Brock and Hommes (1998), LeBaron (2006) or Westerhoff (2008).

the house price. The evolutionary selection of the two different forecasting rules by the agents causes waves in the relative shares of the two groups of agents who amplify and protract initial shocks. We use the partial equilibrium model of Adam, Marcet and Kuang (2011), because it allows us to find a closed form solution for the housing demand function and thus to identify and analyze the relevant feedback and amplification mechanisms in the model.

Moreover, the model is able to replicate the recent boom-and-bust cycle in the US house prices. We are able to discuss three determinants that the literature suggests as potential sources of the recent boom-and-bust cycle in the US house prices. First, the "Greenspan put" explanation that claims that the house price boom would have been caused by persistent low levels in the interest rate. Second, an explanation based on an overall loosening of credit standards that allowed more borrowing from the households, followed by a sudden freeze of credit at the onset of the crisis (e.g., Favilukis, Ludvigson and Van Nieuwerburgh, 2010, Mian and Sufi, 2009). Following Justiniano, Primiceri and Tambalotti (2013) we can call it a "credit liberalization" cycle narrative. The third, instead, refers to an explanation not based on fundamentals, but on exogenous forces modeled as a change in households' preference and housing demand. Justiniano, Primiceri and Tambalotti (2013) call it a "valuation" story; we could also name it a "behavioral" story because it is based on a change of behavior of agents unrelated to fundamentals. Our model identifies the shock to the preference rate for houses as the main driving force behind the recent behavior of house price in the US. Using the Michigan Survey of Consumers to calibrate such shock, the model captures quite well the persistence and hump shaped behavior of the boom and bust in the house price. On the contrary, narratives based on "fundamentals", as the interest rate behavior or the credit market liberalization, appear to be unimportant in explaining the house price movements.

Finally, we also show that an interest rate policy that reacts either to the deviations of the house price from steady state or to the rate of growth in the house price can substantially stabilize the house price.

Our paper is mainly related to Adam, Marcet and Kuang (2011) and to Lengnick and Wohltmann (2010). As said, from the former we take the model setup and from the latter the Agent-Based part of the model. Another related paper is Lengnick and Wohltmann (2010), who combine the chartist-fundamentalist model of financial markets and a standard New Keynesian macroeconomic model to generate endogenous business cycles and stock price bubble.

Adam, Marcet and Kuang (2011) develop a model able to replicate quantitatively the house price dynamics and the associated current account dynamics from 2001 to 2008 in the G7 economies, relaxing the rational expectations hypothesis and allowing households to be uncertain about how house prices are related to the economic fundamentals. To reach this goal, they use the concept of *internal rationality*, previously developed by Adam and Marcet (2010, 2011), where utility maximizer agents do not fully understand how price are formed, so that their subjective probability distribution about prices may not exactly be equal to the true equilibrium distribution. Contrary to us, they find that the boom in the housing market in the U.S. economy would have been largely avoided if the interest rate had fallen by less at the beginning of the 2000's.

Other works incorporates bounded rationality into housing market.[3]

Bolt, M. Demertzis, C. Diks, M. van der Leij (2011) develop a behavioral model of the housing market where agents have heterogenous expectations on the rate of return of holding a house. Similar to our model, agents are simple optimizers who rely on past performance to evaluate and revise their beliefs. They show that such a model generates bifurcations and multiple equilibria and they investigate to which extent these non-linearities could help explaining the boom-bust dynamics in the housing market.

Burnside, Eichenbaum and Rebelo (2011) build an "epidemiological" model in which agents have heterogeneous expectations about long-run fundamentals, but they can infect each other by social interaction. Social dynamics can then generate waves of infectious optimism that vanishes as soon as people become more certain about fundamentals.

Tomura (2012) presents a business cycle model capturing the stylized features of housing market boom-bust cycle in developed countries. In particular, he focuses on the role of over-optimism and the role of monetary easing in generating strong booms in housing market. Over-optimism of mortgage borrowers can cause boom-bust cycles, if mortgage borrowers are credit-constrained and savers who supply mortgage loans do not share the over-optimism of mortgage borrowers. In the presence of price stickiness, the model generates a low policy interest rate during a housing boom as an endogenous reaction through the Taylor rule to a low inflation rate.

Gelain, Lansing and Mendicino (2012) evaluate various policy actions that might be used to dampen the excess volatility in house prices in a DSGE model where the introduction of simple moving-average forecast rules for a subset of agents can significantly magnify the volatility and persistence of house prices and household debt, relative to an

[3]See also Schiller (2007a,b) and Piazzesi and Schneider (2009).

otherwise similar model with fully rational expectations. They find that macroprudential policies that modify the borrowing constraint are the most effective tool for dampening overall excess volatility in the model economy.

Dieci and Westerhoff (2013) investigate the impact of speculative behavior on house price dynamics. Their approach is inspired by recent work on Agent-Based financial market models. Speculative demand for housing is modeled through expectation formation mechanisms and behavioral rules of boundedly rational heterogeneous interacting agents. Real and speculative forces determine excess demand in each period and house price adjustments. The speculative behavior of heterogenous market participants repeatedly destabilize the housing market and endogenous switches between bullish and bearish markets may occur, possibly leading to bifurcations and multiple equilibria. These dynamics imply lasting and significant price swings around the fundamental steady state.

Iacoviello (2005) is the seminal contribution in the rational expectation DSGE literature. He develops a New-Keynesian business cycle model with two types of agents: borrowers and lenders. Borrowers receive nominal loans, but they are subject to a borrowing constraint where houses can be used as collateral. A recent refinement of such a model is proposed by Justiniano, Primiceri and Tambalotti (2013).[4]

The paper is constructed as follows. Section 2 presents the first building block of our model based on Adam, Marcet and Kuang (2011). Section 3 explains the Agent-Based block of our model and the chartists and fundamentalists behavioral rules. Section 4 shows the results of simulations and comprises a thorough investigation of the model mechanics via sensitivity analysis. Section 5 investigates the ability of the model to replicate the recent house price dynamics taking into consideration the real interest rate dynamics, the credit tightness and a preference shock for housing demand. Section 5 analyzes the role of policy in dampening house price volatility. Section 6 concludes.

2.2 The Model

The structure of the dynamic partial equilibrium model is based on Adam, Marcet and Kuang (2011). This framework is very convenient because it allows us to find a closed form solution for housing demand.

We differ from Adam, Marcet and Kuang (2011) in two respects. First, we adopt a different framework for expectation formation, assuming a chartist-fundamentalist mechanism

[4]See also Iacoviello and Neri (2010) and Kiyotaki et al. (2010).

with endogenous selection of forecasting strategies. Considering house as an asset, we rely on a classical and simple framework in the Agent-Based analysis of the financial markets to describe the dynamics of house price and the process of expectations formation. Households belong to two different groups: chartists expect that the trend on the house price will continue in the next periods, while fundamentalists expect that the price will revert towards its perceived fundamental value. Moreover, the size of the two groups is not fixed but it changes across time according to the relative past performance of these two differing strategies. The model is thus able to generate endogenous waves of chartism and fundamentalism that could move the price away from its fundamental value.

Second, another important difference with respect to Adam, Marcet and Kuang (2011) is the timing of actions and decisions. Households solve their problem daily, which we suppose to be the smallest fraction of time for the real economy. House builders, instead, are assumed to operate on a quarterly basis, reflecting a time-to-build effect due to the necessary time that elapses for the construction of a house. They thus will base their production decision on the average of the house price in the past quarter.

From the last assumption it follows that demand and supply are not simultaneous. The house price thus does not emerge from the equality between supply and demand, but its dynamics is driven by excess demand/supply. In this sense, this is a model of *dis*-equilibrium, the price reacts to the difference between demand and supply by increasing (decreasing) when demand is larger (lower) than supply.

2.2.1 The Household problem

The economy is populated by a unit mass of households with identical preferences but potentially different and not rational believes, indicated by $\tilde{E}_t(\cdot)$, that we will specify later in Section 2.3.1 when we will introduce the distinction between chartists and fundamentalists. Households take daily decisions (t stands for days) and their preferences are described by the following intertemporal quasi-linear utility function:

$$\tilde{E}_t \sum_{t=0}^{\infty} \delta^t (c_t + j_t \log h_t), \tag{2.1}$$

where $c_t > 0$ is the daily consumption of goods, h_t is the daily consumption of housing services, $\delta \in (0, 1)$ is the discount factor, and j_t is a preference shock for housing demand.

The household period-by-period budget constraint is:

$$c_t + [h_t - (1 - \tilde{d})h_{t-1}]Q_t + R_t b_{t-1} + k_t = y_t + b_t + k_{t-1} p_t, \tag{2.2}$$

where Q_t is the house price at time t, $\tilde{d} \in [0,1)$ is the daily depreciation rate of a house, b_t is the household new loans, R_t is the gross real interest rate on loans and y_t is income, which is exogenous.[5] The capital stock, k_t, is owned by the households who rent it to house builders for production. Capital fully depreciates in one period and its remuneration is p_t.

As in Kiyotaki and Moore (1997), households are allowed to borrow from banks subject to a borrowing constraint:

$$b_t \leq \theta \frac{Q_t}{R_t} h_t. \tag{2.3}$$

The parameter θ represents the share of assets that can be collateralized. It is fixed and cannot exceed the house value after the depreciation: hence $\theta \in (0, 1-\tilde{d}]$. Kiyotaki and Moore (1997) interprets a value of θ lower than one as reflecting the cost the lenders suffer in case of default. A growing house price relaxes the collateral constraint, implying that the households will have greater access to credit.

Households maximize their utility function (2.1) subject to the sequence of budget and borrowing constraints (3.8-3.9):

$$\max_{c_t,h_t,b_t,k_t} \tilde{E}_t \sum_{t=0}^{\infty} \delta^t \left\{ \begin{array}{c} (c_t + j_t \ln(h_t)) + \\ -\lambda_t \left(c_t + \left(h_t - \left(1-\tilde{d}\right) h_{t-1} \right) Q_t + R_t b_{t-1} + k_t - y_t - b_t - k_{t-1} p_t \right) + \\ +\gamma_t \left(\theta Q_t h_t - R_t b_t\right) + \mu_t c_t + \kappa k_t \end{array} \right\} \tag{2.4}$$

where p_0, k_{-1}, b_{-1} are given initial conditions.

The first order conditions with respect to c_t, h_t, b_t and k_t are:[6]

$$(\partial c_t) : 1 - \lambda_t + \mu_t = 0 \qquad (\mu_t \geq 0; \;\; \mu_t c_t = 0), \tag{2.5}$$

$$(\partial h_t) : \frac{j_t}{h_t} - \lambda_t Q_t + (1-\tilde{d})\delta \tilde{E}_t \lambda_{t+1} Q_{t+1} + \gamma_t \theta Q_t = 0, \tag{2.6}$$

$$(\partial b_t) : \lambda_t - R_t \delta \tilde{E}_t \lambda_{t+1} - \gamma_t R_t = 0 \qquad (\gamma_t \geq 0; \;\; \gamma_t(\theta Q_t h_t - R_t b_t) = 0), \tag{2.7}$$

$$(\partial k_t) : -\lambda_t + \kappa_t + \delta \tilde{E}_t \lambda_{t+1} p_{t+1} = 0 \qquad (\kappa_t \geq 0; \;\; \kappa_t k_t = 0). \tag{2.8}$$

Assuming that the non-negativity of consumption holds ($\mu_t = 0$) and $R_t \delta < 1$, households will borrow as much as possible: hence the borrowing constraint is binding, so $b_t = \frac{\theta Q_t h_t}{R_t}$

[5] We can assume income follows an exogenous stochastic process. In any case, the only role played by income is to pin down consumption (see equation (2.11)), which is not relevant for the analysis in the paper which focuses on house price dynamics.

[6] The utility function is linear in consumption as in Adam, Marcet and Kuang (2010). As them, we assume that the utility from consumption is bounded for high level of c. The first order conditions are thus necessary and sufficient for a maximum due to the linearity of the constraint and the concavity of the objective function in the households' choice variable.

and $\gamma_t > 0$. From equation (2.5) $\lambda_t = 1$; thus from (2.7): $\gamma_t = \frac{1}{R_t} - \delta > 0$. Using these results, it is possible to derive the households' demand for housing services from equation (2.6):

$$h_t^d = j_t \left[\left(1 + \delta\theta - \frac{\theta}{R_t} \right) Q_t - \left(1 - \tilde{d} \right) \delta \tilde{E}_t Q_{t+1} \right]^{-1}. \qquad (2.9)$$

The capital rented by the consumers to house builders should satisfy:

$$(1 - \delta p_{t+1}) k_t = 0, \qquad (2.10)$$

so that either $p_t = \delta^{-1}$ or $k_t = 0$. Given the non-negativity constraint on capital, the quasi-linear utility function implies that at that price capital and consumption are not uniquely determined and agents are indifferent between increasing slightly the capital sold to firms at time t in exchange for δ^{-1} more units of consumption at $t+1$. The capital supply offered by consumers is thus perfectly elastic, so that k_t is determined by firm's demand at the market price of $p_t = \frac{1}{\delta}$.

Finally consumption can be obtained residually using the flow budget constraint:

$$c_t = y_t + b_t - \left(h_t - (1 - \tilde{d}) h_{t-1} \right) Q_t - b_{t-1} R_t - k_t - k_{t-1} \delta^{-1}. \qquad (2.11)$$

Housing Supply

As said above, we assume that house builders operate quarterly (q). The difference in the timing of the action among households and house builders reflects the time that elapses in creating new houses. The house builders borrow capital from the households in a competitive market and employ it as input in a simple decreasing return to scale production function:

$$S_q^h = (\alpha\delta)^{-1} k_q^{\alpha}. \qquad (2.12)$$

k_q is the sum over a quarter of the daily capital received from households and $\alpha \in (0, 1)$. The market for input is always in equilibrium and the price for capital is $p_t = \delta^{-1} \;\; \forall t, q$.

The firm chooses k_q to maximize its profits, i.e., $\max_{k_q \geq 0} \tilde{E}_q \left(S_q^h Q_{q+1} - \delta^{-1} k_q \right)$, where Q_{q+1} is the quarterly house price in the next quarter.[7] The first order condition is:

$$k_q = \left(\tilde{E}_q Q_{q+1} \right)^{\frac{1}{1-\alpha}}. \qquad (2.13)$$

In maximizing profits, house builders need to form expectations about the next quarter house price. We assume house builders have static expectations, so that: $\tilde{E}_q[Q_{q+1}] = Q_q$.

[7]The quarterly house price is defined as an average of the daily prices over the quarter.

The profit-maximizing input demand therefore becomes:

$$k_q = (Q_q)^{\frac{1}{1-\alpha}}, \tag{2.14}$$

and substituting it into the production function we obtain the quarterly supply of new houses:

$$S_q^h = (\alpha\delta)^{-1} Q_q^{\frac{\alpha}{1-\alpha}}. \tag{2.15}$$

The housing stock evolves according to:

$$h_q = (1-d)h_{q-1} + S_q^h. \tag{2.16}$$

Note that this is an end-of-period definition of the housing stock. The stock of houses at the end of quarter q, thus available for consumption in the next quarter $q+1$, depends on the existing stock in the previous quarter, h_{q-1}, net of depreciation, plus the production of new houses in the quarter.[8] It follows that the stock of houses available for consumption in quarter q is equal to the stock at the beginning of quarter q, that is, h_{q-1}.

2.2.2 The Log-Linearized Model

In this part we log-linearize the model around its steady state, where the variables are constant. In our case this implies that the timing of actions does not matter (e.g., if $Q_t = Q\ \forall t$ then $Q_q = Q$) and we equalize demand and supply in a timeless fashion. The main steady state equations for our purposes are:

$$h^d = \frac{j}{Q\left(1 + \delta\theta - \frac{\theta}{R} - \left(1 - \tilde{d}\right)\delta\right)}, \tag{2.17}$$

$$S^h = \frac{1}{\alpha\delta} Q^{\frac{\alpha}{1-\alpha}}, \tag{2.18}$$

$$dh = S^h.$$

Solving for Q we obtain the steady state value for the house price:

$$Q = \left(\frac{dj\alpha\delta}{1 + \theta\delta - \frac{\theta}{R} - \left(1 - \tilde{d}\right)\delta}\right)^{1-\alpha}. \tag{2.19}$$

The log-linearized equations thus are:

$$\widehat{h}_t^d = \widehat{j}_t + \frac{Qh^d}{j}\left[\left(1 - \tilde{d}\right)\delta \tilde{E}_t \hat{Q}_{t+1} - \left(1 + \delta\theta - \frac{\theta}{R}\right)\hat{Q}_t - \frac{\theta}{R}\hat{R}_t\right], \tag{2.20}$$

[8] Note that d is the quarterly depreciation rate, so that: $1 - d = (1 - \tilde{d})^{64}$.

$$\hat{S}_q^h = \frac{\alpha}{1-\alpha}\hat{Q}_q. \tag{2.21}$$

The demand function in (2.20) depends positively from the preference for houses and from the expected future price, while negatively from the current price[9] and from the interest rate. The housing supply is a positive function of the quarterly price.

Finally, log-linearizing (2.16) yields:

$$\hat{h}_q = (1-d)h_q\hat{h}_{q-1} + S^h\hat{S}_q^h. \tag{2.22}$$

2.3 An Agent-Based Approach to House Price

In this Section we present the Agent-Based part of our model, where we adapt the Agent-Based framework in Lengnick and Wohltmann (2010) and Westerhoff (2008) to the housing market.

2.3.1 Expectations

Adam, Marcet and Kuang (2011) conclude that it is difficult to account for the U.S. house price dynamics assuming rational expectations. The empirical evidence on house price behavior, which alternates periods of persistently increasing and decreasing prices motivates the relaxation of the rational expectation hypothesis regarding beliefs on house prices. As Adam, Marcet and Kuang (2011), in order to concentrate on the effects of the Agent-Based part of the model on the housing market dynamics, we assume that households have correct beliefs (i.e., rational expectations) about all variables affecting their demand for housing services, except for the house price. Regarding the latter, they hold non rational beliefs.

More precisely, they can be either chartists $\tilde{E}_t^c(\cdot)$ or fundamentalists $\tilde{E}_t^f(\cdot)$. Chartists expect the price trend will continue, so their forecasting rule is given by:

$$\tilde{E}_t^c[\hat{Q}_{t+1}] = \hat{Q}_t + l^c(\hat{Q}_t - \hat{Q}_{t-1}), \tag{2.23}$$

where $\hat{Q}_t$ is the percentage deviation of house price from its steady state value at t and the parameter $l^c > 0$ represents the degree of "persistence" or trend-chasing in the house price expected by the chartists.

Fundamentalists, instead, believe that a fraction of the actual mis-pricing will be corrected in the future, so their forecasting rule is given by:

$$\tilde{E}_t^f[\hat{Q}_{t+1}] = \hat{Q}_t + l^f(\hat{Q}_t^{fd} - \hat{Q}_t), \tag{2.24}$$

[9] Given our calibration $1 + \delta\theta - \frac{\theta}{R} > 0$.

where the parameter $l^f > 0$ represents the fraction of the mis-pricing that fundamentalists expect to be corrected in the next period, and $\hat{Q}_t^{fd}$ is the perceived fundamental value, that we need to define.

Economic theory would suggest that the value of a house, as that of any asset, should be equal to the present discounted value of the expected returns from holding that asset. In our model, for fundamentalists this could be captured by equation (2.6):

$$Q_t = \left(\frac{j_t}{h_t^{d,f}} + \gamma_t \theta Q_t \right) + (1 - \tilde{d}) \delta \tilde{E}_t^f Q_{t+1}, \tag{2.25}$$

that has the usual asset pricing interpretation: the value of the asset is given by the dividend today plus the expected capital gain/loss tomorrow. The benefit today of holding the house is equal to the marginal utility provided by the housing services plus the utility deriving from the relaxation of the borrowing constraint, that is, $\left(\frac{j_t}{h_t} + \gamma_t \theta Q_t \right)$, while the capital gain tomorrow net of depreciation and discounting is $(1 - \tilde{d}) \delta \tilde{E}_t^f Q_{t+1}$. One could then think of iterating forward equation (2.25) and find a value of the asset that depends on future expected "fundamentals", as one would do in a rational expectation framework. However, in our Agent-Based context it would not make sense to iterate this equation forward, simply because $\tilde{E}_t^f Q_{t+1}$ does not obey rational expectations. So we can not use equation (2.25) to calculate $\tilde{E}_t^f Q_{t+1}$ and then to substitute it iteratively forward in the same equation. In our Agent-Based framework, $\tilde{E}_t^f \hat{Q}_{t+1}$ is defined by the forecasting rule of the fundamentalists, equation (2.24). Instead, equation (2.25) pins down the demand given the particular forecasting rule of the agent.

