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# Money Illusion and TIPS Demand

The market demand for the Treasury Inflation-Protected Securities (TIPS) is rather small. This is puzzling, as we show that an agent, who derives utility from real wealth and dynamically invests into multiple asset classes over a 30-year horizon, incurs a certainty equivalent loss of 1.6% per annum from not investing in inflation-indexed bonds. However, if the investor suffers from money illusion, the perceived loss is only 0.5% per annum. Furthermore, the perceived loss is totally negligible for an unsophisticated money-illusioned investor ignoring the time variation of risk premia. Money illusion causes significant portfolio shifts from inflation-indexed toward nominal bonds, with little effects on equity allocations, contributing to the low market demand for TIPS.

*JEL* codes E43, E52, G11, G12
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Introduced in the U.S. in 1997, after several periods of high and fluctuating inflation, Treasury Inflation-Protected Securities (TIPS) should have been quickly acclaimed by market participants. Owing to the indexation to the consumer price index (CPI), they were supposed to be a close-to-perfect instrument to hedge against inflation and would have offered an observable measure of the term

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structure of inflation expectations. Early works in asset allocation strongly supported the welfare improvements entailed by the inclusion of such securities (Campbell and Viceira 2001, Brennan and Xia 2002, Wachter 2003). However, a few years later, it was obvious that the TIPS market had not delivered in terms of market quality, as the liquidity was still rather low if compared to nominal Treasury Bonds (Fleming and Krishnan 2012). Figure 1(a) reveals that the fraction of TIPS relative to the total of outstanding marketable Treasuries was about 9% at the end of 2019, and has never been higher than 11% in the last two decades. In the course of 2021, a growing inflationary pressure is causing a moderate increase of market attention to inflation, but, while this could happen in the future, it has not translated yet into a significant growth of the TIPS market. Inflation-indexed bonds represent a niche market also in other countries. For instance, at the end of 2016, only 12.3% of the French and 12.7% of the Italian public debts were represented by inflation-protected securities. Among the largest issuers, the main exception is the United Kingdom, where inflation-indexed debt, massively held by pension funds offering inflation protection, has been issued since 1981 and represented 27.3% of the total outstanding debt in 2016. [1]

García and Van Rixel (2007) and Campbell, Shiller, and Viceira (2009) review several arguments in favor of the issuance of inflation-indexed bonds from the perspective of a central bank, as well as the benefits to investors. Ultimately, in the presence of a high market demand for inflation-indexed debt, the issuance of TIPS could reduce the cost of funding. [2] As pointed out in Fleckenstein, Longstaff, and Lustig (2014), TIPS are typically underpriced with respect to nominal bonds, the sign of the mispricing being consistent with a lack of demand. While substantial progress has been made in identifying the TIPS liquidity premium, few authors have investigated why investors shun this asset class. [3] To the contrary, several contributions in the asset allocation literature advocate massive investments in TIPS rather than in nominal bonds. Campbell, Chan, and Viceira (2003) analyze the asset allocation of an infinitely lived investor, while Kothari and Shanken (2004) and Cartea, Saúl, and Toro (2012) study a one-period mean–variance-utility problem, considering a rational risk-averse individual who cares about real returns and invests in several asset classes. They all find that nominal bonds should be crowded out by TIPS. Illeditsch (2017) finds that a rational long-term investor perceives a large utility cost from not

1. In the United Kingdom, as of March 2015, pension funds held 82% of inflation-linked gilts. The remaining market share was therefore rather small. As for the U.S., Andonov, Bauer, and Cremers (2013) report that in 2010 only 56% of public pension funds, and only 10% of private pension funds, offered inflation protection. It is important to highlight that sophisticated investors, other than through TIPS, are able to hedge inflation exposure also through inflation swaps.

2. Martellini, Milhau, and Tarelli (2018) also highlight that firms generating cashflows positively correlated with inflation would significantly benefit from issuing inflation-indexed debt. The market for corporate inflation-indexed bonds is, however, even smaller than the TIPS market.

3. As a matter of fact, most inflation-related instruments have encountered little success. In addition to TIPS and inflation swaps, CPI-related futures were also introduced in 2004 by the Chicago Mercantile Exchange (CME), but failed to attract investors. Fleming and Sporn (2013) document a very low trading activity on the inflation swap market, consisting of just over two trades per day on average since 2010.

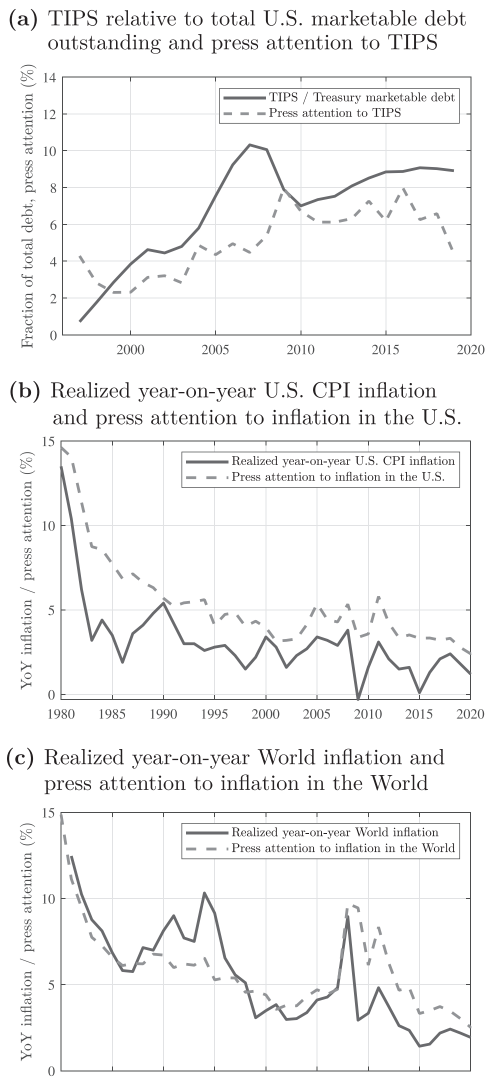


Fig 1. TIPS Outstanding, Press Attention to TIPS, and Press Attention to Inflation.

NOTES: Panel (a) shows the year-end ratio between the notional amounts of TIPS and total Treasury marketable debt outstanding, as reported in the Treasury Bulletin of the Bureau of the Fiscal Service. The press attention to TIPS (number of records in Dow Jones Factiva containing the terms *inflation linked* or *inflation protected* or *inflation indexed*, and *treasury* or *treasuries*) relative to Treasuries (records containing the terms *treasury bond* or *treasury note* or *treasury bill* or *treasuries*) is also shown. Panel (b) shows the year-on-year U.S. CPI inflation (U.S. Bureau of Labor Statistics, Consumer Price Index for All Urban Consumers: All Items, FRED item `CPIAUCSL`), as well as the press attention to inflation in the U.S., measured as the relative number of Factiva records in the U.S. containing the word *inflation* among the records containing any of the words *investments*, *investing*, *portfolio choice*, and *asset allocation*. Panel (c) shows the year-on-year World inflation (World Bank, consumer prices for the World, FRED item `FPCPITOTLZGWLD`), as well as the press attention to inflation in the World.

investing in inflation-indexed bonds. These results are in contrast with a TIPS market representing only a small fraction of the nominal bond market.

This paper belongs to the aforementioned asset allocation literature and shows that money illusion may significantly contribute to the low demand for TIPS. Money illusion can be generally described as a bias in the assessment of the real value of economic transactions, caused by the fact that agents tend to think in nominal amounts. [4] This bias has been studied in monetary economics (e.g., Marschak 1950, 1974, Dusansky and Kalman 1974), as well as in behavioral and experimental economics (e.g., Shafir, Diamond, and Tversky 1997, Fehr and Tyran 2001, 2007, 2014, Luhan and Scharler 2014), and has been found to be useful in solving several asset pricing puzzles.

Despite the common thinking of money illusion mostly being harmful in hyperinflation environments, Brunnermeier and Julliard (2008) and Cavallo, Cruces, and Perez-Truglia (2017) suggest that money illusion and rational inattention toward inflation can be even stronger in low-inflation environments. [5] Investors tend to be misled by current low inflation rates, which are wrongly associated with low inflation risk. This phenomenon could have been particularly relevant in the last couple of decades, when the level of inflation has been low despite a nonnegligible inflation risk. Indeed, as we show in Figure 1(b) for the U.S. and in Figure 1(c) at the global level, the press attention to inflation, which could be considered as a proxy for the agents' attention to inflation, closely follows the year-on-year realized inflation in the corresponding economy. According to Gabaix (2014), economic agents allocate attention purposefully toward salient features, using a sparse representation of reality that ignores information perceived to be less relevant. As we show in Figure 1(a) through a measure of press attention to TIPS, Treasuries get far more attention among market agents than TIPS. Interestingly, the relative press attention to TIPS mimics the fraction of TIPS outstanding relative to the total amount of Treasuries.

In this work, we consider the distortions on the demand side implied by money illusion in a market where multiple asset classes (nominal and indexed bonds, stocks, and cash) are available to a long-term investor with a finite horizon. At first, we consider an infinitely risk-averse investor and show that severe money illusion drives conservative investors out of the indexed bond market. While nonillusioned investors attempt to hedge future variations of the real rate, money-illusioned investors ignore expected inflation, hedging only variations of the nominal rate. We then consider an investor having moderate risk aversion, who takes advantage also of the risk/return trade-off

4. In this work, money illusion can be either considered as a manifestation of irrationality, which might be the case for a significant fraction of households in the economy, or as a rational choice of the agent, which is the case, for instance, for fund managers whose compensation is based on nominal returns, or for pension funds which liabilities are not inflation-linked.

5. Piazzesi and Schneider (2008), accounting for agents with different money illusion, explain the house-price booms and stock market undervaluation in both the 1970s and 2000s, which occurred in opposite interest rate and inflation regimes.

of the tradable assets, beyond their capacity to hedge inflation. We consider both the cases where risk premia are constant and time-varying, discovering that, again, money illusion significantly shifts the portfolio from indexed to nominal bonds. In contrast, the impact on equity investments is negligible, because the inflation-hedging properties of both nominal and inflation-indexed bonds are quantitatively more relevant than those of stocks.

