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Signed magic arrays: existence and constructions

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ABSTRACT

Let m, n, s, k be four integers such that $1 \leqslant s \leqslant n$, $1 \leqslant k \leqslant m$ and $ms = nk$. A signed magic array $\mathrm{SMA}(m, n; s, k)$ is an $m \times n$ partially filled array whose entries belong to the subset $\Omega \subset \mathbb{Z}$, where $\Omega = \{0, \pm 1, \pm 2, \dots, \pm(nk-1)/2\}$ if nk is odd and $\Omega = \{\pm 1, \pm 2, \dots, \pm nk/2\}$ if nk is even, satisfying the following requirements: (a) every $\omega \in \Omega$ appears once in the array; (b) each row contains exactly s filled cells and each column contains exactly k filled cells; (c) the sum of the elements in each row and in each column is 0. In this paper we construct these arrays when n is even and $s, k \geqslant 5$ are coprime integers. This allows us to provide a complete answer to a problem posed in 2017 by Khodkar, Schulz and Wagner, giving the necessary and sufficient conditions for the existence of an $\mathrm{SMA}(m, n; s, k)$ for all admissible values of m, n, s, k.

1. Introduction

In recent years, many authors have worked with magic squares and various generalizations of these objects. For instance, one can consider magic rectangles [8,9], magic hypercubes [18], or magic squares on abelian groups [19]. In particular, Abdollah Khodkar, Christian Schulz and Nathan Wagner introduced in [12] a class of partially filled arrays, called *signed magic arrays*.

Definition 1.1. Let m, n, s, k be four integers such that $1 \leqslant s \leqslant n$, $1 \leqslant k \leqslant m$ and $ms = nk$. A *signed magic array* $\mathrm{SMA}(m, n; s, k)$ is an $m \times n$ partially filled array with entries in $\Omega \subset \mathbb{Z}$, where $\Omega = \{0, \pm 1, \pm 2, \dots, \pm(nk-1)/2\}$ if nk is odd and $\Omega = \{\pm 1, \pm 2, \dots, \pm nk/2\}$ if nk is even, such that

(a) every $\omega \in \Omega$ appears once in the array;
(b) each row contains exactly s filled cells and each column contains exactly k filled cells;
(c) the sum of the elements in each row and in each column is 0.

Replacing in the previous definition the subset Ω with any subset $\Psi \subset \mathbb{Z}$ such that $\{\Psi, -\Psi\}$ is a partition of $\{\pm 1, \pm 2, \dots, \pm nk\}$, one obtains an *integer* Heffter array $\mathrm{H}(m, n; s, k)$, an object introduced by Dan Archdeacon in [2]. In fact, as shown in [15], magic rectangles, signed magic arrays and Heffter arrays are all members of the broader family of the *magic partially filled arrays*. Speaking of integer Heffter arrays, we recall that Archdeacon conjectured (see [2, Conjecture 6.3]) that there exists an integer Heffter array $\mathrm{H}(m, n; s, k)$ for all m, n, s, k with $s, k \geqslant 3$ and $ms = nk$, provided that the

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necessary condition $ms \equiv 0, 3 \pmod 4$ is satisfied. The reader interested in constructing these objects can find a survey on almost all known results in [16]. Further methods for constructing rectangular integer Heffter arrays have been recently described in [14,17]. These methods, adapted here to work with signed magic arrays, allow us to completely solve the existence problem of an $\mathrm{SMA}(m, n; s, k)$.

Theorem 1.2. *Let m, n, s, k be four integers such that $1 \leqslant s \leqslant n$, $1 \leqslant k \leqslant m$ and $ms = nk$. There exists an $\mathrm{SMA}(m, n; s, k)$ if and only if one of the following cases occurs:*

(1) $k = m = s = n = 1$;
(2) $k = 2$, $m = 2$ *and* $s = n \equiv 0, 3 \pmod 4$;
(3) $k = 2$ *and* $m, s \geqslant 3$;
(4) $s = 2$, $n = 2$ *and* $k = m \equiv 0, 3 \pmod 4$;
(5) $s = 2$ *and* $n, k \geqslant 3$;
(6) $s, k \geqslant 3$.

Clearly, if A is an $\mathrm{SMA}(m, n; s, k)$, then its transpose A^{t} is an $\mathrm{SMA}(n, m; k, s)$. Furthermore, an $\mathrm{SMA}(m, n; s, 1)$ exists only when $m = s = n = 1$, in which case the matrix one is looking for is $\boxed{0}$. The existence problem of an $\mathrm{SMA}(m, n; s, 2)$ and of an $\mathrm{SMA}(m, n; s, 3)$ has been already solved in [10] and in [11], respectively. Constructions of signed magic arrays $\mathrm{SMA}(m, n; s, k)$ have been described firstly assuming that $s, k \geqslant 4$ are even integers (see [13, Theorem 1.6]), and then assuming the weaker hypothesis that $\gcd(s, k) \geqslant 2$ (see [14, Theorem 1.9]). The case when $s \equiv 0 \pmod 4$ has been dealt with in [14, Theorem 1.10], while the case when nk is odd has been completely solved in [15, Corollary 3.5]. We can resume the results obtained in [13–15] as follows.

Proposition 1.3. *Let m, n, s, k be four integers such that $3 \leqslant s \leqslant n$, $3 \leqslant k \leqslant m$ and $ms = nk$. There exists an $\mathrm{SMA}(m, n; s, k)$ in each of the following cases:*

(1) $\gcd(s, k) \geqslant 2$;
(2) $s \equiv 0 \pmod 4$;
(3) *nk is odd.*

Once more, we remark that all the previous results on signed magic arrays [10,11,13–15] have been obtained with constructive methods. So, to prove Theorem 1.2 it suffices to consider the case when nk is even, $s, k \geqslant 5$ and $\gcd(s, k) = 1$. Under these assumptions, s and k cannot be two even integers, so we may assume that k is odd and $s \not\equiv 0 \pmod 4$. If s and k are odd coprime integers, from $ms = nk$ even, it follows that $m = 2ck$ and $n = 2cs$ for some positive integer c. On the other hand, if $s \equiv 2 \pmod 4$, we can write $s = 2\bar{s}$ for some odd integer $\bar{s} \geqslant 3$ such that $\gcd(\bar{s}, k) = 1$. Again, from $ms = nk$, we obtain $m = ck$ and $n = 2c\bar{s}$ for some positive integer c. In both cases, we prove the existence of an $\mathrm{SMA}(m, n; s, k)$ by constructing a signed magic array set, see [15].

Definition 1.4. A *signed magic array set* $\mathrm{SMAS}(a, b; e)$ is a set of e arrays of size $a \times b$ with entries in $\Omega \subset \mathbb{Z}$, where $\Omega = \{0, \pm 1, \pm 2, \dots, \pm(abe - 1)/2\}$ if abe is odd and $\Omega = \{\pm 1, \pm 2, \dots, \pm abe/2\}$ if abe is even, such that

(a) every $\omega \in \Omega$ appears once and in a unique array;
(b) for every array, the sum of the elements in each row and in each column is 0.

In the next sections, we prove the following results that, together with Proposition 1.3, will imply the validity of Theorem 1.2, as explained in Section 6.

Proposition 1.5. *Given a positive integer c and two odd integers $a, b \geqslant 5$ such that $(a, b) \neq (5, 5)$, there exists an $\mathrm{SMAS}(a, b; 2c)$.*

Proposition 1.6. *Given an odd integer $b \geqslant 5$, there exists an $\mathrm{SMAS}(6, b; c)$ for all $c \geqslant 1$.*

We recall that, when nk is even, an $\mathrm{SMA}(m, n; s, k)$ is an integer 2-fold Heffter array ${}^2\mathrm{H}(m, n; s, k)$ with some additional properties.

Definition 1.7. [5,13] Let m, n, s, k be four integers such that $1 \leqslant s \leqslant n$, $1 \leqslant k \leqslant m$ and $ms = nk$. Set $v = nk + 1$ and $\Phi = \left\{1, \dots, \left\lfloor \frac{v}{2} \right\rfloor\right\} \subset \mathbb{Z}$. An *integer 2-fold Heffter array* ${}^2\mathrm{H}(m, n; s, k)$ is an $m \times n$ partially filled array with elements from $\Phi \cup -\Phi$ such that

(a) each row contains exactly s filled cells and each column contains exactly k filled cells;

(b) denoting by $\rho(x)$ the number of the entries whose absolute value is equal to a positive integer x, if v is odd, then $\rho(x)=2$ for every $x\in\Phi$; if v is even, then $\rho(x)=2$ for every $x\in\Phi\setminus\left\{\frac{v}{2}\right\}$ while $\rho\left(\frac{v}{2}\right)=1$;
(c) the sum of the elements in each row and in each column is 0.

As a consequence of Theorem 1.2, we have the following.

Corollary 1.8. *Let m,n,s,k be four integers such that $2\leqslant s\leqslant n$, $2\leqslant k\leqslant m$ and $ms=nk$. Suppose $nk\not\equiv 3 \pmod 4$. There exists an integer ${}^2\mathrm{H}(m,n;s,k)$ if and only if nk is even and*

$$(m,n,s,k)\notin\{(\ell,2,2,\ell),(2,\ell,\ell,2):\ell\geqslant 1 \text{ is such that } \ell\equiv 1,2 \pmod 4\}.$$

Signed magic arrays can be interpreted as zero-sum magic partially filled arrays, according to the following.

Definition 1.9. [15] A *zero-sum magic partially filled array* ${}^0\mathrm{MPF}_\Omega(m,n;s,k)$ on a subset Ω of an abelian group $(\Gamma,+)$ is a partially filled array of size $m\times n$ with entries in Ω such that

(a) every $\omega\in\Omega$ appears once in the array;
(b) each row contains s filled cells and each column contains k filled cells;
(c) the sum of the elements in each row and in each column is 0_Γ.

The entries of an $\mathrm{SMA}(m,n;s,k)$ can be also viewed as elements of a finite cyclic group. Setting $\Gamma=\mathbb{Z}_{nk}$ if nk is odd and $\Gamma=\mathbb{Z}_{nk+1}$ if nk is even, an $\mathrm{SMA}(m,n;s,k)$ is a zero-sum magic partially filled array on Γ if nk is odd and on $\Gamma\setminus\{0_\Gamma\}$ if nk is even. Consequently, signed magic arrays may prove to be potentially useful for graph theorists working on graph labeling. We now briefly describe a possible connection between these arrays (and signed magic array sets) and magic labelings (see [7, Sections 5.1 and 5.6]).

A bipartite biregular graph $G(V,E)$ is a graph whose vertex set can be written as a disjoint union $V=V_1\cup V_2$ with $|V_1|=m$, $|V_2|=n$, and where each vertex of V_1 is connected with exactly s vertices of V_2, and each vertex of V_2 is connected with exactly k vertices of V_1. Now, let M be an $\mathrm{SMA}(m,n;s,k)$. We associate to M a bipartite biregular graph $\Phi_M=G(V,E)$ by taking a set V_1 of m points, a set V_2 of n points, and drawing an edge $e_{i,j}$ between the i-th vertex of V_1 and the j-vertex of V_2 if the cell (i,j) of M is not empty. Define the labeling $f_M:E\to\Gamma$, where $f_M(e_{i,j})$ is the entry of the corresponding cell (i,j) of M.

A $\mathbb{Z}_{nk}$*-supermagic labeling* of a graph $G(V,E)$ with $|E|=nk$ is a bijection from E to $\mathbb{Z}_{nk}$ such that the sum of labels of all incident edges of every vertex $v\in V$ is equal to the same element $x\in\mathbb{Z}_{nk}$, see [6]. If M is an $\mathrm{SMA}(m,n;s,k)$ with nk odd, then function f_M is a $\mathbb{Z}_{nk}$-supermagic labeling of Φ_M, where the constant x is $0_{\mathbb{Z}_{nk}}$.

A graph G is said to be *zero-sum* $\mathbb{Z}_{nk+1}$*-magic* if there exists a labeling of the edges of G with elements of $\mathbb{Z}_{nk+1}\setminus\{0\}$ such that, for each vertex v, the sum of the labels of the edges incident with v is equal to $0_{\mathbb{Z}_{nk+1}}$, see [1]. If M is an $\mathrm{SMA}(m,n;s,k)$ with nk even, then Φ_M is a zero-sum $\mathbb{Z}_{nk+1}$-magic graph.

Finally, given a graph $G=(V,E)$, a bijection $f:V\to\mathbb{Z}_{abe+1}\setminus\{0_{\mathbb{Z}_{abe+1}}\}$ is said to be a $\mathbb{Z}_{abe+1}$*-distance magic labeling* of G if there exists $\mu\in\mathbb{Z}_{abe+1}$ such that $\omega(v)=\sum\limits_{u\in N(v)} f(u)$ is μ for all vertices $v\in V$, see [3,4]. If S is an $\mathrm{SMAS}(a,b;e)$ with abe even, then we can consider the complete a-partite graph $K_{be\times a}$ consisting of be parts of the same cardinality a. Labeling the vertices of each part with the a entries of the be columns of the elements of S, we obtain a $\mathbb{Z}_{abe+1}$-distance magic labeling of $K_{be\times a}$ where the constant μ is $0_{\mathbb{Z}_{abe+1}}$.

Explicit constructions of our signed magic array sets are given in Sections 3, 4 and 5. The reader interested only in the proofs of our main results can refer directly to Section 6. Those, who want to understand how we have truly solved this decade-long problem, will find similar constructions (and statements) each requiring small adjustments or different approaches, due to the presence of various parameters. We have unified some of them while trying to maintain a certain level of readability. To help these readers, explicit algorithmic implementations in GAP are available up to request to the authors.

2. Notation and preliminary results

Given an array A, we denote by $\mathcal{E}(A)$ the list of its entries; $\sigma_r(A)$ and $\sigma_c(A)$ denote, respectively, the sequence of the sums of the elements of each row and column of A. If $\sigma_r(A)=(0,\dots,0)$ and $\sigma_c(A)=(0,\dots,0)$, the array A is said to be a zero-sum block. Given a set $\mathfrak{S}=\{A_1,A_2,\dots,A_r\}$ of arrays, we set $\mathcal{E}(\mathfrak{S})=\cup_i\mathcal{E}(A_i)$.

Given an integer $d\geqslant 1$, if a,b are two integers such that $a\equiv b \pmod d$, then we use the notation

$$[a,b]_d=\left\{a+id \mid 0\leqslant i\leqslant \frac{b-a}{d}\right\}\subset\mathbb{Z},$$

whenever $a\leqslant b$. If $a>b$, then $[a,b]_d=\varnothing$. If $d=1$, we simply write $[a,b]$. Given a subset Ω consisting of positive integers, we write $\pm\Omega=\{\pm\omega:\omega\in\Omega\}$. Given two positive integers ε and x, a set $\{x,x+\varepsilon\}$ will be called a 2-set of type ε.

10	−4	−6	5	−5	7	−7
−2	19	−17	−32	32	−33	33
−8	−15	23	27	−27	26	−26
1	−30	29	14	−16	−18	20
−1	30	−29	−14	16	18	−20

−10	4	6	9	−9	11	−11
2	−19	17	−34	34	−35	35
8	15	−23	25	−25	24	−24
3	−31	28	12	−13	−21	22
−3	31	−28	−12	13	21	−22

Fig. 1. An SMAS$(5, 7; 2)$.

Following [17], we consider the following objects.

Definition 2.1. An *integer Heffter array set* IHS$(a, b; c)$ is a set of c arrays of size $a \times b$ such that

(a) the entries of all these arrays constitute a subset $\Omega \subset \mathbb{Z}$ such that $\{\Omega, -\Omega\}$ is a partition of $\{\pm 1, \pm 2, \dots, \pm abc\}$;
(b) every $\omega \in \Omega$ appears once and in a unique array;
(c) for every array, the sum of the elements in each row and in each column is 0.

Theorem 2.2. *[17] Let a, b, c be positive integers such that $abc \equiv 0, 3 \pmod 4$. Suppose that $a, b \geqslant 7$ are odd integers. Then, there exists an* IHS$(a, b; c)$.

If $\{A_1, A_2, \dots, A_c\}$ is an IHS$(a, b; c)$, then $\{A_1, -A_1, A_2, -A_2, \dots, A_c, -A_c\}$ is clearly an SMAS$(a, b; 2c)$. Hence, from Theorem 2.2 we obtain the following.

Corollary 2.3. *Let a, b, c be positive integers such that $abc \equiv 0, 3 \pmod 4$. Suppose that $a, b \geqslant 7$ are odd integers. Then, there exists an* SMAS$(a, b; 2c)$.

Remark 2.4. An SMAS$(a, b; e)$ exists if and only if there exists an SMAS$(b, a; e)$.

In the next two sections, we will construct SMAS$(a, b; 2c)$ when either $a = 5$ or $abc \equiv 1, 2 \pmod 4$. Roughly speaking, to build the elements of our signed magic array set we will use some basic zero-sum blocks of size 3×3, 2×3, 2×4 or 2×6 (see, for instance, Fig. 1). To this purpose, we start with the following constructions.

Lemma 2.5. *Given three nonnegative integers α, β, u such that $u \leqslant \alpha + 1$, there exists a set $\mathfrak{A} = \mathfrak{A}(\alpha, \beta, u)$ consisting of $2u$ square zero-sum blocks of size 3 such that $\mathcal{E}(\mathfrak{A})$ consists of the following disjoint subsets:*

$$\begin{aligned} \mathcal{E}(\mathfrak{A}) \;\; = \;\; & \pm[2\alpha + 2, 2\alpha + 8u]_2 \sqcup \pm[4\alpha + 6u + 4, 4\alpha + 10u]_4 \sqcup \\ & \pm[2\beta + 1, 2\beta + 4u - 1]_2 \sqcup \pm[2\alpha + 2\beta + 2u + 3, 2\alpha + 2\beta + 10u - 1]_4. \end{aligned}$$

Proof. For all $i \in [0, u-1]$, define

$$A_i = \begin{array}{|c|c|c|} \hline 4\alpha + 6u + 4 + 4i & -(2\alpha + 2u + 2 + 2i) & -(2\alpha + 4u + 2 + 2i) \\ \hline -(2\alpha + 2 + 2i) & 2\alpha + 2\beta + 2u + 3 + 4i & -(2\beta + 2u + 1 + 2i) \\ \hline -(2\alpha + 6u + 2 + 2i) & -(2\beta + 1 + 2i) & 2\alpha + 2\beta + 6u + 3 + 4i \\ \hline \end{array}.$$

The set $\mathfrak{A} = \{+A_i, -A_i : i \in [0, u-1]\}$ has the required properties. $\square$

Lemma 2.6. *Given three positive integers α, β, u such that $\beta \geqslant \alpha + 3u - 2$, there exists a set $\mathfrak{B} = \mathfrak{B}(\alpha, \beta, u)$ consisting of u zero-sum blocks of size 2×3 such that $\mathcal{E}(\mathfrak{B})$ consists of the following disjoint subsets:*

$$\mathcal{E}(\mathfrak{B}) = \pm[\alpha, \alpha + 2u - 2]_2 \sqcup \pm[\beta - u + 1, \beta] \sqcup \pm[\alpha + \beta, \alpha + \beta + u - 1].$$

Proof. For every $j \in [0, u-1]$, define

$$B_j = \begin{array}{|c|c|c|} \hline \alpha + 2j & -(\alpha + \beta + j) & \beta - j \\ \hline -(\alpha + 2j) & \alpha + \beta + j & -(\beta - j) \\ \hline \end{array}.$$

The set $\mathfrak{B} = \left\{B_j : j \in [0, u-1]\right\}$ has the required properties. $\square$

Let $Q_1 = Q_1(\{x_1, x_1 + 1\}, \{x_2, x_2 + 1\})$, $Q_2 = Q_2(\{y_1, y_1 + 2\}, \{y_2, y_2 + 2\})$, $Q_3 = Q_3(\{z_1, z_1 + 4\}, \{z_2, z_2 + 4\})$, $R_1 = R_1(\{y_1, y_1 + 2\}, \{x_1, x_1 + 1\}, \{x_2, x_2 + 1\})$ and $R_2 = R_2(\{z_1, z_1 + 4\}, \{y_1, y_1 + 2\}, \{y_2, y_2 + 2\})$ be the following zero-sum blocks:

$$Q_1 = \begin{array}{|c|c|c|c|}\hline x_1 & -(x_1+1) & -x_2 & x_2+1 \\ \hline -x_1 & x_1+1 & x_2 & -(x_2+1) \\ \hline\end{array},$$
$$Q_2 = \begin{array}{|c|c|c|c|}\hline y_1 & -(y_1+2) & -y_2 & y_2+2 \\ \hline -y_1 & y_1+2 & y_2 & -(y_2+2) \\ \hline\end{array},$$
$$Q_3 = \begin{array}{|c|c|c|c|}\hline z_1 & -(z_1+4) & -z_2 & z_2+4 \\ \hline -z_1 & z_1+4 & z_2 & -(z_2+4) \\ \hline\end{array},$$
$$R_1 = \begin{array}{|c|c|c|c|c|c|}\hline y_1 & -(y_1+2) & -x_1 & x_1+1 & -x_2 & x_2+1 \\ \hline -y_1 & y_1+2 & x_1 & -(x_1+1) & x_2 & -(x_2+1) \\ \hline\end{array},$$
$$R_2 = \begin{array}{|c|c|c|c|c|c|}\hline z_1 & -(z_1+4) & -y_1 & y_1+2 & -y_2 & y_2+2 \\ \hline -z_1 & z_1+4 & y_1 & -(y_1+2) & y_2 & -(y_2+2) \\ \hline\end{array}.$$

Lemma 2.7. *Given three positive integers α, β, γ such that $\beta > \gamma + 3 > \alpha + 12 > 22$, there exists a set $\mathfrak{C} = \mathfrak{C}(\alpha, \beta, \gamma)$ consisting of two square blocks of size 5 such that*

$$\sigma_r(C) = (0,0,0,1,-1) \quad \textit{and} \quad \sigma_c(C) = (0,0,0,0,0) \textit{ for all } C \in \mathfrak{C}$$

and $\mathcal{E}(\mathfrak{C})$ consists of the following disjoint subsets:

$$\mathcal{E}(\mathfrak{C}) = \pm[1,7]_2 \sqcup \pm[2,10]_2 \sqcup \pm\{\alpha, \alpha+4, \alpha+6, \alpha+9\} \sqcup \pm[\gamma, \gamma+3] \sqcup \pm[\beta, \beta+7].$$

Proof. Let C_1 and C_2 be the following arrays:

$$\begin{array}{|c|c|c|c|c|}\hline 10 & -8 & -2 & 5 & -5 \\ \hline -6 & -\alpha & \alpha+6 & -(\beta+6) & \beta+6 \\ \hline -4 & \alpha+9 & -(\alpha+4) & \beta & -(\beta+1) \\ \hline 1 & -(\beta+5) & \beta+3 & -(\gamma+1) & \gamma+3 \\ \hline -1 & \beta+4 & -(\beta+3) & \gamma+2 & -(\gamma+3) \\ \hline\end{array}, \quad \begin{array}{|c|c|c|c|c|}\hline -10 & 8 & 2 & 7 & -7 \\ \hline 6 & \alpha & -(\alpha+6) & -(\beta+7) & \beta+7 \\ \hline 4 & -(\alpha+9) & \alpha+4 & \beta+1 & -\beta \\ \hline -3 & \beta+5 & -(\beta+2) & \gamma+1 & -\gamma \\ \hline 3 & -(\beta+4) & \beta+2 & -(\gamma+2) & \gamma \\ \hline\end{array}.$$

The set $\mathfrak{C} = \{C_1, C_2\}$ has the required properties. $\square$

The blocks in the previous set $\mathfrak{C}$ will require the use of some special 2×4 and 2×2 blocks. Let $U_1(\{y_1, y_1+2\}, \{x_1, x_1+1\})$ and $U_2(\{x_1, x_1+1\})$ be the blocks

$$U_1 = \begin{array}{|c|c|c|c|}\hline y_1 & -(y_1+2) & -x_1 & x_1+1 \\ \hline -y_1 & y_1+2 & x_1 & -(x_1+1) \\ \hline\end{array} \quad \text{and} \quad U_2 = \begin{array}{|c|c|}\hline x_1 & -(x_1+1) \\ \hline -x_1 & x_1+1 \\ \hline\end{array}.$$

We have $\sigma_r(U_1) = \sigma_r(U_2) = (-1, +1)$, $\sigma_c(U_1) = (0,0,0,0)$ and $\sigma_c(U_2) = (0,0)$.