So we need to find another route that it is coherent with the irrational beliefs of our agent based framework, where agents do not know the correct (and complex) determinants of house price dynamics. In addition, even expert economists in the real world rarely agree on the actual mechanism linking fundamentals to house prices. Hence, we assume that households can not identify the true fundamental price. In (2.24), fundamentalists thus use a *perceived* fundamental price. Similarly to Lengnick and Wohltmann (2010), that link the perceived steady state of the financial market to the aggregate economic activity, we link the fundamentalists' perceived fundamental price to the sectoral output, that is, to the supply of housing services. The idea is that fundamentalists understand that house supply depends on the expected price (see equation (2.21) in the previous Section). Hence, an increase in housing supply will then be interpreted by fundamentalists as a signal of the house builders reaction to an expected increase in the fundamental house price.[10] Therefore, we assume that the fundamental price perceived by fundamentalists at the beginning of each quarter is proportional to the amount of construction works that they observe. The latest available observation to fundamentalists at the beginning of each quarter is the new housing built during the previous quarter, S^h_{q-1}. Hence:

$$Q^{fd}_t = const * S^h_{q-1} \qquad q = floor\left(\frac{t-1}{64}\right) \tag{2.26}$$

The function $floor(\cdot)$ simply rounds its argument to the nearest integers less than or equal to the argument itself, and it is used to divide the daily time scale into quarters, mapping the days t into the respective quarters q. Note that Q^{fd}_t is "fundamental" in the sense that both it depends on "fundamentals" and it moves at low frequency in the model, because it is constant over the quarter, being $\hat{Q}^{fd}_t = \hat{Q}^{fd}_q$. In log-deviations, (2.26) simply becomes

$$\hat{Q}^{fd}_t = \hat{S}^h_q. \tag{2.27}$$

Inserting (2.23) and (2.24) into (2.20) yields the chartists' demand function:

$$\hat{h}^{d,c}_t = \widehat{j}_t + \frac{Qh^d}{j}\left[\left(1-\tilde{d}\right)\delta\left(\hat{Q}_t + l^c(\hat{Q}_t - \hat{Q}_{t-1})\right) - \left(1+\delta\theta-\frac{\theta}{R}\right)\widehat{Q}_t - \frac{\theta}{R}\widehat{R}_t\right], \tag{2.28}$$

and the fundamentalists' one:

$$\hat{h}^{d,f}_t = \widehat{j}_t + \frac{Qh^d}{j}\left[\left(1-\tilde{d}\right)\delta\left(\hat{Q}_t + l^f(\hat{Q}^{fd}_t - \hat{Q}_t)\right) - \left(1+\delta\theta-\frac{\theta}{R}\right)\widehat{Q}_t - \frac{\theta}{R}\widehat{R}_t\right]. \tag{2.29}$$

The relative shares of chartists and fundamentalists are endogenously determined. Households learn about the past, and they change their believes according to the past performances of the two forecasting rules. Following Lengnick and Wolthmann (2010), each group evaluates the attractiveness, A^i_t, of each forecasting rule on the basis of the following equation[11]:

$$A^i_t = [\exp(\hat{Q}_t) - \exp(\hat{Q}_{t-1})]\hat{h}^{d,i}_{t-2} + \eta A^i_{t-1} \qquad i = c, f. \tag{2.30}$$

The attractiveness is thus partly related to the recent performance of the rule, measured by the term $[\exp(\hat{Q}_t) - \exp(\hat{Q}_{t-1})]\hat{h}^{d,i}_{t-2}$, and partly by its past attractiveness, where $\eta \in [0,1]$ is the memory parameter which defines the strength with which agents discount past performances.

[10] One can also argue that fundamentalists assume that house producers have superior information on the housing market, so they link Q^{fd} to the observed S^h.

[11] Recall that $\hat{Q}_t$ is the logarithm of house price. In order to calculate nominal performance, it has to be delogarithmized.

The fraction of agents that adopts a particular strategy (W_t^i) then is updated every day thanks to the Gibbs Probability, as in the framework of adaptive belief system proposed by Brock and Hommes (1997, 1998):

$$W_t^i = \frac{\exp(eA_{t-1}^i)}{\sum_i \exp(eA_{t-1})} \qquad i = c, f. \tag{2.31}$$

The more attractive is a strategy, the higher is the fraction of agents using it. The parameter e is called the rationality parameter: other things equal, the higher is e, the larger will be the number of agents that switches towards the strategy with the highest attractiveness.

2.3.2 House Price Dynamics

The deviation of the house price from its steady state, $\hat{Q}_t$, evolves according to:

$$\hat{Q}_{t+1} = \hat{Q}_t + a(W_t^c \hat{h}_t^{d,c} + W_t^f \hat{h}_t^{d,f} - \hat{h}_{q-1}) + \varepsilon_t^Q. \tag{2.32}$$

(3.4) states that the change in the house price ($\hat{Q}_{t+1} - \hat{Q}_t$) reacts to the excess of demand over supply in the housing market. This is given by the difference between the sum of the demand deviations of chartists $\hat{h}_t^{d,c}$ and fundamentalists $\hat{h}_t^{d,f}$ from the relative steady state, weighted by their relative shares (W_t^c and W_t^f given by (3.11)), and the available supply of housing services, $\hat{h}_{t-1}$.[12] a measures the responsiveness of the house price to excess demand in the housing market. The noise term ε_t^Q is an i.i.d. normally distributed shock with standard deviation σ_Q^2. It captures the idea that the two strategies are not the only possible strategies that exist into the market.

The quantity actually exchanged in the market obeys to the short side of the market, so it is given by the minimum between the sum of the chartists and fundamentalists demand in the correspondent quarter and the relative existing stock:

$$G = \min \left\{ \sum_{t=64(q-1)+1}^{64q} \left(W_t^f \hat{h}_t^{d,f} + W_t^c \hat{h}_t^{d,c} \right); \hat{h}_{t-1} \right\} \tag{2.33}$$

The actions of households and house builders are not synchronized, because demand and supply run on a different time scale. We assume that a quarter is composed by 64 days. Hence, within one increment on house supply's time index q, households perform 64 times their maximization problem generating their daily demand for houses. Therefore

[12] Recall that the housing stock is constant over the quarter, because it changes only at quarterly frequencies. So: $h_t = h_q$ where $q = floor\left(\frac{t-1}{64}\right)$.

the model is implemented as follows: i) we run the daily demand for a quarter, given the shocks that hit the model; ii) then the quarterly price is equal to the mean of the daily price over that quarter; iii) we insert it into the supply equation to find the reaction of house builders and the new fundamental price; iv) we iterate.

A quarter is defined to contain days $64(q-1)+1,...,64q$, for $q=1,2,...$:

$$\hat{Q}_q = \frac{1}{64}\sum_{t=64(q-1)+1}^{64q}\hat{Q}_t. \tag{2.34}$$

2.4 The Model Simulation

In this part of the paper we analyze the performance of the model. First, we use numerical simulations to investigate the ability of the model to generate fluctuations in the house price driven by endogenous waves of chartism and fundamentalism. Second, we inspect the transmission mechanism of the model by means of sensitivity analysis.

2.4.1 Calibration

The parameter calibration is reported in Table 1. As in Adam, Marcet and Kuang (2011), the annual discount factor δ is fixed at 0.96 and the annual depreciation rate at 3%, implying: $d = 0.0076$ and $\tilde{d} = 1,19 \cdot 10^{-4}$. $\alpha = 0.65$ implies decreasing returns in production and captures the fact that housing is a capital-intensive sector. Parameter θ in the borrowing constraint is calibrated as in Iacoviello (2005). η, l^c and l^c are set as in Lengnick and Wohltmann (2010). The value of a and e are particularly problematic since we have no much guidance on how to calibrate them. We calibrate the rational parameter (intensity of choice) e in a way of minimizing the distance between the real data and the model generated time series in the simulation exercise in the next Section. The implied value is lower than in Lengnick and Wohltmann (2010), but we think this is consistent on our paper focusing on the housing market rather than the financial market. While the financial market can be thought as a system populated mostly by institutional, professional, or at least educated, investors, the housing market comprises a bigger class of participants which is consistent with a lower rationality parameter. Following a similar logic, the parameter a in equation (3.4) is set to a lower value than in Lengnick and Wohltmann (2010). a measures the elasticity of the daily house price to the daily excess demand in the housing market. We argue that price elasticity to excess demand is much less in the housing market with respect to the financial market, because the house price is much more inertial and less volatile than asset prices in the financial market, especially on a daily basis. We calibrate the variance of the shock to the evolution of the house price in equation (3.4), σ_Q^2, so that the variance of the simulated quarterly price is the same as the variance of real quarterly price, collected by Federal Housing Finance Agency (http://www.fhfa.gov/).

Calibration	
$\alpha = 0.65$	$a = 0.001$
$\delta = 0.96$	$l^c = 0.04$
$\theta = 0.55$	$l^f = 0.04$
$d = 0.0076$	$\eta = 0.975$
$e = 100$	$\sigma_Q^2 = 0.011$

Table 1: Calibration of the model

2.4.2 Waves of Chartism and Fundamentalism

Following Lengnick and Wohltmann (2010), we simulate a *representative* run for a period of 40 quarters to show how the model works. We have two different sources of shocks in the model: the noise term on the house price equation (3.4), ε_t^Q, and a shock to the preference for housing services in the utility function, j_t. In this part of the paper, we analyze only the response of the system to repeated draw realizations of the noise term ε_t^Q, while keeping $\hat{\jmath}_t$ fixed at zero. The aim is to investigate how our Agent-Based framework interacts with the more standard partial dynamic model of households' choice.

Figure 1.1 shows the dynamics of the relevant variables: the top left panel displays the quarterly house prices; the top right shows the evolution of housing stock. The two middle panels display: i) daily house price along with the perceived fundamental value; ii) waves of chartism (black) and fundamentalism (white), labelled *animal spirits*. Finally the bottom left panel shows the quantity actually exchanged on the housing market and the bottom right panel shows the supply, $\hat{S}_q^h$, and total demand, $W_t^c \hat{h}_t^{d,c} + W_t^f \hat{h}_t^{d,f}$. The model is able to generate endogenous waves of chartism and fundamentalism, that in turn cause fluctuations in both the house price and quantity. The continuous evaluation of past relative performance induces an endogenous competition between the two forecasting strategies that assures that none dominates forever. While fundamentalists dominate for most of the time, in some particular periods the vast majority of agents follows the chartists' rule. When chartists prevail, the house price departs from its perceived funda-

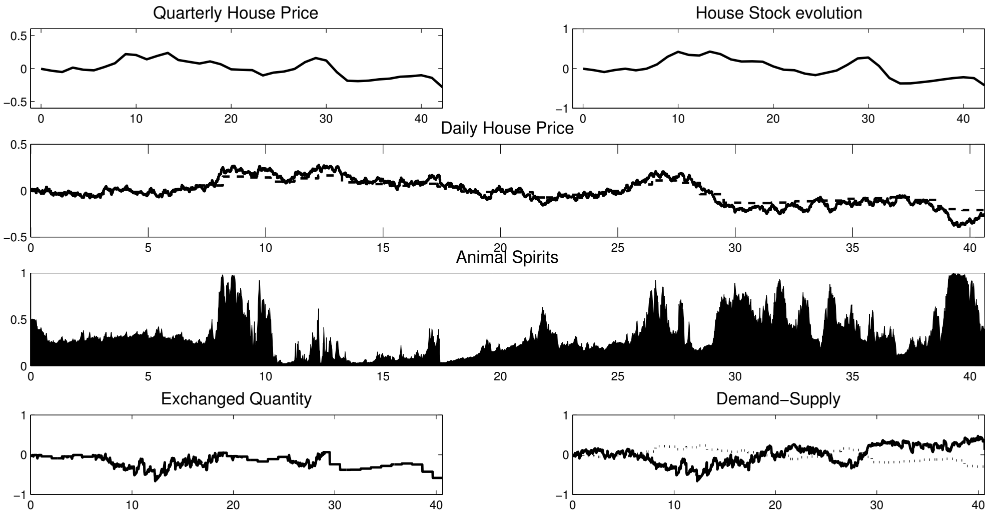


Figure 2.1: Representative run of the relevant variables

mental value. As an example, quarters $q = 7 - 10$ and $q = 26 - 28$ exhibit a boom in the house price driven by chartists, while quarters $q = 30 - 34$ and $q = 38 - 40$ exhibit a bust. In phases dominated by fundamentalists, on the contrary, the house price tends to go back to its fundamental value, which is evident for $q = 11 - 20$.

Assume the house price starts trending upward, then the chartists' rule may outperform the fundamentalists' one strengthening and protracting the upward movement in the house price, and thus creating a boom. However, this sows the seeds for the subsequent bust. First, the supply of new houses increases with the price, and so does the available stock of housing. Therefore the excess demand tends to decrease, because the supply has a standard direct negative effect on the house price through equation (3.4). Second, the effect on demand of an increase in the house price depends on whether the negative substitution effect on demand (due to the increase in the current price) is offset by the positive wealth effect on demand (due to the increase in the expected future price). For chartists, the increase in the expected future price is due to the extrapolative expectations (equation (2.23)), while for fundamentalists this is due to the increase in the fundamental price caused the increase in the supply of new housing (equation (2.26)). Note the twofold role of an increase in supply: on the one hand, it forces a price decrease by increasing the stock of available houses, on the other hand it affects fundamentalists' expectations by increasing their perceived fundamental value, and thus their demand. The fact that demand decreases when the daily house price is higher than its fundamental value, whereas it decreases otherwise seems to suggest that the substitution effect is prevailing on demand. Hence, on the one hand, the forces counteracting the lengthening of the boom phase are standard: the substitution effect on demand and the contribution of the supply of new houses to excess demand. On the other hand, the Agent-Based mechanisms of expectations formation both of chartists, who are trend follower on a daily basis, and fundamentalists, through the perceived fundamental price, contributes to the prolonging of the boom phase. Note, however, that the fundamentalists expectations tend to stabilize the price at high frequencies, because they tend to anchor the price to the perceived fundamental one which is fixed over the quarter. Indeed, Figure 1.1 shows that when fundamentalists dominate, the price tends to move towards the fundamental value.

It is interesting to note that the two expectation formation mechanisms of our model tend to reinforce the trend in the house price, but at different frequencies: daily the chartists, while quarterly the fundamentalists.

Finally, the supply, and thus the evolution of existing houses, are closely influenced by the path of the quarterly house price. The exchanged quantity is the minimum between demand and the existing housing stock. When the former is greater than the latter, the time series is more volatile, on the contrary when supply prevails the time series is represented by a broken line because this variable changes only at the end of each quarter.

2.4.3 Inspecting the Mechanism

We now perform a sensitivity analysis regarding the parameters of the model. With respect to our benchmark calibration in Table 1, we then change one parameter at a time in the log-linearized model, keeping however fixed the particular sequence of the realizations of the shock ε_t^Q that generates Figure 1. The purpose of this exercise is twofold. First, it is a robustness check on the mechanism of the model just described. Second, it provides a better understanding of the different effects at work in the model. Table 2 shows how some key statistics of the log-linearized model change with the parameters of the Agent-Based part of our framework.[13] In particular, these statistics are: the standard deviation of the daily house price, $\hat{Q}_t$, of the stock of houses, $\hat{h}_q$, of the excess demand, $\left(W_t^c \hat{h}_t^{d,c} + W_t^f \hat{h}_t^{d,f} - \hat{S}_t^h\right)$, and of the fundamental price, $\hat{Q}_t^{fd}$, and the average value of the shares of fundamentalists, W_t^f, and chartists, W_t^c. The first column shows the results

[13] We just consider either an increase or a decrease since the effects are symmetric.

employing the benchmark calibration in Table 1.

	Benchmark	l^c= 0.4	l^f= 0.4	$a = 0.01$	$e = 300$	$\alpha = 0.5$
$std\left(\hat{Q}_t\right)$	0.1721	0.1729	0.1249	0.0558	0.1812	0.1830
$std\left(\hat{h}_q\right)$	0.6151	0.6180	0.4392	0.1681	0.6486	0.3528
$std\left(W_t^c \hat{h}_t^{d,c} + W_t^f \hat{h}_t^{d,f} - \hat{S}_t^h\right)$	0.5052	0.5242	1.4461	0.1537	0.5393	1.4542
$std\left(\hat{Q}_t^{fd}\right)$	0.3079	0.3093	0.2199	0.0842	0.3247	0.1766
$mean(W_t^f)$	0.5173	0.5279	0.9050	0.6175	0.5845	0.6356
$mean(W_t^c)$	0.4827	0.4721	0.0949	0.3825	0.4154	0.3643

Table 2: Sensitivity analysis

An increase in l^c in equation (2.23) strengthens the trend following behavior of the chartist's expectations of the house price. Not surprisingly, this causes an increase in the volatility of all the relevant variables. These effects, however, are very minor pointing to the fact that such a parameter does not play a key role in the model. The share of fundamentalists increases but only slightly.

The effect of an increase in l^f in equation (2.24) instead has quite large stabilizing effect on the model. The bigger l^f, the more the fundamentalists' expectations react to the difference between the daily price and the fundamental price. As a consequence, the daily price is closer to the fundamental value and less volatile. So it is supply and hence the fundamental price. The fundamentalists' strategy become more effective, and indeed the average share of fundamentalists is overwhelming: 90%. Note, however, that the volatility of the excess demand increases. This is because the fundamentalists' demand is more volatile since the expectation of the future price moves substantially to correct the deviation of the current price from the fundamental price. Therefore, in the case of high l^f is the demand of fundamentalists that plays the stabilizing role on the price through excess demand.

An increase in the elasticity of the daily house price to the excess demand on the market, a, in equation (3.4) has a very intuitive stabilizing effect on all the variables: the price move on a daily basis to clear the excess demand on the market. Intuitively, the average share of fundamentalists rise to 61.7%, because the market is more stable.

The rationality parameter, e, in equation (3.11) determines the sensitivity of the shares of the two agents with respect to the relative attractiveness of the two forecasting rules.

The higher is e, the greater the number of agents selecting the more attractive strategy. In the extreme case when $e = 0$, agents do not switch strategy, so the shares of the two agents are constant. In the other limiting case when $e = \infty$, all agents have always the same strategy, because they simply pick the forecasting rule with the best performance in the previous period. It is quite intuitive therefore that an increase in e amplifies all the volatilities, even if the quantitative effects are quite minor in our setup.

Finally, it is interesting to analyze also the effects of the return to scale parameter. A decrease in the return to scale parameter makes marginal costs steeper, and hence supply is less elastic with respect to the price. As a consequence, the volatility of supply, and thus of the fundamental price, diminishes. This increases the volatility of excess demand because supply do not move to counteract the movement in demand, and hence also the volatility of the daily price level slightly increases. Since the fundamental price is quite stable, on average the fundamentalists' strategy outperforms the chartists' one and fundamentalists are on average the majority in the market.

2.5 Matching House Price Data

In this Section we ask if the model is able to replicate the recent boom-and-bust house price dynamics in the US data. The aim is to identify the main driving forces of the dynamics of the house price according to our model.[14]

We want to see if our model can match the behavior of the quarterly house price for the period going from Q1:2004 to Q1:2009. The data, Seasonally Adjusted Purchase-only Index, are taken from the Federal Housing Finance Agency. We assume that in the initial period, i.e., Q1:2004, the system is in steady state. We then compute the percentage deviation of the quarterly house price from its steady state value.

The debates among economists have focused on three main possible causes of the recent boom and bust in housing prices. The first narrative identifies the so-called "Greenspan put" as one possible causes of the crisis, that is a persistent low level of interest rates inducing excessive leveraging on the part of both the households, through debt accumulation (especially mortgage debt), and the financial intermediaries through excessive risk/taking. A second narrative points instead to a "liberalization" cycle as the main cause of the crisis, that is, an overall loosening of lending standards that allows more borrowing from

[14] Other contributions, based on non-rational expectation, are able to match the data quite well: see, for example, Adam, Kuang, Marcet (2011).

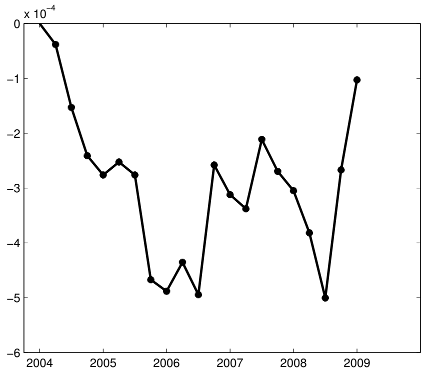


Figure 2.2: Percentage deviation of 30-Years Conventional Mortgage Rate

the households for a given value of the collateral, followed by an abrupt increase of credit tightening at the onset of the financial crisis. A third narrative, instead, considers the possibility that exogenous and more direct factors drove up the housing price, in the sense of an exogenous increase in the demand due to a preference shift. Justiniano, Primiceri and Tambalotti (2013) call the latter the "valuation" story.[15]

Our framework can accommodate these three possible explanations through, respectively: i) the exogenous interest rate; ii) a time-varying value of θ in equation (3.9), as a proxy for credit tightness; iii) the exogenous preference shock, j_t, in the utility function.[16] In order to evaluate the relative importance of these three possible driving factors in our model, we feed into it one exogenous path for each of these three variables.[17]

Regarding the interest rate we use the percentage deviation in 30-Years Conventional Mortgage Rate from its value in the initial period Q1:2004. As displayed in Figure 1.2, this percentage deviation decreases steadily and substantially from 2004 to 2006, then it moves up and then down again, till it sharply increases from the Q2:2008. However, the

[15]One may call it the "Bush-push", in the sense that the demand for housing was surely strongly promoted by the George W. Bush administration. *"We can put light where there's darkness, and hope where there's despondency in this country. And part of it is working together as a nation to encourage folks to own their own home."* President George W. Bush, Oct. 15, 2002. See for example: http://www.nytimes.com/2008/12/21/business/worldbusiness/21iht-admin.4.18853088.html?pagewanted=all&_r=0

[16]A preference shock for housing services as ours is also present in Iacoviello and Neri (2010), Liu, Wang, and Zha (2011) and Justiniano, Primiceri and Tambalotti (2013).

[17]In this Section, shocks to the evolution of the house price, ε_t^Q, are muted in performing this impulse response type of exercise.

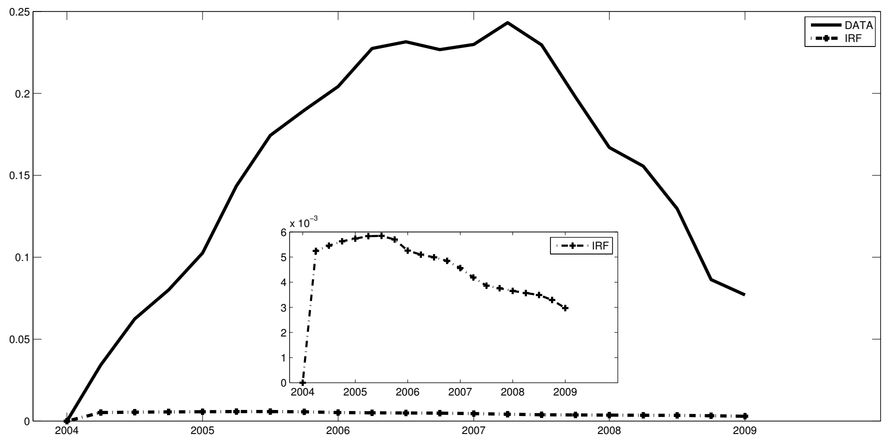


Figure 2.3: Model reaction to interest rate change

change in this variable is rather small. We then fit this series into the demand functions of the two agents to see the resulting path of the house price. Results are shown in Figure 1.3: the reaction of the model (dashed line) is very small compared with real data (solid line). Moreover in the small box, where the simulated price series is shown more clearly, it is possible to note that the series abruptly increases on impact and then decreases slowly; a pattern not completely in accordance with the data.

To test for the "liberalization" story that views the tightness of credit as the main source of the boom-and-bust behavior of the house price, we construct a path for the parameter θ in the borrowing constraint in the following way. First, we calibrate this parameter to be equal to Iacoviello (2005): $\theta = 0.55$. Second, we consider the net percentage of banks reporting tightening credit standards in the US according to *The January 2012 Senior Loan Officer Opinion Survey on Bank Lending Practices*[18].