A key contribution of our work is that we quantify the economic impact of money illusion under different perspectives. First, we evaluate the welfare loss, as perceived by a nonillusioned agent, for implementing the portfolio strategy of a partially or" totally money-illusioned investor. Although the estimation uses TIPS data and is thus based on the low-inflation post-1999 period, we find the welfare loss to be substantial, with a certainty equivalent reduction of up to about 1% per annum (p.a.) for investment horizons longer than 10 years. Second, and most importantly, we provide a rationale for the low demand for inflation-protected bonds, by quantifying the *ex ante* opportunity cost, as perceived by both a nonillusioned and a money-illusioned investors, of not accessing the inflation-indexed bond market. We find that a nonillusioned investor, who ignores time variations in risk premia, perceives a significant utility cost for not investing in inflation-indexed bonds. The certainty equivalent loss is about 0.5% p.a. for a 10-year horizon, 0.85% p.a. for a 20-year horizon, and 1.25% p.a. for a 30-year horizon, representing thus a large incentive to enter the TIPS market. However, we find that a money-illusioned investor perceives a loss smaller than 0.1% p.a. when TIPS are not accessible. The losses are only slightly higher when investors are more sophisticated and account for time variations in risk premia, but the conclusion is the same: the opportunity cost suffered by a money-illusioned agent prevented from investing in inflation-protected instruments is very small, as her perceived expected utility achievable by substituting real with nominal bonds is nearly unchanged. It thus seems that money-illusioned investors have little incentive to enter the inflation-indexed bond market. These findings are mostly driven by a hedging rationale, rather than by speculative motives. Indeed, our results are robust to different levels of the inflation risk premium, and they are confirmed by an additional analysis based on data from the UK market, where large pension fund positions in inflation-indexed gilts cause the inflation risk premium to largely differ from that in the U.S. market.

Our framework is based on a dynamic affine term structure model of the nominal and real term structures, as in Christensen, Lopez, and Rudebusch (2010), D'Amico, Kim, and Wei (2018), Breach, D'Amico, and Orphanides (2020), and Andreasen, Christensen, and Riddell (2021), which is then applied to an asset allocation problem with multiple asset classes, as for instance in Sangvinatsos and Wachter (2005). We contribute to the asset allocation literature relying on dynamic term structure models, providing a different perspective to the empirical issues of the sensitivity to estimation errors and the in-sample overfitting of essentially affine models discussed by Duffee (2011), Feldhütter et al. (2012), and Sarno, Schneider, and Wagner (2016). These works show that unconstrained term structure models with time-varying risk premia entail unrealistic portfolio positions and overstated certainty equivalent returns. We mitigate these issues by imposing reasonable bounds to the volatility of the risk premia and by accounting for survey-based long-run expectations of asset returns.

The preferences that we adopt are inspired by the contribution of Basak and Yan (2010), who analyze the implications of money illusion for asset prices in a market with inflationary fluctuations. Building on their work, David and Veronesi (2013) develop a general equilibrium model, featuring money illusion, that captures historical stock and bond comovements and explains the low P/E ratio and high long-term yields in the late 1970s. They find evidence for a significant degree of money illusion affecting asset prices in the U.S. market. Finally, Miao and Xie (2013) include money illusion in a monetary model of endogenous growth, showing that the impact of money illusion on long-run growth is significant also when expected inflation is close to its long-run mean. While these contributions highlight the distortions of the equilibrium introduced by money illusion, our focus is on the individual behavior of an investor with a long-term finite horizon.

This work is also related to the strand of the literature attempting to explain the mispricing and liquidity puzzles of TIPS. In particular, Fleckenstein, Longstaff, and Lustig (2014) find a massive mispricing of TIPS, highlighted by replicating the payoff of nominal Treasury bonds using inflation swaps and TIPS. Their explanation invokes the near-money characteristic of nominal Treasuries and justifies the persistence of the mispricing as an effect of the slow-moving-capital phenomenon. This interpretation is reinforced by Gromb and Vayanos (2018), who find that expected returns of arbitrage portfolios are negatively related to shocks to the amount of arbitrage capital. Christensen and Gillan (2018) provide additional evidence, showing that quantitative easing reduced the liquidity premium in the TIPS market. The puzzling illiquidity of TIPS has led several authors to identify a liquidity factor (e.g., Abrahams et al. 2016, Pflueger and Viceira 2016, D'Amico, Kim, and Wei 2018, Andreasen, Christensen, and Riddell 2021). Given the evidence of a significant degree of money illusion in the U.S. market (David and Veronesi 2013), our model provides a rationale for the low demand for TIPS, which may contribute to their illiquidity.

The remainder of the paper is organized as follows. In Section 1, we set up the economic framework and derive the optimal portfolio strategy of a money-illusioned long-term investor. Section 2 describes the estimation methodology, presents the data set and discusses the parameter estimates. In Section 3, we discuss the results of the optimal portfolio strategy obtained considering constant asset risk premia, at first for a conservative investor (infinite risk aversion), and then for an investor with a moderate risk aversion. In Section 4, we present additional findings obtained considering time-varying risk premia. Section 5 discusses our findings, in light of other factors potentially contributing to the low market demand for TIPS, and summarizes the robustness checks. Section 6 concludes. A separate Online Appendix contains several technical details, as well as robustness checks and additional empirical findings.

## 1. OPTIMAL PORTFOLIO CHOICE

We consider a long-term investor trading nominal and inflation-indexed bonds, a stock index, and a nominal money market account (cash). We do not include inflation swaps in the investable universe because, in the absence of further frictions, these would be redundant. Furthermore, inflation swaps have a very low trading frequency (Fleming and Sporn 2013) and are not available to a large fraction of investors. As in most asset allocation problems, we are in a situation of partial equilibrium, whereby the prices of the assets available for trade are given to the investor. In the following, we describe the economy where the agent trades and we then derive the optimal portfolio strategy.

### *1.1 The Economy*

*Stochastic discount factor (SDF).* We assume the existence of an SDF that prices nominal and inflation-indexed default-free bonds, as well as a stock index.[6] The nominal SDF dynamics is

$$\frac{d\Phi_t}{\Phi_t} = -R_t \mathrm{d}t - \mathbf{\Lambda}_t' d\mathbf{z}_t, \tag{1}$$

where $R_t$ is the nominal short-term interest rate and $\mathbf{\Lambda}_t$ is the $n \times 1$ vector of market prices of the $n$ systematic sources of risk, $\mathbf{z}_t$. $\mathbf{z}_t$ is a vector of $n \times 1$ standard Brownian motions. Following the literature on dynamic term structure models, we assume an affine functional form for the nominal rate and for the market prices of risk:

$$R_t = R_0 + \mathbf{R}_1' \mathbf{X}_t, \tag{2}$$

$$\mathbf{\Lambda}_t = \mathbf{\Lambda}_0 + \mathbf{\Lambda}_1 \mathbf{X}_t, \tag{3}$$

where $R_0$ is a scalar, $\mathbf{R}_1$ is an $m \times 1$ vector, $\mathbf{\Lambda}_0$ is an $n \times 1$ vector, and $\mathbf{\Lambda}_1$ is an $n \times m$ matrix. Finally, $\mathbf{X}_t$ is the $m \times 1$ vector of state variables, which follows an autoregressive process *à la* Ornstein–Uhlenbeck:

$$d\mathbf{X}_t = \mathbf{\Theta}\left(\bar{\mathbf{X}} - \mathbf{X}_t\right)dt + \mathbf{\Sigma}_{\mathbf{X}}' d\mathbf{z}_t, \tag{4}$$

where $\mathbf{\Theta}$ is an $m \times m$ mean reversion matrix, $\bar{\mathbf{X}}$ is the $m \times 1$ vector of the long-run means of the state variables, and $\mathbf{\Sigma}_{\mathbf{X}}$ is the $n \times m$ volatility matrix.

6. Our SDF is specified in reduced form and is therefore compatible with any investor who can trade such instruments, irrespective of preferences and, in particular, of their degree of money illusion.

Nominal quantities are converted into real quantities using the price level, which dynamics is

$$\frac{dP_t}{P_t} = \pi_t dt + \boldsymbol{\sigma}_P' d\mathbf{z}_t, \tag{5}$$

where $\pi_t$ stands for the instantaneous expected inflation and $\boldsymbol{\sigma}_P$ is the $n \times 1$ volatility vector of unexpected inflation. As for $R_t$ and $\boldsymbol{\Lambda}_t$, we assume that $\pi_t$ is affine in the state variables:

$$\pi_t = \pi_0 + \boldsymbol{\pi}_1' \mathbf{X}_t, \tag{6}$$

where $\pi_0$ is a scalar and $\boldsymbol{\pi}_1$ is an $m \times 1$ vector.

*Traded assets.* We assume that the long-term investor can trade a *nominally* risk-free asset (cash) yielding the short-term interest rate $R_t$. In addition, a stock can be traded and its price, $S_t$, is assumed to have the following dynamics:

$$\frac{dS_t}{S_t} = \left(R_t + \boldsymbol{\sigma}_S' \boldsymbol{\Lambda}_t\right) dt + \boldsymbol{\sigma}_S' d\mathbf{z}_t, \tag{7}$$

where $\boldsymbol{\sigma}_S$ is the $n \times 1$ volatility vector of the stock.

On the fixed-income side, the investor can trade both nominal and real zero-coupon bonds. The nominal bond delivers one unit of the currency at maturity, while the real bond delivers one unit of the *numéraire*. As in Duffie and Kan (1996), nominal discount bond prices are exponential affine functions of the state variables. As shown in Online Appendix A.1, the *nominal* price of a nominal discount bond with a maturity $T$ and a time-to-maturity $\tau = T - t$ is given by

$$B(\mathbf{X}_t, \tau) = e^{-y^n(\mathbf{X}_t, \tau)\tau} = e^{A_0^B(\tau) + \mathbf{A}_1^B(\tau)\mathbf{X}_t}, \tag{8}$$

where $y^n(\mathbf{X}_t, \tau) = -A_0^B(\tau)/\tau - \mathbf{A}_1^B(\tau)/\tau \cdot \mathbf{X}_t$ is the nominal yield. The scalar $A_0^B$ and the $1 \times m$ vector $\mathbf{A}_1^B$ solve a system of ordinary differential equations given in Online Appendix A.1. The dynamics of the nominal bond price is thus:

$$\frac{dB(\mathbf{X}_t, \tau)}{B(\mathbf{X}_t, \tau)} = \left( R_t + \underbrace{\mathbf{A}_1^B(\tau) \boldsymbol{\Sigma}_{\mathbf{X}}' \boldsymbol{\Lambda}_t}_{\substack{\text{nominal yield} \\ \text{risk premium}}} \right) dt + \mathbf{A}_1^B(\tau) \boldsymbol{\Sigma}_{\mathbf{X}}' d\mathbf{z}_t, \tag{9}$$

where we have highlighted the instantaneous risk premium $\mathbf{A}_1^B(\tau)\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Lambda}_t$ that compensates for the exposure to the nominal yield risk.