3. The general case

In this section we construct an SMAS$(a, b; 2c)$ when $a, b \geqslant 5$ are odd integers such that $(a, b) \neq (5,5), (5,7), (7,5)$, and c is a positive integer such that $c \not\equiv 0 \pmod 4$.

3.1. Subcase $c \equiv 3 \pmod 4$

Lemma 3.1. *Suppose that $b \geqslant 9$ is such that $b \equiv 1 \pmod 4$. There exists an SMAS$(5, b; 2c)$ for all $c \geqslant 3$ such that $c \equiv 3 \pmod 4$.*

Proof. Write $c = 4t + 3$ where $t \geqslant 0$. Our SMAS$(5, b; 2c)$ will consist of $8t + 6$ arrays of type

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline X_3 & X_2^t & X_2^t & X_2^t & X_2^t & X_2^t & \cdots & X_2^t & X_2^t \\ \hline X_2 & \multicolumn{3}{c|}{Z_6} & \multicolumn{2}{c|}{Z_4} & \cdots & \multicolumn{2}{c|}{Z_4} \\ \hline\end{array}, \tag{3.1}$$

where each X_r is a zero-sum block of size $r \times 3$ and each Z_ℓ is a zero-sum block of size $2 \times \ell$.

Write $b = 4w + 9$, where $w \geqslant 0$. Hence, we will construct $8t + 6$ blocks X_3, $4(w+2)(4t+3)$ blocks X_2, $2w(4t+3)$ blocks Z_4 and $8t + 6$ blocks Z_6. So, take

$$\begin{aligned} \mathfrak{A} &= \mathfrak{A}(4t+2,\ 4(w+2)(4t+3),\ 4t+3), \\ \mathfrak{B} &= \mathfrak{B}(1,\ 4(16w+37)t + 48w + 111,\ 4(w+2)(4t+3)) \end{aligned}$$

as in Lemmas 2.5 and 2.6, respectively. Then,

$$\mathcal{E}(\mathfrak{A}) \sqcup \mathcal{E}(\mathfrak{B}) = \pm[1, 20w(4t+3) + 180t + 135] \setminus \pm(\mathcal{M}_1 \sqcup \ldots \sqcup \mathcal{M}_6),$$

where

$$\begin{array}{rcl} \mathcal{M}_1 &=& [2, 8t+4]_2, \\ \mathcal{M}_2 &=& [40t+32, 56t+36]_4, \\ \mathcal{M}_3 &=& [56t+40, 8w(4t+3)+112t+78]_2, \\ \mathcal{M}_4 &=& [8w(4t+3)+80t+63, 8w(4t+3)+112t+75]_4, \\ \mathcal{M}_5 &=& [8w(4t+3)+112t+79, 8w(4t+3)+112t+80], \\ \mathcal{M}_6 &=& [8w(4t+3)+112t+82, 12w(4t+3)+116t+87]. \end{array}$$

So, we replace the $8t+6$ instances of X_3 with the arrays in $\mathfrak{A}$ and the $4(w+2)(4t+3)$ instances of X_2 with the arrays in $\mathfrak{B}$.

Next, write

$$\mathcal{M}_1 = \mathcal{M}_1^1 \sqcup \mathcal{M}_1^2 \sqcup \mathcal{M}_1^3 \quad \text{and} \quad \mathcal{M}_6 = \mathcal{M}_6^1 \sqcup \mathcal{M}_6^2 \sqcup \mathcal{M}_6^3 \sqcup \mathcal{M}_6^4 \sqcup \mathcal{M}_6^5,$$

where

$$\begin{array}{rcl} \mathcal{M}_1^1 &=& [2, 8t-2]_4, \\ \mathcal{M}_1^2 &=& [4, 8t]_4, \\ \mathcal{M}_1^3 &=& [8t+2, 8t+4]_2, \\ \mathcal{M}_6^1 &=& [8w(4t+3)+112t+82, 8w(4t+3)+112t+86]_4, \\ \mathcal{M}_6^2 &=& [8w(4t+3)+112t+83, 8w(4t+3)+112t+87]_4, \\ \mathcal{M}_6^3 &=& [8w(4t+3)+112t+84, 8w(4t+3)+112t+85], \\ \mathcal{M}_6^4 &=& [8w(4t+3)+112t+88, 12w(4t+3)+116t+86]_2, \\ \mathcal{M}_6^5 &=& [8w(4t+3)+112t+89, 12w(4t+3)+116t+87]_2. \end{array}$$

The set $\mathcal{M}_1^1 \sqcup \mathcal{M}_1^2 \sqcup \mathcal{M}_2 \sqcup \mathcal{M}_4 \sqcup \mathcal{M}_6^1 \sqcup \mathcal{M}_6^2$ can be written as a disjoint union $F_1 \sqcup \ldots \sqcup F_\alpha$, where $\alpha = 8t+5$ and each F_i is a 2-set of type 4. The set $\mathcal{M}_1^3 \sqcup \mathcal{M}_3 \sqcup \mathcal{M}_6^4 \sqcup \mathcal{M}_6^5$ can be written as a disjoint union $G_1 \sqcup \ldots \sqcup G_\beta$, where $\beta = 4w(4t+3) + 16t + 11$ and each G_j is a 2-set of type 2. Finally, the set $\mathcal{M}_5 \sqcup \mathcal{M}_6^3$ can be written as a disjoint union $H_1 \sqcup H_2$, where each H_h is a 2-set of type 1. Call

$$\begin{array}{rcl} \mathfrak{R}_1 &=& \{R_2(F_i, G_{2i-1}, G_{2i}) : i \in [1, 8t+5]\}, \\ \mathfrak{R}_2 &=& \{R_1(G_{16t+11}, H_1, H_2)\}, \\ \mathfrak{S} &=& \left\{Q_2(G_{2j}, G_{2j+1}) : j \in [8t+6, 2w(4t+3)+8t+5]\right\}. \end{array}$$

Then $\mathfrak{R} = \mathfrak{R}_1 \sqcup \mathfrak{R}_2$ consists of $8t+6$ zero-sum blocks of size 2×6 and $\mathfrak{S}$ consists of $2w(4t+3)$ zero-sum blocks of size 2×4. Furthermore, $\mathcal{E}(\mathfrak{R}) \sqcup \mathcal{E}(\mathfrak{S}) = \pm(\mathcal{M}_1 \sqcup \ldots \sqcup \mathcal{M}_6)$. Hence, we replace the instances of Z_6 with the arrays in $\mathfrak{R}$ and the instances of Z_4 with the arrays in $\mathfrak{S}$. □

Example 3.2. We construct an SMAS(5, 13; 6) following the proof of the previous lemma. Setting $t = 0$ and $w = 1$, we need 6 blocks X_3, 36 blocks X_2, 6 blocks Z_4 and 6 blocks Z_6. In particular, we have $\mathcal{M}_1^1 = \mathcal{M}_1^2 = \varnothing$, $\mathcal{M}_2 \sqcup \mathcal{M}_4 \sqcup \mathcal{M}_6^1 \sqcup \mathcal{M}_6^2 = [32, 36]_4 \sqcup [87, 99]_4 \sqcup [106, 110]_4 \sqcup [107, 111]_4$, $\mathcal{M}_1^3 \sqcup \mathcal{M}_3 \sqcup \mathcal{M}_6^4 \sqcup \mathcal{M}_6^5 = [2, 4]_2 \sqcup [40, 102]_2 \sqcup [112, 122]_2 \sqcup [113, 123]_2$ and $\mathcal{M}_5 \sqcup \mathcal{M}_6^3 = [103, 104] \sqcup [108, 109]$. The 6 elements of our set are the following ones.

30	−12	−18	3	−3	5	−5	7	−7	9	−9	11	−11
−6	85	−79	−161	161	−162	162	−163	163	−164	164	−165	165
−24	−73	97	158	−158	157	−157	156	−156	155	−155	154	−154
1	−160	159	32	−36	−2	4	−40	42	80	−82	−84	86
−1	160	−159	−32	36	2	−4	40	−42	−80	82	84	−86

,

34	−14	−20	15	−15	17	−17	19	−19	21	−21	23	−23
−8	89	−81	−167	167	−168	168	−169	169	−170	170	−171	171
−26	−75	101	152	−152	151	−151	150	−150	149	−149	148	−148
13	−166	153	87	−91	−44	46	−48	50	88	−90	−92	94
−13	166	−153	−87	91	44	−46	48	−50	−88	90	92	−94

,

38	−16	−22	27	−27	29	−29	31	−31	33	−33	35	−35
−10	93	−83	−173	173	−174	174	−175	175	−176	176	−177	177
−28	−77	105	146	−146	145	−145	144	−144	143	−143	142	−142
25	−172	147	95	−99	−52	54	−56	58	96	−98	−100	102
−25	172	−147	−95	99	52	−54	56	−58	−96	98	100	−102

,

−30	12	18	39	−39	41	−41	43	−43	45	−45	47	−47
6	−85	79	−179	179	−180	180	−181	181	−182	182	−183	183
24	73	−97	140	−140	139	−139	138	−138	137	−137	136	−136
37	−178	141	106	−110	−60	62	−64	66	112	−114	−116	118
−37	178	−141	−106	110	60	−62	64	−66	−112	114	116	−118

,

−34	14	20	51	−51	53	−53	55	−55	57	−57	59	−59
8	−89	81	−185	185	−186	186	−187	187	−188	188	−189	189
26	75	−101	134	−134	133	−133	132	−132	131	−131	130	−130
49	−184	135	107	−111	−68	70	−72	74	120	−122	−113	115
−49	184	−135	−107	111	68	−70	72	−74	−120	122	113	−115

,

−38	16	22	63	−63	65	−65	67	−67	69	−69	71	−71
10	−93	83	−191	191	−192	192	−193	193	−194	194	−195	195
28	77	−105	128	−128	127	−127	126	−126	125	−125	124	−124
61	−190	129	76	−78	−103	104	−108	109	117	−119	−121	123
−61	190	−129	−76	78	103	−104	108	−109	−117	119	121	−123

.

Lemma 3.3. *Suppose that $a \geqslant 5$ and $b \geqslant 11$ are such that $a \equiv 1 \pmod 4$ and $b \equiv 3 \pmod 4$. There exists an* $\mathrm{SMAS}(a, b; 2c)$ *for all $c \geqslant 3$ such that $c \equiv 3 \pmod 4$.*

Proof. Write $c = 4t + 3$ where $t \geqslant 0$. Our $\mathrm{SMAS}(a, b; 2c)$ will consist of $8t + 4$ arrays of type

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline X_3 & X_2^t & X_2^t & X_2^t & X_2^t & \cdots & X_2^t & X_2^t \\ \hline X_2 & \multicolumn{2}{c|}{Z_4} & \multicolumn{2}{c|}{Z_4} & \cdots & \multicolumn{2}{c|}{Z_4} \\ \hline X_2 & \multicolumn{2}{c|}{Z_4} & \multicolumn{2}{c|}{Z_4} & \cdots & \multicolumn{2}{c|}{Z_4} \\ \hline \vdots & \multicolumn{2}{c|}{\vdots} & \multicolumn{2}{c|}{\vdots} & \ddots & \multicolumn{2}{c|}{\vdots} \\ \hline X_2 & \multicolumn{2}{c|}{Z_4} & \multicolumn{2}{c|}{Z_4} & \cdots & \multicolumn{2}{c|}{Z_4} \\ \hline \end{array} \tag{3.2}$$

and two arrays of type

$$\begin{array}{|c|c|c|c|c|c|c|}\hline \multirow{2}{*}{Y_5} & X_2^t & X_2^t & X_2^t & \cdots & X_2^t & X_2^t \\ \cline{2-7} & W_2 & \multicolumn{2}{c|}{Z_4} & \cdots & \multicolumn{2}{c|}{Z_4} \\ \hline X_2 & Z_4 & \multicolumn{2}{c|}{Z_4} & \cdots & \multicolumn{2}{c|}{Z_4} \\ \hline \vdots & \vdots & \multicolumn{2}{c|}{\vdots} & \ddots & \multicolumn{2}{c|}{\vdots} \\ \hline X_2 & Z_4 & \multicolumn{2}{c|}{Z_4} & \cdots & \multicolumn{2}{c|}{Z_4} \\ \hline \end{array}, \tag{3.3}$$

where each X_r is a zero-sum block of size $r \times 3$, each Z_4 is a zero-sum block of size 2×4, each Y_5 is a block of size 5×5 and each W_2 is a block of size 2×2 such that

$$\begin{array}{|c|c|}\hline \multirow{2}{*}{Y_5} & X_2^t \\ \cline{2-2} & W_2 \\ \hline \end{array} \tag{3.4}$$

is a zero-sum block of size 5×7.

Write $a = 4v + 5$ and $b = 4w + 11$, where $v, w \geqslant 0$. Hence, we will construct $8t + 4$ blocks X_3, $4(v + w)(4t + 3) + 40t + 26$ blocks X_2, $(8t + 4)(w + 2)(2v + 1) + 2(w + 1 + 2v(w + 2))$ blocks Z_4, 2 blocks Y_5 and 2 blocks W_2. So, take

$$\begin{array}{rcl} \mathfrak{A} & = & \mathfrak{A}(4t + 5,\ 4(v + w)(4t + 3) + 40t + 30,\ 4t + 2), \\ \mathfrak{B} & = & \mathfrak{B}(9,\ 8(2vw + 5v + 2w)(4t + 3) + 180t + 131,\ 4(v + w)(4t + 3) + 40t + 26), \\ \mathfrak{C} & = & \mathfrak{C}(8(v + w)(4t + 3) + 140t + 92,\ 8(2vw + 5v + 2w)(4t + 3) + 180t + 132, \\ & & 8(v + w)(4t + 3) + 140t + 102) \end{array}$$

as in Lemmas 2.5, 2.6 and 2.7, respectively. We have

$$\mathcal{E}(\mathfrak{A}) \sqcup \mathcal{E}(\mathfrak{B}) \sqcup \mathcal{E}(\mathfrak{C}) = \pm[1, 4(4vw + 11v + 5w)(4t + 3) + 220t + 165] \setminus \pm(\mathcal{M}_1 \sqcup \ldots \sqcup \mathcal{M}_{12}),$$

where

$$\begin{array}{rcl}
\mathcal{M}_1 & = & [12, 8t+10]_2, \\
\mathcal{M}_2 & = & [40t+28, 40t+34]_2, \\
\mathcal{M}_3 & = & [40t+38, 56t+42]_4, \\
\mathcal{M}_4 & = & [56t+44, 8(v+w)(4t+3)+96t+66]_2, \\
\mathcal{M}_5 & = & [8(v+w)(4t+3)+96t+68, 8(v+w)(4t+3)+96t+75], \\
\mathcal{M}_6 & = & [8(v+w)(4t+3)+96t+76, 8(v+w)(4t+3)+128t+90]_2, \\
\mathcal{M}_7 & = & [8(v+w)(4t+3)+96t+79, 8(v+w)(4t+3)+128t+91]_4, \\
\mathcal{M}_8 & = & [8(v+w)(4t+3)+128t+92, 8(v+w)(4t+3)+140t+91], \\
\mathcal{M}_9 & = & [8(v+w)(4t+3)+140t+95, 8(v+w)(4t+3)+140t+97]_2, \\
\mathcal{M}_{10} & = & [8(v+w)(4t+3)+140t+106, 4(4vw+9v+3w)(4t+3)+140t+105], \\
\mathcal{M}_{11} & = & [8(v+w)(4t+3)+140t+93, 8(v+w)(4t+3)+140t+94], \\
\mathcal{M}_{12} & = & [8(v+w)(4t+3)+140t+99, 8(v+w)(4t+3)+140t+100].
\end{array}$$

So, we replace the $8t+4$ instances of X_3 with the arrays in $\mathfrak{A}$, the 2 instances of Y_5 with the arrays in $\mathfrak{C}$ and the $4(v+w)(4t+3)+40t+26$ instances of X_2 with the arrays in $\mathfrak{B}$.

Define

$$\mathfrak{R} = \{U_2(\mathcal{M}_{11}), U_2(\mathcal{M}_{12})\}.$$

Then, $\mathcal{E}(\mathfrak{R}) = \pm(\mathcal{M}_{11} \sqcup \mathcal{M}_{12})$: we replace the 2 instances of W_2 with the arrays in $\mathfrak{R}$. Next, write

$$\mathcal{M}_5 = \mathcal{M}_5^1 \sqcup \mathcal{M}_5^2 \sqcup \mathcal{M}_5^3 \sqcup \mathcal{M}_5^4,$$

where

$$\begin{array}{rcl}
\mathcal{M}_5^1 & = & [8(v+w)(4t+3)+96t+68, 8(v+w)(4t+3)+96t+72]_4, \\
\mathcal{M}_5^2 & = & [8(v+w)(4t+3)+96t+69, 8(v+w)(4t+3)+96t+70], \\
\mathcal{M}_5^3 & = & [8(v+w)(4t+3)+96t+71, 8(v+w)(4t+3)+96t+73]_2, \\
\mathcal{M}_5^4 & = & [8(v+w)(4t+3)+96t+74, 8(v+w)(4t+3)+96t+75].
\end{array}$$

The set $\mathcal{M}_3 \sqcup \mathcal{M}_5^1 \sqcup \mathcal{M}_7$ can be written as a disjoint union $F_1 \sqcup \ldots \sqcup F_{2\alpha}$, where $\alpha = 3t+2$ and each F_i is a 2-set of type 4. The set $\mathcal{M}_1 \sqcup \mathcal{M}_2 \sqcup \mathcal{M}_4 \sqcup \mathcal{M}_5^3 \sqcup \mathcal{M}_6 \sqcup \mathcal{M}_9$ can be written as a disjoint union $G_1 \sqcup \ldots \sqcup G_{2\beta}$, where $\beta = (v+w)(4t+3)+10t+7$ and each G_j is a 2-set of type 2. Finally, the set $\mathcal{M}_5^2 \sqcup \mathcal{M}_5^4 \sqcup \mathcal{M}_8 \sqcup \mathcal{M}_{10}$ can be written as a disjoint union $H_1 \sqcup \ldots \sqcup H_{2\gamma}$, where $\gamma = (4vw+7v+w)(4t+3)+3t+1$ and each H_h is a 2-set of type 1. Call

$$\begin{array}{rcl}
\mathfrak{S}_1 & = & \{Q_3(F_{2i-1}, F_{2i}) : i \in [1, 3t+2]\}, \\
\mathfrak{S}_2 & = & \left\{Q_2(G_{2j-1}, G_{2j}) : j \in [1, (v+w)(4t+3)+10t+7]\right\}, \\
\mathfrak{S}_3 & = & \left\{Q_1(H_{2h-1}, H_{2h}) : h \in [1, (4vw+7v+w)(4t+3)+3t+1]\right\}.
\end{array}$$

Then $\mathfrak{S} = \mathfrak{S}_1 \sqcup \mathfrak{S}_2 \sqcup \mathfrak{S}_3$ consists of $2(2vw+4v+w)(4t+3)+16t+10$ zero-sum blocks of size 2×4 and $\mathcal{E}(\mathfrak{S}) = \pm(\mathcal{M}_1 \sqcup \ldots \sqcup \mathcal{M}_{10})$. Hence, we replace the instances of Z_4 with the arrays in $\mathfrak{S}$. □

Lemma 3.4. *Suppose that $a \geqslant 9$ is such that $a \equiv 1 \pmod 4$. There exists an* SMAS$(a, 7; 2c)$ *for all $c \geqslant 3$ such that $c \equiv 3 \pmod 4$.*

Proof. Write $c = 4t+3$ where $t \geqslant 0$. Our SMAS$(a, 7; 2c)$ will consist of $8t+4$ arrays of type

<table>
<tr><td>X_3</td><td>X_2^t</td><td>X_2^t</td></tr>
<tr><td>X_2</td><td colspan="2">Z_4</td></tr>
<tr><td>X_2</td><td colspan="2">Z_4</td></tr>
<tr><td>$\vdots$</td><td colspan="2">$\vdots$</td></tr>
<tr><td>X_2</td><td colspan="2">Z_4</td></tr>
</table>

(3.5)

and two arrays of type

<table>
<tr><td rowspan="2" colspan="2">Y_5</td><td>X_2^t</td></tr>
<tr><td>W_2</td></tr>
<tr><td>X_2</td><td colspan="2">Z_4</td></tr>
<tr><td>$\vdots$</td><td colspan="2">$\vdots$</td></tr>
<tr><td>X_2</td><td colspan="2">Z_4</td></tr>
</table>

, (3.6)

where each X_r is a zero-sum block of size $r\times 3$, each Z_4 is a zero-sum block of size 2×4, each Y_5 is a block of size 5×5 and each W_2 is a block of size 2×2 such that (3.4) is a zero-sum block of size 5×7.