This measure (see Figure 1.4) remains stable from 2004 to the third quarter of 2006, when access to credit starts tightening quite substantially till the second quarter of 2008. Finally, we define θ_t as: $\theta_t = 0.55 - 0.55 * tight\ credit$, because the credit availability is an increasing function of θ. The results of feeding this time-varying value of θ in our model are shown in Figure 1.5. The simulated data do not match the real data both because the size of the changes in the house price is much smaller and because the shape of the path is different. The small box reproduces the simulated time series for the house price:

[18] http://www.federalreserve.gov/boarddocs/snloansurvey/201201/default.htm

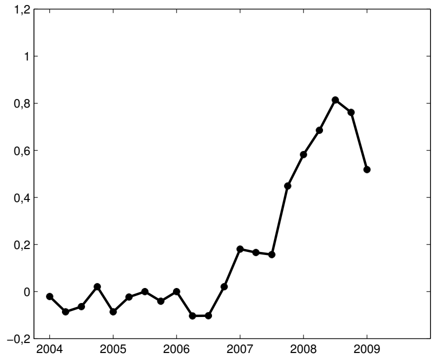


Figure 2.4: Net percentage of banks reporting tightening credit

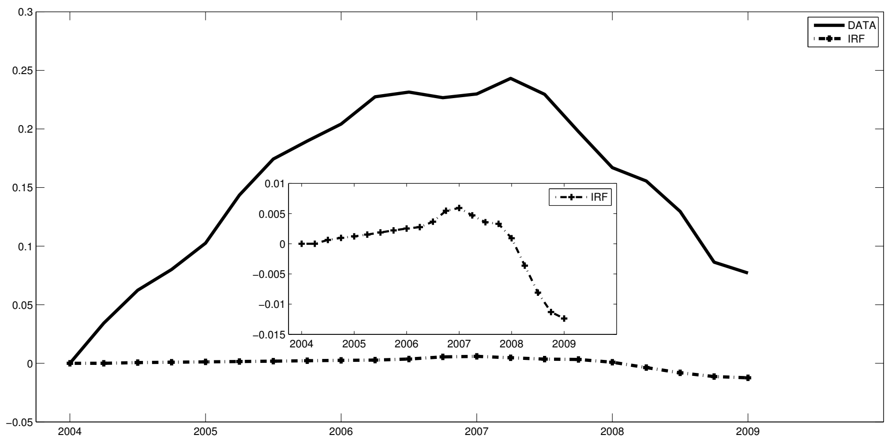


Figure 2.5: Model reaction to credit tightening change

it increases quite mildly up to 2007 and then continuously decreases more strongly than the initial increase. The rationale behind this behavior comes from the impact of θ on the demand equation. When θ decreases, houses are less valuable for the consumer because they can be less collateralized, so their demand decreases (see equation (2.6), where γ_t is positive since the borrowing constraint is binding in our model being $\frac{1}{R_t} - \delta > 0$). This generates a reduction in the house price. It is clear that our model does not suggest this mechanism as the driving force of house price dynamics.
Finally we consider the shock on house preferences $\hat{\jmath}_t$. To build a time series for $\hat{\jmath}_t$, we look at the quarterly table showing the Buying Condition for Houses in the *Michigan Consumers Surveys*. This measure is build from answers to the following question: *Generally speaking, do you think now is a good time or a bad time to buy a house?* We focus on the percentage of positive answers, and we normalized it to generate a series in a way to have figures in the subset $(-1, 1)$. Figure 1.6 visualizes the resulting path for $\hat{\jmath}_t$. Values of the series higher than zero mean a positive preference shock, and vice versa. The shock is thus positive from 2004 to the first half of 2005, when it becomes negative and there remains till the end of our sample, even if reverts toward zero from the middle of the 2007. Figure 1.7 shows the response of the model to this path of the exogenous preference shock. In this case, our model economy is able to replicate quite well the real price dynamics. The house price increase builds up similarly to the data during the first two years and a maximum percentage deviation of house price is reached in the first half of 2006. After that the simulated time series exhibits a constant decline, while the data decrease slightly and then rise up again to a second maximum before starts decreasing. While the simulated time series is not able to reproduce the mild twin-peaks in the data, the two series exhibit very similar behavior after mid-2007 and almost coincide at the end of the sample period. The final exercise consists in putting together all the three effects (see Figure 8). As evident also from the previous figures, basically all of the dynamics in house price is generated by the preference shock. Adding the interest rate and the credit tightness has basically no effects with respect to the case considering only the preference shock. Our analysis therefore suggests that by far the most important factor in the recent boom-bust dynamics of the house price in the US is a change in households' preference and housing demand. As such our analysis emphasizes the importance of the behavioral approach and of the selection mechanism among different expectation rules as determinant factors of the boom and bust cycle in the housing market. On the contrary, narratives based on "fundamentals" as the interest rate behavior or the credit market liberalization appear to

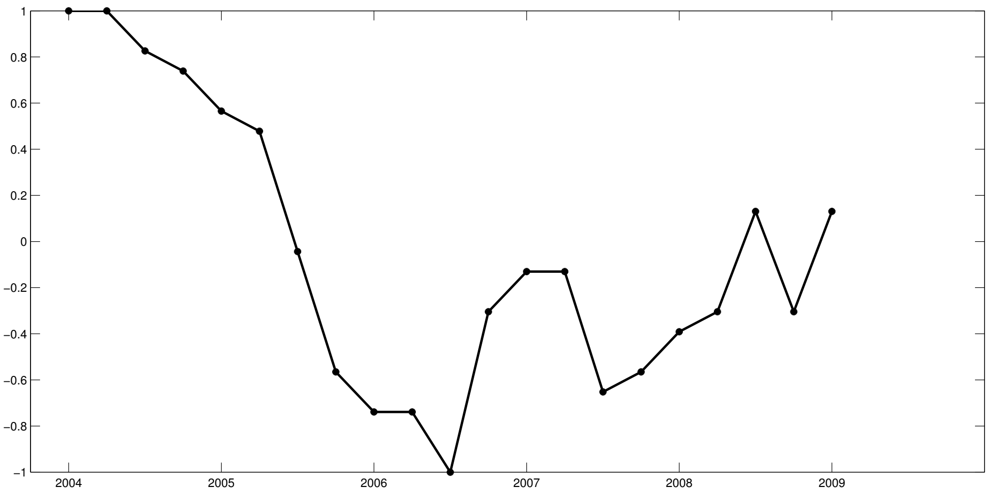


Figure 2.6: Preference shock

be unimportant in our framework.

It is interesting to note that our result is consistent with the findings of Justiniano, Primiceri and Tambalotti (2013)[19], that employ a very different modelling strategy: more structural and less behavioral. Using a quantitative dynamic general equilibrium model with occasionally binding constraint and an asymmetric borrowing constraint, they find that the dynamics of house price in the US could be explained by a "valuation" effect, that is an exogenous shock to preference for housing. Similarly to our framework, a positive shock leads to an increase in the demand and then in the price of houses, that are used as collateral by the households, thereby expanding their ability to borrow, and ultimately generating a boom. On the contrary, like us, they find little role for the "liberalization cycle".

Finally, as stated in the calibration Section 2.4.1, we set the rational parameter (intensity of choice) e in a way of minimizing the distance between the real data and the model output in Figure 1.8. However, the results are quite robust as long as the parameter value increases, as shown by Figure 1.9 and Table 3. The sum of the absolute value of the difference between real data and the generated time series is minimized for $e = 100$, but it is not very sensitive to perturbations of this parameter. Only very low levels of e noticeably worsen the fit of the model. Recall that the lower is e, the less chartists and

[19] See also Iacoviello and Neri (2010) and Kiyotaki, Michaelidies, and Nikolov (2010).

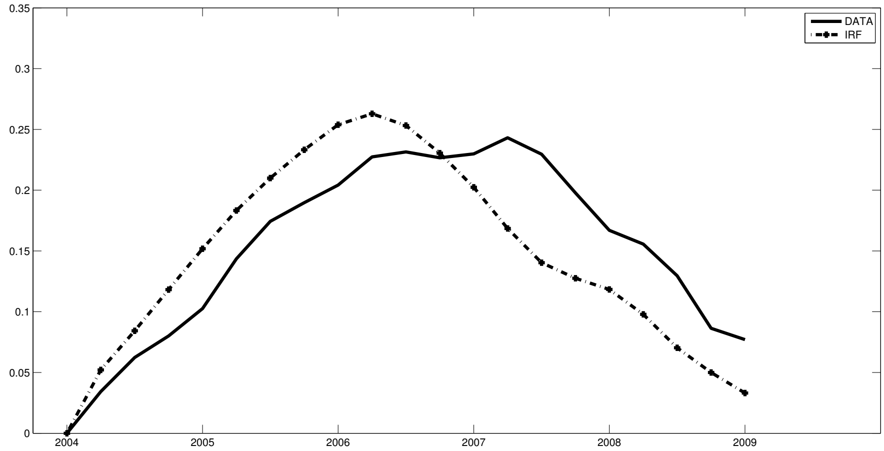


Figure 2.7: Model reaction to preference shock

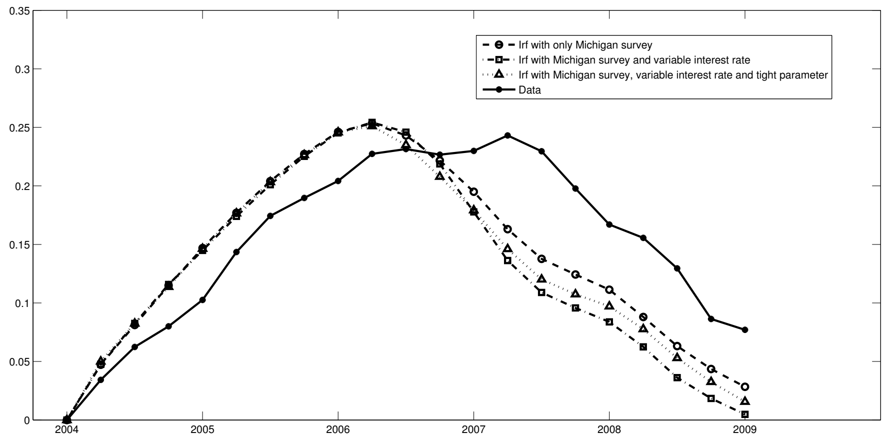


Figure 2.8: Model reaction to the three effects

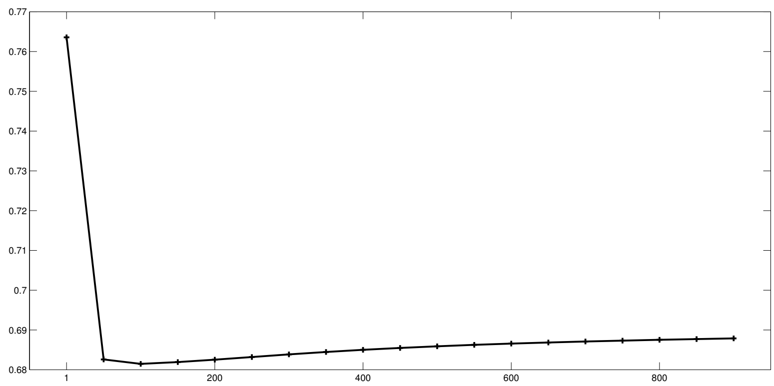


Figure 2.9: Sum of the absolute value of the difference between the data and the model

fundamentalists switch strategy according to their relative attractiveness. Thus, Figure 1.9 suggests that the Agent-Based part of our framework that allows for the endogenous shift between the two forecasting rule in the model plays an important role in the fit of the model. The heterogeneous agent framework gives the quite right persistence in the house price dynamics: the hump shape of the series is given by the self fulfilled mechanism induced by the backward looking expectation. This type of endogenous inertia due to agents' informational problem is typical of boundedly rational models in which agents do not fully understand the nature of the shock or its transmission mechanism and therefore they apply trial and error simple learning rules. Despite its simple structure, the model matches the data quite well, stressing the importance of incorporating behavioral features in economic models.[20]

e	1	100	300	400	500	700	900
$\sum \lvert data - output \rvert$	.7636	.6815	.6839	.6850	.6859	.6871	.6879

Tab 3: Robustness check of the parameter e

2.6 Leaning against the Wind Policies

In this Section we tackle the following question: "Could the boom in house price have been avoided with an appropriate interest rate policy?" This question fosters great discussion in

[20]For recent critiques to rational expectation hypothesis see Bouchaud (2009), Colander (2009) and Farmer (2009).

the policy circles and central banks. Should monetary policy be concerned with financial stability and thus react to asset price, i.e., "leaning against the wind", or should it just focus on inflation (and possibly output) stabilization?

Here we are more concerned about house prices rather than financial asset. In this respect, the Adam, Kuang and Marcet (2011) model predicts that the recent house price dynamics would have been avoided and the current account deficit would have been considerably smaller, if the interest rate had fallen by less at the beginning of the 2000's.

Gelain, Lansing and Mendicino (2013) develop a DSGE model where simple moving-average forecast rules for a subset of agents significantly magnify the volatility and persistence of house prices and household debt. In their framework, a direct response of the central bank's interest rate rule to either house-price growth or credit growth would have the important drawback of substantially magnifies the volatility of inflation.[21]

We would like to ask the same type of question in our framework. To do so, we assume that the real interest rate (i.e., our policy variable) responds to the house price. We then solve for the optimal value of this response. Optimal is here defined as minimizing two different measures of the house price fluctuations: the distortion and the volatility of the house price.

The distortion measure is the average deviation of the house price from its steady state:

$$dis(Q) = \frac{1}{T} \sum_{t=1}^{T} |\hat{Q}_t|. \tag{2.35}$$

The volatility measure is the average change in the time series of the deviation of the house price from its steady state (which coincides with the average rate of growth of the house price):[22]

$$vol(Q) = \frac{1}{T-1} \sum_{t=1}^{T} |\hat{Q}_t - \hat{Q}_{t-1}|. \tag{2.36}$$

Moreover, for this exercise, we adopt two policy rules. First, we assume that the real interest rate responds to the quarterly house price:

$$\hat{R}_q = r^q \hat{Q}_q. \tag{2.37}$$

[21] They show that macroprudential policies that directly affects the borrowing constraint are more effective tools in reducing the volatilities of house prices and household debt.

[22] See Lengnick and Wohltmann (2010). For the same reasons as theirs, we do not use the variance of the simulated series. Our time series shows long deviations from the mean (which we interpret as boom and busts), so that when calculating the variance, one would not measure the volatility but rather the mean squared distortion.

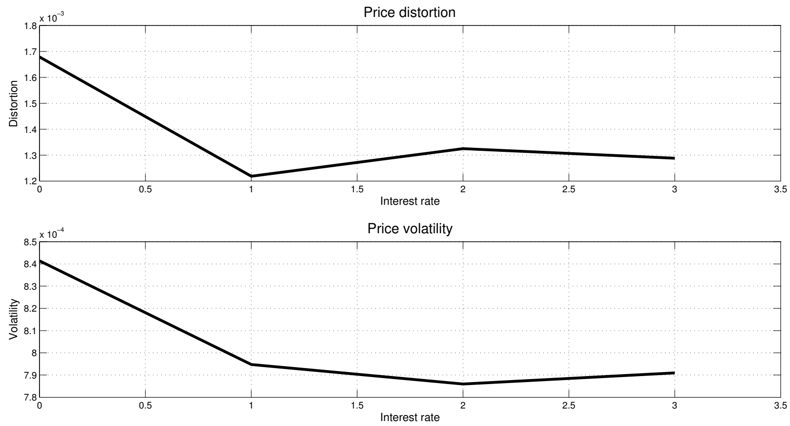


Figure 2.10: House price distortion and volatility with the interest rate reacting at house price.

Second, we modify the target making the interest rate to respond to the rate of growth in the house price:

$$\hat{R}_q = r^q \left(\hat{Q}_q - \hat{Q}_{q-1}\right). \tag{2.38}$$

Then we set the preference shock in house demands equal to zero. To find the optimal value of r^q, we do a grid search on the set $r^q \in [0, 3]$. For each value of r^q we simulate 1000 runs of 100 quarters each, drawing different realizations of the shock to the house price ε_t^Q and then we average across runs. The results are shown in Figures 1.10 and 1.11. Figures 1.10 and 1.11 display the distortion and the volatility of the house price as a function of r^q for rule (2.37) and rule (2.38), respectively. In the case of rule (2.37) both the distortion and the volatility measures are minimized for $r^q = 1$. Hence, policy can minimize both distortion and volatility, but the reaction of the interest rate should not be too strong, because a higher than one-to-one response has the opposite effect. Recall that in our model the real interest rate has a negative effect on demand (see equation (2.20)). It is thus not so surprising that an overreaction to house prices could lead to increase, rather than decrease, the instability of the house price dynamics, amplifying our Agent-Based mechanism of evolutionary selection of expectation rules.

Figure 1.11 shows the results about distortion and volatility using the policy rule described by (2.38). Now the distortion measure is still minimized for $r^q = 1$ but the volatility measure exhibits a different behavior with respect to Figure 1.10. A stronger reaction is needed to minimize such measure. Interestingly, Figure 1.11 exhibits a trade-off between the two measures for $r^q > 1$.

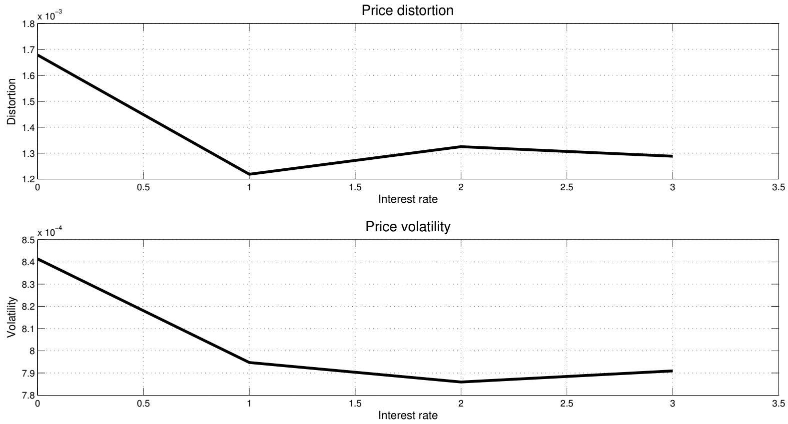


Figure 2.11: House price distortion and volatility with the interest rate reacting at the rate of change in house price.

Admittedly, given the partial equilibrium nature of our model, our policy analysis and implications should be taken with care. First, we are forced to assume that the real, rather than the nominal, interest rate is the policy variable. Second, contrary to a general equilibrium model as in Gelain, Lensing and Mendicino (2013), our framework can not say anything about the possible deleterious effect of a policy rule on other policy relevant variables, as inflation, for example. Third, for the same reason, our framework can not take into account the feedback effects of the fluctuations in the house price on changes in other variables (as income for example), that are taken as exogenous in our partial equilibrium structure.

Although we acknowledge the limits of the policy analysis in our context, it seems that a policy suggestion emerges clearly from the experiments, in line with Adam, Kuang and Marcet (2011): using the interest rate to influence the house prices, the government could avoid the movements at the heart of booms and busts in house prices. However, to assess whether this is an optimal or sound thing to do, a general equilibrium analysis is needed.

2.7 Conclusion

We developed a model to study the housing market starting from an Agent-Based perspective. We showed that it is possible to generate an endogenous creation of boom-and-bust dynamics in the house price by relaxing the rational expectation hypothesis, and embedding into the model an Agent-Based mechanism of evolutionary selection of expectation rules based on backward-looking behavior. The framework is based on the chartist-fundamentalist mechanism, and, despite its simplicity, it is able to match the behavior of the US house price quite well. The interaction between chartists and fundamentalists is sufficient to create endogenous movements in the house price with a large influence on the dynamics of the economic system.

The results point to the exogenous preference shock as the main driving force behind the house price dynamics. On the contrary, the model suggests that other competing hypothesis, as a prolonged period of low interest rates or the liberalization in credit standards, have only minor effects on house price dynamics.

Finally, the model provides a rationale for monetary policy to lean against the wind in order to reduce the fluctuations in house prices.

Our framework can be expanded in several directions. The model is clearly still rather simple in incorporating a really psychological foundation of expectations. Moreover, our framework could be embedded into a general equilibrium model. This step would be particularly welcome to analyze the policy implications of our framework and the robustness of the results in the last Section.

Chapter 3

Heuristics Selection and Heterogeneity

3.1 Introduction

In the last decades economics and finance has been moving inside an important paradigm change, from a representative and rational agent approach to a behavioral, agent-based approach, in which economic environment is populated with boundedly rational, heterogeneous agents using rule of thumb strategies. The approach of neoclassical economics, the *rational* approach, assumes that the decision maker is able to select the alternative that maximizes the utility or profit, given his beliefs of other economic factors; moreover it is assumed that decision maker's original beliefs are self-fulfilling, in the sense that he is able to predict exogenous as well as endogenous variables.