As shown in Online Appendix A.2, the *nominal* price of a real zero-coupon bond, that is a bond that pays the price index level at maturity, is given by

$$I(\mathbf{X}_t, P_t, \tau) = P_t e^{-y^r(\mathbf{X}_t,\tau)\tau} = P_t e^{A_0^I(\tau)+\mathbf{A}_1^I(\tau)\mathbf{X}_t}, \tag{10}$$

where $y^r(\mathbf{X}_t, \tau) = -A_0^I(\tau)/\tau - \mathbf{A}_1^I(\tau)/\tau \cdot \mathbf{X}_t$ is the real yield. The price dynamics is thus:

$$\frac{dI(\mathbf{X}_t, P_t, \tau)}{I(\mathbf{X}_t, P_t, \tau)} = \left( R_t + \underbrace{\mathbf{A}_1^I(\tau)\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Lambda}_t}_{\substack{\text{real yield} \\ \text{risk premium}}} + \underbrace{\boldsymbol{\sigma}_P'\boldsymbol{\Lambda}_t}_{\substack{\text{unexpected inflation} \\ \text{risk premium}}} \right) dt + \left(\mathbf{A}_1^I(\tau)\boldsymbol{\Sigma}_{\mathbf{X}}' + \boldsymbol{\sigma}_P'\right) d\mathbf{z}_t, \tag{11}$$

where we have highlighted the risk premium $\mathbf{A}_1^I(\tau)\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Lambda}_t + \boldsymbol{\sigma}_P'\boldsymbol{\Lambda}_t$. The first term compensates for the real yield risk and the second for unexpected inflation risk. These risks correspond to the two terms appearing in the diffusion coefficient in (11).

We consider $y^r(\mathbf{X}_t, \tau)$ in (10) to be the model-implied yield of a tradable inflation-indexed bond, for example, a TIPS in the U.S. market. As in Christensen, Lopez, and Rudebusch (2010), and differing from D'Amico, Kim, and Wei (2018) and other works, we employ a reduced-form description of the TIPS yield curve, without decomposing the TIPS premium into an inflation premium component and a liquidity premium component. As identifying the components of the TIPS premium is not relevant for our purposes, we opt for a more parsimonious and robust specification, showing in the estimation phase that it provides a good fit of the TIPS yield curve. In the empirical analysis, we check that our asset allocation and welfare results are robust with respect to different specifications of the inflation premium. Furthermore, in order to keep the asset allocation model tractable, and consistent with the aggregate data set that we use, we disregard the 2.5-month indexation lag of TIPS. For the estimation and the asset allocation exercise, we use TIPS yields corresponding to 5-year or longer maturities, and our asset allocation and welfare results are particularly strong for 10-year or longer horizons, where we expect the indexation lag to have very little effect. Previous studies estimated that the 8-month indexation lag of UK inflation-indexed gilts has a tiny impact on yields, that is, 1.5 basis points according to Evans (1998), and between 0 and 6 basis points according to Risa (2001). D'Amico, Kim, and Wei (2018) estimated an indexation-lag premium for U.S. TIPS comprised between −5 and 3 basis points most of the time, with a variation accounting only for a tiny fraction of TIPS yield variance. Finally, in line with all the aforementioned literature, in order to be able to use aggregate yield curve data and for analytical tractability, we also disregard the deflation option in modeling TIPS prices. The deflation option may have a small price impact for recently issued TIPS in a deflationary environment, but D'Amico, Kim, and Wei (2018), who also did not model the deflation floor, showed that its price impact on TIPS yields is below five basis points most of the time. Concerning our analysis, we expect the effect of the deflation option to be negligible for long investment horizons, which is where most of our results turn out to be relevant.

We denote the nominal price of a generic risky asset by $Y_t^i$. Its dynamics takes the following form:

$$\frac{dY_t^i}{Y_t^i} = \left(R_t + \boldsymbol{\sigma}'_{Y^i}\boldsymbol{\Lambda}_t\right)dt + \boldsymbol{\sigma}'_{Y^i}d\mathbf{z}_t. \tag{12}$$

*Preferences.* Consider a long-term investor endowed with utility from real terminal wealth:

$$U(w_T) = \frac{w_T^{1-\gamma}}{1-\gamma}, \tag{13}$$

where $w_T$ stands for the real wealth at the investor's horizon $T$. When the investor is money-illusioned, we assume that her objective function modifies as follows:

$$U(w_T, W_T) = \frac{\left(w_T^{1-\alpha}W_T^{\alpha}\right)^{1-\gamma}}{1-\gamma}, \tag{14}$$

where $W_T$ stands for the nominal terminal wealth and $0 \leq \alpha \leq 1$ measures the degree of money illusion. When $\alpha = 0$, the investor is not money-illusioned, in the sense that she maximizes her expected utility from real terminal wealth. When $\alpha = 1$, the investor is completely money-illusioned, as she thinks only in nominal terms. This specification is inspired by Basak and Yan (2010), Miao and Xie (2013), and David and Veronesi (2013), although we choose to define utility over terminal wealth only (no intermediate consumption) for analytical tractability. In fact, when markets are complete, a quasi-closed-form optimal solution can be derived also for the case of utility over consumption. However, a specification with utility over terminal wealth allows us to obtain analytical results in some relevant cases of suboptimal portfolio strategies and under market incompleteness. These cases include the evaluation of welfare when the investor builds a suboptimal strategy considering $\alpha > 0$, or when the investment universe includes fewer nonredundant assets than the number of sources of risk in the market.

As the relation between real and nominal wealth is $w_T = W_T P_T^{-1}$, the objective function becomes:

$$U(w_T, W_T) = \frac{\left(w_T^{1-\alpha}W_T^{\alpha}\right)^{1-\gamma}}{1-\gamma} = \frac{W_T^{1-\gamma}P_T^{-(1-\alpha)(1-\gamma)}}{1-\gamma} \equiv U(W_T), \tag{15}$$

with the understanding that the case where no money illusion is at play is nested by setting $\alpha = 0$.

*Budget constraint.* The investor allocates her wealth to $N$ risky assets and the money market account. The dynamics of nominal wealth reads as follows:

$$\frac{dW_t}{W_t} = \sum_{i=1}^{N} \omega_t^i \frac{dY_t^i}{Y_t^i} + \left(1 - \sum_{i=1}^{N} \omega_t^i\right) R_t dt, \tag{16}$$

where $\omega_t^i$ stands for the proportion of (indifferently) real or nominal wealth invested in risky asset $i$. Using (12), this dynamics can be written as follows:

$$\frac{dW_t}{W_t} = (R_t + \boldsymbol{\omega}_t' \boldsymbol{\Sigma}' \boldsymbol{\Lambda}_t) dt + \boldsymbol{\omega}_t' \boldsymbol{\Sigma}' d\mathbf{z}_t, \tag{17}$$

where $\boldsymbol{\omega}_t$ is the $N \times 1$ vector of weights and $\boldsymbol{\Sigma}$ is a matrix which columns are the volatility vectors of the risky assets, $\boldsymbol{\sigma}_{Y^i}$. The dynamics of real wealth is obtained by applying Itô's lemma to $w_t = W_t P_t^{-1}$:

$$\frac{dw_t}{w_t} = (R_t - \pi_t + \boldsymbol{\omega}_t' \boldsymbol{\Sigma}' (\boldsymbol{\Lambda}_t - \boldsymbol{\sigma}_P) + \boldsymbol{\sigma}_P' \boldsymbol{\sigma}_P) dt + (\boldsymbol{\omega}_t' \boldsymbol{\Sigma}' - \boldsymbol{\sigma}_P') d\mathbf{z}_t. \tag{18}$$

*Market completeness.* Concerning the estimation of the model, which we address in the next section, the market is complete, as the set of observed asset returns is sufficient to pin down the market prices for all risks introduced in the economy, and therefore to fully characterize the dynamics of the SDF (1). In order to derive the optimal portfolio strategy, however, it is important to take into account the issue of market (in)completeness from the point of view of the investor. The market is complete if there are at least $N = n$ nonredundant assets available for trade. As we consider $n = 5$ sources of risk, we could meet this condition, but, in order to obtain results more easily interpretable from an economic point of view, we prefer to study the optimal allocation considering at most one nominal bond, one inflation-indexed bond, and the stock market, on top of the nominal risk-less asset. When utility depends only on terminal wealth, as is well known since Kim and Omberg (1996), it is possible to solve the optimal asset allocation problem even when markets are incomplete. In this case, knowing the dynamics of traded assets does not allow investors to reverse engineer the dynamics of a unique SDF. To solve this issue, along the lines of He and Pearson (1991) and Sangvinatsos and Wachter (2005), we write the market prices of risk $\boldsymbol{\Lambda}_t$ as the sum of two components: the first, $\boldsymbol{\Lambda}_t^*$, corresponding to their projection onto the returns of the assets available for trade, and the second, $\boldsymbol{\nu}_t$, being orthogonal to the traded asset returns. In principle, there exists an infinity of vectors $\boldsymbol{\nu}_t$ which are compatible with the prices of the traded assets. We show in Online Appendix B.1 how, among all the possible values of $\boldsymbol{\nu}_t$, it is possible to impose that the dynamics of optimal wealth is actually spanned by the traded assets, pinning down the unique vector $\boldsymbol{\nu}_t^*$ that makes the optimal wealth achievable with the traded assets. The nominal SDF of the investor is then:

$$\frac{d\Phi_t^{\nu^*}}{\Phi_t^{\nu^*}} = -R_t dt - (\boldsymbol{\Lambda}_t^* + \boldsymbol{\nu}_t^*)' d\mathbf{z}_t. \tag{19}$$

Equivalently, it is possible to define a real SDF, denoted as $\phi_t^{\nu^*} \equiv P_t \Phi_t^{\nu^*}$.