Write $a=4v+9$, where $v\geqslant 0$. Hence, we will construct $8t+4$ blocks X_3, $4v(4t+3)+40t+26$ blocks X_2, $(8t+4)(2v+3)+2(2v+2)$ blocks Z_4, 2 blocks Y_5 and 2 blocks W_2. So, take

$$\begin{array}{rcl}\mathfrak{A} &=& \mathfrak{A}(4t+5,\ 4v(4t+3)+40t+30,\ 4t+2),\\ \mathfrak{B} &=& \mathfrak{B}(9,\ 24v(4t+3)+212t+155,\ 4v(4t+3)+40t+26),\\ \mathfrak{C} &=& \mathfrak{C}(8v(4t+3)+128t+92,\ 24v(4t+3)+212t+156,\ 8v(4t+3)+128t+102)\end{array}$$

as in Lemmas 2.5, 2.6 and 2.7, respectively. We have

$$\mathcal{E}(\mathfrak{A})\sqcup\mathcal{E}(\mathfrak{B})\sqcup\mathcal{E}(\mathfrak{C})=\pm[1,28v(4t+3)+252t+189]\setminus\pm(\mathcal{M}_1\sqcup\ldots\sqcup\mathcal{M}_{12}),$$

where

$$\begin{array}{rcl}\mathcal{M}_1 &=& [12,8t+10]_2,\\ \mathcal{M}_2 &=& [40t+28,40t+34]_2,\\ \mathcal{M}_3 &=& [40t+38,56t+42]_4,\\ \mathcal{M}_4 &=& [56t+44,8v(4t+3)+96t+66]_2,\\ \mathcal{M}_5 &=& [8v(4t+3)+96t+68,8v(4t+3)+96t+75],\\ \mathcal{M}_6 &=& [8v(4t+3)+96t+76,8v(4t+3)+128t+90]_2,\\ \mathcal{M}_7 &=& [8v(4t+3)+96t+79,8v(4t+3)+128t+91]_4,\\ \mathcal{M}_8 &=& [8v(4t+3)+128t+106,8v(4t+3)+172t+129],\\ \mathcal{M}_9 &=& [8v(4t+3)+128t+93,8v(4t+3)+128t+94],\\ \mathcal{M}_{10} &=& [8v(4t+3)+128t+95,8v(4t+3)+128t+97]_2,\\ \mathcal{M}_{11} &=& [8v(4t+3)+128t+99,8v(4t+3)+128t+100],\\ \mathcal{M}_{12} &=& [8v(4t+3)+172t+130,20v(4t+3)+172t+129].\end{array}$$

So, we replace the $8t+4$ instances of X_3 with the arrays in $\mathfrak{A}$, the 2 instances of Y_5 with the arrays in $\mathfrak{C}$ and the $4v(4t+3)+40t+26$ instances of X_2 with the arrays in $\mathfrak{B}$.

Define

$$\mathfrak{R}=\{U_2(\mathcal{M}_9),U_2(\mathcal{M}_{11})\}.$$

Then, $\mathcal{E}(\mathfrak{R})=\pm(\mathcal{M}_9\sqcup\mathcal{M}_{11})$: we replace the 2 instances of W_2 with the arrays in $\mathfrak{R}$. Next, write

$$\mathcal{M}_5=\mathcal{M}_5^1\sqcup\mathcal{M}_5^2\sqcup\mathcal{M}_5^3\sqcup\mathcal{M}_5^4,$$

where

$$\begin{array}{rcl}\mathcal{M}_5^1 &=& [8v(4t+3)+96t+68,8v(4t+3)+96t+72]_4,\\ \mathcal{M}_5^2 &=& [8v(4t+3)+96t+69,8v(4t+3)+96t+70],\\ \mathcal{M}_5^3 &=& [8v(4t+3)+96t+71,8v(4t+3)+96t+73]_2,\\ \mathcal{M}_5^4 &=& [8v(4t+3)+96t+74,8v(4t+3)+96t+75].\end{array}$$

The set $\mathcal{M}_3\sqcup\mathcal{M}_5^1\sqcup\mathcal{M}_7$ can be written as a disjoint union $F_1\sqcup\ldots\sqcup F_{2\alpha}$, where $\alpha=3t+2$ and each F_i is a 2-set of type 4. The set $\mathcal{M}_1\sqcup\mathcal{M}_2\sqcup\mathcal{M}_4\sqcup\mathcal{M}_5^3\sqcup\mathcal{M}_6\sqcup\mathcal{M}_{10}$ can be written as a disjoint union $G_1\sqcup\ldots\sqcup G_{2\beta}$, where $\beta=v(4t+3)+10t+7$ and each G_j is a 2-set of type 2. Finally, the set $\mathcal{M}_5^2\sqcup\mathcal{M}_5^4\sqcup\mathcal{M}_8\sqcup\mathcal{M}_{12}$ can be written as a disjoint union $H_1\sqcup\ldots\sqcup H_{2\gamma}$, where $\gamma=3v(4t+3)+11t+7$ and each H_h is a 2-set of type 1. Call

$$\begin{array}{rcl}\mathfrak{S}_1 &=& \{Q_3(F_{2i-1},F_{2i}):i\in[1,3t+2]\},\\ \mathfrak{S}_2 &=& \left\{Q_2(G_{2j-1},G_{2j}):j\in[1,v(4t+3)+10t+7]\right\},\\ \mathfrak{S}_3 &=& \left\{Q_1(H_{2h-1},H_{2h}):h\in[1,3v(4t+3)+11t+7]\right\}.\end{array}$$

Then $\mathfrak{S}=\mathfrak{S}_1\sqcup\mathfrak{S}_2\sqcup\mathfrak{S}_3$ consists of $4v(4t+3)+24t+16$ zero-sum blocks of size 2×4 and $\mathcal{E}(\mathfrak{S})=\pm(\mathcal{M}_1\sqcup\ldots\sqcup\mathcal{M}_8\sqcup\mathcal{M}_{10}\sqcup\mathcal{M}_{12})$. Hence, we replace the instances of Z_4 with the arrays in $\mathfrak{S}$. □

3.2. Subcase $c\equiv 1 \pmod 4$

Lemma 3.5. *Suppose that $a\geqslant 5$ and $b\geqslant 9$ are such that $a\equiv b\equiv 1 \pmod 4$. There exists an* SMAS$(a,b;2)$.

Proof. Our $\mathrm{SMAS}(a, b; 2)$ will consist of two arrays of type

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
\multicolumn{2}{|c|}{\multirow{2}{*}{Y_5}} & X_2^{\mathrm{t}} & X_2^{\mathrm{t}} & X_2^{\mathrm{t}} & X_2^{\mathrm{t}} & \cdots & X_2^{\mathrm{t}} & X_2^{\mathrm{t}} \\ \cline{3-9}
\multicolumn{2}{|c|}{} & \multicolumn{2}{c|}{W_4} & \multicolumn{2}{c|}{Z_4} & \cdots & \multicolumn{2}{c|}{Z_4} \\ \hline
X_2 & \multirow{2}{*}{Z_4^{t}} & \multicolumn{2}{c|}{Z_4} & \multicolumn{2}{c|}{Z_4} & \cdots & \multicolumn{2}{c|}{Z_4} \\ \cline{1-1}\cline{3-9}
X_2 & & \multicolumn{2}{c|}{Z_4} & \multicolumn{2}{c|}{Z_4} & \cdots & \multicolumn{2}{c|}{Z_4} \\ \hline
\vdots & \vdots & \multicolumn{2}{c|}{\vdots} & \multicolumn{2}{c|}{\vdots} & \ddots & \multicolumn{2}{c|}{\vdots} \\ \hline
X_2 & \multirow{2}{*}{Z_4^{t}} & \multicolumn{2}{c|}{Z_4} & \multicolumn{2}{c|}{Z_4} & \cdots & \multicolumn{2}{c|}{Z_4} \\ \cline{1-1}\cline{3-9}
X_2 & & \multicolumn{2}{c|}{Z_4} & \multicolumn{2}{c|}{Z_4} & \cdots & \multicolumn{2}{c|}{Z_4} \\ \hline
\end{array}, \tag{3.7}$$

where each X_2 is a zero-sum block of size 2×3, each Z_4 is a zero-sum block of size 2×4, each Y_5 is a block of size 5×5 and each W_4 is a block of size 2×4 such that

$$\begin{array}{|c|c|c|}
\hline
\multirow{2}{*}{Y_5} & X_2^{\mathrm{t}} & X_2^{\mathrm{t}} \\ \cline{2-3}
 & \multicolumn{2}{c|}{W_4} \\ \hline
\end{array} \tag{3.8}$$

is a zero-sum block of size 5×9.

Write $a = 4v + 5$ and $b = 4w + 9$, where $v, w \geqslant 0$. Hence, we will construct $4(v + w + 1)$ blocks X_2, $2(3v + (2v + 1)w)$ blocks Z_4, 2 blocks Y_5 and 2 blocks W_4. So, take

$$\begin{array}{rcl}
\mathfrak{B} & = & \mathfrak{B}(9,\ 16vw + 32v + 16w + 33,\ 4(v + w + 1)), \\
\mathfrak{C} & = & \mathfrak{C}(8(v + w) + 16,\ 16vw + 32v + 16w + 34,\ 8(v + w) + 26)
\end{array}$$

as in Lemmas 2.6 and 2.7, respectively. We have

$$\mathcal{E}(\mathfrak{B}) \sqcup \mathcal{E}(\mathfrak{C}) = \pm[1, 16vw + 36v + 20w + 45] \setminus \pm(\mathcal{M}_1 \sqcup \ldots \sqcup \mathcal{M}_5),$$

where

$$\begin{array}{rcl}
\mathcal{M}_1 & = & [12, 8(v + w) + 14]_2, \\
\mathcal{M}_2 & = & [8(v + w) + 17, 8(v + w) + 18], \\
\mathcal{M}_3 & = & [8(v + w) + 19, 8(v + w) + 21]_2, \\
\mathcal{M}_4 & = & [8(v + w) + 23, 8(v + w) + 24], \\
\mathcal{M}_5 & = & [8(v + w) + 30, 16v(w + 1) + 12(v + w) + 29].
\end{array}$$

So, we replace the 2 instances of Y_5 with the arrays in $\mathfrak{C}$ and the $4(v + w + 1)$ instances of X_2 with the arrays in $\mathfrak{B}$.

Write

$$\mathcal{M}_1 = \mathcal{M}_1^1 \sqcup \mathcal{M}_1^2, \quad \text{where} \quad \mathcal{M}_1^1 = [12, 14]_2 \quad \text{and} \quad \mathcal{M}_1^2 = [16, 8(v + w) + 14]_2.$$

Define

$$\mathfrak{R} = \{U_1(\mathcal{M}_1^1, \mathcal{M}_2), U_1(\mathcal{M}_3, \mathcal{M}_4)\}.$$

Then, $\mathcal{E}(\mathfrak{R}) = \pm(\mathcal{M}_1^1 \sqcup \mathcal{M}_2 \sqcup \mathcal{M}_3 \sqcup \mathcal{M}_4)$: we replace the 2 instances of W_4 with the arrays in $\mathfrak{R}$.

The set $\mathcal{M}_1^2$ can be written as a disjoint union $G_1 \sqcup \ldots \sqcup G_{2\alpha}$, where $\alpha = v + w$ and each G_j is a 2-set of type 2. The set $\mathcal{M}_5$ can be written as a disjoint union $H_1 \sqcup \ldots \sqcup H_{2\gamma}$, where $\gamma = 4v(w + 1) + (v + w)$ and each H_h is a 2-set of type 1. Call

$$\begin{array}{rcl}
\mathfrak{S}_1 & = & \left\{Q_2(G_{2j-1}, G_{2j}) : j \in [1, v + w]\right\}, \\
\mathfrak{S}_2 & = & \left\{Q_1(H_{2h-1}, H_{2h}) : h \in [1, 4v(w + 1) + (v + w)]\right\}.
\end{array}$$

Then $\mathfrak{S} = \mathfrak{S}_1 \sqcup \mathfrak{S}_2$ consists of $4vw + 6v + 2w$ zero-sum blocks of size 2×4 and $\mathcal{E}(\mathfrak{S}) = \pm(\mathcal{M}_1^2 \sqcup \mathcal{M}_5)$. Hence, we replace the instances of Z_4 with the arrays in $\mathfrak{S}$. $\square$

Lemma 3.6. *Suppose that $a \geqslant 5$ and $b \geqslant 9$ are such that $a \equiv b \equiv 1 \pmod 4$. There exists an $\mathrm{SMAS}(a, b; 2c)$ for all $c \geqslant 1$ such that $c \equiv 1 \pmod 4$.*

Proof. By Lemma 3.5 we can write $c = 4t + 5$ where $t \geqslant 0$. Our SMAS$(a, b; 2c)$ will consist of $8t + 8$ arrays of type

$$
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
X_3 & X_2^{\mathrm{t}} & X_2^{\mathrm{t}} & X_2^{\mathrm{t}} & X_2^{\mathrm{t}} & X_2^{\mathrm{t}} & \cdots & X_2^{\mathrm{t}} & X_2^{\mathrm{t}} \\
\hline
X_2 & \multicolumn{3}{c|}{Z_6} & \multicolumn{2}{c|}{Z_4} & \cdots & \multicolumn{2}{c|}{Z_4} \\
\hline
X_2 & Z_4^{\mathrm{t}} & \multicolumn{2}{c|}{Z_4} & \multicolumn{2}{c|}{Z_4} & \cdots & \multicolumn{2}{c|}{Z_4} \\
\cline{1-1}\cline{3-9}
X_2 & & \multicolumn{2}{c|}{Z_4} & \multicolumn{2}{c|}{Z_4} & \cdots & \multicolumn{2}{c|}{Z_4} \\
\hline
\vdots & \vdots & \multicolumn{2}{c|}{\vdots} & \multicolumn{2}{c|}{\vdots} & \ddots & \multicolumn{2}{c|}{\vdots} \\
\hline
X_2 & Z_4^{\mathrm{t}} & \multicolumn{2}{c|}{Z_4} & \multicolumn{2}{c|}{Z_4} & \cdots & \multicolumn{2}{c|}{Z_4} \\
\cline{1-1}\cline{3-9}
X_2 & & \multicolumn{2}{c|}{Z_4} & \multicolumn{2}{c|}{Z_4} & \cdots & \multicolumn{2}{c|}{Z_4} \\
\hline
\end{array}
\tag{3.9}
$$

and two arrays of type (3.7), where each X_r is a zero-sum block of size $r \times 3$, each Z_ℓ is a zero-sum block of size $2 \times \ell$, each Y_5 is a block of size 5×5 and each W_4 is a block of size 2×4 such that (3.8) is a zero-sum block of size 5×9.

Write $a = 4v + 5$ and $b = 4w + 9$, where $v, w \geqslant 0$. Hence, we will construct $8t + 8$ blocks X_3, $4(v + w)(4t + 5) + 32t + 36$ blocks X_2, $(8t + 10)(3v + w(2v + 1))$ blocks Z_4, $8t + 8$ blocks Z_6, 2 blocks Y_5 and 2 blocks W_4. So, take

$$
\begin{array}{rcl}
\mathfrak{A} & = & \mathfrak{A}(4t+5,\ 4(v+w)(4t+5)+32t+40,\ 4t+4),\\
\mathfrak{B} & = & \mathfrak{B}(9,\ (16v(w+1)+16(v+w))(4t+5)+148t+181,\ 4(v+w)(4t+5)+32t+36),\\
\mathfrak{C} & = & \mathfrak{C}(8(v+w)(4t+5)+112t+130,\ (16v(w+1)+16(v+w))(4t+5)+148t+182,\\
 & & 8(v+w)(4t+5)+112t+140)
\end{array}
$$

as in Lemmas 2.5, 2.6 and 2.7, respectively. We have

$$
\mathcal{E}(\mathfrak{A}) \sqcup \mathcal{E}(\mathfrak{B}) \sqcup \mathcal{E}(\mathfrak{C}) = \pm[1, (16v(w+1)+20(v+w))(4t+5)+180t+225] \setminus \pm(\mathcal{M}_1 \sqcup \ldots \sqcup \mathcal{M}_{12}),
$$

where

$$
\begin{array}{rcl}
\mathcal{M}_1 & = & [12, 8t+10]_2,\\
\mathcal{M}_2 & = & [40t+44, 40t+46]_2,\\
\mathcal{M}_3 & = & [40t+50, 56t+62]_4,\\
\mathcal{M}_4 & = & [56t+64, 8(v+w)(4t+5)+80t+94]_2,\\
\mathcal{M}_5 & = & [8(v+w)(4t+5)+80t+96, 8(v+w)(4t+5)+80t+97],\\
\mathcal{M}_6 & = & [8(v+w)(4t+5)+80t+98, 8(v+w)(4t+5)+112t+128]_2,\\
\mathcal{M}_7 & = & [8(v+w)(4t+5)+80t+99, 8(v+w)(4t+5)+112t+127]_4,\\
\mathcal{M}_8 & = & [8(v+w)(4t+5)+112t+131, 8(v+w)(4t+5)+112t+135]_4,\\
\mathcal{M}_9 & = & [8(v+w)(4t+5)+112t+132, 8(v+w)(4t+5)+112t+133],\\
\mathcal{M}_{10} & = & [8(v+w)(4t+5)+112t+137, 8(v+w)(4t+5)+112t+138],\\
\mathcal{M}_{11} & = & [8(v+w)(4t+5)+112t+144, 8(v+w)(4t+5)+116t+145],\\
\mathcal{M}_{12} & = & [8(v+w)(4t+5)+116t+146, (16v(w+1)+12(v+w))(4t+5)+116t+145].
\end{array}
$$

So, we replace the $8t + 8$ instances of X_3 with the arrays in $\mathfrak{A}$, the 2 instances of Y_5 with the arrays in $\mathfrak{C}$ and the $4(v + w)(4t + 5) + 32t + 36$ instances of X_2 with the arrays in $\mathfrak{B}$.

Write

$$
\mathcal{M}_1 = \mathcal{M}_1^1 \sqcup \mathcal{M}_1^2, \quad \mathcal{M}_6 = \mathcal{M}_6^1 \sqcup \mathcal{M}_6^2 \sqcup \mathcal{M}_6^3 \quad \text{and} \quad \mathcal{M}_{11} = \mathcal{M}_{11}^1 \sqcup \mathcal{M}_{11}^2 \sqcup \mathcal{M}_{11}^3,
$$

where

$$
\begin{array}{rcl}
\mathcal{M}_1^1 & = & [12, 8t+8]_4,\\
\mathcal{M}_1^2 & = & [14, 8t+10]_4,\\
\mathcal{M}_6^1 & = & [8(v+w)(4t+5)+80t+98, 8(v+w)(4t+5)+80t+100]_2,\\
\mathcal{M}_6^2 & = & [8(v+w)(4t+5)+80t+102, 8(v+w)(4t+5)+80t+104]_2,\\
\mathcal{M}_6^3 & = & [8(v+w)(4t+5)+80t+106, 8(v+w)(4t+5)+112t+128]_2,\\
\mathcal{M}_{11}^1 & = & [8(v+w)(4t+5)+112t+144, 8(v+w)(4t+5)+116t+142]_2,\\
\mathcal{M}_{11}^2 & = & [8(v+w)(4t+5)+112t+145, 8(v+w)(4t+5)+116t+143]_2,\\
\mathcal{M}_{11}^3 & = & [8(v+w)(4t+5)+116t+144, 8(v+w)(4t+5)+116t+145].
\end{array}
$$

Define

$$
\mathfrak{R} = \{U_1(\mathcal{M}_6^1, \mathcal{M}_5), U_1(\mathcal{M}_6^2, \mathcal{M}_9)\}.
$$

Then, $\mathcal{E}(\mathfrak{R}) = \pm(\mathcal{M}_5 \sqcup \mathcal{M}_6^1 \sqcup \mathcal{M}_6^2 \sqcup \mathcal{M}_9)$: we replace the 2 instances of W_4 with the arrays in $\mathfrak{R}$.