The *bounded rationality* approach (see Sargent, 1993) relies on different requirements because, from this point of view, what the rational approach requires is too demanding: generally speaking, a boundedly rational agent is modeled as able to choose what he perceives as the best for himself, but doesn't own a perfect knowledge of the environment structure. This behavioral approach fits much better with agent based-simulation models and challenges the traditional, representative, rational agent framework; but many ideas of this new approach have a quite long history: Keynes' view that "expectations matter" is related to some elements of the behavioral agent-based model, as well as Simon's idea that economic man is "boundedly rational" and to the view of Kahneman and Tversky (1973, 1974) in psychology that individual behavior can be best described by *judgmental heuristic-representativeness.* By this heuristic, people predict the outcome that appears

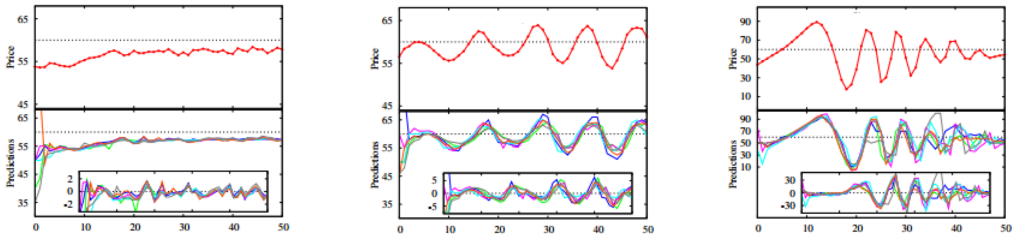
most representative of the evidence.
Indeed the evolution of economic variables, such as stock prices, interest rates, exchange rates and inflation rates, is affected by expectations of agents operating in the financial and real markets: for this reason it's possible to think to the market as an *expectations feedback system*: market history shapes individual expectations which, in turn, determine current aggregate market variables and so on.
Laboratory experiments with human subjects have shown that individuals do not behave in a full rational way but follow simple *heuristics* which can account for persistent biases in taking decisions. This occurrence explains why prices may persistently deviate from fundamentals in laboratory markets, similarly to what can be observed in real stock markets. Moreover heterogeneity is crucial to the aim of expounding a number of evocative findings of the recent learning to forecast experiments.
In this work we develop a simple nonlinear model which is able to exhibit path-dependance explaining how both stable steady states and attracting curves can arise endogenously in the model. In particular we present a simple model with evolutionary selection among different simple forecasting strategies and the economic environment is seen as a complex evolutionary system between competing boundedly rational trading strategies. The choice of heuristics will be governed by an evolutionary selection mechanism, based on the principle that more successful strategies will attract more followers. In this multi-agent model, endogenous fluctuations is caused by a generic phenomenon, that is coexistence of two attractors (a steady state and a periodic or quasi-periodic orbit). In particular the model tries to generalize the *adapted belief system* (ABS) introduced by Brock and Hommes (1997, 1998) and then developed by Anufriev and Hommes (2012). The ABS here considered is a present discounted value asset pricing model with heterogeneous beliefs: there are two trader types and the fractions of these types change over time according to evolutionary fitness, as measured by utility from realized profit. The economic intuition behind this different outcomes (persistent oscillations and convergence) could be explained by the interaction and the evolutionary switching between trend extrapolation and stabilizing fundamental analysis that may lead to coexistence of locally stable fundamental steady state and an attracting closed curve far from the steady state. Indeed asset price fluctuations are characterized by high volatility with large price changes irregularly interchanged by episodes of low volatility with small price changes. In particular there's empirical evidence that many "stylized facts" observed in financial time series recall the presence of endogenous fluctuations that cannot be explained uniquely by external factors

or by fundamentals. For these reasons, the purpose of this paper is to show these facts simultaneously by a simple behavioral model of individual learning.
The paper is organized as follows: Section 2 briefly describes laboratory experiments with human subjects; in Section 3 the evolutionary model is presented along with its assumptions; Section 4 is devoted to the stability analysis of the steady state of the model; simulations are performed in Section 5 and Section 6 concludes.

3.2 Laboratory experiments

A number of computerized *learning to forecast experiments* have been performed in the CREED laboratory at the University of Amsterdam. In this work we try to explain theoretically and analytically the results described in the works of Hommes et al. (2005, 2007). In each session of the experiments, 6 participants were advisers to large pension fund and had to submit point forecasts for the price of a risky asset for 50 consecutive periods. The pension fund can invest money either in a risk-free asset with real interest rate r per period or in shares of an infinitely lived risky asset. In each period the risky asset pays uncertain dividend which is a random variable, independent and identically distributed, with mean $\bar{y}$. The price of the risky asset, p_t, is determined every period by a market clearing equation, as an aggregation of individual forecasts of all participants. The exact functional form of the strategies and the equilibrium equation were unknown to the participants, but they are informed that the higher their forecasts are, the larger the demand for the risky asset is. Participants also know the values of the parameters $r = 0.05$ and $\bar{y} = 3$, and therefore they have enough information to compute the rational fundamental price of the risky asset $p^f = \bar{y}/r = 60$.
Every session of the experiment last 51 periods and in every period each of the six participants provide a two-period-ahead forecast for the price of the risky asset, given the available information which consists of past prices (up to two lags) of the risky asset and own past predictions (up to one lag) made by the participant. The predictions of other participants are unknown neither how each forecast affects the equilibrium price. When all six predictions for the price in period $t+1$ is submitted, the current market clearing price can be computed according with a standard mean-variance asset pricing model with heterogeneous beliefs:

$$p_t = \frac{1}{1+r}((1-n_t)\bar{p}^e_{t+1} + n_t p^f + \bar{y} + \varepsilon_t) \qquad t = 0, ..., 50 \tag{3.1}$$



(a) (Almost) monotonic convergence

(b) Persistent oscillations

(c) Dampening fluctuations

Figure 3.1: Time series of price (upper part), individual predictions (lower part) and forecasting errors (inner frame) in laboratory experiments.

where $\bar{p}^e_{t+1} = \frac{1}{6}\sum_{i=1}^{6} p^e_{i,t+1}$ is the (equally weighted) average of the six individual forecasts, r is the risk-free interest rate, $\bar{y}$ is the mean dividend, ε_t is a stochastic term representing small demand/supply shocks and n_t stands for a small fraction of "robot" traders who always submit a fundamental forecast p^f. Robot traders were introduced as a stabilizing force in the experiment to prevent the occurrence of large bubbles. The fraction of robot traders increased as the price moved away form its fundamental equilibrium level, according to

$$n_t = 1 - exp\left(-\frac{1}{200}|p_{t-1} - p^f|\right) \tag{3.2}$$

At the end of each period every participant h was informed about the realized price and his/her earnings were defined by a quadratic scoring rule:

$$e_{t,h} = max\left(1300 - \frac{1300}{49}(p_t - p^e_{t,h})^2, 0\right) \tag{3.3}$$

The data coming from these experiments can be used as a benchmark for different expectations hypotheses, such as rational expectations or adaptive learning models. Furthermore the main achievements of the experiments are as follows:

- human subjects tend to follow simple forecasting predictors and set up their decisions on past observations;
- participants are able to coordinate on a common prediction strategy even if this can be different between sessions;
- three different price patterns were observed, as Figure 3.1 (slow and almost monotonic convergence, persistent oscillations with almost constant amplitude, dampening fluctuations);

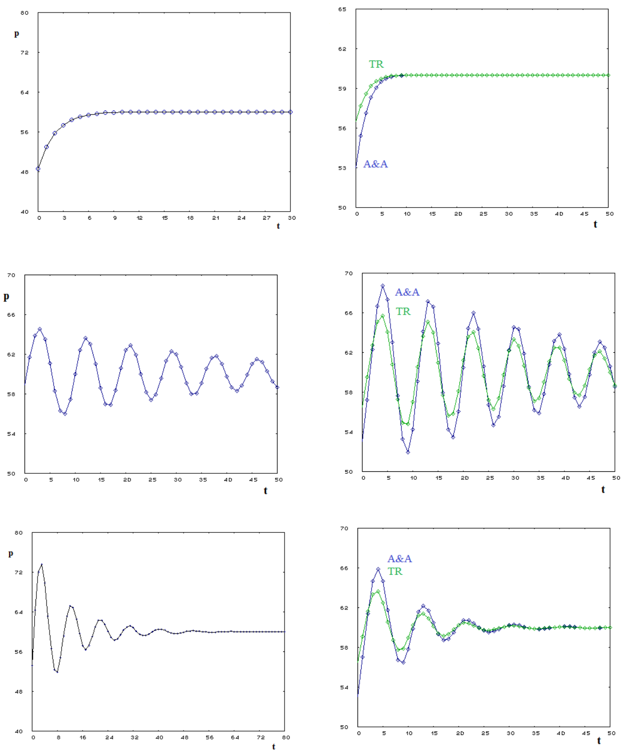


Figure 3.2: Coordination in model simulations. Heuristic switching model simulations (*Left*) and predictions of the two forecasting heuristic in the evolutionary switching model (*Right*). Benchmark parameters are $\beta = 0.4$, $\eta = 0.7$, $\delta = 0.9$ and $m = 0.5$. Monotonic convergence is obtained for $g = 0.2$, $l = 0.1$ (*top panels*), persistent fluctuations for $g = 1.1$, $l = 1$ (*central panels*) and dampening oscillations for $g = 1$, $l = 0.8$ (*bottom panels*)

- realized asset prices are significantly different from the rational fundamental price in every sessions.

The evolutionary switching mechanism matches individual forecasting behavior as well as market outcomes in the experiments. Figure 3.2 shows the simulations of the heuristics-switching model that we will describe in the next section. Coordination of individual forecasts explains the three different aggregate market outcomes, i.e. monotonic convergence to the equilibrium, permanent oscillations and oscillatory convergence. Oscillations are reinforced when the impact of trend followers is relatively large and may be sustained by the anchoring and adjusting heuristic. Thus, similar to the experiments, in our evolutionary framework agents' coordination arises and it supports the main outcomes of learning to forecast laboratory experiments.

3.3 The evolutionary model

In this section we present the model with evolutionary selection between different simple forecasting heuristics. There are some reference points that is worth to underline: there exists a pool of simple forecasting rules (e.g. adaptive or trend-following heuristics) commonly available to the agents and they could be either costly or costless; agents will select rules from this pool. At every time period these heuristics give forecasts for next period's price, and the realized market price depends upon these individual forecasts. Moreover the effects of different forecasting heuristics upon the realized prices are changing over times because individuals are learning based on evolutionary selection: the better a forecasting rule performed in the past, the higher its impact will be in determining next period's price. For these reasons the realized market price and the impact of the forecasting heuristics evolve together in a dynamic process with mutual feedback.

Let $\mathfrak{I}$ denote a set of I heuristics which agents can use for price prediction. In the beginning of period t every rule $i \in \mathfrak{I}$ gives a two period-ahead point prediction for the price p_{t+1}. This prediction is described by a deterministic function f_i of the available information set:

$$p^e_{i,t+1} = f_i(p_{t-1}, p_{t-2}, \dots; p^e_{i,t}, p^e_{i,t-1}, \dots) \tag{3.4}$$

In the market there are only two types of assets, a risk-free and a risky asset: in each period the risk-free asset pays a fixed interest rate r, whereas the risky asset pays stochastic dividends, independently and identically distributed, with mean $\overline{y}$. The price in period t is computed on the basis of these predictions, and it is given by the discounted value of future cash flow, according to equation 3.1 of the previous section (we recall it for sake of simplicity:

$$p_t = \frac{1}{1+r}\big((1-n_t)\overline{p}^e_{t+1} + n_t p^f + \overline{y} + \varepsilon_t\big)$$

where $\overline{p}^e_{t+1}$ denotes an (equally weighted) average of the individual forecast, p^f is the rational fundamental price, $r(=0.05)$ is the risk free interest rate, $\overline{y}(=3)$ is the mean dividend and ε_t is the stochastic term associated with small demand/supply shocks and n_t represents the share of fundamental robot traders as described by equation (3.2). The fraction of robot traders increases in response to the deviations of the asset price from its fundamental level: this mechanism reflects the fact that in real market there is more agreement about over or under evaluation of an asset when the price deviation from the fundamental level is large. Finally the fundamental price is set to $p^f = \overline{y}/r = 3/0.05 = 60$.

The average $\overline{p}^e_{t+1}$ in (3.1) is a population weighted average of the different forecasting rules:

$$\overline{p}^e_{t+1} = \sum_{i=1}^{I} n_{i,t} p^e_{i,t+1} \tag{3.5}$$

The impact of each heuristics, $n_{i,t}$, evolves over time and depends on the past relative performance of all I heuristics, and more successful heuristics attract more followers. The performance measure of a forecasting heuristic in a given period is based on its squared forecasting error (see Anufriev et al. 2012): this means that the performance measure of heuristic i up to (and including) time $t-1$ is

$$U_{i,t-1} = -(p_{t-1} - p^e_{i,t-1})^2 + \eta U_{i,t-2} \tag{3.6}$$

The parameter η represents the memory of the agents, measuring the relative weight that each agent gives to past errors of heuristic i: if $\eta = 0$, the impact of each heuristic is completely determined by the most recent forecast error and for $0 < \eta \leq 1$ all past prediction errors affect the impact of heuristic i, with exponentially declining weights.
Given the performance measure, the fraction of agents using heuristic i or the impact of heuristic i in computing $\overline{p}^e_{t+1}$, is updated according to a discrete choice model with a-synchronous updating (see Hommes et al. 2005 and Diks et al. 2005)

$$n_{i,t} = \delta n_{i,t-1} + (1-\delta)\frac{\exp(\beta U_{i,t-1})}{Z_{t-1}} \tag{3.7}$$

where $Z_{t-1} = \sum_{i=1}^{I} \exp(\beta U_{i,t-1})$ is a normalization factor and $\sum_{i=1}^{I} n_{i,t} = 1$. If $\delta = 0$ the updating rule reduces to the discrete choice model with synchronous updating; the more general case $0 < \delta < 1$, gives some persistence in the impact of rule i, reflecting the fact that not all participants update their rule in every period or at the same time: for this reason δ can be seen as the average per period fraction of agents who doesn't change their previous forecasting rule (if $\delta = 1$ the initial impacts of the rules never change). So if $0 < \delta < 1$ in each period a fraction $1-\delta$ of individuals update their rule according to the well known discrete choice model, used for example in Brock and Hommes (1997). β represents the intensity of choice and measures how sensitive agents are to differences in strategy performance: the higher β, the faster individuals will switch to more successful rules (if $\beta = 0$, the result is an equal distribution of forecasting rules among individuals; if $\beta \to \infty$, the fraction $1-\delta$ who updates its heuristic switch to the most successful predictor).
To keep the model as simple as possible, but rich and complete enough to explain the

different price patterns observed in the experiments, we have selected only two heuristics, which are rather simple and were among the rules estimated on the individual forecasts in the experiments. A behavioral interpretation underlies each heuristic.

The first heuristic is a trend following rule. It extrapolates a trend that can be weak or strong depending on what value is assigned to the parameter g:

$$p^e_{1,t+1} = p_{t-1} + g(p_{t-1} - p_{t-2}) \tag{3.8}$$

This rule means that agents simply predict the last observed price level plus a multiple of the last observed price change ($g > 0$). It generalizes the weak and strong trend following rules developed by Anufriev and Hommes (2012). It is also worth to point out that at the moment when the price forecasting for time $t + 1$ is submitted, price p_t is still unknown and the last observed price is p_{t-1}.

The second rule is a little bit more complicated. It is a combination of the average prediction of the last observed price and an estimate of the long-run equilibrium price level with an extrapolation of the last price change

$$p^e_{t+1} = mp^{av}_{t-1} + (1 - m)p_{t-1} + l(p_{t-1} - p_{t-2}) \tag{3.9}$$

where p^{av}_{t-1} is the sample average of all past prices, i.e. $p^{av}_{t-1} = \frac{1}{t}\sum_{j=0}^{t-1} p_j$. This rule is called *learning anchoring and adjustment heuristic* since it uses an anchor, $mp^{av}_{t-1} + (1 - m)p_{t-1}$ defined as an (equally weighted) average between the last observed price and the sample mean of all past prices, and extrapolates the last price change $l(p_{t-1} - p_{t-2})$ according to the value assigned to the parameter l. The parameter m represents the weight assigned to the fundamental price.

This kind of rule is obtained from a related linear *anchoring and adjustment heuristic*

$$p^e_{2,t+1} = mp^f + (1 - m)p_{t-1} + l(p_{t-1} - p_{t-2}) \tag{3.10}$$

In the experiment subjects did not know p^f explicitly since they were not provided with the fundamental price, but apparently they were able to learn the anchor $mp^f_{t-1} + (1 - m)p_{t-1}$ and extrapolates price change from there. Therefore we replaced p^f in equation (3.10) by a proxy p^{av}_{t-1} given by the observed sample average of past prices[1].

All these rules are first order heuristics in the sense that they only use last observed price level, i.e. the last forecast and/or the last observed price change, but it could be also

[1]However we will use heuristic given by (3.10) in the simulations and stability analysis provided in the following sections

possible to introduce rules with more price lags.
Let us observe that (3.8)-(3.9) can be collected into one general rule, such as:

$$p_{i,t+1} = (1 - \beta_{i,1} - \beta_{i,2})p^f + \beta_{i,1}p_{t-1} + \beta_{i,2}p_{t-2} \qquad i = 1,2 \tag{3.11}$$

In this way the trend extrapolating rule is obtained for $\beta_{1,1} = 1+g$ and $\beta_{1,2} = -g$, whereas the anchoring and adjusting rule is obtained setting $\beta_{2,1} = m + l$ and $\beta_{2,2} = -l$.
In order to analyze the deterministic skeleton of the model, we set $\varepsilon_t = 0$ and obtain:

$$\begin{cases} p^e_{1,t+1} = p_{t-1} + g(p_{t-1} - p_{t-2}) \\ p^e_{2,t+1} = mp^f + (1-m)p_{t-1} + l(p_{t-1} - p_{t-2}) \\ n_t = 1 - \exp\Big(-\frac{1}{200}|p_{t-1} - p^f|\Big) \\ U_{i,t-1} = -(p_{t-1} - p^e_{i,t-1})^2 + \eta U_{i,t-2} \qquad i = 1,2 \\ n_{i,t} = \delta n_{i,t-1} + (1-\delta)\frac{\exp(\beta U_{i,t-1})}{Z_{t-1}} \qquad i = 1,2 \\ p_t = \frac{1}{1+r}\big((1-n_t)(n_{1,t}p^e_{1,t+1} + n_{2,t}p^e_{2,t+1}) + n_t p^f + \overline{y}\big) \end{cases} \tag{3.12}$$

3.4 Analysis of the model

The dynamics described by (3.12) can be re-written in deviations from the fundamental price, setting

$$x_{1,t} = p_t - p^f, \quad x_{2,t} = x_{1,t-1}, \quad x_{3,t} = x_{1,t-2}, \quad x_{4,t} = x_{1,t-3}$$

and plugging the two heuristics into the price equation.
A number of issues arises when analyzing the model and, in order to give an answer to all of these, we consider the deterministic skeleton of the system in (3.3) and analyze its properties.
The following 7-dimensional system of first order equations consists in 2 equations describing the evolution of performance measures, one describing the fractions of different forecasting rules, one equation describes the price dynamics and other 3 equations are used to take lags of price deviations into account.

$$\begin{cases} U_{1,t-1} = \eta U_{1,t-2} - (x_{1,t-1} - (1+g)x_{3,t-1} + g \cdot x_{4,t-1})^2 \\ U_{2,t-1} = \eta U_{2,t-2} - (x_{1,t-1} - (m+l)x_{3,t-1} + l \cdot x_{4,t-1})^2 \\ n_{1,t} = \delta n_{1,t-1} + \frac{1-\delta}{Z_{t-1}} \exp\left(\beta[-(x_{1,t-1} - (1+g)x_{3,t-1} + gx_{4,t-1})^2 + \eta U_{1,t-2}]\right) \\ x_{1,t} = \exp(-\frac{1}{200}|x_{1,t-1}|)\frac{1}{1+r}\Big([\delta n_{1,t-1} + \frac{1-\delta}{Z_{t-1}} \exp\big(\beta\big[-(x_{1,t-1} - (1+g)x_{3,t-1} + gx_{4,t-1})^2 + \eta U_{1,t-2}]\big)\big]((1+g)x_{1,t-1} - gx_{2,t-1}) + \\ +[\delta n_{2,t-1} + \frac{1-\delta}{Z_{t-1}} \exp\big(\beta\big[-(x_{1,t-1} - (m+l)x_{3,t-1} + lx_{4,t-1})^2 + \eta U_{2,t-2}]\big)\big]((m+l)x_{1,t-1} - lx_{2,t-1})\Big) \\ x_{2,t} = x_{1,t-1} \\ x_{3,t} = x_{2,t-1} \\ x_{4,t} = x_{3,t-1} \end{cases}$$

First of all it is worth to notice that it is straightforward to check that the fundamental steady state is a fixed point for the map and we investigate its local stability. Furthermore notice that the local stability of the price dynamics at the steady state $p_t = p^f = p^*$ is not affected by the robot traders. Taking this result into account, we study the local stability of the equilibrium $x_1 = x_2 = x_3 = x_4 = 0$ of this system with price equal to p^f and zero fraction of robot traders.
The Jacobian matrix $\mathbf{J}$ of the system at the steady state is given by

$$J_s = \begin{bmatrix} \eta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 & 0 & 0 \\ \frac{\beta\eta(1-\delta)}{4} & -\frac{\beta\eta(1-\delta)}{4} & \delta & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\delta p^f}{1+r} & \frac{(1+g+m+l)}{2(1+r)} & \frac{-g-l}{2(1+r)} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

This Jacobian matrix has eigenvalues equal to 0 and η (both of multiplicity 2), δ of multiplicity 1 and the remaining two eigenvalues are the roots of characteristic polynomial for the matrix

$$J_\lambda = \begin{bmatrix} \frac{(1+g+m+l)}{2(1+r)} & \frac{-g-l}{2(1+r)} \\ 1 & 0 \end{bmatrix}$$

Since η and δ are supposed to be smaller than 1, they do not determine a change in the stability of the system. We also assume $\delta \neq 1$ that assures an important impact of the heuristics over time and $\eta < 1$ which means agents take into account their past performances. For these reasons the local stability conditions are completely determined by the eigenvalues of J_λ and the coefficients of forecasting heuristics are the main driving forces. Nevertheless the parameters η and δ affect the speed of convergence, being eigenvalues

of the Jacobian matrix. Furthermore the local stability is not influenced by the intensity of choice β, as in Anufriev et al. (2012). So the fundamental steady state of the asset pricing model is locally stable if all the eigenvalues of J_λ lie inside the unit circle. When the heuristic coefficients are specified, the eigenvalues of J_λ can be computed.
Let

$$\mathcal{P}(\lambda) = \lambda^2 - \frac{1+g+m+l}{2(1+r)}\lambda + \frac{g+l}{2(1+r)} \tag{3.13}$$

be the characteristic polynomial of J_λ. The stability region of the fixed point is determined by the following conditions (see the work of Medio 1992 and Medio and Lines 2003):

$$\begin{cases} \mathcal{P}(1) = 1 - Tr + Det > 0 \\ \mathcal{P}(-1) = 1 + Tr + Det > 0 \\ Det < 1 \end{cases}$$

Due to the symmetry of the map, simple computations show that a pitchfork[2] bifurcation of the fundamental steady state arises when $\mathcal{P}(1) = 0$, whose curve has equation $m = 1 + 2r$. It does not depend on g and l and occurs for parameter values that have no economic meaning in our setting. The flip bifurcation curve, $\mathcal{P}(-1) = 0$, has equation $3 + 2(r + g + l) + m = 0$. The Neimark-Sacker bifurcation curve, defined by $Det = 1$, corresponds to $\frac{g+l}{2} = R$ where we have set $(1 + r) = R$.