### *1.2 Optimal Portfolio Choice under Money Illusion*

Our setting allows us to write a separable value function for the investor's problem:

$$\begin{aligned} J(W_t, P_t, T-t) &\equiv \max_{[\omega_s]_{s=t}^T} \mathrm{E}_t \left[ \frac{W_T^{1-\gamma} P_T^{-(1-\alpha)(1-\gamma)}}{1-\gamma} \right] \\ &= \frac{W_t^{1-\gamma} P_t^{-(1-\alpha)(1-\gamma)}}{1-\gamma} [F(\mathbf{X}_t, T-t)]^{\gamma}. \end{aligned} \tag{20}$$

The optimal strategy uncovers the typical structure *à la* Merton:

$$\boldsymbol{\omega}_t = \frac{1}{\gamma}(\boldsymbol{\Sigma}'\boldsymbol{\Sigma})^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Lambda}_t - (1-\alpha)\frac{1-\gamma}{\gamma}(\boldsymbol{\Sigma}'\boldsymbol{\Sigma})^{-1}\boldsymbol{\Sigma}'\boldsymbol{\sigma}_P + (\boldsymbol{\Sigma}'\boldsymbol{\Sigma})^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Sigma}_{\mathbf{X}}\frac{(F_{\mathbf{X}})'}{F}, \tag{21}$$

where $F_{\mathbf{X}}$ is the column vector of the partial derivatives of $F$. As we show in Online Appendix B.1,

$$F(\mathbf{X}_t, T-t) = \exp\left\{\frac{1}{2}\mathbf{X}_t'\mathbf{B}_3(T-t)\mathbf{X}_t + \mathbf{B}_2(T-t)\mathbf{X}_t + B_1(T-t)\right\}, \tag{22}$$

where $B_1(T-t)$, $\mathbf{B}_2(T-t)$, and $\mathbf{B}_3(T-t)$ are the solution of a system of Riccati equations.

Taking $\tilde{\mathbf{B}}_3(T-t) = (\mathbf{B}_3(T-t) + \mathbf{B}_3'(T-t))/2$, we can rewrite the optimal portfolio strategy (21) as:

$$\begin{aligned} \boldsymbol{\omega}_t = &\frac{1}{\gamma}(\boldsymbol{\Sigma}'\boldsymbol{\Sigma})^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Lambda}_t - (1-\alpha)\frac{1-\gamma}{\gamma}(\boldsymbol{\Sigma}'\boldsymbol{\Sigma})^{-1}\boldsymbol{\Sigma}'\boldsymbol{\sigma}_P \\ &+(\boldsymbol{\Sigma}'\boldsymbol{\Sigma})^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Sigma}_{\mathbf{X}}(\tilde{\mathbf{B}}_3(T-t)\mathbf{X}_t + \mathbf{B}_2'(T-t)). \end{aligned} \tag{23}$$

The first component is the mean–variance speculative component, taking advantage of the instantaneous risk/return trade-off offered by the assets available for trade. The second component hedges instantaneous realized inflation risk and reflects a direct impact of money illusion. Notably, this term disappears under severe money illusion ($\alpha = 1$), which means that the investor is no longer concerned by realized inflation risk. The last term is the intertemporal hedging component, which is affected by money illusion in a nonlinear way. The portfolio strategy is still linear in the state variables $\mathbf{X}_t$, driving the short-term rate, the expected inflation, and the market prices of risk.

To better gather the economics behind the strategy in (21), note that it can be rewritten as:

$$\begin{aligned}\boldsymbol{\omega}_t &= \frac{1}{\gamma}(\boldsymbol{\Sigma}'\boldsymbol{\Sigma})^{-1}\boldsymbol{\Sigma}'[\boldsymbol{\Lambda}_t - (1-\alpha)\boldsymbol{\sigma}_P] + (1-\alpha)(\boldsymbol{\Sigma}'\boldsymbol{\Sigma})^{-1}\boldsymbol{\Sigma}'\boldsymbol{\sigma}_P \\ &\quad + (\boldsymbol{\Sigma}'\boldsymbol{\Sigma})^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Sigma}_{\mathbf{X}}(\tilde{\mathbf{B}}_3(T-t)\mathbf{X}_t + \mathbf{B}_2'(T-t)). \end{aligned} \tag{24}$$

The investor is interested in the real risk/return trade-off, and thus in the real risk premia that traded assets can offer. Consequently, for a nonillusioned investor, the first mean–variance component in (24) involves the real market prices of risk $\boldsymbol{\Lambda}_t - \boldsymbol{\sigma}_P$, which do indeed represent the volatility of the real SDF. However, in the presence of money illusion, the investor accounts to a lesser extent for realized inflation volatility, even ignoring it under severe money illusion ($\alpha = 1$). The same happens to the second term, which does not depend on risk aversion: a nonillusioned investor attempts to hedge (perfectly or imperfectly) the inflation risk exposure, but, once again, money illusion distorts this behavior and leads a perfectly illusioned investor to ignore unexpected inflation risk. Overall, money illusion affects the risk/return trade-off perceived by the investor, as well as the unexpected inflation risk to hedge.

The intertemporal hedging component contributes with horizon-dependent effects to the portfolio strategy. To assess the potential impact of money illusion on this component, we start by looking at the case of an extremely conservative investor, that is, an investor with an infinite risk aversion. We show in Online Appendix B.2 that, when the market prices of risk are constant ($\boldsymbol{\Lambda}_1 = \mathbf{0}$) and bonds with maturities equal to the remaining horizon, $T-t$, are available, the optimal strategy is

$$\boldsymbol{\omega}_t^{\gamma\to\infty} = (1-\alpha)(\boldsymbol{\Sigma}'\boldsymbol{\Sigma})^{-1}\boldsymbol{\Sigma}'\left(\boldsymbol{\Sigma}_{\mathbf{X}}\mathbf{A}_1^I(T-t) + \boldsymbol{\sigma}_P\right) + \alpha(\boldsymbol{\Sigma}'\boldsymbol{\Sigma})^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Sigma}_{\mathbf{X}}\mathbf{A}_1^B(T-t) \tag{25}$$

$$\begin{aligned} &= (\boldsymbol{\Sigma}'\boldsymbol{\Sigma})^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Sigma}_{\mathbf{X}}[\mathbf{R}_1'(\mathbf{e}^{-\boldsymbol{\Theta}(T-t)} - \mathbf{I})\boldsymbol{\Theta}^{-1}]' \\ &\quad + (1-\alpha)(\boldsymbol{\Sigma}'\boldsymbol{\Sigma})^{-1}\boldsymbol{\Sigma}'\left(\boldsymbol{\sigma}_P - \boldsymbol{\Sigma}_{\mathbf{X}}\left[\boldsymbol{\pi}_1'\left(\mathbf{e}^{-\boldsymbol{\Theta}(T-t)} - \mathbf{I}\right)\boldsymbol{\Theta}^{-1}\right]'\right). \end{aligned} \tag{26}$$

According to (25), recalling that the volatility vector of an indexed bond is $\boldsymbol{\Sigma}_{\mathbf{X}}\mathbf{A}_1^I(T-t) + \boldsymbol{\sigma}_P$ and that the volatility vector of a nominal bond is $\boldsymbol{\Sigma}_{\mathbf{X}}\mathbf{A}_1^B(T-t)$, it is clear that a nonillusioned investor ($\alpha = 0$) avoids speculating through the risky assets (bonds and stocks) and invests all the wealth in an indexed bond, which maturity coincides with the investment horizon. A money-illusioned investor, conversely, combines an indexed and a nominal bond, both maturing at her investment horizon. The relative weight of the two depends on the degree of money illusion. An extremely illusioned investor ($\alpha = 1$) invests only in a nominal bond with the appropriate maturity. Representing the portfolio weights as in (26), it appears that nonillusioned investors, other than hedging unexpected inflation, aim to hedge the real rate, that is, the difference between the nominal rate (loading on the state variables with the vector $\mathbf{R}_1$) and expected inflation (loading on the state variables with the vector $\boldsymbol{\pi}_1$). Conversely, money-illusioned investors focus on hedging the short-term nominal rate only, since the two components related to realized and expected inflation vanish for $\alpha = 1$. Clearly, if expected inflation affects the nominal short-term rate (for example, through a monetary policy rule), the illusioned investor indirectly still partially hedges expected inflation, but expected inflation is not a source of risk that matters *per se*.

When risk aversion is finite and risk premia are allowed to vary over time, there is no immediately interpretable explicit solution. However, it is worth recalling that:

$$F(\mathbf{X}_t, T-t) = E_t\left[\left(\frac{\Phi_T^{\nu^*}}{\Phi_t^{\nu^*}}\left(\frac{P_T}{P_t}\right)^{1-\alpha}\right)^{1-\frac{1}{\gamma}}\right]. \tag{27}$$

This representation provides an interesting perspective on the intertemporal hedging component. A nonillusioned investor focuses on the quantities driving the real pricing kernel $\phi_t^{\nu^*} = \Phi_t^{\nu^*} P_t$, which are the real rate, unexpected inflation, and the market prices of risk. A severely money-illusioned investor instead focuses on the risks driving the nominal pricing kernel, which are the nominal short-term rate and the market prices of risk. Expected and realized inflation are not relevant to a money-illusioned investor. In the empirical analysis, we assess the quantitative importance of these effects.

## 2. ESTIMATION

In this section, we estimate the model. After describing the data set used, we present the methodology employed and highlight the characteristics of the different specifications for the risk premia that we consider. Finally, we discuss the estimates of the parameters.

### *2.1 Data*

We estimate the model using U.S. monthly data from January 31, 1999 until December 31, 2019. We consider zero-coupon nominal yields for the following maturities: 3 and 6 months, and 1, 2, 3, 5, 7, and 10 years. The 3- and 6-month yields are obtained from the Treasury Bill rates, available on the Federal Reserve Economic Data website [7] (series `GS3M` and `GS6M`). The other nominal zero-coupon yields are the

7. https://research.stlouisfed.org/fred2/.

series fitted by Gürkaynak, Sack, and Wright (2007), [8] available on the website of the Federal Reserve Board. [9] We use zero-coupon real yields for the maturities of 5, 7, and 10 years, as fitted in Gürkaynak, Sack, and Wright (2010), which are also available on the website of the Federal Reserve Board. [10] As a broad U.S. stock market index, we consider the CRSP NYSE/Amex/NASDAQ/ARCA Value-Weighted Market Index. To compute monthly realized inflation, we use the Consumer Price Index for All Urban Consumers: All Items, available on the Federal Reserve Economic Data website (item `CPIAUCSL`). Finally, we consider the 10-year stock return forecast and the 10-year forward average of the 10-year bond yield from the Survey of Professional Forecasters (median responses for `STOCK10` and `BOND10`, respectively). [11]

## 2.2 *Methodology*

Along the lines of Joslin, Singleton, and Zhu (2011), we estimate the model by maximum likelihood, choosing as pricing factors the first three principal components of the whole set of the time series of observed nominal and real yields, which we collect in the vector of state variables $\mathbf{X}_t$. [12][13] We stack the state variables $\mathbf{X}_t$, the log price index $\log(P_t)$, and the log stock index level $\log(S_t)$ into a column vector $\mathbf{Z}_t$:

$$\mathbf{Z}_t = \begin{bmatrix} X_t^1 & X_t^2 & X_t^3 & \log(P_t) & \log(S_t) \end{bmatrix}'. \tag{28}$$

The joint dynamics of the elements of $\mathbf{Z}_t$ is

$$d\mathbf{Z}_t = \mathbf{B}dt + \mathbf{A}\mathbf{Z}_t dt + \mathbf{\Sigma}'_{\mathbf{Z}} d\mathbf{z}_t, \tag{29}$$

8. The yield curves are fitted for maturities between 1 and 30 years. As the authors discourage extrapolations outside of this range, for the short end of the yield curve we complement their data set using Treasury Bill rates.