The set $\mathcal{M}_1^1 \sqcup \mathcal{M}_1^2 \sqcup \mathcal{M}_3 \sqcup \mathcal{M}_7 \sqcup \mathcal{M}_8$ can be written as a disjoint union $F_1 \sqcup \ldots \sqcup F_\alpha$, where $\alpha = 8t+7$ and each F_i is a 2-set of type 4. The set $\mathcal{M}_2 \sqcup \mathcal{M}_4 \sqcup \mathcal{M}_6^3 \sqcup \mathcal{M}_{11}^1 \sqcup \mathcal{M}_{11}^2$ can be written as a disjoint union $G_1 \sqcup \ldots \sqcup G_\beta$, where $\beta = 2(v+w)(4t+5)+16t+15$ and each G_j is a 2-set of type 2. Finally, the set $\mathcal{M}_{10} \sqcup \mathcal{M}_{11}^3 \sqcup \mathcal{M}_{12}$ can be written as a disjoint union $H_1 \sqcup \ldots \sqcup H_\gamma$, where $\gamma = (8v(w+1)+2(v+w))(4t+5)+2$ and each H_h is a 2-set of type 1. Call

$$\begin{array}{rcl} \mathfrak{S}_1 & = & \{R_2(F_i, G_{2i-1}, G_{2i}) : i \in [1, 8t+7]\}, \\ \mathfrak{S}_2 & = & \left\{R_1(G_\beta, H_{\gamma-1}, H_\gamma)\right\}, \\ \mathfrak{T}_1 & = & \left\{Q_2(G_{2j-1}, G_{2j}) : j \in [8t+8, (v+w)(4t+5)+8t+7]\right\}, \\ \mathfrak{T}_2 & = & \left\{Q_1(H_{2h-1}, H_{2h}) : h \in [1, (4v(w+1)+(v+w))(4t+5)]\right\}. \end{array}$$

Then $\mathfrak{S} = \mathfrak{S}_1 \sqcup \mathfrak{S}_2$ consists of $8t+8$ zero-sum blocks of size 2×6 and the set $\mathfrak{T} = \mathfrak{T}_1 \sqcup \mathfrak{T}_2$ consists of $(4v(w+1)+2(v+w))(4t+5)$ zero-sum blocks of size 2×4. Furthermore, $\mathcal{E}(\mathfrak{S}) \sqcup \mathcal{E}(\mathfrak{T}) = \pm(\mathcal{M}_1 \sqcup \ldots \sqcup \mathcal{M}_4 \sqcup \mathcal{M}_6^3 \sqcup \mathcal{M}_7 \sqcup \mathcal{M}_8 \sqcup \mathcal{M}_{10} \sqcup \mathcal{M}_{11} \sqcup \mathcal{M}_{12})$. Hence, we replace the instances of Z_6 with the arrays in $\mathfrak{S}$ and the instances of Z_4 with the arrays in $\mathfrak{T}$. □

Lemma 3.7. *Suppose that $b \geqslant 11$ is such that $b \equiv 3 \pmod 4$. There exists an* SMAS$(5, b; 2c)$ *for all $c \geqslant 1$ such that $c \equiv 1 \pmod 4$.*

Proof. Write $c = 4t+1$ where $t \geqslant 0$. Our SMAS$(5, b; 2c)$ will consist of $8t+2$ arrays of type

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline X_3 & X_2^t & X_2^t & X_2^t & X_2^t & \cdots & X_2^t & X_2^t \\ \hline X_2 & \multicolumn{2}{c|}{Z_4} & \multicolumn{2}{c|}{Z_4} & \cdots & \multicolumn{2}{c|}{Z_4} \\ \hline \end{array}. \tag{3.10}$$

Write $b = 4w+11$, where $w \geqslant 0$. Hence, we will construct $8t+2$ blocks X_3, $2(2w+5)(4t+1)$ blocks X_2 and $2(w+2)(4t+1)$ blocks Z_4. So, take

$$\begin{array}{rcl} \mathfrak{A} & = & \mathfrak{A}(4t,\ 2(2w+5)(4t+1),\ 4t+1), \\ \mathfrak{B} & = & \mathfrak{B}(1,\ (16w+45)(4t+1),\ 2(2w+5)(4t+1)) \end{array}$$

as in Lemmas 2.5 and 2.6, respectively. Then,

$$\mathcal{E}(\mathfrak{A}) \sqcup \mathcal{E}(\mathfrak{B}) = \pm[1, 5(4w+11)(4t+1)] \setminus \pm(\mathcal{M}_1 \sqcup \ldots \sqcup \mathcal{M}_5),$$

where

$$\begin{array}{rcl} \mathcal{M}_1 & = & [2, 8t]_2, \\ \mathcal{M}_2 & = & [40t+12, 56t+8]_4, \\ \mathcal{M}_3 & = & [56t+12, 8w(4t+1)+128t+30]_2, \\ \mathcal{M}_4 & = & [8w(4t+1)+96t+27, 8w(4t+1)+128t+31]_4, \\ \mathcal{M}_5 & = & [8w(4t+1)+128t+32, 12w(4t+1)+140t+35]. \end{array}$$

So, we replace the $8t+2$ instances of X_3 with the arrays in $\mathfrak{A}$ and the $2(2w+5)(4t+1)$ instances of X_2 with the arrays in $\mathfrak{B}$.

Next, write

$$\mathcal{M}_3 = \mathcal{M}_3^1 \sqcup \mathcal{M}_3^2 \quad \text{and} \quad \mathcal{M}_4 = \mathcal{M}_4^1 \sqcup \mathcal{M}_4^2,$$

where

$$\begin{array}{rcl} \mathcal{M}_3^1 & = & [56t+12, 8w(4t+1)+128t+26]_2, \\ \mathcal{M}_3^2 & = & [8w(4t+1)+128t+28, 8w(4t+1)+128t+30]_2, \\ \mathcal{M}_4^1 & = & [8w(4t+1)+96t+27, 8w(4t+1)+128t+23]_4, \\ \mathcal{M}_4^2 & = & [8w(4t+1)+128t+27, 8w(4t+1)+128t+31]_4. \end{array}$$

Note that $\mathcal{M}_3^2 \sqcup \mathcal{M}_4^2 = [8w(4t+1)+128t+27, 8w(4t+1)+128t+28] \sqcup [8w(4t+1)+128t+30, 8w(4t+1)+128t+31]$.

The set $\mathcal{M}_2 \sqcup \mathcal{M}_4^1$ can be written as a disjoint union $F_1 \sqcup \ldots \sqcup F_{6t}$, where each F_i is a 2-set of type 4. The set $\mathcal{M}_1 \sqcup \mathcal{M}_3^1$ can be written as a disjoint union $G_1 \sqcup \ldots \sqcup G_{2\beta}$, where $\beta = (w+2)(4t+1)+2t$ and each G_j is a 2-set of type 2. Finally, the set $\mathcal{M}_3^2 \sqcup \mathcal{M}_4^2 \sqcup \mathcal{M}_5$ can be written as a disjoint union $H_1 \sqcup \ldots \sqcup H_{2\gamma}$, where $\gamma = w(4t+1)+3t+2$ and each H_h is a 2-set of type 1. Call

$$\begin{array}{rcl} \mathfrak{R}_1 & = & \{Q_3(F_{2i-1}, F_{2i}) : i \in [1, 3t]\}, \\ \mathfrak{R}_2 & = & \left\{Q_2(G_{2j-1}, G_{2j}) : j \in [1, (w+2)(4t+1)+2t]\right\}, \\ \mathfrak{R}_3 & = & \left\{Q_1(H_{2h-1}, H_{2h}) : h \in [1, w(4t+1)+3t+2]\right\}. \end{array}$$

Then $\mathfrak{R} = \mathfrak{R}_1 \sqcup \mathfrak{R}_2 \sqcup \mathfrak{R}_3$ consists of $2(w+2)(4t+1)$ zero-sum blocks of size 2×4 and $\mathcal{E}(\mathfrak{R}) = \pm(\mathcal{M}_1 \sqcup \ldots \sqcup \mathcal{M}_5)$. Hence, we replace the instances of Z_4 with the arrays in $\mathfrak{R}$. □

Lemma 3.8. *Suppose that $a, b \geqslant 7$ are such that $a \equiv b \equiv 3 \pmod 4$. There exists an* SMAS$(a, b; 2)$.

Proof. Our SMAS$(a, b; 2)$ will consist of two arrays of type (3.3). Write $a = 4v + 7$ and $b = 4w + 7$, where $v, w \geqslant 0$. Hence, we will construct $4(v + w + 1)$ blocks X_2, $2(w + (2v + 1)(w + 1))$ blocks Z_4, 2 blocks Y_5 and 2 blocks W_2. So, take

$$\begin{array}{rcl}\mathfrak{B} & = & \mathfrak{B}(9,\ 16vw + 24(v + w) + 37,\ 4(v + w + 1)),\\ \mathfrak{C} & = & \mathfrak{C}(8(v + w) + 16,\ 16vw + 24(v + w) + 38,\ 8(v + w) + 26)\end{array}$$

as in Lemmas 2.6 and 2.7, respectively. We have

$$\mathcal{E}(\mathfrak{B}) \sqcup \mathcal{E}(\mathfrak{C}) = \pm[1, 16vw + 28(v + w) + 49] \setminus \pm(\mathcal{M}_1 \sqcup \ldots \sqcup \mathcal{M}_5),$$

where

$$\begin{array}{rcl}\mathcal{M}_1 & = & [12, 8(v + w) + 14]_2,\\ \mathcal{M}_2 & = & [8(v + w) + 17, 8(v + w) + 18],\\ \mathcal{M}_3 & = & [8(v + w) + 19, 8(v + w) + 21]_2,\\ \mathcal{M}_4 & = & [8(v + w) + 23, 8(v + w) + 24],\\ \mathcal{M}_5 & = & [8(v + w) + 30, 16vw + 20(v + w) + 33].\end{array}$$

So, we replace the 2 instances of Y_5 with the arrays in $\mathfrak{C}$ and the $4(v + w + 1)$ instances of X_2 with the arrays in $\mathfrak{B}$. Call

$$\mathfrak{R} = \{U_2(\mathcal{M}_2), U_2(\mathcal{M}_4)\}.$$

Then, $\mathcal{E}(\mathfrak{R}) = \pm(\mathcal{M}_2 \sqcup \mathcal{M}_4)$: we replace the 2 instances of W_2 with the arrays in $\mathfrak{R}$.

The set $\mathcal{M}_1 \sqcup \mathcal{M}_3$ can be written as a disjoint union $G_1 \sqcup \ldots \sqcup G_{2\beta}$, where $\beta = v + w + 1$ and each G_j is a 2-set of type 2. The set $\mathcal{M}_5$ can be written as a disjoint union $H_1 \sqcup \ldots \sqcup H_{2\gamma}$, where $\gamma = 4vw + 3(v + w) + 1$ and each H_h is a 2-set of type 1. Call

$$\begin{array}{rcl}\mathfrak{S}_1 & = & \{Q_2(G_{2j-1}, G_{2j}) : j \in [1, v + w + 1]\},\\ \mathfrak{S}_2 & = & \{Q_1(H_{2h-1}, H_{2h}) : h \in [1, 4vw + 3(v + w) + 1]\}.\end{array}$$

Then, $\mathfrak{S} = \mathfrak{S}_1 \sqcup \mathfrak{S}_2$ consists of $4vw + 4(v + w) + 2$ zero-sum blocks of size 2×4 and $\mathcal{E}(\mathfrak{S}) = \pm(\mathcal{M}_1 \sqcup \mathcal{M}_3 \sqcup \mathcal{M}_5)$. Hence, we replace the instances of Z_4 with the arrays in $\mathfrak{S}$. $\square$

Lemma 3.9. *Suppose that $a, b \geqslant 7$ are such that $a \equiv b \equiv 3 \pmod 4$. There exists an* SMAS$(a, b; 2c)$ *for all $c \geqslant 1$ such that $c \equiv 1 \pmod 4$.*

Proof. By Lemma 3.8 we can write $c = 4t + 5$ where $t \geqslant 0$. Our SMAS$(a, b; 2c)$ will consist of $8t + 8$ arrays of type (3.2) and two arrays of type (3.3). Write $a = 4v + 7$ and $b = 4w + 7$, where $v, w \geqslant 0$. Hence, we will construct $8t + 8$ blocks X_3, $4(v + w)(4t + 5) + 32t + 36$ blocks X_2, $2(8t + 8)(v + 1)(w + 1) + 2(w + (2v + 1)(w + 1))$ blocks Z_4, 2 blocks Y_5 and 2 blocks W_2. So, take

$$\begin{array}{rcl}\mathfrak{A} & = & \mathfrak{A}(4t + 5,\ 4(v + w)(4t + 5) + 32t + 40,\ 4t + 4),\\ \mathfrak{B} & = & \mathfrak{B}(9,\ (16vw + 24(v + w))(4t + 5) + 164t + 201,\ 4(v + w)(4t + 5) + 32t + 36),\\ \mathfrak{C} & = & \mathfrak{C}(8(v + w)(4t + 5) + 112t + 132,\ (16vw + 24(v + w))(4t + 5) + 164t + 202,\\ & & 8(v + w)(4t + 5) + 112t + 142)\end{array}$$

as in Lemmas 2.5, 2.6 and 2.7, respectively. We have

$$\mathcal{E}(\mathfrak{A}) \sqcup \mathcal{E}(\mathfrak{B}) \sqcup \mathcal{E}(\mathfrak{C}) = \pm[1, (16vw + 28(v + w))(4t + 5) + 196t + 245] \setminus \pm(\mathcal{M}_1 \sqcup \ldots \sqcup \mathcal{M}_{12}),$$

where

$$\begin{array}{rcl}\mathcal{M}_1 & = & [12, 8t + 10]_2,\\ \mathcal{M}_2 & = & [40t + 44, 40t + 46]_2,\\ \mathcal{M}_3 & = & [40t + 50, 56t + 62]_4,\\ \mathcal{M}_4 & = & [56t + 64, 8(v + w)(4t + 5) + 80t + 94]_2,\\ \mathcal{M}_5 & = & [8(v + w)(4t + 5) + 80t + 96, 8(v + w)(4t + 5) + 80t + 99],\\ \mathcal{M}_6 & = & [8(v + w)(4t + 5) + 80t + 100, 8(v + w)(4t + 5) + 112t + 130]_2,\\ \mathcal{M}_7 & = & [8(v + w)(4t + 5) + 80t + 103, 8(v + w)(4t + 5) + 112t + 131]_4,\\ \mathcal{M}_8 & = & [8(v + w)(4t + 5) + 112t + 133, 8(v + w)(4t + 5) + 112t + 134],\\ \mathcal{M}_9 & = & [8(v + w)(4t + 5) + 112t + 135, 8(v + w)(4t + 5) + 112t + 137]_2,\\ \mathcal{M}_{10} & = & [8(v + w)(4t + 5) + 112t + 139, 8(v + w)(4t + 5) + 112t + 140],\\ \mathcal{M}_{11} & = & [8(v + w)(4t + 5) + 112t + 146, 8(v + w)(4t + 5) + 132t + 165],\\ \mathcal{M}_{12} & = & [8(v + w)(4t + 5) + 132t + 166, (16vw + 20(v + w))(4t + 5) + 132t + 165].\end{array}$$

So, we replace the $8t+8$ instances of X_3 with the arrays in $\mathfrak{A}$, the 2 instances of Y_5 with the arrays in $\mathfrak{C}$ and the $4(v+w)(4t+5)+32t+36$ instances of X_2 with the arrays in $\mathfrak{B}$.

Define

$$\mathfrak{R} = \{U_2(\mathcal{M}_8), U_2(\mathcal{M}_{10})\}.$$

Then, $\mathcal{E}(\mathfrak{R}) = \pm(\mathcal{M}_8 \sqcup \mathcal{M}_{10})$: we replace the 2 instances of W_2 with the arrays in $\mathfrak{R}$.

The set $\mathcal{M}_3 \sqcup \mathcal{M}_7$ can be written as a disjoint union $F_1 \sqcup \ldots \sqcup F_{2\alpha}$, where each $\alpha = 3t+3$ and F_i is a 2-set of type 4. The set $\mathcal{M}_1 \sqcup \mathcal{M}_2 \sqcup \mathcal{M}_4 \sqcup \mathcal{M}_6 \sqcup \mathcal{M}_9$ can be written as a disjoint union $G_1 \sqcup \ldots \sqcup G_{2\beta}$, where $\beta = (v+w)(4t+5)+8t+9$ and each G_j is a 2-set of type 2. Finally, the set $\mathcal{M}_5 \sqcup \mathcal{M}_{11} \sqcup \mathcal{M}_{12}$ can be written as a disjoint union $H_1 \sqcup \ldots \sqcup H_{2\gamma}$, where $\gamma = (4vw+3(v+w))(4t+5)+5t+6$ and each H_h is a 2-set of type 1. Call

$$\begin{array}{rcl}
\mathfrak{S}_1 & = & \{Q_3(F_{2i-1}, F_{2i}) : i \in [1, 3t+3]\}, \\
\mathfrak{S}_2 & = & \left\{Q_2(G_{2j-1}, G_{2j}) : j \in [1, (v+w)(4t+5)+8t+9]\right\}, \\
\mathfrak{S}_3 & = & \left\{Q_1(H_{2h-1}, H_{2h}) : h \in [1, (4vw+3(v+w))(4t+5)+5t+6]\right\}.
\end{array}$$

Then $\mathfrak{S} = \mathfrak{S}_1 \sqcup \mathfrak{S}_2 \sqcup \mathfrak{S}_3$ consists of $(4vw+4(v+w))(4t+5)+16t+18$ zero-sum blocks of size 2×4 and $\mathcal{E}(\mathfrak{S}) = \pm(\mathcal{M}_1 \sqcup \ldots \sqcup \mathcal{M}_7 \sqcup \mathcal{M}_9 \sqcup \mathcal{M}_{11} \sqcup \mathcal{M}_{12})$. Hence, we replace the instances of Z_4 with the arrays in $\mathfrak{S}$. $\square$

3.3. *Subcase $c \equiv 2 \pmod 4$*

Lemma 3.10. *Suppose that $a \geqslant 5$ and $b \geqslant 9$ are such that $a \equiv 1 \pmod 4$ and b is odd. There exists an* $\mathrm{SMAS}(a, b; 2c)$ *for all $c \geqslant 2$ such that $c \equiv 2 \pmod 4$.*

Proof. Write $c = 4t+2$ where $t \geqslant 0$. Also, write $a = 4v+5$ and $b = 4w+9+2\delta$, where $v, w \geqslant 0$ and $\delta \in \{0, 1\}$. If $b \equiv 1 \pmod 4$, then our $\mathrm{SMAS}(a, b; 2c)$ will consist of $8t+2$ arrays of type (3.9) and two arrays of type (3.7); if $b \equiv 3 \pmod 4$, it will consist of $8t+2$ arrays of type (3.2) and two arrays of type (3.3). So, we will construct $8t+2$ blocks X_3, $(8(v+w)+4\delta)(2t+1)+32t+12$ blocks X_2, $(4t+2)(v(6+2\delta)+w(4v+2))+(16t+6)\delta$ blocks Z_4, and 2 blocks Y_5. Furthermore, we will also construct $8t+2$ blocks Z_6 and 2 blocks W_4 if $\delta = 0$, or 2 blocks W_2 if $\delta = 1$. So, take

$$\begin{array}{rcl}
\mathfrak{A} & = & \mathfrak{A}(4t+5,\ (8(v+w)+4\delta)(2t+1)+32t+16,\ 4t+1), \\
\mathfrak{B} & = & \mathfrak{B}(9,\ (32v(w+1)+32(v+w)+16\delta(v+1))(2t+1)+148t+70, \\
& & (8(v+w)+4\delta)(2t+1)+32t+12), \\
\mathfrak{C} & = & \mathfrak{C}((16(v+w)+8\delta)(2t+1)+80t+37,\ (32v(w+1)+32(v+w)+16\delta(v+1))(2t+1)+148t+71, \\
& & (16(v+w)+8\delta)(2t+1)+116t+55)
\end{array}$$

as in Lemmas 2.5, 2.6 and 2.7, respectively. We have

$$\begin{array}{rcl}
\mathcal{E}(\mathfrak{A}) \sqcup \mathcal{E}(\mathfrak{B}) \sqcup \mathcal{E}(\mathfrak{C}) & = & \pm[1, (32v(w+1)+40(v+w)+4\delta(4v+5))(2t+1)+180t+90] \setminus \\
& & \pm(\mathcal{M}_1 \sqcup \ldots \sqcup \mathcal{M}_{12}),
\end{array}$$

where

$$\begin{array}{rcl}
\mathcal{M}_1 & = & [12, 8t+10]_2, \\
\mathcal{M}_2 & = & [40t+20, 40t+26]_2, \\
\mathcal{M}_3 & = & [40t+28, 56t+32]_4, \\
\mathcal{M}_4 & = & [56t+34, 8(2v+2w+\delta)(2t+1)+80t+36]_2, \\
\mathcal{M}_5 & = & [8(2v+2w+\delta)(2t+1)+80t+38, 8(2v+2w+\delta)(2t+1)+80t+39], \\
\mathcal{M}_6 & = & [8(2v+2w+\delta)(2t+1)+80t+40, 8(2v+2w+\delta)(2t+1)+80t+42]_2, \\
\mathcal{M}_7 & = & [8(2v+2w+\delta)(2t+1)+80t+44, 8(2v+2w+\delta)(2t+1)+80t+45], \\
\mathcal{M}_8 & = & [8(2v+2w+\delta)(2t+1)+80t+48, 8(2v+2w+\delta)(2t+1)+80t+49], \\
\mathcal{M}_9 & = & [8(2v+2w+\delta)(2t+1)+80t+50, 8(2v+2w+\delta)(2t+1)+112t+52]_2, \\
\mathcal{M}_{10} & = & [8(2v+2w+\delta)(2t+1)+80t+53, 8(2v+2w+\delta)(2t+1)+112t+49]_4, \\
\mathcal{M}_{11} & = & [8(2v+2w+\delta)(2t+1)+112t+53, 8(2v+2w+\delta)(2t+1)+116t+54], \\
\mathcal{M}_{12} & = & [8(2v+2w+\delta)(2t+1)+116t+59, \\
& & (12(2v+2w+\delta)+16v(2w+2+\delta))(2t+1)+116t+58].
\end{array}$$

We replace the $8t+2$ instances of X_3 with the arrays in $\mathfrak{A}$, the 2 instances of Y_5 with the arrays in $\mathfrak{C}$ and the $(8(v+w)+4\delta)(2t+1)+32t+12$ instances of X_2 with the arrays in $\mathfrak{B}$.