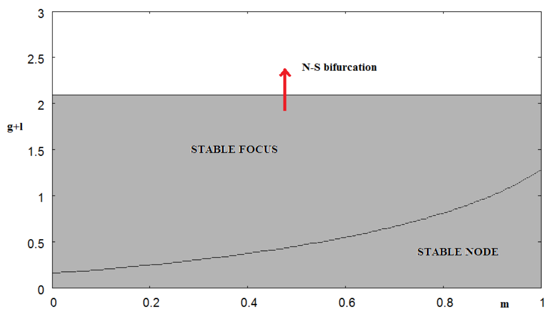


Figure 3.3: Stability region in the $(m, g + l)$ parameter space

Figure 4.4 depicts the stability region in the parameter space $(m, g + l)$. As the picture shows, the only bifurcation that can occur is a Neimark-Sacker, represented by the horizontal line $\frac{g+l}{2} = R$: this situation appears when the average extrapolation coefficient exceeds R. Moreover pitchfork and flip bifurcations may occur but only for parameter

[2]Recall that the fundamental steady state always exists.

values that have not economic meaning in our setting.
This evolutionary switching model with only two heuristics is able to generate dynamics with an unstable fundamental steady state. As we have already mentioned, the switching model with competing trend extrapolating and anchoring and adjusting heuristics undergoes a Neimark-Sacker bifurcation when the average extrapolation coefficient exceeds the value of the gross interest rate R. Moreover it is important to stress that even if the fundamental steady state is locally stable, other attractors, such a stable cycle, may co-exist. Hence the local stability properties of the fundamental steady state do not imply global stability, depending on the coefficients of the forecasting heuristics.

3.5 Model simulations

In this section we show some interesting properties that the model is able to exhibit[3]. First of all, different behaviors can arise from our setting and the explanation is that heterogeneous learning has a kind of path-dependance behavior, as Anufriev and Hommes have shown in their works (2012). For this reason the model is capable of reproducing different price courses, consistent with the outcomes of the laboratory experiments with human subjects. Moreover this path-dependent distinction of the model remains effective for a large range of parameters. As shown in Figure 3.2 of Section 2, the three qualitatively different price patterns observed in the experiment can be reproduced, that is monotonic convergence, persistent oscillation and dampening fluctuations.
A particular feature of the model is the possibility in which prices do not necessarily converge to the fundamental steady state. This means that in this heterogeneous agent model with evolutionary learning, prices could settle down to a stable periodic or quasi-periodic orbit and not to their fundamental value, even if this remains locally stable.
Figure 4.9 shows coexistence of attractors in our heterogeneous framework: indeed we can observe either convergence to the fundamental steady state (Figure 4.9, left box) or convergence to a stable periodic attractor (Figure 4.9, right box) for the same parameter values. Note that we are considering parameter values in the stability region of the fundamental steady state, i.e. $(g + l)/2 = 1.04 < R$.
It has also to be noticed that the convergence to the fundamental steady state requires more. In this eventuality agents start with a strong trend extrapolation and only after

[3]Here we restrict our analysis to the case of no memory in the performance measure ($\eta = 0$) and no stickiness in the fraction of individuals who keep fixed the previous strategy ($\delta = 0$).

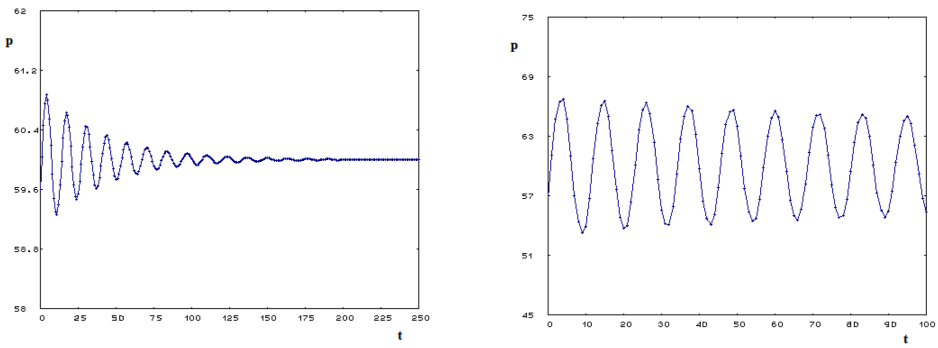


Figure 3.4: LHS: convergence to the fundamental steady state ($p_0 = 59, q_0 = 59$). RHS: convergence to a periodic attractor ($p_0 = 55, q_0 = 55$). Parameter values: $g = 1.06, l = 1.02, m = 0.35, \beta = 0.4$

some periods some kind of fundamental expectations were used, due to the initial large prediction errors and thus low performances.

We further deepen the analysis showing the existence of two kinds of bifurcation and the importance of this aspect is related to the fact that there exists an open region in the parameter space where a stable steady state and a stable attracting closed curve coexist, as the previous pictures showed. Even when the fundamental price p^f is locally stable, price need not converge to this fundamental value, but may focus on an attracting closed curve. Indeed, as shown by Gaunersdorfer et al. (2008), an open region in the parameter space with a stable steady state and an attracting closed curve may exist. This phenomenon is due to the existence of a particular codimension two bifurcation, the so-called Chenciner bifurcation (see Hommes, 2013).

Generally a steady state may lose stability through a Neimark-Sacker bifurcation when its Jacobian matrix has two eigenvalues lying on the unit circle with all other eigenvalues inside the unit circle. Two types of Neimark-Sacker bifurcation are distinguished:

- a *supercritical* Neimark-Sacker bifurcation where the stable steady state becomes unstable and the unstable steady state is surrounded by an *attracting closed curve* with periodic or quasi-periodic dynamics;

- a *subcritical* Neimark-Sacker bifurcation where the stable equilibrium becomes unstable and the stable equilibrium is surrounded by a *repelling closed curve* with periodic or quasi-periodic dynamics.

A subcritical Neimark-Sacker bifurcation is generally associated with explosive dynamics that occur after its appearance. Here we do not have such kind of dynamics but we can observe bounded and wide oscillations. From the view point of a policy maker, it could be reasonable prevent this kind of dynamics. Accordingly, policy maker should reveal clear information about fundamentals so that anchoring or fundamental beliefs become stronger and Neimark-Sacker bifurcation becomes supercritical.
Following Gaunersdorfer et al. (2008), we compute the curve in the parameter space (m, g) which indicates the transition from a *supercritical* bifurcation to a *subcritical* one.[4]
Recall that at this stage we have restricted our analysis to the case of no memory ($\eta = 0$) and synchronous updating in fractions ($\delta = 0$). Then the parameter space P is equal to

$$P = \{(\beta, g, m, l, R) : \beta > 0, \ \ g > 0, \ \ 0 < m < 1, \ \ l > 0, \ \ R > 1\}$$

The Neimark-Sacker bifurcation manifold is given by

$$H = \{(\beta, g, m, l, R) \in P : \frac{g + l}{2} = R\}$$

In our evolutionary adaptive learning model, the locus of degenerate Neimark-Sacker (Chenciner) bifurcation points in the (m, g) parameter plane, within the Neimark-Sacker bifurcation manifold $H = \{\frac{g+l}{2} = R\}$ is reported in Figure 3.5. From Figure 3.5 we can see that for small g values the steady state loses stability through a subcritical Neimark-Sacker bifurcation. When g is sufficiently large, as m increases the bifurcation becomes subcritical, the steady state becomes unstable and, when it is stable, it co-exists with a repelling closed curve. Moreover there exists a so called *volatility clustering*[5] region where a locally stable fundamental steady state and an attracting closed curve coexist. Note that for $g < 0.95$ the bifurcation is always subcritical but it arises for $l = 2R - g > 1.15$, i.e. when the anchoring and adjusting rule becomes destabilizing.
In Figure 3.6 the bifurcation diagram in the parameter space (l, m) shows the transition from a supercritical bifurcation to a subcritical. How can it be read? Let us move from figure 3.5 and fix a g-value (for example $g = 1.5$). For this value there exists a m^*-value ($m^* = 0.496232$) on the x-axis that marks the transition from supercritical to subcritical bifurcation. Furthermore the bifurcation diagram in Figure 3.6 clarifies this mechanism:

[4] In the appendix the main steps to reach this goal are shown.

[5] We refer to volatility clustering as an endogenous phenomenon due to the presence of different kind of agents in which price irregularly move between different regimes where periods of low volatility are interchanged by periods of persistent deviations from the fundamentals and high volatility.

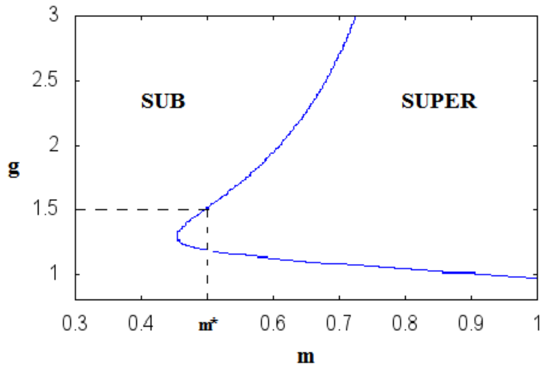


Figure 3.5: Plot of the curve that marks the transition from a supercritical to a subcritical bifurcation

indeed on the bifurcation manifold $\frac{g+l}{2} = R$ we reported the m^*-value for which the transition happens and we can observe that for values of m lower than m^* the bifurcation is supercritical whereas for $m > m^*$ the bifurcation is subcritical. Indeed in such a case the periodicity regions appear before the Neimark-Sacker bifurcation.
We can provide a simple economic intuition about this phenomena, i.e. in the presence of strong extrapolators, price deviations are reinforced by trend followers. This leads the fundamental steady to be locally unstable and prices depart from their fundamental value. But when deviation becomes too large, trend followers extrapolate less and condition their forecast on market fundamentals: in this way a fundamental rule is prevailing and prices quickly return to the fundamental value. This is a cyclic process that repeats. So when trend followers are weakly extrapolators (i.e. g close to 1), the interaction among agents can lead them to coordinate on a periodic orbit around the locally stable fundamental steady state.
Up to now it has been shown that the fundamental steady state p^* is stable for small g and l values, and it loses its stability in a Neimark-Sacker bifurcation at $g + l = 2(1 + r)$: indeed we have seen that for $g + l < 2(1 + r)$ a stable invariant curve may exist when the steady state is still locally stable.
Our model can also show some more complex dynamical behavior for which chaos arises. We show this behavior by two different time series: in the left box of Figure 3.7, obtained with i.c. $p_0 = 31.6$ and $q_0 = 31.6$, we can observe a cycle of period 8 around the fundamental steady state (there is not a long transient to converge to the cycle) whereas in the right box of Figure 3.7, obtained with i.c. $p_0 = 31.7$ and $q_0 = 31.7$, the longer transient

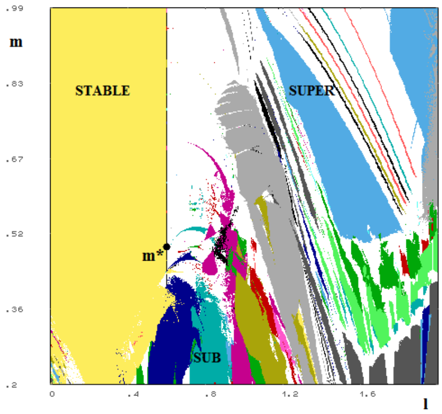


Figure 3.6: Bifurcation diagram in the (l, m) parameter plane

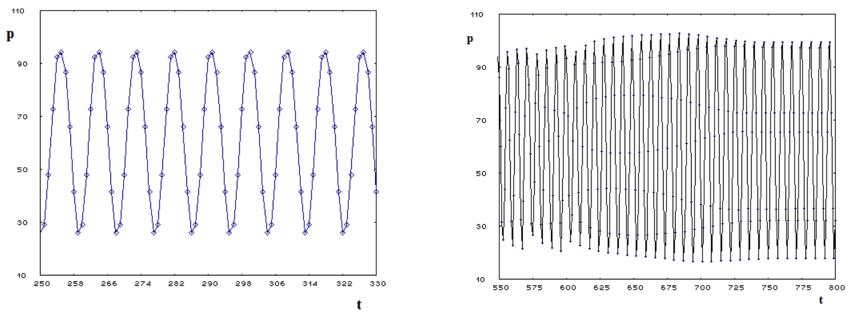


Figure 3.7: LHS: periodic cycle with a short transient. RHS: chaotic repeller with a longer transient

part reveals the existence of a chaotic repeller.

This dynamical change is due to a homoclinic bifurcation that occurs when trend extrapolators are strong (large g values) and the second rule gives greater importance to the last price change (large l values). Indeed trend followers strongly extrapolate small deviations from the fundamental steady state leading to oscillating prices. Moreover, this learning system shows the existence of a homoclinic orbit which is associated to complex dynamic behavior. This evolutionary interaction is connected to the existence of a homoclinic bifurcation which generates chaotic dynamics (for better explanation see the works of Agliari et al. 2001, 2005).

3.6 A more general case: $\delta \neq 0$ and $\eta \neq 0$

Previous simulations have been conducted by setting $\delta = 0$ and $\eta = 0$ which means respectively synchronous updating and no memory. Similar simulations can be provided showing that the same qualitative results can be reached even if we re-introduce a kind of inertia in fractions dynamics (a-synchronous updating) and memory in the fitness measure.

The three price patterns observed in laboratory experiments can be reproduced thank to the heterogeneous learning mechanism (see figure 3.2 in Section 2). We simulate the model for a fixed set of parameters, i.e. $\beta = 0.4$, $\delta = 0.9$, $\eta = 0.7$ and $m = 0.5$. Note that the parameter values of g, l and m are selected within the stability region displayed in figure 4.4. As previously shown, local stability does not depend on the intensity of choice β. Furthermore, its dependance on the other two parameters of the learning mechanism, η and δ is also limited. Indeed the local stability conditions are completely determined by polynomial (3.13) and only depend on the coefficients of forecasting heuristics. The parameters η and δ, being eigenvalues of the Jacobian matrix, affect the speed of the convergence.

In the previous section it has been shown that the fundamental steady state is stable for some g-values, it loses stability in a Neimark-Sacker bifurcation for $\frac{g+l}{2} = R$ but a stable limit cycle can even exist for $g + l < 2R$, i.e. when the fundamental equilibrium is still locally stable. The same qualitative behavior exists also in the presence of a-synchronous updating and memory in the fitness measure. Furthermore, by numerical simulations, in Figure 3.8 we shows an example of 3 coexisting attractors in the $(p_t - p_{t+1})$ plane for different initial conditions and with the same parameter values ($g = 1$, $l = 1.062$ and $m = 0.01$). Since $\frac{g+l}{2} < R$, these attractors coexist with the locally stable fundamental steady state. We can observe the fundamental price along with a stable 6-cycle and a quasi-periodic circle.

In the left panel of Figure 3.9 it is illustrated what happens before and after the Neimark-Sacker bifurcation. Numerical simulations suggest that for $g = 1$ and $l = 1$ the fundamental steady state is unique and locally stable since it still lies in the stability region $\frac{g+l}{2} < R$ (m is set equal to 0.5 in both pictures). As g increases, the right panel of Figure 4.1 displays that an invariant attracting curve (a quasi-periodic attractor) appears via a supercritical bifurcation. We can claim that the fundamental value p^f is a unique locally stable equilibrium if the price does not differ too much from it and the dynamics converges towards it. Moreover the fundamental price p^f can be destabilized via a

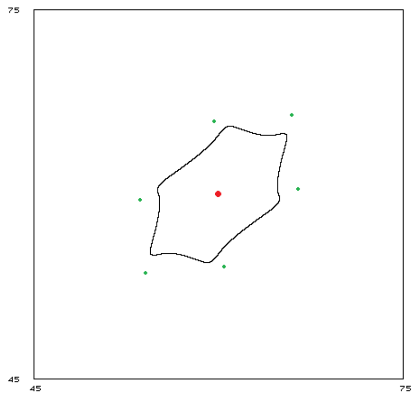
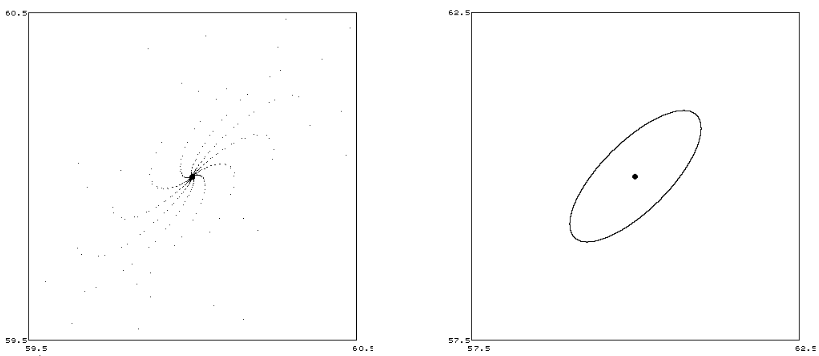


Figure 3.8: Projections of attractors in the $p_t - p_{t+1}$ plane

Neimark-Sacker bifurcation at $\frac{g+l}{2} = R$ and a stable closed curve with periodic or quasi-periodic behavior occurs. This closed curve may be associated with a chaotic repeller via homoclinic bifurcation as well (as in the case of no memory and synchronous updating of fractions), meaning that price will not converge to the fundamental value but it will fluctuate around it after a long transient part. Finally, the bifurcation diagram in Figure



(a) $g + l = 2$ (b) $g + l = 2.2$

Figure 3.9: Projection of a stable and a quasi-periodic attractor in the $p_t - p_{t+1}$ plane

3.10 depicts how the dynamics of the switching model with two heuristics depends on the memory parameter η. Small values of the memory parameter η imply that agents do not give great importance to past performance of both heuristics and they forget it rapidly. Since the trend extrapolating rule is generally self-reinforcing on a short time scale, this heuristic will often dominate even if some occasional errors can occur when trend reverses. Consequently, oscillations are especially wide for small η values. When memory increase (larger η values), the evolutionary switching model tends to generate smaller amplitude fluctuations. This evidence is related to a lower usage of the trend following rule due to its poor performance.

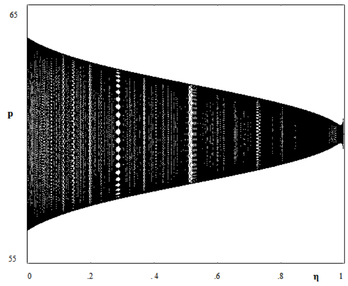


Figure 3.10: Bifurcation diagram in the switching model with respect to the memory parameter η. The amplitude of the quasi-periodic oscillations becomes smaller as η increases. Benchmark parameters are $\beta = 0.4$, $\delta = 0.9$, $r = 0.05$ and extrapolation coefficients $g = 1.3$, $l = 1$

3.7 Conclusions

Neoclassical economic thinking have been assuming that individuals form expectations rationally leading to an efficient allocation of resources. On the other hand laboratory experiments with human subjects have shown that agents do not behave fully rational, but follow simple heuristics (rules of thumb). For this reason the variables, such as prices or interest rates, may deviate from fundamentals, similar to the large fluctuations observed in financial markets. Moving from these stylized facts, we addressed the question whether it is possible to express this evidence by means of a simple model and whether this model is able to generate both persistent excess volatility and convergence to the fundamental equilibrium.

The model is made up with simple and different forecasting rules, each of which can generate its own type of dynamics. In every period the forecasting strategy is selected among the population of heuristics and agents adapt their selection over time, based on the relative performance of the heuristics (evolutionary selection mechanism). The outcomes we came up with is quite interesting: the dynamics exhibits path-dependence feature, i.e. the capability to generate both persistent oscillating and converging patterns for the same parameter values. Indeed path dependance implies that initial conditions are responsible for differences in aggregate price patterns. We can claim that even if the path-dependence feature remains valid for a large range of parameters, some quantitative aspects may change when parameters vary, such as the speed of convergence or the amplitude and frequency of oscillations.

We have also shown that the fundamental steady state can be locally unstable with our set of heuristics and, although the fundamental equilibrium is locally stable, other attractors may co-exist. There is a simple economic intuition of this phenomena, depending on the strength of the trend extrapolation and the role of stabilization forces. Indeed evolutionary interaction among strong extrapolators and stabilizing agents leads to other attractors with irregular price fluctuations, switching between periods of low and high volatility. Thus the interaction among different kind of agents may lead to a stable closed curve (or a more complicated attractor) around a locally stable steady state. Irregular price oscillations and coexistence of attractors can be considered as an explanation of some stylized facts, such as volatility clustering observed in real financial markets.

We are aware the model proposed here is simple and should be considered as stylized behavioral model, but excess volatility and a kind of volatility clustering are created by the trading process, and this seems to be in line with financial practices. If the evolutionary interaction of boundedly rational agents with different trading strategies extols volatility, there are important consequences for regulatory policy in financial markets. Good or bad news in the markets can be amplified by the evolutionary mechanism. Since we are embedded in an increasingly globalized world, and of course this is true for financial markets, small changes in fundamentals in one country may generate changes in asset price in other countries.

Appendix: Normal form analysis

In everything that follows we set $\delta = 0$ ad $\eta = 0$. Then prices are written in deviations from the fundamental price, as introduced in Section 3. The fundamental fixed point p^f corresponds with $x = 0$.

Considering the family of maps $\Phi(\mu, x) = (\varphi(x, \mu), x_1, x_2, x_3)$, the evolution map φ is assumed to be of the form:

$$\varphi(x, \mu) = \frac{1}{R}\Big[\Big(\tilde{s}(x_1 + g(x_1 - x_2)) + (1 - \tilde{s})(l(x_2 - x_1) - mx_1)\Big)\Big]$$

with

$$\tilde{s} = \frac{\exp(-\beta[(x_1 - x_3 - g(x_3 - x_4))]^2)}{\exp(-\beta[(x_1 - x_3 - g(x_3 - x_4))]^2) + \exp(-\beta[x_1 - mx_3 - l(x_3 - x_4)]^2)}$$

and

$$\mu = (g, l, m, R)$$

Now we focus on the bifurcations of the origin and, for this aim, the linearization $D\Phi(0)$ of Φ at 0 has to be computed:

$$J = D\Phi(0) = \frac{d\Phi}{dx}(0) = \begin{pmatrix} \frac{1}{2R}(m + l + g + 1) & -\frac{(l+g)}{2R} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2a & -b & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

introducing parameters a and b by

$$a = \frac{1}{4R}(m + l + g + 1) \text{ and } b = \frac{l+g}{2R}$$

and with steady state values $x = 0$ and $\tilde{s} = \frac{1}{2}$.