9. https://www.federalreserve.gov/pubs/feds/2006/200628/200628abs.html.

10. https://www.federalreserve.gov/pubs/feds/2008/200805/200805abs.html.

11. https://www.philadelphiafed.org/research-and-data/real-time-center/survey-of-professional-forecasters/data-files.

12. We also tried considering two separate level factors for the nominal and real yields, as in Christensen, Lopez, and Rudebusch (2010), but we did not notice any significant difference in the quality of the estimates, nor in the implications in terms of portfolio choice. This is probably because of the fact that, unlike in their work, we do not impose any constraints on the mean-reversion matrix, as we are not interested in obtaining a representation *à la* Nelson–Siegel of the term structure.

13. The choice of the number of state variables comes from a trade-off between goodness-of-fit and parsimony. Duffee (2011) pointed out that fully fledged four- or five-factor essentially affine term structure models may lead to unreasonably high portfolio positions in asset allocation contexts. As D'Amico, Kim, and Wei (2018) found, the first three principal components of the joint set of nominal and TIPS yields explain more than 94% of their variance, which we deem to be enough to capture the joint price dynamics of the assets in our investable universe.

where the column vector $\mathbf{B}$ and the matrix $\mathbf{A}$ can be compactly written as

$$\mathbf{B} = \begin{bmatrix} \boldsymbol{\Theta}\bar{\mathbf{X}} \\ \pi_0 - \frac{\|\boldsymbol{\sigma}_P\|^2}{2} \\ R_0 + \boldsymbol{\sigma}_S' \boldsymbol{\Lambda}_0 - \frac{\|\boldsymbol{\sigma}_S\|^2}{2} \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} & 0 & 0 \\ -\boldsymbol{\Theta} & 0 & 0 \\ & 0 & 0 \\ \boldsymbol{\pi}_1' & 0 & 0 \\ \mathbf{R}_1' + \boldsymbol{\sigma}_S' \boldsymbol{\Lambda}_1 & 0 & 0 \end{bmatrix}, \tag{30}$$

where we set $\bar{\mathbf{X}} = \mathbf{0}$, as the state variables are the principal components of bond yields and are centered on zero. Note that the last two columns of $\mathbf{A}$ are zeros, as only the three state variables $\mathbf{X}_t$, loaded by the nonzero columns of $\mathbf{A}$, determine the drift of the quantities in $\mathbf{Z}_t$. The volatility matrix $\boldsymbol{\Sigma}_{\mathbf{Z}}$ is obtained by juxtaposing $\boldsymbol{\Sigma}_{\mathbf{X}}$, $\boldsymbol{\sigma}_P$, and $\boldsymbol{\sigma}_S$:

$$\boldsymbol{\Sigma}_{\mathbf{X}} = \begin{bmatrix} \Sigma_{\mathbf{X}}(1,1) & \Sigma_{\mathbf{X}}(1,2) & \Sigma_{\mathbf{X}}(1,3) \\ 0 & \Sigma_{\mathbf{X}}(2,2) & \Sigma_{\mathbf{X}}(2,3) \\ 0 & 0 & \Sigma_{\mathbf{X}}(3,3) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\sigma}_P = \begin{bmatrix} \sigma_P(1) \\ \sigma_P(2) \\ \sigma_P(3) \\ \sigma_P(4) \\ 0 \end{bmatrix}, \quad \boldsymbol{\sigma}_S = \begin{bmatrix} \sigma_S(1) \\ \sigma_S(2) \\ \sigma_S(3) \\ \sigma_S(4) \\ \sigma_S(5) \end{bmatrix}. \tag{31}$$

Applying an Euler scheme, we perform an exact discretization of the multivariate continuous-time process $\mathbf{Z}_t$, which determines the first contribution to the log-likelihood function. [14] The second contribution is obtained by imposing the bond pricing restrictions, which relate the current value of the state variables $\mathbf{X}_t$ to the observed nominal and real bond yields. We allow for Gaussian observation errors, uncorrelated both in time series and cross-sectionally, with a constant standard deviation $\sigma_\epsilon^B$ for the nominal yields and $\sigma_\epsilon^I$ for the real yields. We also impose that the 10-year forward expected average of the 10-year bond yield and the 10-year stock return forecast from the Survey of Professional Forecasters are equal to the model-implied quantities (derived in Online Appendices A.1 and A.3) plus an observation error with standard deviation, respectively, $\sigma_\epsilon^{BF}$ and $\sigma_\epsilon^{SF}$.

We numerically maximize the likelihood function with respect to the whole set of model parameters at the same time, by considering different alternatives to the restrictions that can be imposed on the dynamics of the market prices of risk. First, we consider a specification with the restriction that the asset risk premia are constant, which we obtain imposing that $\boldsymbol{\Lambda}_1 = \mathbf{0}$. Second, we consider a specification with time-varying risk premia, initially with no restrictions imposed on the matrix $\boldsymbol{\Lambda}_1$. As in Christensen, Lopez, and Rudebusch (2010), we then recursively iterate the estimation, by imposing, at each step, a zero restriction on the element of $\boldsymbol{\Lambda}_1$ with the lowest

14. For details on the exact discretization of the continuous-time process and on the construction of the likelihood function, we refer the reader to Sangvinatsos and Wachter (2005), Koijen, Nijman, and Werker (2010), or Lioui and Tarelli (2019).

$t$-statistics, stopping when all the elements of $\mathbf{\Lambda}_1$ have a $t$-stat higher than 2. [15] Third, we consider a specification where we let the risk premia of the risky assets vary, but constraining their volatility to some reasonable values. In particular, we impose that the volatility of the risk premia of the nominal bonds, $\|\mathbf{A}_1^B(\tau)\mathbf{\Sigma}_\mathbf{X}'\mathbf{\Lambda}_1\mathbf{\Sigma}_\mathbf{X}'\|$, and the real bonds, $\|(\mathbf{A}_1^I(\tau)\mathbf{\Sigma}_\mathbf{X}' + \boldsymbol{\sigma}_P')\mathbf{\Lambda}_1\mathbf{\Sigma}_\mathbf{X}'\|$, are not higher than the volatility of the short-term interest rate, which is equal to 0.59% per annum. [16] We also impose that the volatility of the risk premium of unexpected inflation, $\|\boldsymbol{\sigma}_P'\mathbf{\Lambda}_1\mathbf{\Sigma}_\mathbf{X}'\|$, is not higher than 0.5% per annum and, finally, that the volatility of the equity premium, $\|\boldsymbol{\sigma}_S'\mathbf{\Lambda}_1\mathbf{\Sigma}_\mathbf{X}'\|$, is not higher than 1% per annum. By imposing economically reasonable restrictions on the time variation of risk premia, this methodology attempts to implement, as suggested by Sarno, Schneider, and Wagner (2016), a modeling approach that is flexible but limits overfitting.

For the empirical study, we consider as base case the results obtained for constant risk premia, which is for our purpose the most reliable framework, allowing us to focus on the roles of interest rate, expected inflation, and unexpected inflation hedging. We also present the results obtained for the case of volatility-constrained time-varying risk premia. We relegate to Online Appendix E the results obtained in the case where the statistically significant elements of $\mathbf{\Lambda}_1$ are left unconstrained, justifying why we deem that this framework is not appropriate for the analysis. In particular, Online Appendix E.2 presents an out-of-sample forecasting exercise, where we compare the three specifications for the risk premia. As shown in Online Appendix Table A.4, the model based on constant risk premia better predicts nominal and real bond yield variations for horizons of one year or longer, followed by the volatility-constrained model. Overall, the unconstrained specification shows poor predictive ability.

### *2.3 Parameter Estimates*

Table 1 shows the parameter estimates for the specification with constant risk premia (Panel A) and with volatility-constrained risk premia (Panel B). In the second case, as standard inference methodologies do not apply when binding constraints are imposed to the parameter space, we adopt the bootstrapping methodology outlined in Tibshirani and Efron (1993) and employed by Ireland (2015) to estimate constrained dynamic term structure models. We simulate 1,000 trajectories of artificial data using the point estimates of the parameters obtained from the true data, and then repeat the estimation for each one of the trajectories, considering a sample size equal to the original data set. The standard errors in Table 1, Panel B are the standard deviations of the estimates obtained over the 1,000 replications. The estimates of the parameters

15. Similar procedures have been followed by several authors, such as Duffee (2002), Sangvinatsos and Wachter (2005) and Christensen, Lopez, and Rudebusch (2010). Joslin, Priebsch, and Singleton (2014) performed a model selection among all possible sets of zero restrictions on the elements of $\mathbf{\Lambda}_1$. This would entail very long computation times in our continuous-time framework.