Write

$$\mathcal{M}_9 = \mathcal{M}_9^1 \sqcup \mathcal{M}_9^2 \sqcup \mathcal{M}_9^3 \quad \text{and} \quad \mathcal{M}_{11} = \mathcal{M}_{11}^1 \sqcup \mathcal{M}_{11}^2 \sqcup \mathcal{M}_{11}^3,$$

where

$$\begin{array}{rcl}
\mathcal{M}_9^1 & = & [8(2v+2w+\delta)(2t+1)+80t+50, 8(2v+2w+\delta)(2t+1)+104t+52]_2,\\
\mathcal{M}_9^2 & = & [8(2v+2w+\delta)(2t+1)+104t+54, 8(2v+2w+\delta)(2t+1)+112t+50]_4,\\
\mathcal{M}_9^3 & = & [8(2v+2w+\delta)(2t+1)+104t+56, 8(2v+2w+\delta)(2t+1)+112t+52]_4,\\
\mathcal{M}_{11}^1 & = & [8(2v+2w+\delta)(2t+1)+112t+53, 8(2v+2w+\delta)(2t+1)+112t+54],\\
\mathcal{M}_{11}^2 & = & [8(2v+2w+\delta)(2t+1)+112t+55, 8(2v+2w+\delta)(2t+1)+116t+53]_2,\\
\mathcal{M}_{11}^3 & = & [8(2v+2w+\delta)(2t+1)+112t+56, 8(2v+2w+\delta)(2t+1)+116t+54]_2.
\end{array}$$

The set $\mathcal{M}_3 \sqcup \mathcal{M}_9^2 \sqcup \mathcal{M}_9^3 \sqcup \mathcal{M}_{10}$ can be written as a disjoint union $F_1 \sqcup \ldots \sqcup F_\alpha$, where $\alpha = 8t+1$ and each F_i is a 2-set of type 4. The set $\mathcal{M}_1 \sqcup \mathcal{M}_2 \sqcup \mathcal{M}_4 \sqcup \mathcal{M}_9^1 \sqcup \mathcal{M}_{11}^2 \sqcup \mathcal{M}_{11}^3$ can be written as a disjoint union $G_1 \sqcup \ldots \sqcup G_{2\beta+2(8t+2)}$, where $\beta = (4t+2)(v+w)+(2t+1)\delta$ and each G_j is a 2-set of type 2. Finally, the set $\mathcal{M}_7 \sqcup \mathcal{M}_8 \sqcup \mathcal{M}_{11}^1 \sqcup \mathcal{M}_{12}$ can be written as a disjoint union $H_1 \sqcup \ldots \sqcup H_{2\gamma+3}$, where $\gamma = (4t+2)(v(5+2\delta)+w(4v+1))+(2t+1)\delta$ and each H_h is a 2-set of type 1.

Call

$$\begin{array}{rcl}
\mathfrak{Q}_1 & = & \left\{Q_2(G_{2j-1}, G_{2j}) : j \in [1,\beta]\right\},\\
\mathfrak{Q}_2 & = & \left\{Q_1(H_{2h-1}, H_{2h}) : h \in [1,\gamma]\right\}.
\end{array}$$

Then $\mathfrak{Q} = \mathfrak{Q}_1 \sqcup \mathfrak{Q}_2$ consists of $\beta+\gamma = (4t+2)(v(6+2\delta)+w(4v+2))+(4t+2)\delta$ zero-sum blocks of size 2×4.

Suppose $\delta = 0$. First of all, we replace the instances of Z_4 with the arrays in $\mathfrak{Q}$. Then, we need still to construct $8t+2$ blocks Z_6 and 2 blocks W_4. To this purpose, define

$$\begin{array}{rcl}
\mathfrak{R}_1 & = & \left\{R_2(F_i, G_{2\beta+2i-1}, G_{2\beta+2i}) : i \in [1,8t+1]\right\},\\
\mathfrak{R}_2 & = & \left\{R_1(G_{2\beta+2(8t+1)+1}, H_{2\gamma+1}, H_{2\gamma+2})\right\},\\
\mathfrak{S} & = & \{U_1(\mathcal{M}_6, \mathcal{M}_5), U_1(G_{2\beta+2(8t+2)}, H_{2\gamma+3})\}.
\end{array}$$

Then $\mathfrak{R} = \mathfrak{R}_1 \sqcup \mathfrak{R}_2$ consists of $8t+2$ zero-sum blocks of size 2×6. Hence, we replace the instances of Z_6 with the arrays in $\mathfrak{R}$ and the instances of W_4 with the arrays in $\mathfrak{S}$.

Next, suppose $\delta = 1$. Write $\mathcal{M}_5 \sqcup \mathcal{M}_6$ as $\mathcal{N}_1 \sqcup \mathcal{N}_2$, where

$$\begin{array}{rcl}
\mathcal{N}_1 & = & [8(2v+2w+\delta)(2t+1)+80t+38, 8(2v+2w+\delta)(2t+1)+80t+42]_4,\\
\mathcal{N}_2 & = & [8(2v+2w+\delta)(2t+1)+80t+39, 8(2v+2w+\delta)(2t+1)+80t+40].
\end{array}$$

Define

$$\begin{array}{rcl}
\mathfrak{Q}_3 & = & \{Q_3(F_{2i-1}, F_{2i}) : j \in [1,4t]\} \sqcup \{Q_3(F_{8t+1}, \mathcal{N}_1)\},\\
\mathfrak{Q}_4 & = & \left\{Q_2(G_{2\beta+2j-1}, G_{2\beta+2j}) : i \in [1,8t+2]\right\},\\
\mathfrak{Q}_5 & = & \left\{Q_1(H_{2\gamma+1}, H_{2\gamma+2})\right\}.
\end{array}$$

Then $\mathfrak{Q}' = \mathfrak{Q}_3 \sqcup \mathfrak{Q}_4 \sqcup \mathfrak{Q}_5$ consists of $12t+4$ zero-sum blocks of size 2×4. So, we replace the instances of Z_4 with the arrays in $\mathfrak{Q} \sqcup \mathfrak{Q}'$. To conclude, we construct the two blocks W_2 by taking $U_2(\mathcal{N}_2)$ and $U_2(H_{2\gamma+3})$. $\square$

Lemma 3.11. *Suppose that $a \geqslant 9$ is such that $a \equiv 1 \pmod 4$. There exists an* SMAS$(a,7;2c)$ *for all $c \geqslant 2$ such that $c \equiv 2 \pmod 4$.*

Proof. Write $c = 4t+2$ where $t \geqslant 0$. Our SMAS$(a,7;2c)$ will consist of $8t+2$ arrays of type (3.5) and two arrays of type (3.6). Write $a = 4v+9$, where $v \geqslant 0$. Hence, we will construct $8t+2$ blocks X_3, $8v(2t+1)+40t+16$ blocks X_2, $(8t+2)(2v+3)+2(2v+2)$ blocks Z_4, 2 blocks Y_5 and 2 blocks W_2. So, take

$$\begin{array}{rcl}
\mathfrak{A} & = & \mathfrak{A}(4t+5,\ 8v(2t+1)+40t+20,\ 4t+1),\\
\mathfrak{B} & = & \mathfrak{B}(9,\ 48v(2t+1)+212t+102,\ 8v(2t+1)+40t+16),\\
\mathfrak{C} & = & \mathfrak{C}(16v(2t+1)+96t+42,\ 48v(2t+1)+212t+103,\ 16v(2t+1)+128t+61)
\end{array}$$

as in Lemmas 2.5, 2.6 and 2.7, respectively. We have

$$\mathcal{E}(\mathfrak{A}) \sqcup \mathcal{E}(\mathfrak{B}) \sqcup \mathcal{E}(\mathfrak{C}) = \pm[1, 56v(2t+1)+252t+126] \setminus \pm(\mathcal{M}_1 \sqcup \ldots \sqcup \mathcal{M}_{11}),$$

where

$$\begin{array}{rcl}
\mathcal{M}_1 & = & [12, 8t+10]_2, \\
\mathcal{M}_2 & = & [40t+20, 40t+26]_2, \\
\mathcal{M}_3 & = & [40t+28, 56t+32]_4, \\
\mathcal{M}_4 & = & [56t+34, 16v(2t+1)+96t+40]_2, \\
\mathcal{M}_5 & = & [16v(2t+1)+96t+44, 16v(2t+1)+96t+45], \\
\mathcal{M}_6 & = & [16v(2t+1)+96t+47, 16v(2t+1)+96t+49]_2, \\
\mathcal{M}_7 & = & [16v(2t+1)+96t+50, 16v(2t+1)+96t+52]_2, \\
\mathcal{M}_8 & = & [16v(2t+1)+96t+53, 16v(2t+1)+128t+57]_4, \\
\mathcal{M}_9 & = & [16v(2t+1)+96t+54, 16v(2t+1)+128t+60]_2, \\
\mathcal{M}_{10} & = & [16v(2t+1)+128t+65, 16v(2t+1)+172t+86], \\
\mathcal{M}_{11} & = & [16v(2t+1)+172t+87, 40v(2t+1)+172t+86].
\end{array}$$

So, we replace the $8t+2$ instances of X_3 with the arrays in $\mathfrak{A}$, the 2 instances of Y_5 with the arrays in $\mathfrak{C}$ and the $8v(2t+1)+40t+16$ instances of X_2 with the arrays in $\mathfrak{B}$.

Write

$$\mathcal{M}_{10} = \mathcal{M}_{10}^1 \sqcup \mathcal{M}_{10}^2,$$

where

$$\begin{array}{rcl}
\mathcal{M}_{10}^1 & = & [16v(2t+1)+128t+65, 16v(2t+1)+128t+66], \\
\mathcal{M}_{10}^2 & = & [16v(2t+1)+128t+67, 16v(2t+1)+172t+86].
\end{array}$$

Define

$$\mathfrak{R} = \{U_2(\mathcal{M}_5), U_2(\mathcal{M}_{10}^1)\}.$$

Then, $\mathcal{E}(\mathfrak{R}) = \pm(\mathcal{M}_5 \sqcup \mathcal{M}_{10}^1)$: we replace the 2 instances of W_2 with the arrays in $\mathfrak{R}$.

The set $\mathcal{M}_3 \sqcup \mathcal{M}_8$ can be written as a disjoint union $F_1 \sqcup \ldots \sqcup F_{2\alpha}$, where $\alpha = 3t+1$ and each F_i is a 2-set of type 4. The set $\mathcal{M}_1 \sqcup \mathcal{M}_2 \sqcup \mathcal{M}_4 \sqcup \mathcal{M}_6 \sqcup \mathcal{M}_7 \sqcup \mathcal{M}_9$ can be written as a disjoint union $G_1 \sqcup \ldots \sqcup G_{2\beta}$, where $\beta = 2v(2t+1)+10t+4$ and each G_j is a 2-set of type 2. Finally, the set $\mathcal{M}_{10}^2 \sqcup \mathcal{M}_{11}$ can be written as a disjoint union $H_1 \sqcup \ldots \sqcup H_{2\gamma}$, where $\gamma = 6v(2t+1)+11t+5$ and each H_h is a 2-set of type 1. Call

$$\begin{array}{rcl}
\mathfrak{S}_1 & = & \{Q_3(F_{2i-1}, F_{2i}) : i \in [1, 3t+1]\}, \\
\mathfrak{S}_2 & = & \left\{Q_2(G_{2j-1}, G_{2j}) : j \in [1, 2v(2t+1)+10t+4]\right\}, \\
\mathfrak{S}_3 & = & \left\{Q_1(H_{2h-1}, H_{2h}) : h \in [1, 6v(2t+1)+11t+5]\right\}.
\end{array}$$

Then $\mathfrak{S} = \mathfrak{S}_1 \sqcup \mathfrak{S}_2 \sqcup \mathfrak{S}_3$ consists of $8v(2t+1)+24t+10$ zero-sum blocks of size 2×4 and $\mathcal{E}(\mathfrak{S}) = \pm(\mathcal{M}_1 \sqcup \ldots \sqcup \mathcal{M}_4 \sqcup \mathcal{M}_6 \sqcup \ldots \sqcup \mathcal{M}_9 \sqcup \mathcal{M}_{10}^2 \sqcup \mathcal{M}_{11})$. Hence, we replace the instances of Z_4 with the arrays in $\mathfrak{S}$. □

Lemma 3.12. *Suppose that $a, b \geqslant 7$ are such that $a \equiv b \equiv 3 \pmod 4$. There exists an* SMAS$(a, b; 2c)$ *for all $c \geqslant 2$ such that $c \equiv 2 \pmod 4$.*

Proof. Write $c = 4t+2$ where $t \geqslant 0$. Our SMAS$(a, b; 2c)$ will consist of $8t+2$ arrays of type (3.2) and two arrays of type (3.3). Write $a = 4v+7$ and $b = 4w+7$, where $v, w \geqslant 0$. Hence, we will construct $8t+2$ blocks X_3, $8(v+w)(2t+1)+32t+12$ blocks X_2, $2(8t+2)(v+1)(w+1)+2(w+(2v+1)(w+1))$ blocks Z_4, 2 blocks Y_5 and 2 blocks W_2. So, take

$$\begin{array}{rcl}
\mathfrak{A} & = & \mathfrak{A}(4t+5,\ 8(v+w)(2t+1)+32t+16,\ 4t+1), \\
\mathfrak{B} & = & \mathfrak{B}(9,\ (32vw+48(v+w))(2t+1)+164t+78,\ 8(v+w)(2t+1)+32t+12), \\
\mathfrak{C} & = & \mathfrak{C}(16(v+w)(2t+1)+80t+34,\ (32vw+48(v+w))(2t+1)+164t+79, \\
& & 16(v+w)(2t+1)+112t+55)
\end{array}$$

as in Lemmas 2.5, 2.6 and 2.7, respectively. We have

$$\mathcal{E}(\mathfrak{A}) \sqcup \mathcal{E}(\mathfrak{B}) \sqcup \mathcal{E}(\mathfrak{C}) = \pm[1, (32vw+56(v+w))(2t+1)+196t+98] \setminus \pm(\mathcal{M}_1 \sqcup \ldots \sqcup \mathcal{M}_{12}),$$

where

$$\begin{array}{rcl}
\mathcal{M}_1 & = & [16(v+w)(2t+1)+80t+36, 16(v+w)(2t+1)+80t+37],\\
\mathcal{M}_2 & = & [16(v+w)(2t+1)+80t+45, 16(v+w)(2t+1)+80t+46],\\
\mathcal{M}_3 & = & [12, 8t+10]_2,\\
\mathcal{M}_4 & = & [40t+20, 40t+26]_2,\\
\mathcal{M}_5 & = & [40t+28, 56t+32]_4,\\
\mathcal{M}_6 & = & [56t+34, 16(v+w)(2t+1)+80t+32]_2,\\
\mathcal{M}_7 & = & [16(v+w)(2t+1)+80t+39, 16(v+w)(2t+1)+80t+41]_2,\\
\mathcal{M}_8 & = & [16(v+w)(2t+1)+80t+42, 16(v+w)(2t+1)+80t+44]_2,\\
\mathcal{M}_9 & = & [16(v+w)(2t+1)+80t+48, 16(v+w)(2t+1)+112t+54]_2,\\
\mathcal{M}_{10} & = & [16(v+w)(2t+1)+80t+49, 16(v+w)(2t+1)+112t+53]_4,\\
\mathcal{M}_{11} & = & [16(v+w)(2t+1)+112t+59, 16(v+w)(2t+1)+132t+66],\\
\mathcal{M}_{12} & = & [16(v+w)(2t+1)+132t+67, (32vw+40(v+w))(2t+1)+132t+66].
\end{array}$$

So, we replace the $8t+2$ instances of X_3 with the arrays in $\mathfrak{A}$, the 2 instances of Y_5 with the arrays in $\mathfrak{C}$ and the $8(v+w)(2t+1)+32t+12$ instances of X_2 with the arrays in $\mathfrak{B}$.

Define

$$\mathfrak{R} = \{U_2(\mathcal{M}_1), U_2(\mathcal{M}_2)\}.$$

Then, $\mathcal{E}(\mathfrak{R}) = \pm(\mathcal{M}_1 \sqcup \mathcal{M}_2)$: we replace the 2 instances of W_2 with the arrays in $\mathfrak{R}$.

The set $\mathcal{M}_5 \sqcup \mathcal{M}_{10}$ can be written as a disjoint union $F_1 \sqcup \ldots \sqcup F_{2\alpha}$, where $\alpha = 3t+1$ and each F_i is a 2-set of type 4. The set $\mathcal{M}_3 \sqcup \mathcal{M}_4 \sqcup \mathcal{M}_6 \sqcup \mathcal{M}_7 \sqcup \mathcal{M}_8 \sqcup \mathcal{M}_9$ can be written as a disjoint union $G_1 \sqcup \ldots \sqcup G_{2\beta}$, where $\beta = 2(v+w)(2t+1)+8t+3$ and each G_j is a 2-set of type 2. Finally, the set $\mathcal{M}_{11} \sqcup \mathcal{M}_{12}$ can be written as a disjoint union $H_1 \sqcup \ldots \sqcup H_{2\gamma}$, where $\gamma = (8vw+6(v+w))(2t+1)+5t+2$ and each H_h is a 2-set of type 1. Call

$$\begin{array}{rcl}
\mathfrak{S}_1 & = & \{Q_3(F_{2i-1}, F_{2i}) : i \in [1, 3t+1]\},\\
\mathfrak{S}_2 & = & \left\{Q_2(G_{2j-1}, G_{2j}) : j \in [1, 2(v+w)(2t+1)+8t+3]\right\},\\
\mathfrak{S}_3 & = & \left\{Q_1(H_{2h-1}, H_{2h}) : h \in [1, (8vw+6(v+w))(2t+1)+5t+2]\right\}.
\end{array}$$

Then $\mathfrak{S} = \mathfrak{S}_1 \sqcup \mathfrak{S}_2 \sqcup \mathfrak{S}_3$ consists of $(8vw+8(v+w))(2t+1)+16t+6$ zero-sum blocks of size 2×6 and $\mathcal{E}(\mathfrak{S}) = \pm(\mathcal{M}_3 \sqcup \ldots \sqcup \mathcal{M}_{12})$. Hence, we replace the instances of Z_4 with the arrays in $\mathfrak{S}$. $\square$

4. The case $\{a, b\} = \{5, 7\}$ and the case $c \equiv 0 \pmod 4$

In this section, we consider the case of an SMA$(a, b; 2c)$ where $c \equiv 0 \pmod 4$ and the exceptional case of an SMAS$(5, 7; 2c)$ that require slightly different constructions.

Lemma 4.1. *Given four nonnegative integers α, β, γ, u such that $\alpha + 1 \geqslant 2u$ and $2\gamma \geqslant \alpha + 8u$, there exists a set $\overline{\mathfrak{A}} = \overline{\mathfrak{A}}(\alpha, \beta, \gamma, u)$ consisting of $4u$ square zero-sum blocks of size 3 such that $\mathcal{E}(\overline{\mathfrak{A}})$ consists of the following disjoint subsets:*

$$\begin{array}{rcl}
\mathcal{E}(\overline{\mathfrak{A}}) & = & \pm[2\alpha+2, 2\alpha+16u]_2 \sqcup \pm[4\alpha+12u+4, 4\alpha+20u]_4 \sqcup \pm[2\beta+1, 2\beta+2u-1]_2 \sqcup\\
& & \pm[2\beta+4u+1, 2\beta+6u-1]_2 \sqcup \pm[2\alpha+2\beta+4u+3, 2\alpha+2\beta+8u-1]_4 \sqcup\\
& & \pm[2\alpha+2\beta+12u+3, 2\alpha+2\beta+16u-1]_4 \sqcup \pm[4\gamma+2, 4\gamma+8u-2]_4 \sqcup\\
& & \pm[2\alpha+4\gamma+8u+2, 2\alpha+4\gamma+10u]_2 \sqcup \pm[2\alpha+4\gamma+16u+2, 2\alpha+4\gamma+18u]_2.
\end{array}$$

Proof. For all $i \in [0, u-1]$, define

$$A_{2i} = \begin{array}{|c|c|c|}\hline
4\alpha+12u+4+4i & -(2\alpha+4u+2+2i) & -(2\alpha+8u+2+2i)\\ \hline
-(2\alpha+2+2i) & 2\alpha+2\beta+4u+3+4i & -(2\beta+4u+1+2i)\\ \hline
-(2\alpha+12u+2+2i) & -(2\beta+1+2i) & 2\alpha+2\beta+12u+3+4i\\ \hline
\end{array},$$

$$A_{2i+1} = \begin{array}{|c|c|c|}\hline
4\alpha+16u+4+4i & -(2\alpha+6u+2+2i) & -(2\alpha+10u+2+2i)\\ \hline
-(2\alpha+2u+2+2i) & 2\alpha+4\gamma+10u-2i & -(4\gamma+8u-2-4i)\\ \hline
-(2\alpha+14u+2+2i) & -(4\gamma+4u-2-4i) & 2\alpha+4\gamma+18u-2i\\ \hline
\end{array}.$$

The set $\overline{\mathfrak{A}} = \{+A_{2i}, -A_{2i}, +A_{2i+1}, -A_{2i+1} : i \in [0, u-1]\}$ has the required properties. $\square$

Lemma 4.2. *Suppose that $b \geqslant 7$ is an odd integer. There exists an* SMAS$(5, b; 2c)$ *for all $c \geqslant 4$ such that $c \equiv 0 \pmod 4$.*

Proof. Write $c = 4t+4$ where $t \geqslant 0$. Our SMAS$(5, b; 2c)$ will consist of $8t+8$ arrays of type (3.1) if $b \equiv 1 \pmod 4$, or of type (3.10) if $b \equiv 3 \pmod 4$, where each X_r is a zero-sum block of size $r \times 3$ and each Z_ℓ is a zero-sum block of size $2 \times \ell$.