The characteristic equation of the matrix turns out to be:

$$\lambda^2(\lambda^2 - 2a\lambda + b) = 0$$

The root $\lambda = 0$ has multiplicity 2 whereas the other two roots are

$$\lambda_\pm = a \pm \sqrt{a^2 - b}$$

A necessary condition for the occurrence of a Neimark-Sacker bifurcation is that $|\lambda_\pm| = 1$ and $\lambda_\pm \notin \{-1, 1\}$ or, equivalently, that:

$$b = 1 \text{ and } a^2 - 1 < 0.$$

In terms of original parameters:

$$g = 2R \text{ and } -6R - 1 < m < 2R - 1$$

We'll assume in the following that these conditions are fulfilled and let $\lambda = \lambda_+$; then $\lambda_- = \bar{\lambda}$.
Complex eigenvectors of J are q and $\bar{q} \in \mathbb{C}^4$, given by:

$$q = (1, \bar{\lambda}, \bar{\lambda}^2, \bar{\lambda}^3)$$

satisfying $Jq = \lambda q$, $J\bar{q} = \bar{\lambda}\bar{q}$, and e_3 and e_4, satisfying $Je_3 = e_4$, $Je_4 = 0$, where e_j denotes the j'th unit vector.
The subspace spanned by eigenvectors of J with eigenvalues having norm equal to 1 is called the linear *center* space E^c whereas the subspace spanned by eigenvectors of$J$ with eigenvalues with norm not equal to 1 is called *hyperbolic* space E^h. In this case E^c is spanned by linear combinations of q and $\bar{q}$ of the form $zq + \bar{z}\bar{q}$; the space spanned by e_3 and e_4 is the space E^h. The spaces E^c and E^h span together the tangent space of X at zero.
Now the Neimark bifurcation can be studied reducing the system to a center manifold which is tangent to the span of eigenvectors q and $\bar{q}$. Since every vector $x \in X$ can be written as

$$x = zq + \bar{z}\bar{q} + y$$

where $z \in \mathbb{C}$ and $y \in E^h$. The center manifold can be described by a map $w : E^c \to E^h$, $w = w(z, \bar{z})$ with the property that

$$w(0,0) = \tfrac{\partial w}{\partial z}(0,0) = \tfrac{\partial w}{\partial \bar{z}}(0,0) = 0$$

Furthermore it is convenient to introduce the notion of adjoint eigenvectors (see [62]). If the complex inner product is given by

$$\langle x, y \rangle = \sum_{i=1}^{n} \bar{x}_i y_i$$

then there is a unique vector $p \in \mathbb{C}^4$, called the adjoint eigenvector of q, satisfying:

$$J^T p = \bar{\lambda} \text{ and } \langle p, q \rangle = 1$$

where J^T denotes the transpose of J and with the properties $\langle p, \bar{q} \rangle = 0$ and any real vector y satisfies $\langle p, y \rangle = 0$ if and only if $y \in E^h$.

For $q = (1, \bar{\lambda}, \bar{\lambda}^2, \bar{\lambda}^3)$, the vector p is equal to

$$p = \frac{1}{1-\lambda^2}(1, -\lambda, 0, 0)$$

Recall that the map defining the system is of the form $\Phi(\mu, x) = (\varphi(x, \mu), x_1, x_2, x_3)$ with

$$\varphi(x, \mu) = \frac{1}{R}\Big[\Big(\tilde{s}(x_1 + g(x_1 - x_2)) + (1 - \tilde{s})(l(x_2 - x_1) - mx_1)\Big)\Big]$$

where

$$\tilde{s} = e^{-x_1^2}\frac{e^{-\beta u_2}}{e^{-\beta u_1} + e^{-\beta u_2}}$$

u_1 and u_2 are of the form

$$u_1 = -\left(x_1 - x_3 - g(x_3 - x_4)\right)^2$$

$$u_2 = -\left(x_1 - l(x_3 - x_4) - mx_3\right)^2$$

Moreover, in order to compute the normal form of a Chenciner bifurcation, a Taylor development of φ is required. This leads to

$$\begin{aligned}\varphi(x_1, x_2, x_3, x_4) = \quad & \frac{1}{R}\Big[l(x_1 - x_2) + mx_1 + \Big(\frac{1}{2} + \frac{\beta(u_1(x_1, x_3, x_4) - u_2(x_1, x_3, x_4))}{4}\Big) \\ & \cdot(x_1 + g(x_1 - x_2) - l(x_1 - x_2) - mx_1)\Big]\end{aligned}$$

The following step is due to find an expression of Φ in $(z, \bar{z}, x)$ variables. Note that:

$$(x_1, x_2, x_3, x_4) = (z + \bar{z}, \bar{\lambda}z + \lambda\bar{z}, x_3 + \bar{\lambda}^2 z + \lambda^2\bar{z}, x_4 + \bar{\lambda}^3 z + \lambda^3\bar{z})$$

In the new coordinates the factors of φ have the form:

$$l(x_1 - x_2) + mx_1 = c_0 z + \overline{c_0 z}$$

$$x_1 + g(x_1 - x_2) - l(x_1 - x_2) - mx_1 = z(c_0 - d_0) + \bar{z}(\bar{c_0} - \bar{d_0})$$

$$\tfrac{\beta}{4}(u_1 - u_2) = c_1 z^2 + c_2|z|^2 + \bar{c_1}\bar{z}^2$$

where c_j $(j = 0, 1, 2)$ and d_0 are used to collect the parameters.
Following the approach of the GHW-model, the dynamics on the center manifold up to fourth order are determined by:

$$\psi(z, \bar{z}) = \zeta(z, \bar{z}, w(z, \bar{z})) = \lambda z + \frac{1}{1-\bar{\lambda}^2}\sum_{m+n=3}\zeta_{mn} z^m z^n + O(|z|^5)$$

with $m, n > 0$. The Neimark bifurcation at 0 is degenerate if

$$Re\zeta_{21}\lambda - \bar{\lambda} = 0$$

or equivalently if

$$Im\zeta_{21} = 0$$

Chapter 4

Macroeconomic Stability and Heterogeneous Expectations

4.1 Introduction

From the sixties onwards, *rational expectations* (RE) emerged as the dominant paradigm in economics. Nowadays RE are the main mathematical formulation of the agents' expectations. During the last decades indeed, after the works of Muth (1961) and Lucas (1972), the RE hypothesis have been widely applied in all the different field of economics and finance modeling, becoming the leading and more appealing paradigm.

Even if the RE hypothesis has the advantage of being more easily tractable, today it seems quite unrealistic to assume that agents have perfect knowledge of the whole economic system; moreover, as emphasized also by Sargent (1993), rational expectations imply not only that individuals are perfectly aware of the mechanisms moving the economy, but also that they are able to solve all the computational problems which arise in the model.

Of course real people often act on the basis of overconfidence, fear and peer pressure - topics that behavioral economics is now addressing. As stated by Hommes (2005), the characteristic of an economic system is the fact that it is an expectations feedback system, therefore expectations play a central role in all the modern macroeconomic theory. The equilibrium models that were developed, by assumption, do not consider most of the structure of a real economy because this implies too much nonlinearity and complexity for equilibrium methods to be easily tractable.

Modeling economy as a bottom-up system is an approach in which individuals understand the whole framework but only a very small part of it (see De Grauwe, 2010).

These systems work as a result of the application of simple rules of thumb by a variety of individuals populating the system.
As suggested by Hommes (2006), the tremendous volume of trading operations that can be observed every day in all the real markets, reinforces the idea of heterogeneous expectations (HE) and the idea that differences of opinions among market participants are necessary for trade to occur.
Heterogeneous agent models mimic important observed stylized facts in asset returns, such as fat tails, volatility clustering and long memory, as discussed e.g. in the extensive surveys of LeBaron (2006) and Hommes (2005).

Some recent examples of macro and financial models with heterogeneous expectations include Evans and Honkapohja (2003, 2006) Bullard and Mitra (2002), Hommes (2001, 2002), Ascari et al. (2012). Moreover Hommes (2011), Assenza et al. (2013) and some others studies, provided evidence in favor of heterogeneous expectations using laboratory experiments with human subjects.

Recently Anufriev et al. (2012) introduce heterogeneous expectations in macroeconomics employing a simple frictionless DSGE model to investigate inflation dynamics under alternative interest rate rules where agents have heterogeneous expectations and update their beliefs based on past performance as in Brock and Hommes (1997).
In this work we follow this line of research but we enrich the framework introducing output dynamics. We investigate the stabilizing effects of different monetary policy rules in an economy described by the 3-equations New Keynesian model where agents switch between different forecasting rules on the basis of their past performances. We consider a parsimonious model with simple heuristics which is able to generate endogenous waves of optimism and pessimism (animal spirits). The analysis of monetary policy is conducted to investigate the role of inflation and output gap in business cycle movements. We also give some policy suggestions analyzing the role of the Central Bank in reducing the volatility and the distortion of output and inflation.
The model is composed by the IS curve, a New Keynesian Phillips curve and a Taylor rule, as in the work of Clarida et al. (1997). According to the benchmark model of Branch and McGough (2009, 2010), our setting has the same functional form as the standard formulation except for the homogeneous expectation hypothesis which is replaced with a combination of heterogenous expectations. We show that incorporating heterogeneous expectations into the New Keynesian model may significantly alter the stability properties of the fundamental equilibrium: the model may have multiple equilibria in the presence

of expectations heterogeneity.
The business cycle dynamics depends on the expectations environment and the coefficients of the interest rate rule as in Anufriev et al. (2012) but, differently from their findings, we conclude that the monetary authority has to react even more aggressively in order to guarantee the uniqueness (and global stability) of the RE equilibrium. The Central Bank has to send correct and stronger signals to agents in order to break down the reinforcing process that arises not only between past and current inflation but also between inflation and output. If the monetary policy reacts weakly to inflation, a cumulative process of rising inflation and output appears. On the other hand, with an aggressive monetary policy, the monetary authority is able to induce stable dynamics. Finally the Taylor principle is sufficient to guarantee convergence to the fundamental steady state but it is no longer enough to avoid multiple equilibria and it can reduce volatility and distortion of output and inflation. Additionally we show that in our setting there exists the possibility of two supercritical Neimark-Sacker bifurcations to occur which give rise to the appearance of two invariant curves around the respective focus. The convergence to the non rational steady states is no more monotonic but there are quasi-periodic oscillations.
The paper is organized as follow: Sections 2 and 3 present the model economy and the heterogeneous expectations framework; the main contributions of the paper are presented in Sections 4 and 5 where the dynamical properties of the model are investigated. Firstly we consider the simplest scenario in which agents can choose among only three different forecasting rules, then we apply the concept of *large type limit* (LTL) to investigate the dynamics in the case of a continuum of forecasting rules. In Section 6 we perform a policy analysis to investigate how monetary policy should be designed in order to minimize volatility and distortion of the simulated time series. Section 7 concludes.

4.2 The model economy

The model is made up of a standard aggregate demand and supply, augmented with a Taylor rule. Heterogeneity is introduced because agents use different rules of thumb (heuristics) to forecast the future values of economic variables; moreover these rules are subjected to a learning mechanism which is able to create endogenous business cycle.

The aggregate demand is presented as

$$y_t = a_1 E_t y_{t+1} + a_2 \left(i_t - E_t \pi_{t+1} \right) \tag{4.1}$$

where y_t and y_{t+1} are respectively the output gap in period t and $t+1$, i_t is the nominal interest at t rate, π_{t+1} is the rate of inflation at $t+1$ and E is the expectations operator.

The aggregate supply can be interpreted as a New Keynesian Phillips curve:

$$\pi_t = f_1 E_t \pi_{t+1} + f_2 y_t \tag{4.2}$$

Equation 4.2 can be derived from firms' profit maximization under sticky price assumption where inflation at time t is increasing in both output at the current period and expected inflation.

Finally the monetary policy rule is given by

$$i_t = c_1 \pi_t + c_2 y_t \tag{4.3}$$

We adopt a contemporaneous Taylor rule as a simplifying assumption even if we are aware about the criticism in that direction. We set the baseline calibration of the model according to the work of Clarida et al. (1997) and table 1 summarizes the parameter values.

$a_1 = 1$	$a_2 = -1$
$f_1 = 0.99$	$f_2 = 0.3$
$c_1 = 1.5$	$c_2 = 0.5$

Table 1: parameters calibration

Combining (4.1) and (4.3) we have

$$y_t = a_1 E_t y_{t+1} + a_2 c_1 \pi_t + a_2 c_2 y_t - a_2 E_t \pi_{t+1}$$

and substituting (4.2) we get

$$y_t = a_1 E_t y_{t+1} + a_2 c_1 f_1 E_t \pi_{t+1} + a_2 c_1 f_2 y_t + a_2 c_2 y_t - a_2 E_t \pi_{t+1}$$

Solving for y_t

$$y_t = \frac{a_1}{(1 - a_2 c_1 f_2 - a_2 c_2)} E_t y_{t+1} + \frac{a_2 (c_1 f_1 - 1)}{(1 - a_2 c_1 f_2 - a_2 c_2)} E_t \pi_{t+1} \tag{4.4}$$

Then we plug (4.4) into (4.2) and after some algebra we obtain

$$\pi_t = \frac{a_1 f_2}{(1 - a_2 c_1 f_2 - a_2 c_2)} E_t y_{t+1} + \frac{f_1 (1 - a_2 c_2) - f_2 a_2}{(1 - a_2 c_1 f_2 - a_2 c_2)} E_t \pi_{t+1} \tag{4.5}$$

4.2.1 Expectations

In this part we will present the Adaptive Belief System developed by Brock and Hommes (1997) that allows to model expectations by introducing heterogeneity among agents. In particular we refer to the work of Anufriev et al. (2012) to describe the evolutionary part of the model and we will briefly present the updating mechanism for agents' beliefs, addressing the cited paper for major details.

As already stressed by Simon (1955), people have limited knowledge and calculus ability, and if they want to pursue better decision rules, they must bear some search costs; due to these two limitations, agents are endowed only with a bounded rationality and as a consequence, they use simple heuristics when they face a decision that entails some degree of uncertainty.

Assume that agents can form expectations choosing from $2H+1$ different forecasting rules. We denote with $\hat{E}_{h,t}y_{t+1}$ and $\hat{E}_{h,t}\pi_{t+1}$ the forecasts of output and inflation by h-th rule. Moreover, each rule for output and for inflation prediction can be chosen by (eventually) different number of individuals: therefore the fraction of agents using forecasting rule h to forecast output at time t is denoted by $w_{h,t}$ and the one for inflation $z_{h,t}$.

The fractions are updated according to an *evolutionary fitness measure.*The fitness measures, for output and inflation respectively, are publicly available (i.e. assume they can be read in a freely available newspaper) but subject to noises and expressed in utility terms as:

$$\tilde{U}_{h,t} = U_{h,t} + \varepsilon_{h,t} \tag{4.6}$$

$$\tilde{V}_{h,t} = V_{h,t} + \nu_{h,t} \tag{4.7}$$

where $U_{h,t}$ and $V_{h,t}$ are the deterministic parts and $\varepsilon_{h,t}$ and $\nu_{h,t}$ are the stochastic components of the fitness measures.

The fraction of agents choosing strategy h is given by the well known discrete choice model:

$$w_{h,t} = \frac{e^{\gamma_1 U_{h,t-1}}}{\sum_{h=1}^{H} e^{\gamma_1 U_{h,t-1}}} \tag{4.8}$$

$$z_{h,t} = \frac{e^{\gamma_2 V_{h,t-1}}}{\sum_{h=1}^{H} e^{\gamma_2 V_{h,t-1}}} \tag{4.9}$$

where γ_1 and γ_2 are the intensity of choice parameters and reflect the sensitivity of agents in selecting the optimal strategy. The higher the fitness of a forecasting rule h, the higher the probability that an agent will select strategy h.

The functions $U(\cdot)$, $V(\cdot)$ are the past squared forecast errors:

$$U_{h,t-1} = -\left(y_{t-1} - \hat{E}_{h,t-2}y_{t-1}\right)^2 - C_h \tag{4.10}$$

$$V_{h,t-1} = -\left(\pi_{t-1} - \hat{E}_{h,t-2}\pi_{t-1}\right)^2 - C_h \tag{4.11}$$

where C_h is the information cost of predictor h. Since they are boundedly rational, the fundamental steady state predictor for output gap and inflation requires some efforts or some information gathering costs[1] $C_h > 0$. Simon (1955) stressed information gathering and processing costs as an obstacle to fully rational, optimal behavior. Agents must either face search and information gathering costs in using sophisticated, optimal rules or may choose to employ free and easily available simple rules of thumb that perform "reasonably well". Generally, for a simple heuristic predictor, information costs are assumed $C_h = 0$, but for more complex forecasting rules (e.g. rational expectations) information gathering costs may be positive.

4.3 Evolutionary model with constant belief types

We consider a scenario in which agents can choose among $2H + 1$ different symmetric forecasting rules, where positive and negative biases are exactly balanced around the Rational Expectations Equilibrium (henceforth REE). This choice implies that the REE is among the steady states of the model. Notice however that this hypothesis is not essential for most of quantitative results to hold. In this environment agents roughly know the fundamental steady state of the economy whereas they disagree about the correct value of the fundamental output and inflation steady state. The existence of constant predictors is supported by learning to forecast laboratory experiments with human subjects that have shown that individuals use very simple rules (see the work of Assenza et al., 2013).
The rational predictor is:

$$\hat{E}_{0,t}y_{t+1} = \hat{E}_{0,t}\pi_{t+1} = 0 \tag{4.12}$$

[1] It is here necessary to state clearly that a fundamentalist agent is different from the rational one (except in one particular case): as a matter of fact the former does not know that in the economy some disturbing, heterogeneous agents are present and so, he predicts the equilibrium thinking that all the agents are as he is; differently the "true" rational agent, must be aware of the presence of all the disturbing agents and also of their biased predictions, taking them into account while predicting the equilibrium

The other constant belief types are defined by:

$$b_h = \hat{E}_{h,t} y_{t+1} = \begin{cases} \frac{b}{h} & \text{if } 1 \leq h \leq H \\ -\frac{b}{2H+1-h} & \text{if } H+1 \leq h \leq 2H \end{cases}$$

$$d_h = \hat{E}_{h,t} \pi_{t+1} = \begin{cases} \frac{d}{h} & \text{if } 1 \leq h \leq H \\ -\frac{d}{2H+1-h} & \text{if } H+1 \leq h \leq 2H \end{cases}$$

for output and inflation respectively.
The market forecast is obtained as a weighted average of the H predictors:

$$E_t y_{t+1} = \sum_{h=0}^{2H} w_{h,t} \hat{E}_{h,t} y_{t+1} \tag{4.13}$$

$$E_t \pi_{t+1} = \sum_{h=0}^{2H} z_{h,t} \hat{E}_{h,t} \pi_{t+1} \tag{4.14}$$

Substituting (4.8), (4.10) in (4.13) and (4.9), (4.11) in (4.14), and taking also into account that the rational agents have predictor (4.12), we obtain

$$E_t y_{t+1} = \frac{\sum_{h=1}^{2H} b_h \exp\left(-\gamma_1 \left(y_{t-1} - b_h\right)^2\right)}{\sum_{h=1}^{2H} \exp\left(-\gamma_1 \left(y_{t-1} - b_h\right)^2\right) + \exp\left(-\gamma_1 \left(y_{t-1}\right)^2 + C\right)} \tag{4.15}$$

$$E_t \pi_{t+1} = \frac{\sum_{h=1}^{2H} d_h \exp\left(-\gamma_2 \left(\pi_{t-1} - d_h\right)^2\right)}{\sum_{h=1}^{2H} \exp\left(-\gamma_2 \left(\pi_{t-1} - d_h\right)^2\right) + \exp\left(-\gamma_2 \left(\pi_{t-1}\right)^2 + C\right)} \tag{4.16}$$

Now by inserting (4.15) and (4.16) into (4.4) and (4.5), we get the 2-D system

$$\begin{aligned} y_t \quad = \quad & \frac{a_1}{(1 - a_2 c_1 f_2 - a_2 c_2)} \frac{\sum_{h=1}^{2H} b_h \exp\left(-\gamma_1 \left(y_{t-1} - b_h\right)^2\right)}{\sum_{h=1}^{2H} \exp\left(-\gamma_1 \left(y_{t-1} - b_h\right)^2\right) + \exp\left(-\gamma_1 \left(y_{t-1}\right) + C\right)^2} + \\ & + \frac{a_2 (c_1 f_1 - 1)}{(1 - a_2 c_1 f_2 - a_2 c_2)} \frac{\sum_{h=1}^{2H} d_h \exp\left(-\gamma_1 \left(\pi_{t-1} - d_h\right)^2\right)}{\sum_{h=1}^{2H} \exp\left(-\gamma_1 \left(\pi_{t-1} - d_h\right)^2\right) + \exp\left(-\gamma_1 \left(\pi_{t-1}\right)^2 + C\right)} \end{aligned} \tag{4.17}$$

$$\begin{aligned} \pi_t \quad = \quad & \frac{a_1 f_2}{(1 - a_2 c_1 f_2 - a_2 c_2)} \frac{\sum_{h=1}^{2H} b_h \exp\left(-\gamma_2 \left(y_{t-1} - b_h\right)^2\right)}{\sum_{h=1}^{2H} \exp\left(-\gamma_2 \left(y_{t-1} - b_h\right)^2\right) + \exp\left(-\gamma_2 \left(y_{t-1}\right)^2 + C\right)} + \\ & + \frac{f_1 (1 - a_2 c_2) - f_2 a_2}{(1 - a_2 c_1 f_2 - a_2 c_2)} \frac{\sum_{h=1}^{2H} d_h \exp\left(-\gamma_2 \left(\pi_{t-1} - d_h\right)^2\right)}{\sum_{h=1}^{2H} \exp\left(-\gamma_2 \left(\pi_{t-1} - d_h\right)^2\right) + \exp\left(-\gamma_2 \left(\pi_{t-1}\right)^2 + C\right)} \end{aligned} \tag{4.18}$$

We refer to the map expressed by equations (4.17)-(4.18) as

$$L\left(y_t, \pi_t; H, \gamma_1, \gamma_2\right) \tag{4.19}$$

4.4 Model analysis

The analysis of the model described by (4.19) is conducted by fixing $\gamma_1 = \gamma_2 = \gamma$, $H = 1$ and $C = 0$. In this way we consider the simplest scenario in which agents can choose among three different heuristics with bias parameters $b = 1$ and $d = 1$, meaning that type 1 agents expect that inflation and output will be above its fundamental level whereas type 2 agents expect an inflation and output level lower than the fundamental value. Type 3 agents believe that output and inflation rate will be always at its REE value.
Assuming these conditions, the map described in (4.19) always owns the REE $(y^*, \pi^*) = (0, 0)$. This fundamental steady state can be locally stable or even unstable. Furthermore multiple equilibria may appear leading the dynamics to non rational expectations steady states.
We will provide the analysis of the dynamics which depends on parameter γ and the Taylor coefficient c_1. First of all we compute the Jacobian matrix of the map at the RE equilibrium:

$$J = \begin{pmatrix} \frac{4\,a1\,\gamma\,e^{-\gamma}}{(-a_2\,c_1\,f_2 - a_2\,c_2+1)\,(2\,e^{-\gamma}+e^c)} & \frac{4\,a_1\,f_1\,\gamma\,e^{-\gamma}}{(-a_2\,c_1\,f_2 - a_2\,c_2+1)\,(2\,e^{-\gamma}+e^c)} \\ \frac{4\,a_1\,f_2\,\gamma\,e^{-\gamma}}{(-a_2\,c_1\,f_2 - a_2\,c_2+1)\,(2\,e^{-\gamma}+e^c)} & \frac{4\left(\frac{a_1\,f_1\,f_2}{-a_2\,c_1\,f_2 - a_2\,c_2+1}+f_1\right)\gamma\,e^{-\gamma}}{2\,e^{-\gamma}+e^c} \end{pmatrix}$$

The trace and the determinant of matrix J are respectively given by

$$Tr(J) = \frac{4\left(\frac{a_1\,f_1\,f_2}{-a_2\,c_1\,f_2 - a_2\,c_2+1} + f_1\right)\gamma\,e^{-\gamma}}{2\,e^{-\gamma}+e^c} + \frac{4\,a1\,\gamma\,e^{-\gamma}}{(-a_2\,c_1\,f_2 - a_2\,c_2+1)\,(2\,e^{-\gamma}+e^c)}$$

$$Det(J) = -\frac{16\,a_1\,f_1\,\gamma^2}{(a_2\,c_1\,f_2 + a_2\,c_2 - 1)\,(e^{\gamma+c}+2)^2}$$

Following the work of Medio (1996), the stability region of the fundamental steady state is determined by the following conditions:

$$\begin{cases} 1 - Tr(J) + Det(J) > 0 \\ 1 + Tr(J) + Det(J) > 0 \\ Det(J) < 1 \end{cases} \tag{4.20}$$

In our setting the REE can lose stability via pitchfork bifurcation, that occurs when the curve $1 - Tr(J) + Det(J) = 0$ is crossed. We compute this curve letting the inflation coefficient and the intensity of choice vary, and keeping fixed the other parameters at the baseline calibration (see Table 1). Figure (4.1) shows the stability region in the parameter space (γ, c_1): the REE is locally stable for any (γ, c_1) that lie in the red region.