16. We choose to impose this restriction for the 3- and 10-year nominal bonds, and for the 7-year real bond.

TABLE 1

Parameter Estimates

Panel A. Constant risk premia

| $R_0$ | $\mathbf{R}_1$ | $\pi_0$ | $\boldsymbol{\pi}_1$ | | $\boldsymbol{\Theta}$ | | $\sigma_\epsilon^B$ | $\sigma_\epsilon^I$ |
|---|---|---|---|---|---|---|---|---|
| 0.0178 | 0.3584 | 0.0211 | 0.0553 | 0.0666 | −0.5903 | 0.7883 | 0.0013 | 0.0007 |
| (0.0001) | (0.0012) | (0.0018) | (0.0040) | (0.0026) | (0.0116) | (0.0275) | (0.0000) | (0.0000) |
| | −0.4303 | | −0.6751 | 0.0134 | 0.2183 | −0.1604 | | |
| | (0.0045) | | (0.0232) | (0.0017) | (0.0071) | (0.0169) | | |
| | 0.2193 | | −1.0766 | −0.0126 | 0.1001 | 0.1717 | | |
| | (0.0114) | | (0.0474) | (0.0014) | (0.0072) | (0.0159) | | |

| $\boldsymbol{\Lambda}_0$ | | $\boldsymbol{\Lambda}_1$ | | | $\boldsymbol{\Sigma}_X$ | | $\boldsymbol{\sigma}_P$ | $\boldsymbol{\sigma}_S$ |
|---|---|---|---|---|---|---|---|---|
| −0.6408 | 0 | 0 | 0 | 0.0203 | 0.0063 | −0.0033 | 0.0003 | 0.0368 |
| (0.0289) | | | | (0.0009) | (0.0008) | (0.0005) | (0.0006) | (0.0093) |
| 0.4478 | 0 | 0 | 0 | 0 | 0.0113 | 0.0016 | −0.0004 | −0.0267 |
| (0.0449) | | | | | (0.0005) | (0.0005) | (0.0006) | (0.0093) |
| −0.0917 | 0 | 0 | 0 | 0 | 0 | 0.0083 | −0.0038 | −0.0461 |
| (0.0487) | | | | | | (0.0004) | (0.0005) | (0.0087) |
| 1.2215 | 0 | 0 | 0 | 0 | 0 | 0 | 0.0084 | −0.0183 |
| (0.2363) | | | | | | | (0.0004) | (0.0086) |
| 0.8481 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.1350 |
| (0.1046) | | | | | | | | (0.0060) |

Panel B. Time-varying risk premia with volatility constraints

| $R_0$ | $\mathbf{R}_1$ | $\pi_0$ | $\boldsymbol{\pi}_1$ | | $\boldsymbol{\Theta}$ | | $\sigma_\epsilon^B$ | $\sigma_\epsilon^I$ |
|---|---|---|---|---|---|---|---|---|
| 0.0178 | 0.3595 | 0.0210 | 0.0745 | 0.0710 | −0.4387 | 1.5060 | 0.0013 | 0.0007 |
| (0.0001) | (0.0021) | (0.0030) | (0.0472) | (0.0574) | (0.1182) | (0.1576) | (0.0000) | (0.0000) |
| | −0.4296 | | −0.6174 | 0.0319 | 0.1967 | −0.4457 | | |
| | (0.0057) | | (0.1053) | (0.0232) | (0.0575) | (0.1042) | | |
| | 0.2164 | | −1.5915 | −0.0123 | 0.0985 | 0.3965 | | |
| | (0.0110) | | (0.1333) | (0.0244) | (0.0501) | (0.0606) | | |

| $\boldsymbol{\Lambda}_0$ | | $\boldsymbol{\Lambda}_1$ | | | $\boldsymbol{\Sigma}_X$ | | $\boldsymbol{\sigma}_P$ | $\boldsymbol{\sigma}_S$ |
|---|---|---|---|---|---|---|---|---|
| −0.6585 | −0.0493 | −7.5595 | −36.6765 | 0.0198 | 0.0064 | −0.0033 | 0.0002 | 0.0355 |
| (0.0325) | (2.9791) | (6.0711) | (8.1156) | (0.0009) | (0.0007) | (0.0006) | (0.0006) | (0.0096) |
| 0.4755 | −1.8723 | 6.0817 | 47.3778 | 0 | 0.0111 | 0.0016 | −0.0003 | −0.0263 |
| (0.0493) | (3.7726) | (9.1421) | (13.2234) | | (0.0005) | (0.0005) | (0.0005) | (0.0094) |
| −0.1158 | 0.6523 | −3.8847 | −52.6291 | 0 | 0 | 0.0081 | −0.0039 | −0.0456 |
| (0.0572) | (4.0031) | (9.7640) | (10.5519) | | | (0.0004) | (0.0006) | (0.0097) |
| 1.1883 | 3.2380 | 5.6039 | −87.1044 | 0 | 0 | 0 | 0.0081 | −0.0191 |
| (0.3835) | (5.8964) | (13.6410) | (19.7856) | | | | (0.0004) | (0.0088) |
| 0.8476 | −0.7221 | 4.3413 | −20.0573 | 0 | 0 | 0 | 0 | 0.1346 |
| (0.1200) | (2.3976) | (5.0732) | (9.8221) | | | | | (0.0058) |

Note: The tables show the maximum-likelihood estimates of the model parameters. The sample period runs from January 1999 until December 2019. Panel A shows the parameter estimates obtained for the specification with constant risk premia ($\boldsymbol{\Lambda}_1 = \mathbf{0}$). The values in parentheses are the asymptotic standard errors of the estimates. Panel B shows the parameter estimates obtained for the specification with time-varying risk premia and restrictions on the maximum volatility of bond, realized inflation and stock risk premia. The volatility of the risk premia for the 3- and 10-year nominal bonds, and for the 7-year inflation-indexed bond, are imposed to be lower than the volatility of the short-term nominal rate (0.6% per annum). The volatility of the realized inflation risk premium is not higher than 0.5% per annum and the volatility of the equity premium is not higher than 1% per annum. The values in parentheses are the standard errors of the estimates.

related to the instantaneous nominal risk-free rate, $R_0$ and $\mathbf{R}_1$, are, as expected, almost identical in the two settings, as well as the volatility vectors and the vector of constant risk premia $\mathbf{\Lambda}_0$. $R_0$ is very close to the average 3-month nominal yield (1.78% versus 1.84%). The drift of the price index under the objective probability measure, $\pi_0$, is about 2.10% in both settings. The average model-implied instantaneous real rate, $R_0 - \pi_0 + \boldsymbol{\sigma}_P' \mathbf{\Lambda}_0$, is equal to 0.68% and 0.66%, respectively. The vector of loadings $\boldsymbol{\pi}_1$ is different between the two settings, but we verified that, as expected, the corresponding quantities under the pricing measure, $\boldsymbol{\pi}_1 - \mathbf{\Lambda}_1' \boldsymbol{\sigma}_P$, are nearly identical. The same applies to the mean-reversion matrices, $\mathbf{\Theta}$, which are different between the two settings, but the quantities $\mathbf{\Theta} + \mathbf{\Sigma}_{\mathbf{X}}' \mathbf{\Lambda}_1$ are very similar. Finally, the standard deviations of the observation errors of nominal yields, $\sigma_\epsilon^B$, are in both settings equal to 13 basis points, while the corresponding quantities for real yields, $\sigma_\epsilon^I$, are both equal to 7 basis points. [17] The standard deviations of the observation errors of the 10-year average yield forecast, $\sigma_\epsilon^{BF}$, in the two settings are, respectively, 1.04% and 1.01%, while those for the 10-year stock return forecast, $\sigma_\epsilon^{SF}$, are, respectively, 0.84% and 0.50%.

The goodness of fit relative to the realized distributions can be checked in Table 2, where we report the annualized mean values and the volatilities, both historical and model-implied, of bond yields, realized inflation, and realized equity returns. The two specifications fit the historical moments very well. The model-implied means of the risk premia are also very similar in the two settings. The fitted risk premia of the nominal bonds are about 0.2–0.4% higher than the risk premia of inflation-indexed bonds for the same maturities. The unexpected inflation risk premium is nonnegligible and about 1% in both settings, while the equity premium is just above 6%. In the second setting, the volatilities of the risk premia are close to the bounds imposed, that is, the bond premia volatilities are close to the volatility of the historical 3-month rate (0.59%), while the realized inflation and equity risk premia volatilities are, respectively, 0.5% and 1%. Finally, the short-term rate volatilities are similar between the two settings, while the model-implied volatility of the expected inflation is slightly higher when the risk premia are time varying.

Table 3 shows the pairwise correlations between returns and economic variables, both from the historical distribution and as implied by the estimated parameters for the two specifications. The model-implied pairwise correlations are rather similar across models and fit the historical values reasonably well, with some exceptions among the correlations involving short-term nominal yields. Real bond returns, differing from nominal bond returns, are positively correlated with the price index. They also have

17. The standard deviation of the observation errors of TIPS yields that we obtain is in line with the measurement errors reported by D'Amico, Kim, and Wei (2018) for their four-factor model specifications including a TIPS liquidity factor, while it is much lower than in their proposed three-factor specifications that do not include the additional factor. The different performance in pricing TIPS of their three-factor models is probably due to our choice of using as state variables the first principal components of both nominal and real yields over the full sample, while they filter the latent variables from a large set of nominal yields and few TIPS yields, which, additionally, are available only for part of the sample period in their data set.

TABLE 2

Historical and Model-Implied Summary Statistics

Panel A. Constant risk premia

| Time series | Mean value | | Volatility | |
|---|---|---|---|---|
| | Estimation | Data | Estimation | Data |
| 3M nominal yield | 1.84% | 1.81% | 0.61% | 0.61% |
| 6M nominal yield | 1.90% | 1.91% | 0.60% | 0.59% |
| 1Y nominal yield | 2.01% | 2.05% | 0.61% | 0.69% |
| 2Y nominal yield | 2.25% | 2.25% | 0.65% | 0.80% |
| 3Y nominal yield | 2.49% | 2.47% | 0.72% | 0.86% |
| 5Y nominal yield | 2.92% | 2.91% | 0.85% | 0.90% |
| 7Y nominal yield | 3.29% | 3.28% | 0.93% | 0.91% |
| 10Y nominal yield | 3.70% | 3.70% | 0.95% | 0.91% |
| 5Y real yield | 1.10% | 1.09% | 0.85% | 0.91% |
| 7Y real yield | 1.31% | 1.33% | 0.82% | 0.81% |
| 10Y real yield | 1.58% | 1.57% | 0.81% | 0.72% |
| Log realized inflation | 2.11% | 2.15% | 0.92% | 0.99% |
| Equity log returns | 6.72% | 6.91% | 15.09% | 15.10% |
| 3M nominal risk premium | 0.12% | | 0.00% | |
| 6M nominal risk premium | 0.24% | | 0.00% | |
| 1Y nominal risk premium | 0.48% | | 0.00% | |
| 2Y nominal risk premium | 0.96% | | 0.00% | |
| 3Y nominal risk premium | 1.42% | | 0.00% | |
| 5Y nominal risk premium | 2.24% | | 0.00% | |
| 7Y nominal risk premium | 2.88% | | 0.00% | |
| 10Y nominal risk premium | 3.50% | | 0.00% | |
| 5Y real risk premium | 2.05% | | 0.00% | |
| 7Y real risk premium | 2.51% | | 0.00% | |
| 10Y real risk premium | 3.07% | | 0.00% | |
| Unexpected inflation risk premium | 1.02% | | 0.00% | |
| Equity risk premium | 6.08% | | 0.00% | |
| Nominal risk-free rate | 1.78% | | 0.62% | |
| Expected inflation | 2.11% | | 1.29% | |

(*Continued*)

a weak correlation with equity returns, while nominal bond returns are negatively correlated with the equity. Nominal and real bond returns corresponding to the same maturities tend to be strongly positively correlated. Furthermore, it is interesting to observe the model-implied correlations between asset returns and the unobservable economic variables. Nominal bond returns are more (negatively) correlated with the short-term rate $R$ relative to real bond returns for the same maturities. Real bond returns, differing from nominal bond returns, are strongly positively correlated with the innovations in the expected inflation $\pi$ and strongly negatively correlated with the real rate $r$.