Write $b = 4w + 7 + 2\delta$, where $w \geqslant 0$ and $\delta \in \{0, 1\}$. Hence, we will construct $8t + 8$ blocks X_3, $8(2w + 3 + \delta)(t + 1)$ blocks X_2, $8(w + 1 - \delta)(t + 1)$ blocks Z_4 and $8\delta(t + 1)$ blocks Z_6. Take

$$\begin{array}{rcl} \overline{\mathfrak{A}} & = & \overline{\mathfrak{A}}(4(t+1),\ 8(2w+3+\delta)(t+1),\ 10(t+1),\ 2(t+1)), \\ \mathfrak{B} & = & \mathfrak{B}(1,\ 4(16w+29+8\delta)(t+1),\ 8(2w+3+\delta)(t+1)) \end{array}$$

as in Lemmas 4.1 and 2.6, respectively. Then,

$$\mathcal{E}(\overline{\mathfrak{A}}) \sqcup \mathcal{E}(\mathfrak{B}) = \pm[1, 40(2w+\delta)(t+1) + 140t + 140] \setminus \pm(\mathcal{M}_1 \sqcup \ldots \sqcup \mathcal{M}_{10}),$$

where

$$\begin{array}{rcl} \mathcal{M}_1 & = & [2, 8t+8]_2, \\ \mathcal{M}_2 & = & [56t+58, 64t+64]_2, \\ \mathcal{M}_3 & = & [68t+70, 80t+80]_2, \\ \mathcal{M}_4 & = & [84t+86, 16(2w+\delta)(t+1)+88t+88]_2, \\ \mathcal{M}_5 & = & [16(2w+\delta)(t+1)+52t+53, 16(2w+\delta)(t+1)+56t+55]_2, \\ \mathcal{M}_6 & = & [16(2w+\delta)(t+1)+60t+61, 16(2w+\delta)(t+1)+64t+63]_2, \\ \mathcal{M}_7 & = & [16(2w+\delta)(t+1)+64t+65, 16(2w+\delta)(t+1)+72t+69]_4, \\ \mathcal{M}_8 & = & [16(2w+\delta)(t+1)+72t+73, 16(2w+\delta)(t+1)+80t+79]_2, \\ \mathcal{M}_9 & = & [16(2w+\delta)(t+1)+80t+81, 16(2w+\delta)(t+1)+88t+85]_4, \\ \mathcal{M}_{10} & = & [16(2w+\delta)(t+1)+88t+89, 24(2w+\delta)(t+1)+92t+92]. \end{array}$$

So, we replace the $8t + 8$ instances of X_3 with the arrays in $\overline{\mathfrak{A}}$ and the $8(2w + 3 + \delta)(t + 1)$ instances of X_2 with the arrays in $\mathfrak{B}$.

Next, write

$$\mathcal{M}_1 = \mathcal{M}_1^1 \sqcup \mathcal{M}_1^2, \quad \mathcal{M}_2 = \mathcal{M}_2^1 \sqcup \mathcal{M}_2^2 \quad \text{and} \quad \mathcal{M}_{10} = \mathcal{M}_{10}^1 \sqcup \mathcal{M}_{10}^2 \sqcup \mathcal{M}_{10}^3,$$

where

$$\begin{array}{rcl} \mathcal{M}_1^1 & = & [2, 8t+6]_4, \\ \mathcal{M}_1^2 & = & [4, 8t+8]_4, \\ \mathcal{M}_2^1 & = & [56t+58, 64t+62]_4, \\ \mathcal{M}_2^2 & = & [56t+60, 64t+64]_4, \\ \mathcal{M}_{10}^1 & = & [16(2w+\delta)(t+1)+88t+89, 16(2w+\delta)(t+1)+92t+91]_2, \\ \mathcal{M}_{10}^2 & = & [16(2w+\delta)(t+1)+88t+90, 16(2w+\delta)(t+1)+92t+92]_2, \\ \mathcal{M}_{10}^3 & = & [16(2w+\delta)(t+1)+92t+93, 24(2w+\delta)(t+1)+92t+92] \end{array}$$

The set $\mathcal{M}_1^1 \sqcup \mathcal{M}_1^2 \sqcup \mathcal{M}_2^1 \sqcup \mathcal{M}_2^2 \sqcup \mathcal{M}_7 \sqcup \mathcal{M}_9$ can be written as a disjoint union $F_1 \sqcup \ldots \sqcup F_{2\alpha}$, where $\alpha = 3t + 3$ and each F_i is a 2-set of type 4. The set $\mathcal{M}_3 \sqcup \mathcal{M}_4 \sqcup \mathcal{M}_5 \sqcup \mathcal{M}_6 \sqcup \mathcal{M}_8 \sqcup \mathcal{M}_{10}^1 \sqcup \mathcal{M}_{10}^2$ can be written as a disjoint union $G_1 \sqcup \ldots \sqcup G_{2\beta}$, where $\beta = (4w + 5 + 2\delta)(t + 1)$ and each G_j is a 2-set of type 2. Finally, the set $\mathcal{M}_{10}^3$ can be written as a disjoint union $H_1 \sqcup \ldots \sqcup H_{2\gamma}$, where $\gamma = (4w + 2\delta)(t + 1)$ and each H_h is a 2-set of type 1.

Call

$$\begin{array}{rcl} \mathfrak{Q}_1 & = & \left\{Q_2(G_{2j-1}, G_{2j}) : j \in [1, 4w(t+1)]\right\}, \\ \mathfrak{Q}_2 & = & \left\{Q_1(H_{2h-1}, H_{2h}) : h \in [1, 4w(t+1)]\right\}. \end{array}$$

If $\delta = 0$, define $\mathfrak{R} = \varnothing$ and $\mathfrak{Q} = \mathfrak{Q}_1 \sqcup \mathfrak{Q}_2 \sqcup \mathfrak{Q}_3 \sqcup \mathfrak{Q}_4$, where

$$\begin{array}{rcl} \mathfrak{Q}_3 & = & \{Q_3(F_{2i-1}, F_{2i}) : i \in [1, 3t+3]\}, \\ \mathfrak{Q}_4 & = & \left\{Q_2(G_{2j-1}, G_{2j}) : j \in [4w(t+1)+1, (4w+5)(t+1)]\right\}. \end{array}$$

If $\delta = 1$, define $\mathfrak{Q} = \mathfrak{Q}_1 \sqcup \mathfrak{Q}_2$ and $\mathfrak{R} = \mathfrak{R}_1 \sqcup \mathfrak{R}_2$, where

$$\begin{array}{rcl} \mathfrak{R}_1 & = & \left\{R_2(F_i, G_{8w(t+1)+2i-1}, G_{8w(t+1)+2i}) : i \in [1, 6t+6]\right\}, \\ \mathfrak{R}_2 & = & \left\{R_1(G_{(8w+12)(t+1)+j}, H_{8w(t+1)+2j-1}, H_{8w(t+1)+2j}) : j \in [1, 2t+2]\right\}. \end{array}$$

Then $\mathfrak{Q}$ consists of $8(w + 1 - \delta)(t + 1)$ zero-sum blocks of size 2×4, $\mathfrak{R}$ consists of $8\delta(t + 1)$ zero-sum blocks of size 2×6 and $\mathcal{E}(\mathfrak{Q}) \sqcup \mathcal{E}(\mathfrak{R}) = \pm(\mathcal{M}_1 \sqcup \ldots \sqcup \mathcal{M}_{10})$. Hence, we replace the instances of Z_4 with the arrays in $\mathfrak{Q}$ and the instances of Z_6 with the arrays in $\mathfrak{R}$. $\square$

Lemma 4.3. *There exists an* SMAS$(5, 7; 2c)$ *for all* $c \geqslant 1$ *such that* $c \equiv 1 \pmod 4$.

Proof. An SMAS$(5,7;2)$ is shown in Fig. 1. So, write $c=4t+5$ where $t\geqslant 0$. Our SMAS$(5,7;2c)$ will consist of $8t+10$ arrays of type

$$\begin{array}{|c|c|c|}\hline X_3 & X_2^{\mathrm{t}} & X_2^{\mathrm{t}} \\ \hline X_2 & \multicolumn{2}{c|}{Z_4} \\ \hline \end{array}. \tag{4.1}$$

Hence, we will construct $8t+10$ blocks X_3, $6(4t+5)$ blocks X_2 and $2(4t+5)$ blocks Z_4.

Take

$$\begin{array}{rcl} \overline{\mathfrak{A}} &=& \overline{\mathfrak{A}}(4t+5,\ 6(4t+5),\ 10t+12,\ 2t+2),\\ \mathfrak{B} &=& \mathfrak{B}(1,\ 29(4t+5),\ 6(4t+5)) \end{array}$$

as in Lemmas 4.1 and 2.6, respectively. Also, define

$$A'=\begin{array}{|c|c|c|}\hline 10 & -4 & -6 \\ \hline -2 & 92t+111 & -(92t+109) \\ \hline -8 & -(92t+107) & 92t+115 \\ \hline \end{array}$$

and set $\mathfrak{A}'=\overline{\mathfrak{A}}\sqcup\{+A',-A'\}$. Then,

$$\mathcal{E}(\mathfrak{A}')\sqcup\mathcal{E}(\mathfrak{B})=\pm[1,140t+175]\setminus\pm(\mathcal{M}_1\sqcup\ldots\sqcup\mathcal{M}_{17}),$$

where

$$\begin{array}{rclrcl} \mathcal{M}_1 &=& [12,8t+10]_2, & \mathcal{M}_2 &=& [40t+44,40t+46]_2,\\ \mathcal{M}_3 &=& [52t+65,56t+63]_2, & \mathcal{M}_4 &=& [56t+64,56t+67],\\ \mathcal{M}_5 &=& [56t+68,60t+70]_2, & \mathcal{M}_6 &=& [60t+72,64t+75],\\ \mathcal{M}_7 &=& [64t+77,64t+79]_2, & \mathcal{M}_8 &=& [64t+83,72t+87]_4,\\ \mathcal{M}_9 &=& [68t+80,72t+82]_2, & \mathcal{M}_{10} &=& [72t+84,72t+86]_2,\\ \mathcal{M}_{11} &=& [72t+88,80t+91], & \mathcal{M}_{12} &=& [80t+93,80t+95]_2,\\ \mathcal{M}_{13} &=& [80t+99,88t+103]_4, & \mathcal{M}_{14} &=& [84t+96,88t+102]_2,\\ \mathcal{M}_{15} &=& [88t+104,92t+105], & \mathcal{M}_{16} &=& [92t+106,92t+112]_2,\\ \mathcal{M}_{17} &=& [92t+113,92t+114]. & & & \end{array}$$

So, we replace the $8t+10$ instances of X_3 with the arrays in $\mathfrak{A}'$ and the $6(4t+5)$ instances of X_2 with the arrays in $\mathfrak{B}$.

The set $\mathcal{M}_8\sqcup\mathcal{M}_{13}$ can be written as a disjoint union $F_1\sqcup\ldots\sqcup F_{2\alpha}$, where $\alpha=t+1$ and each F_i is a 2-set of type 4. The set $\mathcal{M}_1\sqcup\mathcal{M}_2\sqcup\mathcal{M}_3\sqcup\mathcal{M}_5\sqcup\mathcal{M}_7\sqcup\mathcal{M}_9\sqcup\mathcal{M}_{10}\sqcup\mathcal{M}_{12}\sqcup\mathcal{M}_{14}\sqcup\mathcal{M}_{16}$ can be written as a disjoint union $G_1\sqcup\ldots\sqcup G_{2\beta}$, where $\beta=3t+5$ and each G_j is a 2-set of type 2. Finally, the set $\mathcal{M}_4\sqcup\mathcal{M}_6\sqcup\mathcal{M}_{11}\sqcup\mathcal{M}_{15}\sqcup\mathcal{M}_{17}$ can be written as a disjoint union $H_1\sqcup\ldots\sqcup H_{2\gamma}$, where $\gamma=4t+4$ and each H_h is a 2-set of type 1. Call

$$\begin{array}{rcl} \mathfrak{R}_1 &=& \{Q_3(F_{2i-1},F_{2i}) : i\in[1,t+1]\},\\ \mathfrak{R}_2 &=& \left\{Q_2(G_{2j-1},G_{2j}) : j\in[1,3t+5]\right\},\\ \mathfrak{R}_3 &=& \left\{Q_1(H_{2h-1},H_{2h}) : h\in[1,4t+4]\right\}. \end{array}$$

Then $\mathfrak{R}=\mathfrak{R}_1\sqcup\mathfrak{R}_2\sqcup\mathfrak{R}_3$ consists of $8t+10$ zero-sum blocks of size 2×4 and $\mathcal{E}(\mathfrak{R})=\pm(\mathcal{M}_1\sqcup\ldots\sqcup\mathcal{M}_{17})$. Hence, we replace the instances of Z_4 with the arrays in $\mathfrak{R}$. $\square$

Lemma 4.4. *There exists an* SMAS$(5,7;2c)$ *for all* $c\geqslant 3$ *such that* $c\equiv 3 \pmod 4$.

Proof. An SMAS$(5,7;6)$ is shown in Fig. 2. So, write $c=4t+7$ where $t\geqslant 0$. Our SMAS$(5,7;2c)$ will consist of $8t+12$ arrays of type (4.1) and two arrays of type (3.4). Hence, we will construct $8t+12$ blocks X_3, $24t+38$ blocks X_2, $8t+12$ blocks Z_4, 2 blocks Y_5 and 2 blocks W_2.

Take

$$\begin{array}{rcl} \overline{\mathfrak{A}} &=& \overline{\mathfrak{A}}(4t+9,\ 24t+42,\ 10t+17,\ 2t+3),\\ \mathfrak{B} &=& \mathfrak{B}(9,\ 116t+199,\ 24t+38),\\ \mathfrak{C} &=& \mathfrak{C}(92t+148,\ 116t+200,\ 92t+158) \end{array}$$

as in Lemmas 4.1, 2.6 and 2.7, respectively. We have

$$\mathcal{E}(\overline{\mathfrak{A}})\sqcup\mathcal{E}(\mathfrak{B})\sqcup\mathcal{E}(\mathfrak{C})=\pm[1,140t+245]\setminus\pm(\mathcal{M}_1\sqcup\ldots\sqcup\mathcal{M}_{18}),$$

where

$$
\begin{array}{|c|c|c|c|c|c|c|}\hline 10 & -8 & -2 & 5 & -5 & 17 & -17\\\hline -6 & -59 & 65 & -90 & 90 & -96 & 96\\\hline -4 & 68 & -63 & 84 & -85 & 79 & -79\\\hline 1 & -89 & 87 & -45 & 47 & 49 & -50\\\hline -1 & 88 & -87 & 46 & -47 & -49 & 50\\\hline \end{array}, \quad
\begin{array}{|c|c|c|c|c|c|c|}\hline -10 & 8 & 2 & 7 & -7 & 19 & -19\\\hline 6 & 59 & -65 & -91 & 91 & -97 & 97\\\hline 4 & -68 & 63 & 85 & -84 & 78 & -78\\\hline -3 & 89 & -86 & 45 & -44 & 51 & -52\\\hline 3 & -88 & 86 & -46 & 44 & -51 & 52\\\hline \end{array},
$$

$$
\begin{array}{|c|c|c|c|c|c|c|}\hline 40 & -18 & -22 & 21 & -21 & 23 & -23\\\hline -14 & 48 & -34 & -98 & 98 & -99 & 99\\\hline -26 & -30 & 56 & 77 & -77 & 76 & -76\\\hline 9 & -92 & 83 & 60 & -62 & -64 & 66\\\hline -9 & 92 & -83 & -60 & 62 & 64 & -66\\\hline \end{array}, \quad
\begin{array}{|c|c|c|c|c|c|c|}\hline -40 & 18 & 22 & 25 & -25 & 27 & -27\\\hline 14 & -48 & 34 & -100 & 100 & -101 & 101\\\hline 26 & 30 & -56 & 75 & -75 & 74 & -74\\\hline 11 & -93 & 82 & 55 & -57 & -67 & 69\\\hline -11 & 93 & -82 & -55 & 57 & 67 & -69\\\hline \end{array},
$$

$$
\begin{array}{|c|c|c|c|c|c|c|}\hline 36 & -16 & -20 & 29 & -29 & 31 & -31\\\hline -12 & 53 & -41 & -102 & 102 & -103 & 103\\\hline -24 & -37 & 61 & 73 & -73 & 72 & -72\\\hline 13 & -94 & 81 & 39 & -43 & -54 & 58\\\hline -13 & 94 & -81 & -39 & 43 & 54 & -58\\\hline \end{array}, \quad
\begin{array}{|c|c|c|c|c|c|c|}\hline -36 & 16 & 20 & 33 & -33 & 35 & -35\\\hline 12 & -53 & 41 & -104 & 104 & -105 & 105\\\hline 24 & 37 & -61 & 71 & -71 & 70 & -70\\\hline 15 & -95 & 80 & 28 & -32 & -38 & 42\\\hline -15 & 95 & -80 & -28 & 32 & 38 & -42\\\hline \end{array}.
$$

Fig. 2. An SMAS(5, 7; 6).

$$
\begin{array}{rclrcl}
\mathcal{M}_1 & = & [12, 8t+18]_2, & \mathcal{M}_2 & = & [40t+68, 40t+72]_4,\\
\mathcal{M}_3 & = & [52t+91, 56t+93]_2, & \mathcal{M}_4 & = & [56t+94, 56t+95],\\
\mathcal{M}_5 & = & [56t+98, 60t+100]_2, & \mathcal{M}_6 & = & [60t+102, 64t+111],\\
\mathcal{M}_7 & = & [64t+113, 64t+115]_2, & \mathcal{M}_8 & = & [64t+119, 72t+123]_4,\\
\mathcal{M}_9 & = & [68t+118, 72t+124]_2, & \mathcal{M}_{10} & = & [72t+126, 80t+131],\\
\mathcal{M}_{11} & = & [80t+132, 80t+135], & \mathcal{M}_{12} & = & [80t+143, 88t+147]_4,\\
\mathcal{M}_{13} & = & [80t+137, 80t+139]_2, & \mathcal{M}_{14} & = & [84t+142, 92t+144]_2,\\
\mathcal{M}_{15} & = & [88t+151, 92t+149]_2, & \mathcal{M}_{16} & = & [92t+146, 92t+150]_4,\\
\mathcal{M}_{17} & = & [92t+151, 92t+153]_2, & \mathcal{M}_{18} & = & [92t+155, 92t+156].
\end{array}
$$

So, we replace the $8t+12$ instances of X_3 with the arrays in $\overline{\mathfrak{A}}$, the 2 instances of Y_5 with the arrays in $\mathfrak{C}$ and the $24t+38$ instances of X_2 with the arrays in $\mathfrak{B}$.

Call

$$\mathfrak{R} = \{U_2(\mathcal{M}_4), U_2(\mathcal{M}_{18})\}.$$

Then, $\mathcal{E}(\mathfrak{R}) = \pm(\mathcal{M}_4 \sqcup \mathcal{M}_{18})$: we replace the 2 instances of W_2 with the arrays in $\mathfrak{R}$.

The set $\mathcal{M}_2 \sqcup \mathcal{M}_8 \sqcup \mathcal{M}_{12} \sqcup \mathcal{M}_{16}$ can be written as a disjoint union $F_1 \sqcup \ldots \sqcup F_{2\alpha}$, where $\alpha = t+2$ and each F_i is a 2-set of type 4. The set $\mathcal{M}_1 \sqcup \mathcal{M}_3 \sqcup \mathcal{M}_5 \sqcup \mathcal{M}_7 \sqcup \mathcal{M}_9 \sqcup \mathcal{M}_{13} \sqcup \mathcal{M}_{14} \sqcup \mathcal{M}_{15} \sqcup \mathcal{M}_{17}$ can be written as a disjoint union $G_1 \sqcup \ldots \sqcup G_{2\beta}$, where $\beta = 4t+5$ and each G_j is a 2-set of type 2. Finally, the set $\mathcal{M}_6 \sqcup \mathcal{M}_{10} \sqcup \mathcal{M}_{11}$ can be written as a disjoint union $H_1 \sqcup \ldots \sqcup H_{2\gamma}$, where $\gamma = 3t+5$ and each H_h is a 2-set of type 1. Call

$$
\begin{array}{rcl}
\mathfrak{S}_1 & = & \{Q_3(F_{2i-1}, F_{2i}) : i \in [1, t+2]\},\\
\mathfrak{S}_2 & = & \left\{Q_2(G_{2j-1}, G_{2j}) : j \in [1, 4t+5]\right\},\\
\mathfrak{S}_3 & = & \left\{Q_1(H_{2h-1}, H_{2h}) : h \in [1, 3t+5]\right\}.
\end{array}
$$

Then $\mathfrak{S} = \mathfrak{S}_1 \sqcup \mathfrak{S}_2 \sqcup \mathfrak{S}_3$ consists of $8t+12$ zero-sum blocks of size 2×4 and $\mathcal{E}(\mathfrak{S}) = \pm(\mathcal{M}_1 \sqcup \mathcal{M}_2 \sqcup \mathcal{M}_3 \sqcup \mathcal{M}_5 \sqcup \ldots \sqcup \mathcal{M}_{17})$. Hence, we replace the instances of Z_4 with the arrays in $\mathfrak{S}$. $\square$

Lemma 4.5. *Given three positive integers α, β, γ such that $\beta > \alpha + 15 > \gamma + 18 > 40$, there exists a set $\overline{\mathfrak{C}} = \overline{\mathfrak{C}}(\alpha, \beta, \gamma)$ consisting of two square blocks of size 5 such that*

$$\sigma_r(C) = (0, 0, 0, 1, -1) \quad \text{and} \quad \sigma_c(C) = (0, 0, 0, 0, 0) \text{ for all } C \in \overline{\mathfrak{C}}$$

and $\mathcal{E}(\overline{\mathfrak{C}})$ consists of the following disjoint subsets:

$$\mathcal{E}(\overline{\mathfrak{C}}) = \pm[8, 15] \sqcup \pm\{22\} \sqcup \pm[\gamma, \gamma+3] \sqcup \pm\{\alpha, \alpha+4, \alpha+12, \alpha+15\} \sqcup \pm[\beta, \beta+3] \sqcup \pm[\beta+12, \beta+15].$$

Proof. Let C_1 and C_2 be the following arrays:

$$
\begin{array}{|c|c|c|c|c|}\hline 22 & -14 & -8 & 13 & -13\\\hline -12 & -\alpha & \alpha+12 & -(\beta+14) & \beta+14\\\hline -10 & \alpha+15 & -(\alpha+4) & \beta & -(\beta+1)\\\hline 9 & -(\beta+13) & \beta+3 & -(\gamma+1) & \gamma+3\\\hline -9 & \beta+12 & -(\beta+3) & \gamma+2 & -(\gamma+3)\\\hline \end{array}, \quad
\begin{array}{|c|c|c|c|c|}\hline -22 & 14 & 8 & 15 & -15\\\hline 12 & \alpha & -(\alpha+12) & -(\beta+15) & \beta+15\\\hline 10 & -(\alpha+15) & \alpha+4 & \beta+1 & -\beta\\\hline -11 & \beta+13 & -(\beta+2) & \gamma+1 & -\gamma\\\hline 11 & -(\beta+12) & \beta+2 & -(\gamma+2) & \gamma\\\hline \end{array}.
$$

The set $\overline{\mathfrak{C}} = \{C_1, C_2\}$ has the required properties. $\square$

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 53 & 3 & -56 & 17 & -17 & 19 & -19 \\ \hline -57 & 59 & -2 & -65 & 65 & -66 & 66 \\ \hline 55 & -60 & 5 & 48 & -48 & 47 & -47 \\ \hline -58 & 4 & 54 & 16 & -18 & -26 & 28 \\ \hline 7 & -6 & -1 & -16 & 18 & 26 & -28 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|c|c|c|}\hline -53 & -3 & 56 & 21 & -21 & 23 & -23 \\ \hline 57 & -59 & 2 & -67 & 67 & -68 & 68 \\ \hline -55 & 60 & -5 & 46 & -46 & 45 & -45 \\ \hline 58 & -4 & -54 & 31 & -33 & -34 & 36 \\ \hline -7 & 6 & 1 & -31 & 33 & 34 & -36 \\ \hline \end{array},$$

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 22 & -14 & -8 & 13 & -13 & 25 & -25 \\ \hline -12 & -20 & 32 & -63 & 63 & -69 & 69 \\ \hline -10 & 35 & -24 & 49 & -50 & 44 & -44 \\ \hline 9 & -62 & 52 & -38 & 40 & 29 & -30 \\ \hline -9 & 61 & -52 & 39 & -40 & -29 & 30 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|c|c|c|}\hline -22 & 14 & 8 & 15 & -15 & 27 & -27 \\ \hline 12 & 20 & -32 & -64 & 64 & -70 & 70 \\ \hline 10 & -35 & 24 & 50 & -49 & 43 & -43 \\ \hline -11 & 62 & -51 & 38 & -37 & 41 & -42 \\ \hline 11 & -61 & 51 & -39 & 37 & -41 & 42 \\ \hline \end{array}.$$

Fig. 3. An SMAS(5, 7; 4).