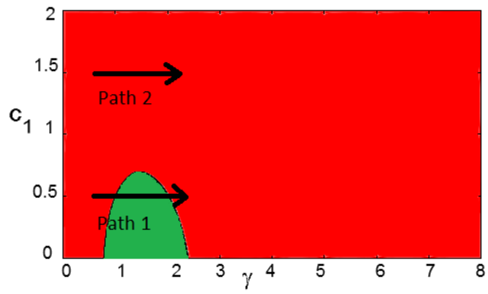


Figure 4.1: Stability region in the (γ, c_1) plane

Moreover the fundamental steady state can also lose stability via flip bifurcation but for parameter values that have no economic meaning in our setting. Finally the Neimark-Sacker bifurcation can not occur because it can be shown that the eigenvalues of the Jacobian matrix at the REE are always real.

Now we will focus on the existence of other equilibria in this simplest scenario. Recall that the steady states of the map are determined by setting

$$y_{t-1} = y_t = y$$
$$\pi_{t-1} = \pi_t = \pi$$

From (4.18) we can obtain the expression for y, given by

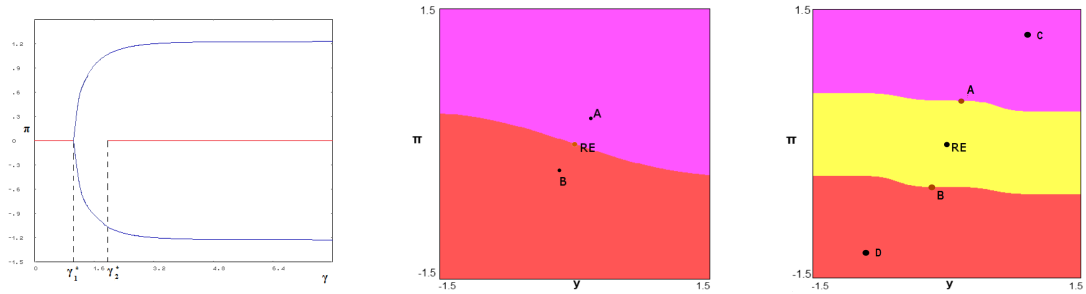
$$y = \frac{\pi - \frac{f_1 \left(d_2\, e^{-(\pi - d_2)^2 \gamma} + d_1\, e^{-(\pi - d_1)^2 \gamma} \right)}{e^{c - \pi^2 \gamma} + e^{-(\pi - d_2)^2 \gamma} + e^{-(\pi - d_1)^2 \gamma}}}{f_2} \tag{4.21}$$

Then by substituting the latter expression into (4.17) we are able to get a function $G(\pi) = 0$ which allows us to compute all the steady states of the system.

In what follows we provide an analysis of the global dynamics of (4.17)-(4.18) and we show how these dynamics depend on parameters γ and c_1.[2]

Let us consider a weak monetary policy scenario ($c_1 = 0.5$): the bifurcation diagram in Figure 4.2-a shows the evolution of the steady states as γ increases and, moving along *Path 1* in Figure 1 we can see how the REE loses stability. There exist values $0 < \gamma_1^* < \gamma_2^*$ such that:

[2]We will mainly focus on the dynamics of the fundamental steady state for a small number of γ values even if we are aware of the existence of further dynamics in addition to what we show. For example in the moderate monetary policy scenario the non fundamental steady states present other bifurcations that can lead to the appearance of stable focus or unstable nodes.



(a) $0 < \gamma < 10$ (b) $\gamma = 1.1$ (c) $\gamma = 5$

Figure 4.2: Bifurcation diagram and basins of attraction in the low information costs scenario and weak monetary policy

- for $\gamma < \gamma_1^*$ there exists only one steady state, the RE equilibrium, which is unique and may be globally stable (recall that the case $\gamma = 0$ corresponds to the circumstance of infinite variance and difference in fitness can not be observed: so agents do not switch among predictors and all fractions are constant and equal to $1/H$);
- for $\gamma = \gamma_1^* \approx 1.062$ a supercritical pitchfork bifurcation occurs: the RE equilibrium loses stability and two new stable steady states are created around it;
- for $\gamma_1^* < \gamma < \gamma_2^*$ there are two stable steady state whose basins are separated by the stable set of the RE equilibrium which is a saddle (Figure 4.2-b);
- for $\gamma = \gamma_2^* \approx 1.9275$ the RE equilibrium becomes stable again via a subcritical pitchfork bifurcation and two saddles appear;
- for $\gamma > \gamma_2^*$ at least three locally stable steady states exist and their basins are separated by the stable sets of the saddles (Figure 4.2-c).

In Figure (4.2-b-c) we have denoted the stable steady states with black circles and the saddle points with brown circles.

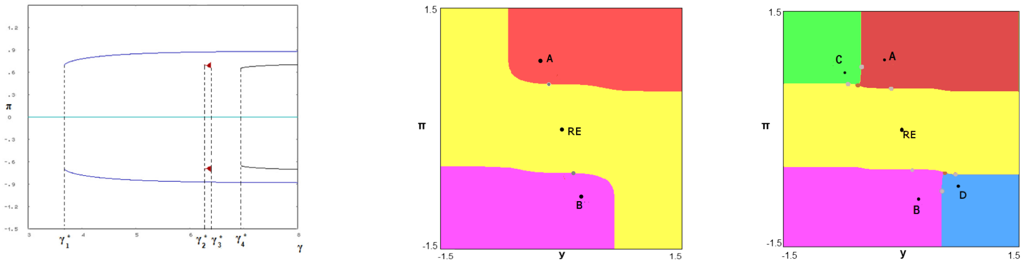
Considering a moderate interest rate rule by setting $c_1 = 1.5$, we follow the bifurcation diagram in Figure (4.3) corresponding to *Path 2* in Figure 1. By numerical simulations, we can approximatively find values $0 < \gamma_1^* < \gamma_2^* < \gamma_3^* < \gamma_4^*$ such that:

- for $\gamma < \gamma_1^*$ the RE equilibrium is the unique steady state which can be globally stable;

- for $\gamma = \gamma_1^* \approx 3.67035$ there are two simultaneous[3] saddle-node bifurcations which give rise to two saddles and two stable nodes;
- for $\gamma_1^* < \gamma < \gamma_2^*$ there are three stable steady states and two saddle points whose stable manifolds separate the basins of the stable fixed points (Figure 4.3-b);
- for $\gamma = \gamma_2^* \approx 6.265$ there are two simultaneous saddle node bifurcations and two nodes (that suddenly become stable focus) along with two saddles appear;
- for $\gamma_2^* < \gamma < \gamma_3^*$ there are two stable nodes along with the RE steady state, two stable nodes or focus and four saddles. Moreover endogenous fluctuations appear: such fluctuations are dampened down when the nodes become stable focus but they can be also self reinforcing. Indeed there exists a particular value $\gamma = \gamma_{NS}$ where two supercritical Neimark-Sacker bifurcations occur and cause the appearance of two invariant curves around the respective focus (Figure 4.4-a);
- for $\gamma = \gamma_3^* \approx 6.395$ there are two simultaneous contact bifurcations in which the invariant curves collide with the border of their basins and disappear;
- $\gamma_3^* < \gamma < \gamma_4^*$ there are three stable steady states, two unstable focus and two saddle points whose stable sets separate the basins of the stable fixed points;
- for $\gamma = \gamma_4^* \approx 6.97$ there are two simultaneous saddle-node bifurcations which creates two saddles and two stable nodes;
- for $\gamma > \gamma_4^*$ at least five locally stable steady states exist along with two unstable nodes. The basins of attractions of these fixed points are delimited by the stable sets of the corresponding saddle points (Figure 4.3-c).

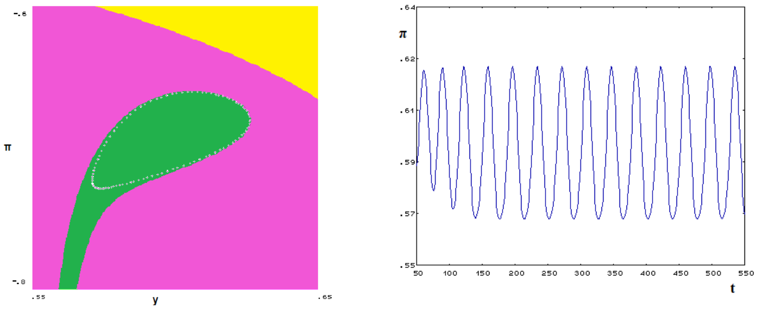
In the moderate monetary policy scenario, even if the RE equilibrium is always locally stable, a variety of global dynamics appears. When the value of γ increases, the RE equilibrium remains always locally stable but multiple equilibria are created via saddle-node bifurcations and this occurrence may have interesting policy implications. As a matter of fact, when the reaction to inflation in a neighborhood of the RE steady state is relatively high, the dynamics converge to the fundamental equilibrium. On the other hand, when inflation and output gap are out of the basin of attraction of the RE equilibrium, the implemented policy is not able to lead the economy to the fundamental steady state.

[3]Recall the map is symmetric w.r.t. the REE.



(a) $0 < \gamma < 10$ (b) $\gamma = 5$ (c) $\gamma = 10$

Figure 4.3: Bifurcation diagram and basins of attraction in the low information costs scenario and moderate monetary policy



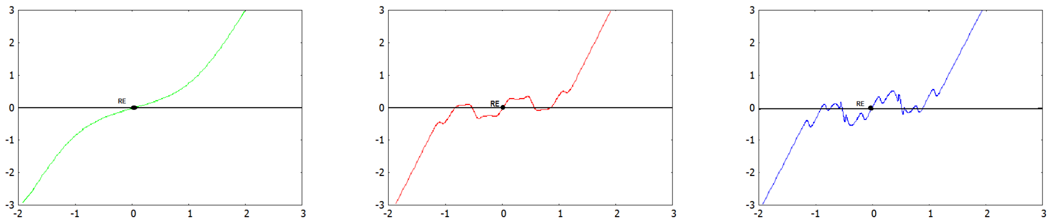
(a) (b)

Figure 4.4: Basin of attraction of the invariant curve and inflation time series

Hence more agents will adopt the positive (negative) bias driving the system to one of the positive (negative) non-fundamental steady states. In Figure 4.4-a we show the existence of an attracting invariant curve for $\gamma = 6.387$: for this γ value the convergence to the non rational steady state is no longer monotonic but there are quasi-periodic oscillations, as reported in Figure 4.4-b. The presence of a stable focus is associated with endogenous and self-reinforcing fluctuations of the variables.[4] Thus it is worth to point out that even if the Taylor principle is sufficient to guarantee convergence to the RE equilibrium, it is no longer enough to avoid the economy settle down to other non fundamental equilibria.

To give a greater insight to the role of the parameters γ and c_1, we show the behavior of the function $G(\pi)$ which gives the equilibrium values for variable π. In Figure 4.5 we fix $c_1 = 1.5$ letting vary γ. We can observe that the number of the roots of $G(\pi)$, and consequently the steady states of the map, increases. The oscillations, as long as the intensity of choice rises, become more frequent and wide. In Figure 4.6 we fix $\gamma = 1000$ as a proxy

[4] Even if we are aware we do not consider all the possible bifurcation that can occur in this map, our focus is on the analysis of the RE equilibrium and its stability properties.



(a) $\gamma = 1$ (b) $\gamma = 5$ (c) $\gamma = 10$

Figure 4.5: Function $G(\pi)$ in the $(\pi, G(\pi))$ plane for different γ values and $c_1 = 1.5$

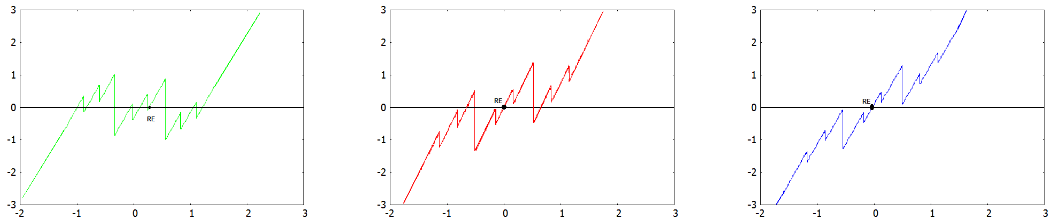
for a high sensitivity of agents, and we let vary c_1. When the monetary policy becomes stronger, the number of the steady states decreases and, in particular, for $c_1 = 4.425$ there exists only one root for function $G(\pi)$.

We define the monetary policy as *aggressive* if $c_1 > c_1^* = 4.425$. Indeed when we look at the limit case $\gamma \to +\infty$, the Jacobian matrix has an eigenvalues equal to zero with multiplicity two. Therefore the REE is always locally stable.

The economic intuition behind this result can be found in the heterogeneous framework of expectations. Differently from the work of Anufriev et al. (2012), in which authors developed a 1-D simple frictionless DSGE model to study the role of heterogeneous expectations about future inflation and the potential de-stabilizing effect of different interest rate rules, our 2D model gives different policy intuitions. In order to avoid multiple equilibria, the policy rule must be aggressive enough to ensure uniqueness and global stability of the REE. Anufriev et al. found evidence of global stability of the REE if the Taylor rule coefficient was greater than 2. Employing a standard 3-equations New-Keynesian model, we conclude that the monetary authority has to react more aggressively in order to guarantee the uniqueness (and global stability) of the REE. Therefore an aggressive monetary policy can influence heterogeneous expectations by sending correct signals to agents that correct their expectations in order to break down the reinforcing process that arises between inflation and output.

4.5 Infinitely many beliefs types

Depending on the strength of the monetary policy and the values that the intensity of choice assumes, we showed that in an economy populated by only 3 types of agents, the fundamental steady state $y^* = \pi^* = 0$ can be locally stable or unstable, and the dynamics displays coexistence of several steady states.



(a) $c_1 = 0.5$ (b) $c_1 = 2.5$ (c) $c_1 = 4.425$

Figure 4.6: Function $G(\pi)$ in the $(\pi, G(\pi))$ plane for different c_1 values and $\gamma = 1000$

The aim of this section is to investigate how the stability conditions change when the number of constant forecasting rules increases and approaches to infinity.

Suppose that there exist $2H+1$ different belief types for output b_h and $2H+1$ for inflation d_h all available at zero cost.

In order to study the dynamics of the system as long as the number of predictors become large, we apply the concept of Large Type Limit (LTL) developed by Brock et al. (2005).

Assume that at $t=0$ the beliefs about output $b = b_h \in \mathbb{R}$ are drawn from a common distribution with density $\psi(b)$, and that the predictors for inflation $d = d_h \in \mathbb{R}$ are also drawn from a common initial distribution with density $\omega(d)$. Now assuming that distributions for predictors are both normal, $\psi(b) \sim N(m, s^2)$ and $\omega(d) \sim N(n, q^2)$, the LTL map becomes

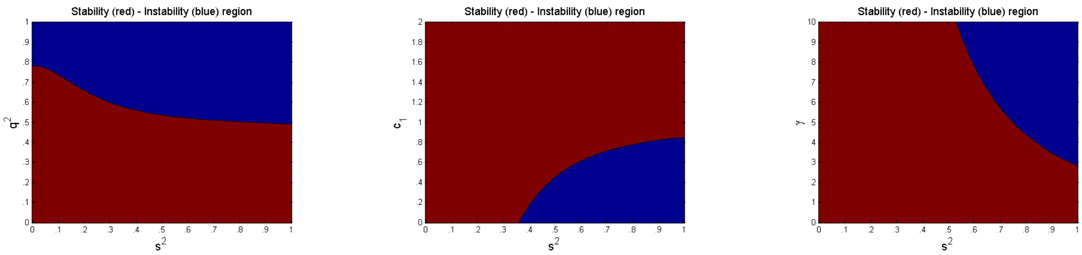
$$y = \frac{a_1}{1 - a_2 c_1 f_2 - a_2 c_2} \frac{m + 2\gamma s^2 y}{1 + 2\gamma s^2} + \frac{a_2 (c_1 f_1 - 1)}{1 - a_2 c_1 f_2 - a_2 c_2} \frac{n + 2\gamma q^2 \pi}{1 + 2\gamma q^2} \tag{4.22}$$

$$\pi = \frac{a_1 f_2}{(1 - a_2 c_1 f_2 - a_2 c_2)} \frac{m + 2\gamma s^2 y}{1 + 2\gamma s^2} + \frac{f_1 (1 - a_2 c_2) - a_2 f_2}{(1 - a_2 c_1 f_2 - a_2 c_2)} \frac{n + 2\gamma q^2 \pi}{1 + 2\gamma q^2} \tag{4.23}$$

Assuming that both distribution functions are centered around zero mean $m = n = 0$ giving the unique steady state $y^* = \pi^* = 0$. Normality and zero mean are simplifying assumptions, indeed the main results hold also for positive distributions and asymmetric predictors. Obviously the beliefs are symmetric with respect to the RE equilibrium only under the previous hypothesis, helping us in investigating the stability properties of the fundamental steady state.

Now we compute the Jacobian matrix of the system given by (4.22)-(4.23) to investigate the stability of the RE equilibrium. If $n \neq m \neq 0$ the following results and the critical values of γ do not change even if the steady state of the LTL map is not the REE.

$$J = \begin{pmatrix} -\frac{2s^2\gamma a_1}{(2s^2\gamma+1)(a_2c_1f_2+a_2c_2-1)} & \frac{2q^2\gamma}{2q^2\gamma+1}\frac{a_2-a_2c_1f_1}{a_2c_1f_2+a_2c_2-1} \\ -\frac{2s^2\gamma a_1 f_2}{(2s^2\gamma+1)(a_2c_1f_2+a_2c_2-1)} & \frac{2q^2\gamma}{2q^2\gamma+1}\left(f_1 + \frac{a_2f_2-f_2a_2c_1f_1}{a_2c_1f_2+a_2c_2-1}\right) \end{pmatrix}$$



(a) Space (s^2, q^2) (b) Space (s^2, c_1) (c) Space (s^2, γ)

Figure 4.7: Stability/instability region of the RE equilibrium in different parameter spaces

The trace is

$$Tr(J) = \frac{2\gamma q^2 \left(f_1 + \frac{a_2 f_2 - f_2 a_2 c_1 f_1}{a_2 c_1 f_2 + a_2 c_2 - 1}\right)}{2\gamma q^2 + 1} - \frac{2\gamma s^2 a_1}{(2\gamma s^2 + 1)(a_2 c_1 f_2 + a_2 c_2 - 1)}$$

The determinant is

$$Det(J) = \frac{-4q^2 s^2 \gamma^2 a_1 f_1}{(2\gamma q^2 + 1)(2\gamma s^2 + 1)(a_2 c_1 f_2 + a_2 c_2 - 1)}$$

The stability conditions in a 2-D system, as we have already shown, are given by (4.20). Substituting our parametrization (see Table 1) we can find a relation between the intensity of choice γ and the variances of the two distributions, s^2 and q^2. By numerical investigation, we can exclude the existence of flip or Neimark-Sacker bifurcations. The pitchfork bifurcation, whose curve is given by $H_{c_1}(\gamma) = 1 - Tr(J) + Det(J) = 0$, is the only bifurcation that can occur in our setting.

We investigate numerically the local stability of the REE assigning different values at q^2, s^2 and γ.