The first row of graphs in Figure 2 shows, for the two specifications, the model-implied short-term interest rate, the expected inflation, and the break-even inflation. The instantaneous expected inflation is slightly higher than the break-even inflation, and the difference between the two is the unexpected inflation risk premium. The second graphs from the top show the model-implied 10-year nominal and real yields,

TABLE 2
(CONTINUED)

| Panel B. Constrained time-varying risk premia | | | | |
|---|---|---|---|---|
| | Mean value | | Volatility | |
| Time series | Estimation | Data | Estimation | Data |
| 3M nominal yield | 1.84% | 1.81% | 0.59% | 0.61% |
| 6M nominal yield | 1.90% | 1.91% | 0.58% | 0.59% |
| 1Y nominal yield | 2.01% | 2.05% | 0.59% | 0.69% |
| 2Y nominal yield | 2.25% | 2.25% | 0.63% | 0.80% |
| 3Y nominal yield | 2.49% | 2.47% | 0.70% | 0.86% |
| 5Y nominal yield | 2.92% | 2.91% | 0.83% | 0.90% |
| 7Y nominal yield | 3.28% | 3.28% | 0.92% | 0.91% |
| 10Y nominal yield | 3.70% | 3.70% | 0.96% | 0.91% |
| 5Y real yield | 1.10% | 1.09% | 0.83% | 0.91% |
| 7Y real yield | 1.31% | 1.33% | 0.81% | 0.81% |
| 10Y real yield | 1.58% | 1.57% | 0.80% | 0.72% |
| Log realized inflation | 2.10% | 2.15% | 0.90% | 0.99% |
| Equity log returns | 6.72% | 6.91% | 15.00% | 15.10% |
| 3M nominal risk premium | 0.12% | | 0.10% | |
| 6M nominal risk premium | 0.24% | | 0.18% | |
| 1Y nominal risk premium | 0.48% | | 0.32% | |
| 2Y nominal risk premium | 0.96% | | 0.51% | |
| 3Y nominal risk premium | 1.42% | | 0.59% | |
| 5Y nominal risk premium | 2.23% | | 0.54% | |
| 7Y nominal risk premium | 2.87% | | 0.42% | |
| 10Y nominal risk premium | 3.52% | | 0.59% | |
| 5Y real risk premium | 2.04% | | 0.60% | |
| 7Y real risk premium | 2.50% | | 0.64% | |
| 10Y real risk premium | 3.05% | | 0.59% | |
| Unexpected inflation risk premium | 0.99% | | 0.50% | |
| Equity risk premium | 6.07% | | 1.00% | |
| Nominal risk-free rate | 1.78% | | 0.60% | |
| Expected inflation | 2.10% | | 1.62% | |

NOTE:The tables show annualized historical and model-implied means and volatilities of bond yields, equity logarithmic returns, and realized inflation. The tables also show the model-implied means and volatilities of bond risk premia and equity risk premium, as well as of the nominal risk-free rate and the expected inflation.

as well as their difference, that is, the 10-year break-even inflation. This can be compared to the model-implied 10-year annualized expected inflation (derived in Online Appendix A.2). The difference between the expected and break-even inflation for a given maturity $\tau$ is constant and nearly zero under constant risk premia, while under time-varying risk premia it depends on the risk premia associated to the sources of risk driving the term structure of nominal and real interest rates, as well as to the unexpected inflation. In particular, the 10-year expected inflation tends to be higher than the 10-year break-even inflation until 2009 and slightly lower after 2009, as in D'Amico, Kim, and Wei (2018).

The third row of graphs in Figure 2 represents the risk premia of the 10-year nominal and real bonds, the stock and the realized inflation. For the model with volatility-constrained time-varying premia, these are centered on the values obtained under constant risk premia and are sensible, ranging between about 2% and 5% for the

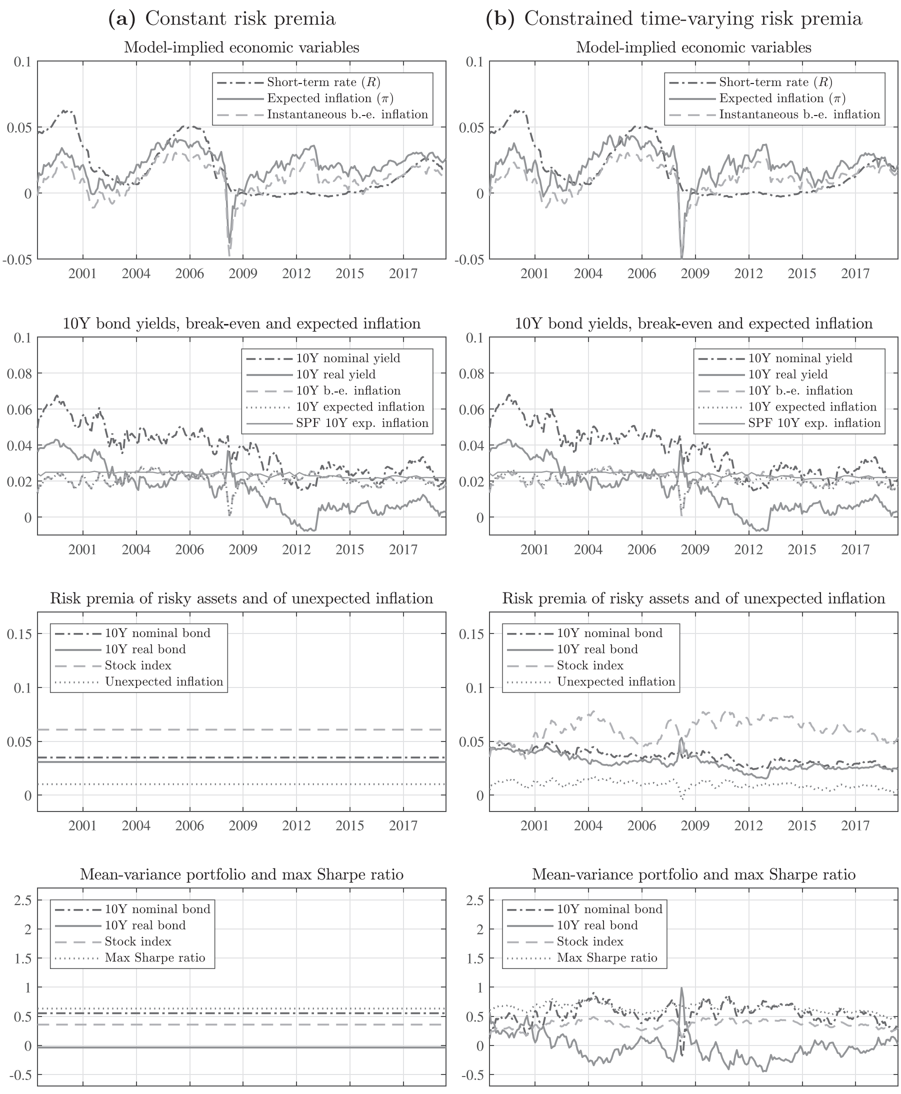


Fig 2. Time Series of Relevant Model-Implied and Observed Variables.

Notes: The top graphs show the model-implied instantaneous nominal rate, real rate, and break-even inflation. The second graphs from the top show the model-implied 10-year nominal yield, real yield, break-even inflation, expected inflation, and the 10-year CPI inflation rate forecast from the Survey of Professional Forecasters. The third graphs from the top show the model-implied risk premia of the risky assets and of the unexpected inflation. The bottom graphs show the portfolio allocation of a nominal mean–variance investor ($\gamma = 10$) and the corresponding maximum *ex ante* Sharpe ratio (considering a 10-year nominal bond, a 10-year inflation-indexed bond, and the stock index).

TABLE 3

Correlations between Asset Returns and Economic Variables

Panel A. Data

| | 3M nom | 1Y nom | 2Y nom | 5Y nom | 10Y nom | 5Y real | 10Y real | Equity | CPI |
|---|---|---|---|---|---|---|---|---|---|
| 3M nom | 1.000 | | | | | | | | |
| 1Y nom | 0.662 | 1.000 | | | | | | | |
| 2Y nom | 0.472 | 0.923 | 1.000 | | | | | | |
| 5Y nom | 0.263 | 0.713 | 0.893 | 1.000 | | | | | |
| 10Y nom | 0.122 | 0.513 | 0.685 | 0.911 | 1.000 | | | | |
| 5Y real | 0.018 | 0.327 | 0.420 | 0.504 | 0.478 | 1.000 | | | |
| 10Y real | 0.016 | 0.341 | 0.453 | 0.628 | 0.692 | 0.901 | 1.000 | | |
| Equity | −0.142 | −0.286 | −0.351 | −0.327 | −0.261 | 0.040 | 0.018 | 1.000 | |
| CPI | −0.106 | −0.144 | −0.142 | −0.197 | −0.237 | 0.344 | 0.140 | 0.073 | 1.000 |

Panel B. Constant risk premia

| | 3M nom | 1Y nom | 2Y nom | 5Y nom | 10Y nom | 5Y real | 10Y real | Equity | CPI | $R$ | $\pi$ | $r$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3M nom | 1.000 | | | | | | | | | | | |
| 1Y nom | 0.958 | 1.000 | | | | | | | | | | |
| 2Y nom | 0.834 | 0.956 | 1.000 | | | | | | | | | |
| 5Y nom | 0.508 | 0.726 | 0.895 | 1.000 | | | | | | | | |
| 10Y nom | 0.282 | 0.531 | 0.753 | 0.965 | 1.000 | | | | | | | |
| 5Y real | 0.222 | 0.315 | 0.399 | 0.503 | 0.575 | 1.000 | | | | | | |
| 10Y real | 0.182 | 0.356 | 0.520 | 0.715 | 0.804 | 0.943 | 1.000 | | | | | |
| Equity | −0.222 | −0.286 | −0.325 | −0.314 | −0.259 | 0.090 | −0.020 | 1.000 | | | | |
| CPI | 0.016 | −0.055 | −0.119 | −0.177 | −0.163 | 0.250 | 0.113 | 0.051 | 1.000 | | | |
| $R$ | −0.984 | −0.924 | −0.772 | −0.412 | −0.177 | −0.183 | −0.111 | 0.192 | −0.044 | 1.000 | | |
| $\pi$ | −0.357 | −0.375 | −0.347 | −0.197 | −0.030 | 0.718 | 0.545 | 0.349 | 0.370 | 0.344 | 1.000 | |
| $r$ | −0.121 | −0.071 | −0.025 | −0.001 | −0.058 | −0.849 | −0.630 | −0.270 | −0.412 | 0.143 | −0.880 | 1.000 |