Lemma 4.6. *There exists an* SMAS$(5, 7; 2c)$ *for all* $c \geqslant 2$ *such that* $c \equiv 2 \pmod 4$.

Proof. An SMAS(5, 7; 4) is shown in Fig. 3. So, write $c = 4t + 6$ where $t \geqslant 0$. Our SMAS$(5, 7; 2c)$ will consist of $8t + 8$ arrays of type (4.1), two arrays of type (3.4) and two of type

$$\begin{array}{|c|c|c|}\hline \multirow{2}{*}{X_5} & X_2^t & X_2^t \\ \cline{2-3} & \multicolumn{2}{c|}{Z_4} \\ \hline \end{array},$$

where each X_r is a zero-sum block of size $r \times 3$ and each Z_4 is a zero-sum block of size 2×4. Hence, we will construct $8t + 8$ blocks X_3, 2 blocks X_5, $24t + 30$ blocks X_2, $8t + 10$ blocks Z_4, 2 blocks Y_5 and 2 blocks W_2.

Take

$$\begin{array}{rcl} \overline{\mathfrak{A}} & = & \overline{\mathfrak{A}}(4t + 12,\ 24t + 38,\ 10t + 14,\ 2t + 2), \\ \mathfrak{B} & = & \mathfrak{B}(17,\ 116t + 164,\ 24t + 30), \\ \overline{\mathfrak{C}} & = & \overline{\mathfrak{C}}(92t + 118,\ 116t + 165,\ 60t + 89) \end{array}$$

as in Lemmas 4.1, 2.6 and 4.5, respectively. Furthermore, take $\mathfrak{D} = \{+D, -D\}$, where

$$D = \begin{array}{|c|c|c|}\hline 116t + 169 & 3 & -(116t + 172) \\ \hline -(116t + 173) & 116t + 175 & -2 \\ \hline 116t + 171 & -(116t + 176) & 5 \\ \hline -(116t + 174) & 4 & 116t + 170 \\ \hline 7 & -6 & -1 \\ \hline \end{array}.$$

Then,

$$\mathcal{E}(\overline{\mathfrak{A}}) \sqcup \mathcal{E}(\mathfrak{B}) \sqcup \mathcal{E}(\overline{\mathfrak{C}}) \sqcup \mathcal{E}(\mathfrak{D}) = \pm[1, 140t + 210] \setminus \pm(\mathcal{M}_1 \sqcup \ldots \sqcup \mathcal{M}_{18}),$$

where

$$\begin{array}{rclrcl} \mathcal{M}_1 & = & [16, 18]_2, & \mathcal{M}_2 & = & [20, 24]_4, \\ \mathcal{M}_3 & = & [26, 8t + 24]_2, & \mathcal{M}_4 & = & [40t + 60, 40t + 72]_4, \\ \mathcal{M}_5 & = & [56t + 74, 56t + 86]_4, & \mathcal{M}_6 & = & [52t + 81, 56t + 83]_2, \\ \mathcal{M}_7 & = & [56t + 90, 60t + 88]_2, & \mathcal{M}_8 & = & [60t + 93, 64t + 96], \\ \mathcal{M}_9 & = & [64t + 97, 64t + 107]_2, & \mathcal{M}_{10} & = & [64t + 109, 72t + 113]_4, \\ \mathcal{M}_{11} & = & [68t + 102, 80t + 112]_2, & \mathcal{M}_{12} & = & [72t + 117, 80t + 123]_2, \\ \mathcal{M}_{13} & = & [80t + 125, 88t + 129]_4, & \mathcal{M}_{14} & = & [84t + 118, 92t + 116]_2, \\ \mathcal{M}_{15} & = & [88t + 133, 92t + 131]_2, & \mathcal{M}_{16} & = & [92t + 120, 92t + 124]_4, \\ \mathcal{M}_{17} & = & [92t + 126, 92t + 128]_2, & \mathcal{M}_{18} & = & [92t + 132, 92t + 134]_2. \end{array}$$

So, we replace the $8t + 8$ instances of X_3 with the arrays in $\overline{\mathfrak{A}}$, the 2 instances of Y_5 with the arrays in $\overline{\mathfrak{C}}$, the 2 instances of X_5 with the arrays in $\mathfrak{D}$ and the $24t + 30$ instances of X_2 with the arrays in $\mathfrak{B}$.

Write $\mathcal{M}_8 = \mathcal{M}_8^1 \sqcup \mathcal{M}_8^2 \sqcup \mathcal{M}_8^3$, where

$$\mathcal{M}_8^1 = [60t + 93, 60t + 94], \quad \mathcal{M}_8^2 = [60t + 95, 60t + 96] \quad \text{and} \quad \mathcal{M}_8^3 = [60t + 97, 64t + 96].$$

Define

$$\mathfrak{R} = \{U_2(\mathcal{M}_8^1), U_2(\mathcal{M}_8^2)\}.$$

Then, $\mathcal{E}(\mathfrak{R}) = \pm(\mathcal{M}_8^1 \sqcup \mathcal{M}_8^2)$: we replace the 2 instances of W_2 with the arrays in $\mathfrak{R}$.

The set $\mathcal{M}_2 \sqcup \mathcal{M}_4 \sqcup \mathcal{M}_5 \sqcup \mathcal{M}_{10} \sqcup \mathcal{M}_{13} \sqcup \mathcal{M}_{16}$ can be written as a disjoint union $F_1 \sqcup \ldots \sqcup F_{2\alpha}$, where $\alpha = t + 4$ and each F_i is a 2-set of type 4. The set $\mathcal{M}_1 \sqcup \mathcal{M}_3 \sqcup \mathcal{M}_6 \sqcup \mathcal{M}_7 \sqcup \mathcal{M}_9 \sqcup \mathcal{M}_{11} \sqcup \mathcal{M}_{12} \sqcup \mathcal{M}_{14} \sqcup \mathcal{M}_{15} \sqcup \mathcal{M}_{17} \sqcup \mathcal{M}_{18}$ can be written as a disjoint union $G_1 \sqcup \ldots \sqcup G_{2\beta}$, where $\beta = 6t+6$ and each G_j is a 2-set of type 2. Finally, the set $\mathcal{M}_8^3$ can be written as a disjoint union $H_1 \sqcup \ldots \sqcup H_{2t}$, where each H_h is a 2-set of type 1. Call

$$\begin{array}{rcl} \mathfrak{S}_1 & = & \{Q_3(F_{2i-1}, F_{2i}) : i \in [1, t+4]\}, \\ \mathfrak{S}_2 & = & \left\{Q_2(G_{2j-1}, G_{2j}) : j \in [1, 6t+6]\right\}, \\ \mathfrak{S}_3 & = & \left\{Q_1(H_{2h-1}, H_{2h}) : h \in [1, t]\right\}. \end{array}$$

Then $\mathfrak{S} = \mathfrak{S}_1 \sqcup \mathfrak{S}_2 \sqcup \mathfrak{S}_3$ consists of $8t+10$ zero-sum blocks of size 2×4 and $\mathcal{E}(\mathfrak{S}) = \pm(\mathcal{M}_1 \sqcup \ldots \sqcup \mathcal{M}_7 \sqcup \mathcal{M}_8^3 \sqcup \mathcal{M}_9 \sqcup \ldots \sqcup \mathcal{M}_{18})$. Hence, we replace the instances of Z_4 with the arrays in $\mathfrak{S}$. □

5. The case $a = 6$

We now consider the construction of an SMAS$(6, b; c)$ when $b \geqslant 5$ is an odd integer.

Lemma 5.1. *If there exists an* SMAS$(6, b; c)$ *for* $b, c \geq 1$, *then there exists an* SMAS$(6, b+4; c)$.

Proof. For every $j \geqslant 0$, let

$$S_j = \begin{array}{|c|}\hline Q_1([j+1, j+2], [j+3, j+4]) \\ \hline Q_1([j+5, j+6], [j+7, j+8]) \\ \hline Q_1([j+9, j+10], [j+11, j+12]) \\ \hline \end{array}.$$

Then, S_j is a 6×4 zero-sum block such that $\mathcal{E}(S_j) = \pm[j+1, j+12]$. Let $\{R_0, \ldots, R_{c-1}\}$ be an SMAS$(6, b; c)$. For every $i \in [0, c-1]$, define the $6 \times (b+4)$ array

$$T_i = \begin{array}{|c|c|}\hline R_i & S_{3bc+12i} \\ \hline \end{array}.$$

Then, the set $\{T_0, \ldots, T_{c-1}\}$ is an SMAS$(6, b+4; c)$. □

Lemma 5.2. *There exist an* SMAS$(6, 5; c)$ *and an* SMAS$(6, 7; c)$ *for all* $c \geqslant 4$ *such that* $c \equiv 0 \pmod 4$.

Proof. Write $c = 4t+4$, where $t \geqslant 0$. Our SMAS$(6, 5; c)$ will consist of $4t+4$ blocks of type

$$\begin{array}{|c|c|}\hline X_2 & \\ \cline{1-1} X_2 & Z_6^{t} \\ \cline{1-1} X_2 & \\ \hline \end{array}, \tag{5.1}$$

whereas our SMAS$(6, 7; c)$ will consist of $4t+4$ blocks of type

$$\begin{array}{|c|c|}\hline X_2 & Z_4 \\ \hline X_2 & Z_4 \\ \hline X_2 & Z_4 \\ \hline \end{array}, \tag{5.2}$$

where each X_2 is a zero-sum block of size 2×3 and each Z_ℓ is a zero-sum block of size $2 \times \ell$. Write $b = 5 + \delta$, where $\delta \in \{0, 2\}$, and take

$$\mathfrak{B} = \mathfrak{B}(1,\ 36t+36,\ 12t+12)$$

as in Lemma 2.6. Then, $\mathcal{E}(\mathfrak{B}) = \pm[1, 12(t+1)(5+\delta)] \setminus \pm(\mathcal{M}_1 \sqcup \mathcal{M}_2)$, where

$$\mathcal{M}_1 = [2, 24t+24]_2 \quad \text{and} \quad \mathcal{M}_2 = [48t+49, 60t+60+12(t+1)\delta].$$

So, we replace the $12t+12$ instances of X_2 with the arrays in $\mathfrak{B}$.

Write

$$\mathcal{M}_1 = \mathcal{M}_1^1 \sqcup \mathcal{M}_1^2 \sqcup \mathcal{M}_1^3 \quad \text{and} \quad \mathcal{M}_2 = \mathcal{M}_2^1 \sqcup \mathcal{M}_2^2 \sqcup \mathcal{M}_2^3,$$

where

$$\begin{array}{rclrcl} \mathcal{M}_1^1 & = & [2, 8t+6]_4, & \mathcal{M}_2^1 & = & [48t+49, 52t+51]_2, \\ \mathcal{M}_1^2 & = & [4, 8t+8]_4, & \mathcal{M}_2^2 & = & [48t+50, 52t+52]_2, \\ \mathcal{M}_1^3 & = & [8t+10, 24t+24]_2, & \mathcal{M}_2^3 & = & [52t+53, 60t+60+12(t+1)\delta]. \end{array}$$

The set $\mathcal{M}_1^1 \sqcup \mathcal{M}_1^2$ can be written as a disjoint union $F_1 \sqcup \ldots \sqcup F_\alpha$, where $\alpha = 2t+2$ and each F_i is a 2-set of type 4. The set $\mathcal{M}_1^3 \sqcup \mathcal{M}_2^1 \sqcup \mathcal{M}_2^2$ can be written as a disjoint union $G_1 \sqcup \ldots \sqcup G_\beta$, where $\beta = 6t+6$ and each G_j is a 2-set of type 2. Finally, the set $\mathcal{M}_2^3$ can be written as a disjoint union $H_1 \sqcup \ldots \sqcup H_\gamma$, where $\gamma = 2(2+3\delta)(t+1)$ and each H_h is a 2-set of type 1.

Suppose $\delta = 0$ and call

$$\begin{array}{rcl} \mathfrak{R}_1 &=& \{R_2(F_i, G_{2i-1}, G_{2i}) : i \in [1, 2t+2]\}, \\ \mathfrak{R}_2 &=& \{R_1(G_{4t+4+j}, H_{2j-1}, H_{2j}) : j \in [1, 2t+2]\}. \end{array}$$

Then $\mathfrak{R} = \mathfrak{R}_1 \sqcup \mathfrak{R}_2$ consists of $4t+4$ zero-sum blocks of size 2×6 and $\mathcal{E}(\mathfrak{R}) = \pm(\mathcal{M}_1 \sqcup \mathcal{M}_2)$. Hence, we replace the $4t+4$ instances of Z_6 with the arrays in $\mathfrak{R}$.

Suppose $\delta = 2$ and call

$$\begin{array}{rcl} \mathfrak{R}_1 &=& \{Q_3(F_{2i-1}, F_{2i}) : i \in [1, t+1]\}, \\ \mathfrak{R}_2 &=& \{Q_2(G_{2j-1}, G_{2j}) : j \in [1, 3t+3]\}, \\ \mathfrak{R}_3 &=& \{Q_1(H_{2h-1}, H_{2h}) : h \in [1, 8t+8]\}. \end{array}$$

Then, $\mathfrak{R} = \mathfrak{R}_1 \sqcup \mathfrak{R}_2 \sqcup \mathfrak{R}_3$ consists of $12t+12$ zero-sum blocks of size 2×4 and $\mathcal{E}(\mathfrak{R}) = \pm(\mathcal{M}_1 \sqcup \mathcal{M}_2)$. Hence, we replace the $3(4t+4)$ instances of Z_4 with the arrays in $\mathfrak{R}$. $\square$

Lemma 5.3. *There exists an* SMAS$(6,5;c)$ *for all* $c \geqslant 1$ *such that* $c \equiv 1 \pmod 4$.

Proof. Write $c = 4t+1$, where $t \geqslant 0$. Our SMAS$(6,5;c)$ will consist of $4t+1$ blocks of type (5.1). Take

$$\mathfrak{B} = \mathfrak{B}(1,\ 36t+8,\ 12t+3)$$

as in Lemma 2.6. Then $\mathcal{E}(\mathfrak{B}) = \pm[1, 15(4t+1)] \setminus \pm(\mathcal{M}_1 \sqcup \mathcal{M}_2)$, where

$$\mathcal{M}_1 = [2, 24t+4]_2 \quad \text{and} \quad \mathcal{M}_2 = [48t+12, 60t+15].$$

Write

$$\mathcal{M}_1 = \mathcal{M}_1^1 \sqcup \mathcal{M}_1^2 \sqcup \mathcal{M}_1^3 \quad \text{and} \quad \mathcal{M}_2 = \mathcal{M}_2^1 \sqcup \mathcal{M}_2^2 \sqcup \mathcal{M}_2^3,$$

where

$$\begin{array}{rclrcl} \mathcal{M}_1^1 &=& [2, 16t+4]_2, & \mathcal{M}_2^1 &=& [48t+12, 52t+10]_2, \\ \mathcal{M}_1^2 &=& [16t+6, 24t+2]_4, & \mathcal{M}_2^2 &=& [48t+13, 52t+11]_2, \\ \mathcal{M}_1^3 &=& [16t+8, 24t+4]_4, & \mathcal{M}_2^3 &=& [52t+12, 60t+15]. \end{array}$$

So, we replace the $12t+3$ instances of X_2 with the arrays in $\mathfrak{B}$.

The set $\mathcal{M}_1^2 \sqcup \mathcal{M}_1^3$ can be written as a disjoint union $F_1 \sqcup \ldots \sqcup F_\alpha$, where $\alpha = 2t$ and each F_i is a 2-set of type 4. The set $\mathcal{M}_1^1 \sqcup \mathcal{M}_2^1 \sqcup \mathcal{M}_2^2$ can be written as a disjoint union $G_1 \sqcup \ldots \sqcup G_\beta$, where $\beta = 6t+1$ and each G_j is a 2-set of type 2. Finally, the set $\mathcal{M}_2^3$ can be written as a disjoint union $H_1 \sqcup \ldots \sqcup H_\gamma$, where $\gamma = 4t+2$ and each H_h is a 2-set of type 1. Call

$$\begin{array}{rcl} \mathfrak{R}_1 &=& \{R_2(F_i, G_{2i-1}, G_{2i}) : i \in [1, 2t]\}, \\ \mathfrak{R}_2 &=& \{R_1(G_{4t+j}, H_{2j-1}, H_{2j}) : j \in [1, 2t+1]\}. \end{array}$$

Then $\mathfrak{R} = \mathfrak{R}_1 \sqcup \mathfrak{R}_2$ consists of $4t+1$ zero-sum blocks of size 2×6 and $\mathcal{E}(\mathfrak{R}) = \pm(\mathcal{M}_1 \sqcup \mathcal{M}_2)$. Hence, we replace the $4t+1$ instances of Z_6 with the arrays in $\mathfrak{R}$. $\square$

Lemma 5.4. *There exists an* SMAS$(6,7;c)$ *for all* $c \geqslant 1$ *such that* $c \equiv 1 \pmod 4$.

Proof. The existence of an SMAS$(6,7;1)$ follows from [12, Lemma 7]. So, write $c = 4t+5$, where $t \geqslant 0$. Our SMAS$(6,7;c)$ will consist of $4t+4$ blocks of type (5.2) and one block

$$\begin{array}{|c|c|} \hline \multirow{3}{*}{X_6} & Z_4 \\ \cline{2-2} & Z_4 \\ \cline{2-2} & Z_4 \\ \hline \end{array}, \tag{5.3}$$

where X_6 is a zero-sum block of size 6×3 and each Z_4 is a zero-sum block of size 2×4. Take the zero-sum block

$$A = \begin{array}{|c|c|c|c|c|c|} \hline 1 & -1 & 2 & -2 & 4 & -4 \\ \hline 8 & 7 & -8 & 5 & -7 & -5 \\ \hline -9 & -6 & 6 & -3 & 3 & 9 \\ \hline \end{array} \tag{5.4}$$

−4	−5	9	11	−11
4	−7	3	−13	13
−2	5	−3	−24	24
2	−8	6	25	−25
−1	7	−6	−26	26
1	8	−9	27	−27

10	−28	17	19	−18
−10	28	−17	−23	22
12	−29	16	−20	21
−12	29	−16	−19	18
14	−30	15	23	−22
−14	30	−15	20	−21

Fig. 4. An SMAS(6, 5; 2).

and the set

$$\mathfrak{B} = \mathfrak{B}(11,\ 36t+45,\ 12t+12)$$

as in Lemma 2.6. Then $\mathcal{E}(A) \sqcup \mathcal{E}(\mathfrak{B}) = \pm[1, 21(4t+5)] \setminus \pm(\mathcal{M}_1 \sqcup \mathcal{M}_2 \sqcup \mathcal{M}_3)$, where

$$\mathcal{M}_1 \ = \ [10, 24t+32]_2, \qquad \mathcal{M}_2 \ = \ [36t+46, 36t+55], \qquad \mathcal{M}_3 \ = \ [48t+68, 84t+105].$$

So, we replace the unique instance of X_6 with A^{t} and the $12t+12$ instances of X_2 with the arrays in $\mathfrak{B}$.