Figure 4.7-a represents the stability/instability region in the space (s^2, q^2) if $c_1 = 0.5$ and $\gamma = 10$. The stable part is marked in red whereas blue color represents the unstable configuration. It has to be noticed that with these calibrations, the stability conditions can be achieved only if the ratio of the variances is not too large, meaning that the agents can choose among a continuum of forecasts that are not *too much distant* from the RE predictor. This underlines the importance of the spread of the initial beliefs as reported in the work of Anufriev et al. (2012). In order to have a better understanding on the conditions under which the system can be stable, we assign the same variance to both output and inflation distribution. Figure 4.7-b displays the stability region in the parameter space (s^2, c_1). According to this plot, if the variance is not too large, the REE always exhibits local stability for any c_1 value.

Finally Figure 4.7-c shows the stability/instability region in the plane $\left(s^2, \gamma\right)$. There exists an inverse relation between the variance and the intensity of choice. This result remarks the importance of the spread among predictors: thus if s^2 is not too large, the system is locally stable because the adopted forecasts are equally distributed around the RE predictor. Therefore the fractions will remain almost constant. On the other hand, if s^2 is large enough, the system is unstable only if γ is sufficiently large: this means that agents can easily switch among different predictors which are not close to the RE equilibrium, leading the system to be unstable.

In the previous analysis we have assumed a normal distribution with zero mean of initial beliefs for both output and inflation. As shown by Hommes (2010), similar conclusions can be derived for fixed strictly positive distribution functions of initial beliefs. To get some intuition for this result, it is useful to look at the limiting case $\gamma = \infty$. If there exists a continuum of beliefs, the best predictor in every period, measured according to the past forecast error, will be the forecast that exactly coincides with the last period's realization of both output and inflation, $b_h = y_{t-1}$ and $d_h = \pi_{t-1}$. For $\gamma = \infty$, the fitness measure for each strategy is perfectly observable and in each period all agents pick the forecasting rule with the higher performance in the previous period, switching to the optimal predictor. Therefore, for the case $\gamma = \infty$, the economy can be represented taking into consideration only one representative naive agent. Hence the system is given by:

$$
\begin{aligned}
y_t &= \frac{a_1}{1 - a_2 c_1 f_2 - a_2 c_2} y_{t-1} + \frac{a_2\left(c_1 f_2 - 1\right)}{1 - a_2 c_1 f_2 - a_2 c_2} \pi_{t-1} \\
\pi_t &= \frac{a_1 f_2}{1 - a_2 c_1 f_2 - a_2 c_2} y_{t-1} + \frac{f_1\left(1 - a_2 c_2\right) - a_2 f_2}{\left(1 - a_2 c_1 f_2 - a_2 c_2\right)} \pi_{t-1}
\end{aligned}
$$

Following the work of Galí (2009), we can rewrite the system in matrix notation obtaining

$$
\begin{bmatrix} y_t \\ \pi_t \end{bmatrix} = A \begin{bmatrix} y_{t-1} \\ \pi_{t-1} \end{bmatrix} \tag{4.24}
$$

$$
A = \Omega \begin{bmatrix} a_1 & a_2\left(c_1 f_2 - 1\right) \\ a_1 f_2 & f_1\left(1 - a_2 c_2\right) - a_2 f_2 \end{bmatrix}
$$

where

$$
\Omega = \frac{1}{1 - a_2 c_1 f_2 - a_2 c_2}
$$

is a parameter aggregation and, under the usual assumptions it is always positive. The solution $y^* = \pi^* = 0$ always satisfies the system (4.24) which is locally stable if the trace and the determinant of A satisfy the usual stability conditions shown in (4.20).

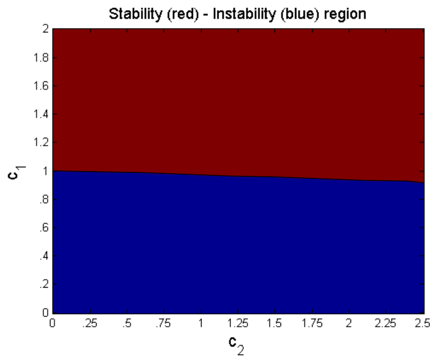


Figure 4.8: Stability and instability region for the case $\gamma = \infty$

The dynamics of (4.24) depends on the policy coefficients (c_1, c_2), in addition to the non-policy parameters.
Let us restrict our attention to the case of rules for which $c_1, c_2 > 0$. Under the assumption of non-negative values for (c_1, c_2), a necessary and sufficient condition for $y^* = \pi^* = 0$ to be globally stable is given by

$$(a_1 + a_2c_2 - a_1a_2c_2 - a_1a_2c_1f_2 - 1)f_1 + a_2f_2 - a_1 + 1 > 0 \tag{4.25}$$

Therefore in an environment with only one representative naive agent the monetary authority should respond to deviations of inflation and output from their target levels by adjusting the nominal interest rate satisfying the Taylor principle: at least in the long run, nominal interest rates should rise by more than the increase in the inflation rate. Indeed figure (4.8) illustrates graphically the regions of parameter space for c_1, c_2 associated with stable and unstable REE, as implied by condition (4.25). Thus, the equilibrium will be unique under interest rate rule (4.3) whenever c_1 and c_2 are sufficiently large enough to guarantee that the real rate eventually rises in the face of an increase in inflation.

4.6 Policy analysis

In this section we consider a different perspective. We adopt an agent-based approach employing our model as an artificial laboratory to carry out computer experiments to improve our insights into the working of certain regulatory mechanisms. Now we are keen in expressing the impact that different policy settings have on the time series of the relevant variables. We follow Westerhoff (2008) introducing two indexes. The first is based on the distortion, calculated as:

$$Dist(x) = \frac{1}{T}\sum_{t=1}^{T} \mid x_t - x_{RE} \mid$$

with $x = y, \pi$.

$Dist(x)$ measures the distortion of the time series, that is the mean of the deviation of the relevant variable from its steady state. We do not use the standard deviation because it considers the distortion as the dispersion of the time series from its mean, while the mean of x_t is not the steady state.

The second index, called volatility index, denotes the rate of change of the time series and it is calculated as:

$$Vol(x) = \frac{1}{T-1}\sum_{t=2}^{T} \mid x_t - x_{t-1} \mid$$

with $x = y, \pi$.

To calculate the distortion and the volatility we add a white noise term ε_t to equation (4.2). This component, as in Clarida et al. (1997), can be interpreted as a *cost push shock*and it can affects marginal costs due to, for example, wage stickiness. We run the model for 1000 quarters with different values of the Taylor rule coefficients, c_1 and c_2. We perform Monte Carlo simulations using 1000 different realizations of the pseudo random number generator for each c_1 as well as each c_2, and then taking the mean. In doing so we also vary the intensity of choice coefficient and, to have a larger overview, we compute the analysis for three values of γ, namely $\gamma = [1, 3, 5]$.

First of all we present the results obtained letting the inflation coefficient in the Taylor rule vary, setting $c_2 = 0.5$ and fixing the number of agents equal to 11 because results with only three agents are not robust enough and display an unpredictable behavior.

Figure 4.9 (top left and bottom left boxes) presents the distortion and the volatility of the output as long as c_1 increases from 0.5 to 1.5. The three curves are computed for different values of the intensity of choice parameter γ. Increasing the rationality parameter, the generated time series deviate much more from the rational expectations equilibrium, displaying also a greater volatility. Output distortion and volatility reach their minimum at $\hat{c}_1 = 1.02$. It has to be noticed that, even if the Taylor principle is only *weakly* satisfied, it minimizes both distortion and volatility of output. On the other hand, both inflation distortion and volatility monotonically decrease as c_1 increases (see Figure 4.9 top right and bottom right boxes). Moreover the series are steeper as long as γ increases, meaning that the Taylor principle mostly affects an economy populated by more reactive agents.

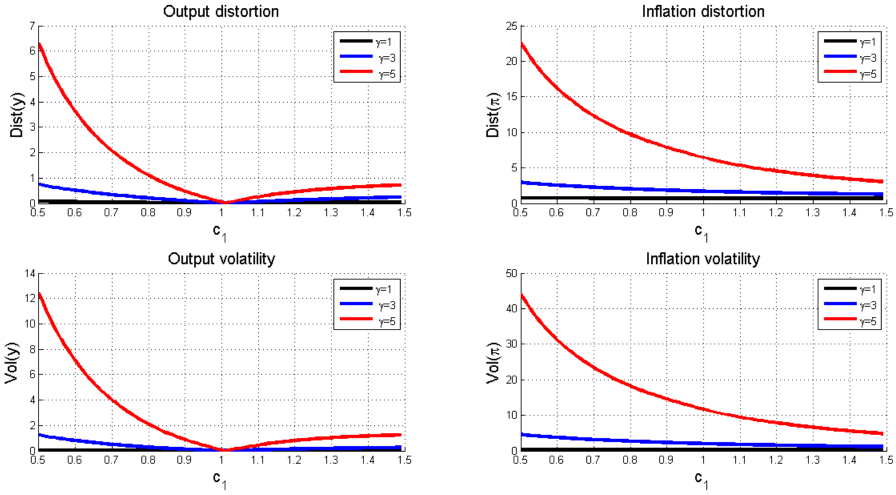


Figure 4.9: The influence of c_1 on volatility and distortion indexes

The non-monotonic decreasing curves of output distortion and volatility suggests the existence of a value $\hat{c}_1$ such that for $c_1 < \hat{c}_1$ inflation targeting is able to reduce both inflation and output distortion and volatility. Indeed for $c_1 > \hat{c}_1$ there exists a trade-off between output and inflation, the lower are $Dist(\pi)$ and $Vol\,(\pi)$, the higher are $Dist\,(y)$ and $Vol\,(y)$.

Inflation targeting does not exclude the role of output stabilization. DSGE modelers underline that price rigidities provide a rationale for output stabilization by Central Bank (see Clarida et al., 1997 and Galí, 2009) or for a flexible inflation targeting (Svensson, 1997). Because of the existence of rigidities, when sufficiently large shocks occur, leading inflation to depart from its target, the Central Bank should follow a strategy of gradual return of inflation to its target. Since too abrupt attempts to bring back inflation to its target, would require such high increases in the interest rate implying too strong declines in output.

We perform a similar exercise, fixing $c_1 = 1.5$ and letting the output coefficient of the Taylor rule vary from 0.1 to 2.1. Results suggest that there exists a trade-off between inflation and output distortion (Figure 4.10 top left and top right panels). Output distortion monotonically decreases as long as the reaction coefficient to output gap increases. On the contrary, inflation distortion increases. Noteworthy that the rise in the inflation distortion is bigger (in absolute value) than the reduction in output distortion.

On the other hand, by increasing the parameter c_2, we observe a reduction in output

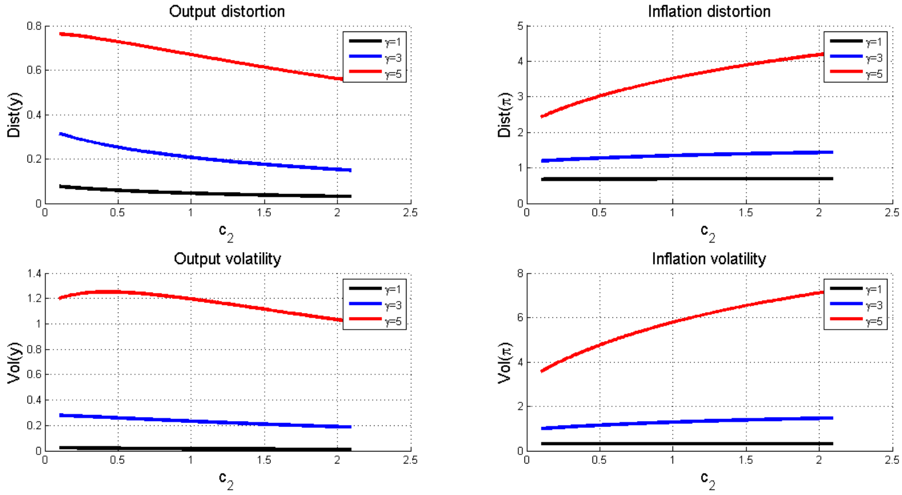


Figure 4.10: The influence of c_2 on volatility and distortion indexes

volatility and an increase in inflation volatility, as Figure 4.10 shows (bottom left and right panels). From Figure 4.10 (bottom left) it can be noticed that output volatility increases for $c_2 < 0.5$ and a sufficiently high value of γ. These trends make an interpretation less clear: indeed there exists a trade-off between output and inflation volatility for $c_2 > 0.5$, if agents switch sufficiently fast between predictors. As in the previous analysis, the intensity of choice affects the policy influence on the economy: if agents hardly switch among predictors, the monetary policy loses most of its influence but distortion and volatility of the two variables are cut down by this occurrence.

To summarize our findings, if the Central Bank is keen in inflation targeting with a monetary policy rule such that $c_1 < \hat{c_1}$, it is possible to reduce both output and inflation variability. The relation is non-linear and, with a too high inflation stabilization parameter, there exists a trade-off where lower inflation variability is obtained at the cost of increased output variability. Even with a Taylor rule coefficient that could lead to a multiplicity of equilibria, the variability of the two variables can be lowered at the same time. Moreover some output stabilization is good because it reduces both output and inflation variability by preventing too large switches in forecasting behavior.

4.7 Conclusions

In this work we studied the stability properties of a macroeconomic model in which agents have heterogeneous expectations. We investigated the stabilizing effects of different monetary policy rules in an economy described by the 3-equations New Keynesian model where agents switch between different forecasting rules on the basis of their past performances. Moreover we analyze the role of the Central Bank in reducing the volatility and the distortion of output and inflation.

We found that the business cycle dynamics depends on the expectations environment and the coefficients of an interest rate rule (e.g. Taylor rule). If the monetary policy reacts weakly to inflation, a cumulative process of rising inflation and output appears. Different signals from the market can lead the economy to non-fundamental steady states, reinforced by self-fulfilling expectations of high inflation. On the contrary, when the response to inflation is moderate, the heterogeneous expectations can be managed in order to correct past forecast error and to conduct the economy towards the REE. Even with an aggressive monetary policy, the monetary authority is able to send correct signals to agents and can induce stable dynamics settling down to the fundamental steady state. It is also worth to point out that even if the Taylor principle is sufficient to guarantee convergence to the fundamental steady state, it is no longer enough to avoid multiple equilibria. Indeed the monetary policy rule must be sufficiently aggressive to guarantee a proximity between the realized inflation and the REE.

Our model suggests that, to avoid multiple equilibria, the policy rule must react more aggressively than in the work of Anufriev et al. to guarantee the uniqueness (and global stability) of the REE, breaking down the reinforcement process that arises between inflation and output.

Furthermore, in the case of many beliefs types (a *continuum of beliefs*), a monetary policy rule that reacts aggressively to current inflation can fully stabilize the system. If the policy rule is not aggressive enough and the intensity of choice is large, the cumulative process of inflation and output appears again.

Finally we considered two summary indexes (i.e. volatility and distortion) that link the impact of the Taylor rule coefficients to distortion and volatility of the fundamental variables. Policy makers can reduce volatility and distortion of output and inflation with a sufficient degree of reaction.

Depending on the target of the monetary authority, inflation volatility and distortion can be minimized but also output stabilization can be taken into account. Indeed, if the Central Bank shifts its target from inflation to output, results suggest that there exists a trade-off between inflation and output distortion but, a strong reaction to output is more likely to stabilize the economy.

Chapter 5

Summary

In this thesis we have analyzed and appraised the effect of modeling human decision making and individual behavior as boundedly rational in a different economic settings. In particular, we focus on situations in which different subjects act and use the same limited resources. In these systems, if agents were rational, they would account for the effects due to their interaction and coordinate their decisions to the equilibrium level where they would all get the same outcome. Modeling agents as boundedly rational could help in explaining the observed size and variability of fluctuations of economic variables, such as prices, output or inflation rate, even when no changes of the underlying fundamentals occur.

Boundedly rational agents are specified as using simple heuristics in their decision making. An important aspect of the type of bounded rationality described in the present work is that the population of agents is heterogeneous, which means that actors can choose different decision strategies to solve the same economic problem. The set of rules is disciplined by a selection mechanism, where the best performing rule, measured according to some fitness rule, attracts the most number of agents. This feature implies that our model are dynamics, with agents switching among the different rules at different periods of time. An important role in triggering switching between rules is played by the dynamics feedback among individual expectations of economic variables and their aggregate realizations. This expectational feedback mechanism, which we call expectational feedback, transforms agents' interaction into a mutual dependence between the individual choices of economic actors and the environment against which these choices are evaluated. At the light of this kind of feedback, rational agents are usually assumed to have rational expectations, that is, to find actions such that expectations and realizations are consistent and the system is at equilibrium. In this respect, as the functional dependence of fractions of agents in terms of economic variables and previous fractions is nonlinear, our systems are nonlinear so that a variety of different types of behaviors other than convergence or divergence can be observed, such as bounded erratic fluctuations.

The contribution of the thesis is twofold. In departing from the traditional approach where a representative rational agent is present, we evaluate when the frequently used argument in favor of rationality (namely that rationality is the outcome of the repetend interaction of heterogeneous boundedly rational agents) is justified. Furthermore, by showing in what respect our results differ from the rational benchmark, we characterize whether the interacting agents framework can reproduce empirically observed phenomena in the different economic settings we consider. Secondly, we develop three models with heterogeneous expectations and evolutionary selection among forecasting strategies and study aggregate behavior under heterogeneous expectations among boundedly rational agents.

In chapter 2 we have developed a model to study the housing market starting from an Agent-Based perspective. Relaxing the rational expectation hypothesis and allowing households to have a backward-looking behavior, we have shown that an endogenous appearance of bubbles can lead the price to long-last deviate from its fundamental steady state. The model with chartist-fundamentalist mechanism matches real data quite well. The exogenous preference shock, calibrated using the Michigan Consumers Surveys, is the main force driving the system. Adding the interest rate and the credit tightness effects, we can observe only a minimum anticipation of the price dynamics. This heterogeneous framework gives the right persistence in the house price dynamics and the hump shape of the price series is given by the self fulfilled mechanism induced by the backward looking expectation. Indeed in a rational expectation model the inertia is the result of a lag transmission of exogenous shocks; in contrast, our behavioral model is capable to reproduce the inertia in the price series without imposing lags in the transmission process. The model has also some space for policy investigation anchoring the interest rate to house price. The distortion and the volatility of prices can be reduced using an appropriate degree of reaction.

In chapter 3 we have developed a simple present discounted value asset pricing model with heterogeneous beliefs which is able to exhibit a kind of path-dependance property explaining how both stable steady states and attracting curves can arise endogenously within the model. The model is made up with simple and different forecasting rules, each of which can generate its own type of dynamics. In every period the forecasting strategy is selected among the population of heuristics and agents adapt their selection over time, based on the relative performance of the heuristics (evolutionary selection mechanism). The outcomes we came up with is quite interesting, i.e. the model can endogenously generate both persistent oscillating and converging patterns within the same economic framework, for the same parameter values. We can claim that even if these dynamic aspects remain valid for a large range of parameters, some quantitative aspects may change when parameters vary, such as the speed of convergence or the amplitude and frequency of oscillations.

We have also shown that the fundamental steady state can be locally unstable with our set of heuristics and, although the fundamental equilibrium is locally stable, other attractors may co-exist. There is a simple economic intuition of this phenomena, depending on the strength of the trend extrapolation and the role of stabilization forces. Indeed evolutionary interaction among strong extrapolators and stabilizing agents leads to other attractors with irregular price fluctuations, switching between periods of low and high volatility. Thus the interaction among different kind of agents may lead to a stable closed curve (or a more complicated attractor) around a locally stable steady state. Irregular price oscillations and coexistence of attractors can be considered as an explanation of some stylized facts, such as volatility clustering observed in real financial markets.

We are aware the model proposed here is simple and should be considered as stylized behavioral model, but excess volatility and a kind of volatility clustering are created by the trading process, and this seems to be in line with financial practices. If the evolutionary interaction of boundedly rational agents with different trading strategies extols volatility, there are important consequences for regulatory policy in financial markets. Good or bad news in the markets can be amplified by the evolutionary mechanism. Since we are embedded in an increasingly globalized world, and of course this is true for financial markets, small changes in fundamentals in one country may generate changes in asset price in other countries.

In chapter 4 we studied the stability properties of a macroeconomic model in which agents have heterogeneous expectations. We investigated the stabilizing effects of different monetary policy rules in an economy described by the 3-equations New Keynesian model where agents switch between different forecasting rules on the basis of their past performances. Moreover we analyzed the role of the Central Bank in reducing the volatility and the distortion of output and inflation.

We found that the business cycle dynamics depends on the expectations environment and the coefficients of the monetary policy. If the monetary policy reacts only weakly to inflation, a cumulative process of rising inflation and output appears. Different signals from the market can lead the economy to non-fundamental steady states, reinforced by self-fulfilling expectations of high inflation. On the contrary, when the response to inflation is moderate, the heterogeneous expectations can be managed in order to correct past forecast errors and to conduct the economy towards the RE equilibrium. Even with an aggressive monetary policy, the monetary authority is able to send correct signals to agents and can induce stable dynamics settling down to the fundamental steady state. It is also worth to point out that even if the Taylor principle is sufficient to guarantee convergence to the fundamental steady state, it is no longer enough to avoid multiple equilibria.

We have also to highlight that, in the case of many beliefs types, a monetary policy rule that reacts aggressively to current inflation can fully stabilize the system. If the policy rule is not aggressive enough and the intensity of choice is large, the cumulative process of inflation and output appears again.

We have also considered two summary indexes (i.e. volatility and distortion) that link the impact of the Taylor rule coefficients to distortion and volatility of the fundamental variables. Policy makers can reduce volatility and distortion of output and inflation with a sufficient degree of reaction. Thus, depending on the target of the monetary authority, inflation volatility and distortion can be minimized but also output stabilization can be taken into account. Indeed, if the Central Bank shifts its target from inflation to output, results suggest that there exists a trade-off between inflation and output variability.

A general conclusion following from the results of this thesis is that non-rational beliefs may survive evolutionary competition among heterogeneous forecasting strategies. Boundedly rational agents' interaction and adaptation can trigger ongoing fluctuations around equilibrium levels in addition to convergence or deviation from the steady state of the model. This is, for example, consistent with excess volatility in financial markets. In general, these endogenous fluctuations can be characterized as irregular cycles along which rules perform better than others in different periods of time. Therefore, policy makers should seriously take into account bounded rationality when designing monetary or fiscal policies, since decisions constructed under the assumption of homogeneous rational expectations can be destabilizing when expectations are heterogeneous.

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