The set $\mathcal{M}_1$ can be written as a disjoint union $G_1 \sqcup \ldots \sqcup G_{2\beta}$, where $\beta = 3t+3$ and each G_j is a 2-set of type 2. The set $\mathcal{M}_2 \sqcup \mathcal{M}_3$ can be written as a disjoint union $H_1 \sqcup \ldots \sqcup H_{2\gamma}$, where $\gamma = 9t+12$ and each H_h is a 2-set of type 1. Call

$$\begin{aligned} \mathfrak{R}_1 &= \{Q_2(G_{2j-1}, G_{2j}) : j \in [1, 3t+3]\}, \\ \mathfrak{R}_2 &= \{Q_1(H_{2h-1}, H_{2h}) : h \in [1, 9t+12]\}. \end{aligned}$$

Then, $\mathfrak{R} = \mathfrak{R}_1 \sqcup \mathfrak{R}_2$ consists of $12t+15$ zero-sum blocks of size 2×4 and $\mathcal{E}(\mathfrak{R}) = \pm(\mathcal{M}_1 \sqcup \mathcal{M}_2 \sqcup \mathcal{M}_3)$. Hence, we replace the $3(4t+5)$ instances of Z_4 with the arrays in $\mathfrak{R}$. $\square$

Lemma 5.5. *There exists an* SMAS(6, 5; c) *for all* $c \geqslant 2$ *such that* $c \equiv 2 \pmod 4$.

Proof. An SMAS(6, 5; 2) is shown in Fig. 4. So, write $c = 4t+6$, where $t \geqslant 0$. Our SMAS(6, 5; c) will consist of $4t+5$ blocks of type (5.1) and one block

$$\begin{array}{|c|c|} \hline X_6 & Z_6^{\mathrm{t}} \\ \hline \end{array}, \tag{5.5}$$

where X_6 is a zero-sum block of size 6×3 and Z_6 is a zero-sum block of size 2×6. Take the zero-sum block A given in (5.4) and the set

$$\mathfrak{B} = \mathfrak{B}(11,\ 36t+55,\ 12t+15)$$

as in Lemma 2.6. Then $\mathcal{E}(A) \sqcup \mathcal{E}(\mathfrak{B}) = \pm[1, 15(4t+6)] \setminus \pm(\mathcal{M}_1 \sqcup \mathcal{M}_2 \sqcup \mathcal{M}_3)$, where

$$\mathcal{M}_1 \ = \ [10, 24t+40]_2, \qquad \mathcal{M}_2 \ = \ [36t+56, 36t+65], \qquad \mathcal{M}_3 \ = \ [48t+81, 60t+90].$$

So, we replace the unique instance of X_6 with A^{t} and the $12t+15$ instances of X_2 with the arrays in $\mathfrak{B}$.

Write

$$\mathcal{M}_1 = \mathcal{M}_1^1 \sqcup \mathcal{M}_1^2 \sqcup \mathcal{M}_1^3, \quad \mathcal{M}_2 = \mathcal{M}_2^1 \sqcup \mathcal{M}_2^2 \sqcup \mathcal{M}_2^3 \quad \text{and} \quad \mathcal{M}_3 = \mathcal{M}_3^1 \sqcup \mathcal{M}_3^2 \sqcup \mathcal{M}_3^3,$$

where

$$\begin{aligned} \mathcal{M}_1^1 &= [10, 8t+24]_2, & \mathcal{M}_2^1 &= [36t+56, 36t+58]_2, \\ \mathcal{M}_1^2 &= [8t+26, 24t+38]_4, & \mathcal{M}_2^2 &= [36t+57, 36t+59]_2, \\ \mathcal{M}_1^3 &= [8t+28, 24t+40]_4, & \mathcal{M}_2^3 &= [36t+60, 36t+65], \\ \mathcal{M}_3^1 &= [48t+81, 60t+87]_2, & \mathcal{M}_3^2 &= [48t+82, 60t+88]_2, \\ \mathcal{M}_3^3 &= [60t+89, 60t+90]. \end{aligned}$$

The set $\mathcal{M}_1^2 \sqcup \mathcal{M}_1^3$ can be written as a disjoint union $F_1 \sqcup \ldots \sqcup F_\alpha$, where $\alpha = 4t+4$ and each F_i is a 2-set of type 4. The set $\mathcal{M}_1^1 \sqcup \mathcal{M}_2^1 \sqcup \mathcal{M}_2^2 \sqcup \mathcal{M}_3^1 \sqcup \mathcal{M}_3^2$ can be written as a disjoint union $G_1 \sqcup \ldots \sqcup G_\beta$, where $\beta = 8t+10$ and each G_j is a 2-set of type 2. Finally, the set $\mathcal{M}_2^3 \sqcup \mathcal{M}_3^3$ can be written as a disjoint union $H_1 \sqcup \ldots \sqcup H_4$, where each H_h is a 2-set of type 1. Call

$$\begin{aligned} \mathfrak{R}_1 &= \{R_2(F_i, G_{2i-1}, G_{2i}) : i \in [1, 4t+4]\}, \\ \mathfrak{R}_2 &= \{R_1(G_{8t+8+j}, H_{2j-1}, H_{2j}) : j \in [1, 2]\}. \end{aligned}$$

Then, $\mathfrak{R} = \mathfrak{R}_1 \sqcup \mathfrak{R}_2$ consists of $4t+6$ zero-sum blocks of size 2×6 and $\mathcal{E}(\mathfrak{R}) = \pm(\mathcal{M}_1 \sqcup \mathcal{M}_2 \sqcup \mathcal{M}_3)$. Hence, we replace the $4t+6$ instances of Z_6 with the arrays in $\mathfrak{R}$. $\square$

Lemma 5.6. *There exists an* SMAS$(6,7;c)$ *for all* $c \geqslant 2$ *such that* $c \equiv 2 \pmod 4$.

Proof. Write $c = 4t+2$, where $t \geqslant 0$. Our SMAS$(6,7;c)$ will consist of $4t+1$ blocks of type (5.2) and one block of type (5.3). Take the zero-sum block A given in (5.4) and the set

$$\mathfrak{B} = \mathfrak{B}(11,\ 36t+19,\ 12t+3)$$

as in Lemma 2.6. Then $\mathcal{E}(A) \sqcup \mathcal{E}(\mathfrak{B}) = \pm[1, 21(4t+2)] \setminus \pm(\mathcal{M}_1 \sqcup \mathcal{M}_2 \sqcup \mathcal{M}_3)$, where

$$\mathcal{M}_1 = [10, 24t+16]_2, \qquad \mathcal{M}_2 = [36t+20, 36t+29], \qquad \mathcal{M}_3 = [48t+33, 84t+42].$$

So, we replace the unique instance of X_6 with A^t and the $12t+3$ instances of X_2 with the arrays in $\mathfrak{B}$.

The set $\mathcal{M}_1$ can be written as a disjoint union $G_1 \sqcup \ldots \sqcup G_{2\beta}$, where $\beta = 3t+1$ and each G_j is a 2-set of type 2. The set $\mathcal{M}_2 \sqcup \mathcal{M}_3$ can be written as a disjoint union $H_1 \sqcup \ldots \sqcup H_{2\gamma}$, where $\gamma = 9t+5$ and each H_h is a 2-set of type 1. Call

$$\begin{aligned} \mathfrak{R}_1 &= \{Q_2(G_{2j-1}, G_{2j}) : j \in [1, 3t+1]\}, \\ \mathfrak{R}_2 &= \{Q_1(H_{2h-1}, H_{2h}) : h \in [1, 9t+5]\}. \end{aligned}$$

Then, $\mathfrak{R} = \mathfrak{R}_1 \sqcup \mathfrak{R}_2$ consists of $12t+6$ zero-sum blocks of size 2×4 and $\mathcal{E}(\mathfrak{R}) = \pm(\mathcal{M}_1 \sqcup \mathcal{M}_2 \sqcup \mathcal{M}_3)$. Hence, we replace the $3(4t+2)$ instances of Z_4 with the arrays in $\mathfrak{R}$. $\square$

Lemma 5.7. *There exists an* SMAS$(6,5;c)$ *for all* $c \geqslant 3$ *such that* $c \equiv 3 \pmod 4$.

Proof. Write $c = 4t+3$, where $t \geqslant 0$. Our SMAS$(6,5;c)$ will consist of $4t+2$ blocks of type (5.1) and one block (5.5). Take the zero-sum block A given in (5.4) and the set

$$\mathfrak{B} = \mathfrak{B}(11,\ 36t+27,\ 12t+6)$$

as in Lemma 2.6. Then $\mathcal{E}(A) \sqcup \mathcal{E}(\mathfrak{B}) = \pm[1, 15(4t+3)] \setminus \pm(\mathcal{M}_1 \sqcup \mathcal{M}_2 \sqcup \mathcal{M}_3)$, where

$$\mathcal{M}_1 = [10, 24t+20]_2, \qquad \mathcal{M}_2 = [36t+28, 36t+37], \qquad \mathcal{M}_3 = [48t+44, 60t+45].$$

So, we replace the unique instance of X_6 with A^t and the $12t+6$ instances of X_2 with the arrays in $\mathfrak{B}$.

Write

$$\mathcal{M}_1 = \mathcal{M}_1^1 \sqcup \mathcal{M}_1^2 \sqcup \mathcal{M}_1^3 \quad \text{and} \quad \mathcal{M}_3 = \mathcal{M}_3^1 \sqcup \mathcal{M}_3^2 \sqcup \mathcal{M}_3^3,$$

where

$$\begin{aligned} \mathcal{M}_1^1 &= [10, 8t+20]_2, & \mathcal{M}_3^1 &= [48t+44, 60t+42]_2, \\ \mathcal{M}_1^2 &= [8t+22, 24t+18]_4, & \mathcal{M}_3^2 &= [48t+45, 60t+43]_2, \\ \mathcal{M}_1^3 &= [8t+24, 24t+20]_4, & \mathcal{M}_3^3 &= [60t+44, 60t+45]. \end{aligned}$$

The set $\mathcal{M}_1^2 \sqcup \mathcal{M}_1^3$ can be written as a disjoint union $F_1 \sqcup \ldots \sqcup F_\alpha$, where $\alpha = 4t$ and each F_i is a 2-set of type 4. The set $\mathcal{M}_1^1 \sqcup \mathcal{M}_3^1 \sqcup \mathcal{M}_3^2$ can be written as a disjoint union $G_1 \sqcup \ldots \sqcup G_\beta$, where $\beta = 8t+3$ and each G_j is a 2-set of type 2. Finally, the set $\mathcal{M}_2 \sqcup \mathcal{M}_3^3$ can be written as a disjoint union $H_1 \sqcup \ldots \sqcup H_6$, where each H_h is a 2-set of type 1. Call

$$\begin{aligned} \mathfrak{R}_1 &= \{R_2(F_i, G_{2i-1}, G_{2i}) : i \in [1, 4t]\}, \\ \mathfrak{R}_2 &= \{R_1(G_{8t+j}, H_{2j-1}, H_{2j}) : j \in [1, 3]\}. \end{aligned}$$

Then, $\mathfrak{R} = \mathfrak{R}_1 \sqcup \mathfrak{R}_2$ consists of $4t+3$ zero-sum blocks of size 2×6 and $\mathcal{E}(\mathfrak{R}) = \pm(\mathcal{M}_1 \sqcup \mathcal{M}_2 \sqcup \mathcal{M}_3)$. Hence, we replace the $4t+3$ instances of Z_6 with the arrays in $\mathfrak{R}$. $\square$

Lemma 5.8. *There exists an* SMAS$(6,7;c)$ *for all* $c \geqslant 3$ *such that* $c \equiv 3 \pmod 4$.

Proof. Write $c = 4t+3$, where $t \geqslant 0$. Our SMAS$(6,7;c)$ will consist of $4t+3$ blocks of type (5.2). Take

$$\mathfrak{B} = \mathfrak{B}(1,\ 36t+26,\ 12t+9)$$

as in Lemma 2.6. Then $\mathcal{E}(\mathfrak{B}) = \pm[1, 21(4t+3)] \setminus \pm(\mathcal{M}_1 \sqcup \mathcal{M}_2)$, where

$$\mathcal{M}_1 = [2, 24t+16]_2 \quad \text{and} \quad \mathcal{M}_2 = [48t+36, 84t+63].$$

So, we replace the $12t+9$ instances of X_2 with the arrays in $\mathfrak{B}$.

Table 1
Proof of Proposition 1.5.

$\mathbf{c \equiv 0 \pmod 4}$	$b=7$	$9 \leqslant b \equiv 1 \pmod 4$	$11 \leqslant b \equiv 3 \pmod 4$
$a=5$	L. 4.2	L. 4.2	L. 4.2
$a \geqslant 7$	C. 2.3	C. 2.3	C. 2.3
$\mathbf{c \equiv 1 \pmod 4}$	$b=7$	$9 \leqslant b \equiv 1 \pmod 4$	$11 \leqslant b \equiv 3 \pmod 4$
$a=5$	L. 4.3	L. 3.6	L. 3.7
$7 \leqslant a \equiv 3 \pmod 4$	L. 3.9	C. 2.3	L. 3.9
$9 \leqslant a \equiv 1 \pmod 4$	C. 2.3	L. 3.6	C. 2.3
$\mathbf{c \equiv 2 \pmod 4}$	$b=7$	$9 \leqslant b \equiv 1 \pmod 4$	$11 \leqslant b \equiv 3 \pmod 4$
$a=5$	L. 4.6	L. 3.10	L. 3.10
$a=7$	L. 3.12	L. 3.11	L. 3.12
$9 \leqslant a \equiv 1 \pmod 4$	L. 3.11	L. 3.10	L. 3.10
$11 \leqslant a \equiv 3 \pmod 4$	L. 3.12	L. 3.10	L. 3.12
$\mathbf{c \equiv 3 \pmod 4}$	$b=7$	$9 \leqslant b \equiv 1 \pmod 4$	$11 \leqslant b \equiv 3 \pmod 4$
$a=5$	L. 4.4	L. 3.1	L. 3.3
$a=7$	C. 2.3	L. 3.4	C. 2.3
$9 \leqslant a \equiv 1 \pmod 4$	L. 3.4	C. 2.3	L. 3.3
$11 \leqslant a \equiv 3 \pmod 4$	C. 2.3	L. 3.3	C. 2.3

The set $\mathcal{M}_1$ can be written as a disjoint union $G_1 \sqcup \ldots \sqcup G_{2\beta}$, where $\beta = 3t+2$ and each G_j is a 2-set of type 2. The set $\mathcal{M}_2$ can be written as a disjoint union $H_1 \sqcup \ldots \sqcup H_{2\gamma}$, where $\gamma = 9t+7$ and each H_h is a 2-set of type 1. Call

$$\begin{aligned}\mathfrak{R}_1 &= \{Q_2(G_{2j-1}, G_{2j}) : j \in [1, 3t+2]\},\\ \mathfrak{R}_2 &= \{Q_1(H_{2h-1}, H_{2h}) : h \in [1, 9t+7]\}.\end{aligned}$$

Then, $\mathfrak{R} = \mathfrak{R}_1 \sqcup \mathfrak{R}_2$ consists of $12t+9$ zero-sum blocks of size 2×4 and $\mathcal{E}(\mathfrak{R}) = \pm(\mathcal{M}_1 \sqcup \mathcal{M}_2)$. Hence, we replace the $3(4t+3)$ instances of Z_4 with the arrays in $\mathfrak{R}$. $\square$

6. Conclusions

We can now prove our main results.

Proof of Proposition 1.5. By Remark 2.4, the statement follows from the results of Sections 3 and 4, according to Table 1. $\square$

Proof of Proposition 1.6. For all $c \geqslant 1$, there exists an SMAS$(6,5;c)$ by Lemmas 5.2, 5.3, 5.5 and 5.7, and there exists an SMAS$(6,7;c)$ by Lemmas 5.2, 5.4, 5.6 and 5.8. Hence, the result follows from Lemma 5.1. $\square$

Proof of Theorem 1.2. By [10, Main Theorem] and [11, Main Theorem 20], we may assume $s,k \geq 4$. Up to transposition, by Proposition 1.3 we may also assume that $k \geqslant 5$ is odd, n is even, $s \not\equiv 0 \pmod 4$ and $\gcd(s,k)=1$. From $ms = nk$ we get that there exists a positive integer e such that $m = ek$ and $n = es$.

Suppose that s is odd. Since n is even, $e = 2c$ for a suitable positive integer c. Let $\{R_1, \ldots, R_{2c}\}$ be an SMAS$(k,s;2c)$: the existence of this set follows from Proposition 1.5, since $s \geqslant 5$ and $(s,k) \neq (5,5)$. In this case, the $m \times n$ block-diagonal matrix $\mathrm{diag}(R_1, \ldots, R_{2c})$ is an SMA$(m,n;s,k)$.

Next, suppose that s is even. By the previous assumptions, we can write $s = 2\bar{s}$, where $\bar{s} \geqslant 3$ is an odd integer, coprime with k. If $\bar{s} = 3$, let $\{R_1, \ldots, R_e\}$ be an SMAS$(k,6;e)$, whose existence follows from Proposition 1.6. In this case, the $m \times n$ block-diagonal matrix $\mathrm{diag}(R_1, \ldots, R_e)$ is an SMA$(m,n;s,k)$. If $\bar{s} \geqslant 5$, let $\{R_1, \ldots, R_{2e}\}$ be an SMAS$(k,\bar{s};2e)$, whose existence follows from Proposition 1.5 since $(k,\bar{s}) \neq (5,5)$. For every $i \in [1,e]$, define the $k \times s$ array $S_i = \boxed{R_{2i-1}}\boxed{R_{2i}}$. In this case, the $m \times n$ block-diagonal matrix $\mathrm{diag}(S_1, \ldots, S_e)$ is an SMA$(m,n;s,k)$. $\square$

Proof of Corollary 1.8. By [5, Proposition 4.2], there is no (integer) ${}^2\mathrm{H}(m,n;s,k)$ when $nk \equiv 1 \pmod 4$. So, assume that nk even. By Theorem 1.2 there exists an SMA$(m,n;s,k)$ (and hence, an integer ${}^2\mathrm{H}(m,n;s,k)$) when (m,n,s,k) does not belong to the set

$$\{(\ell,2,2,\ell),(2,\ell,\ell,2) : \ell \geqslant 1 \text{ is such that } \ell \equiv 1,2 \pmod 4\}.$$

So, we are left to consider the case when $m = k = 2$ and $n = s \equiv 1,2 \pmod 4$. For the sake of contradiction, suppose that there exists an integer ${}^2\mathrm{H}(2,n;n,2)$, say A. Denoting by R_1, R_2 the two rows of A, we must have $R_2 = -R_1$, where the absolute values of the entries of R_1 constitute the set $[1,n]$. Since $n \equiv 1,2 \pmod 4$, the sum of the entries of R_1 cannot be equal to zero, a contradiction. $\square$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.disc.2026.115313.

Data availability

No data was used for the research described in the article.

References

[1] S. Akbari, F. Rahmati, S. Zare, Zero-sum magic labelings and null sets of regular graphs, Electron. J. Comb. 21 (2014) #P2.17.
[2] D.S. Archdeacon, Heffter arrays and biembedding graphs on surfaces, Electron. J. Comb. 22 (2015) #P1.74.
[3] S. Cichacz, A Γ-magic rectangle set and group distance magic labeling, in: J. Kratochvíl, M. Miller, D. Froncek (Eds.), Combinatorial Algorithms. Revised Selected Papers from the 25th International Workshop (IWOCA 2014) Held in Duluth, MN, October 15–17, 2014, in: Lecture Notes in Comput. Sci., vol. 8986, Springer, Cham, 2015.
[4] S. Cichacz, Disjoint zero-sum subsets in Abelian groups and their application: a survey, in: G.O.H. Katona, B. Patkós, C. Tompkins (Eds.), Sum(m)it280. Surveys in Extremal Combinatorics and Combinatorial Geometry, in: Bolyai Society Mathematical Studies, vol. 32, Springer, 2026.
[5] S. Costa, A. Pasotti, On λ-fold relative Heffter arrays and biembedding multigraphs on surfaces, Eur. J. Comb. 97 (2021) 103370, 21 pp.
[6] D. Froncek, J. McKeown, J. McKeown, M. McKeown, Z_{2nm}-supermagic labeling of $C_n \square C_m$, Indones. J. Comb. 2 (2018) 57–71.
[7] J.A. Gallian, A dynamic survey of graph labeling, Electron. J. Comb. 5 (1998) #DS6, Twenty-eighth edition.
[8] T. Harmuth, Über magische Quadrate und ähniche Zahlenfiguren, Arch. Math. Phys. 66 (1881) 286–313.
[9] T. Harmuth, Über magische Rechtecke mit ungeraden Seitenzahlen, Arch. Math. Phys. 66 (1881) 413–447.
[10] A. Khodkar, B. Ellis, Signed magic rectangles with two filled cells in each column, J. Comb. Math. Comb. Comput. 121 (2024) 31–40.
[11] A. Khodkar, D. Leach, B. Ellis, Signed magic rectangles with three filled cells in each column, Bull. Inst. Comb. Appl. 90 (2020) 87–106.
[12] A. Khodkar, C. Schulz, N. Wagner, Existence of some signed magic arrays, Discrete Math. 340 (2017) 906–926.
[13] F. Morini, M.A. Pellegrini, Magic rectangles, signed magic arrays and integer λ-fold relative Heffter arrays, Australas. J. Comb. 80 (2021) 249–280.
[14] F. Morini, M.A. Pellegrini, Rectangular Heffter arrays: a reduction theorem, Discrete Math. 345 (2022) #113073.
[15] F. Morini, M.A. Pellegrini, Magic partially filled arrays on abelian groups, J. Comb. Des. 31 (2023) 347–367.
[16] A. Pasotti, J.H. Dinitz, A survey of Heffter arrays, in: C.J. Colbourn, J.H. Dinitz (Eds.), New Advances in Designs, Codes and Cryptography. NADCC 2022, in: Fields Institute Communications, vol. 86, Springer, Cham, 2024, pp. 353–392.
[17] M.A. Pellegrini, T. Traetta, Towards a solution of Archdeacon's conjecture on integer Heffter arrays, J. Comb. Des. 33 (2025) 310–323.
[18] R.G. Sun, The existence of higher-dimensional magic cubes, J. Inn. Mong. Univ. 19 (1988) 213–223.
[19] Y. Wen, H. Sun, Note on magic squares and magic cubes on abelian groups, J. Math. Res. Expo. 17 (1997) 176–178.