Panel C. Constrained time-varying risk premia

| | 3M nom | 1Y nom | 2Y nom | 5Y nom | 10Y nom | 5Y real | 10Y real | Equity | CPI | $R$ | $\pi$ | $r$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3M nom | 1.000 | | | | | | | | | | | |
| 1Y nom | 0.959 | 1.000 | | | | | | | | | | |
| 2Y nom | 0.839 | 0.957 | 1.000 | | | | | | | | | |
| 5Y nom | 0.517 | 0.730 | 0.896 | 1.000 | | | | | | | | |
| 10Y nom | 0.291 | 0.536 | 0.754 | 0.965 | 1.000 | | | | | | | |
| 5Y real | 0.199 | 0.297 | 0.389 | 0.506 | 0.584 | 1.000 | | | | | | |
| 10Y real | 0.170 | 0.345 | 0.513 | 0.715 | 0.807 | 0.945 | 1.000 | | | | | |
| Equity | −0.225 | −0.286 | −0.322 | −0.310 | −0.255 | 0.091 | −0.018 | 1.000 | | | | |
| CPI | 0.025 | −0.049 | −0.118 | −0.182 | −0.172 | 0.260 | 0.115 | 0.053 | 1.000 | | | |
| $R$ | −0.982 | −0.919 | −0.768 | −0.410 | −0.176 | −0.172 | −0.104 | 0.186 | −0.067 | 1.000 | | |
| $\pi$ | −0.362 | −0.434 | −0.461 | −0.380 | −0.237 | 0.599 | 0.369 | 0.387 | 0.424 | 0.300 | 1.000 | |
| $r$ | −0.093 | −0.054 | −0.019 | −0.012 | −0.076 | −0.853 | −0.638 | −0.269 | −0.432 | 0.137 | −0.874 | 1.000 |

Note: Panel A shows the unconditional correlations between nominal and real bond returns, stock returns, and realized inflation, calculated from the monthly time series. Panels B and C report the one-month conditional pairwise correlations between nominal and real bond returns, stock returns, realized inflation, nominal interest rate, expected inflation, and real interest rate.

two 10-year bonds, between 4% and 8% for the stock market, and between about 0% and 2% for the realized inflation premium. In order to show that our estimates for the time-varying risk premia are able to provide realistic portfolio positions in a benchmark setting, in the bottom graphs we represent the allocation followed by a nominal mean–variance investor with a risk aversion $\gamma = 10$. [18] When risk premia are time-varying, the positions also vary with time. As can be noticed, the positions range from $-50\%$ to about $100\%$, without reaching excessive levels of leverage or short selling, as expected by an investor with a moderate level of risk aversion. We also show the maximum *ex ante* Sharpe ratio achievable with the three assets above, that is the quantity maximized by the strategy in the previous graphs. Its value is sensible, being just above 0.6 when the risk premia are constant, while it ranges between about 0.5 and 0.8 under time-varying risk premia. Our specification is thus robust to the issue raised by Duffee (2011), who shows that unconstrained essentially-affine term structure models may lead to unrealistically high conditional Sharpe ratios.

Figure A.6 in Online Appendix E.1 shows the same time series as in Figure 2, but under unconstrained time-varying risk premia. The time series of risk premia, mean–variance portfolio weights, and maximum achievable Sharpe ratio are subject to variations which are unreasonably large. [19] The unconstrained model overfits the data and, consequently, has the drawback of returning uncontrollably volatile time series for these unobservable quantities.

For the empirical analysis that follows, we adopt the model with constant risk premia as base case and then verify that similar results are obtained considering volatility-constrained risk premia. We discuss in Online Appendix E the findings under unconstrained risk premia, which, although qualitatively confirming most of our conclusions, are not quantitatively sensible. This is unsurprising, in light of the findings by Feldhütter et al. (2012) and Sarno, Schneider, and Wagner (2016), who argue that (unconstrained) essentially affine term structure models are very sensitive to estimation errors and in-sample overfitting, as well as of the in-sample empirical results in Sangvinatsos and Wachter (2005) and Barillas (2011), who find very large optimal portfolio positions and unrealistically high utility losses associated with suboptimal portfolio strategies. [20]

## 3. MAIN EMPIRICAL FINDINGS

We start our empirical investigation considering the case where risk premia are constant. We have already argued that this is a robust setting in an asset allocation context, being less prone to overfitting than models based on time-varying premia

18. The investor chooses the portfolio weights by maximizing the quantity $U = \mu - \frac{\gamma}{2}\sigma^2$, where $\mu$ is the expected nominal portfolio return, $\sigma$ is the volatility of portfolio returns, and $\gamma$ the risk aversion.

19. Duffee (2011) noticed the same empirical fact and attempted to reduce the issue of estimation overfitting by imposing a numerical constraint on the average value of the maximum achievable Sharpe ratio.

20. To the best of our knowledge, only Duffee (2002) documented an acceptable out-of-sample predictive ability of unconstrained essentially affine term structure models. This analysis, however, is based on rather small sample (less than 4 years of out-of-sample monthly observations). Duffee (2002) also found a suspiciously high in-sample model-implied volatility of bond risk premia.

and, as pointed out by Feldhütter et al. (2012), less sensitive to estimation errors. Furthermore, in this context, we can unambiguously associate the intertemporal hedging demands, as well as their variations corresponding to different degrees of money illusion, to the economic variables, such as the nominal short-term rate, the expected inflation, and the real rate.

After assessing the optimal portfolio for agents with different degrees of money illusion, we evaluate the expected utility losses that would be suffered by rational nonillusioned investors if they were forced to implement the suboptimal strategy of a money-illusioned agent. We thus measure the utility cost of money illusion when it represents an irrational behavior. Furthermore, and most importantly, we evaluate the utility losses, as perceived by investors with different degrees of money illusion, of removing inflation-indexed bonds from the investable universe. This is a measure of the subjective economic incentive to enter the TIPS market. For these analyses, we are interested in evaluating the *ex ante* welfare, as perceived by each category of investor, under the assumption that the agent believes that the dynamics of the market is given by the model that is adopted. As already mentioned, in this section we consider investors believing that risk premia are constant. This seems a sensible assumption for a large number of unsophisticated agents.

We first consider a conservative investor, that is, an agent with an infinite risk aversion, and we then consider a moderate investor, that is, an agent with a medium level of risk aversion ($\gamma = 10$). In the baseline analysis, the investor can trade a 10-year nominal bond, a 10-year inflation-indexed bond, a stock index, and a nominal money market account. [21]

### *3.1 Conservative Investor*

*Portfolio strategy.* In the case of $\gamma \to \infty$ and constant risk premia, it is possible to explicitly determine the optimal portfolio strategy, as in (25). Considering the state variables at their long-run means, Figure 3 shows the positions in the four assets for different degrees of money illusion. $\alpha = 0$ corresponds to a nonillusioned investor, $\alpha = 0.5$ to a partially illusioned investor and $\alpha = 1$ to a totally money-illusioned investor. For the U.S. market, there is evidence for a rather high level of money illusion. Through a consumption-based model, David and Veronesi (2013) estimate a degree of money illusion of 0.81 over a 50-year long sample of U.S. data. Similar estimates have been obtained by Maio (2018).

A conservative investor with a very short horizon invests only in the money market account (cash). Increasing the horizon implies an investment in the other assets. A particular case arises when the horizon equals the maturity of the bonds available for

21. As shown by Sangvinatsos and Wachter (2005), when several highly correlated bonds are available for trade, optimal unconstrained portfolio positions, as well as certainty equivalent wealth, may reach unrealistically high values. We follow their approach by making welfare considerations in an incomplete market, where the dynamics of the state variables are not perfectly spanned by the assets available for trade. The choice of having only one nominal and one inflation-indexed bond has the additional advantage of making the results particularly easy to interpret.

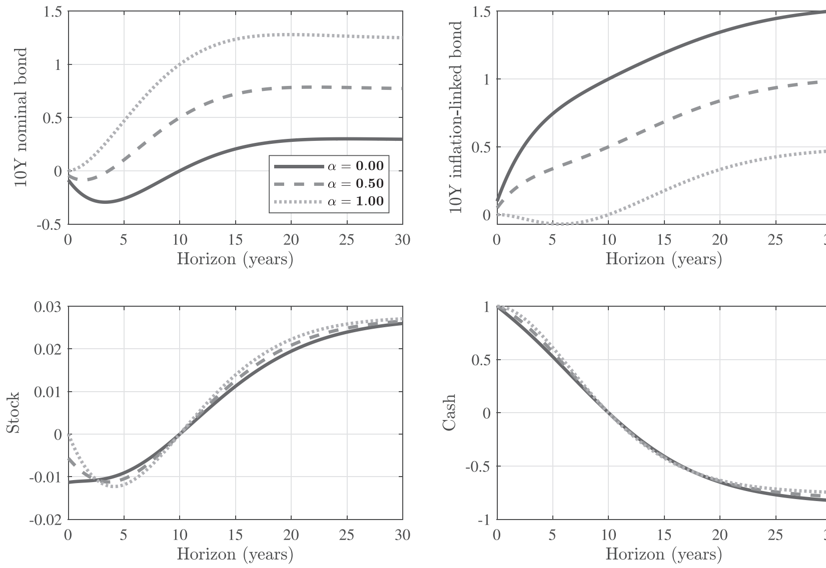


Fig 3. Optimal Portfolio Strategy for $\gamma \to \infty$ and Different Degrees of Money Illusion $\alpha$; Constant Risk Premia.

trade (10 years). In this case, a nonillusioned investor ($\alpha = 0$) allocates all her wealth in the indexed bond and nothing in the other assets. For intermediate levels of money illusion, the investor reduces the position in the real bond and increases the position in the nominal bond. A severely money-illusioned investor ($\alpha = 1$) invests only in the nominal bond. For any investment horizon, the stock and cash positions are barely affected by money illusion.

In terms of intertemporal hedging demands, an interesting pattern emerges: while the position in the nominal bond flattens when the investment horizon is above 10 years, the position in the real bond keeps increasing steadily. Roughly, a nonillusioned investor allocates 0% in the nominal bond if the investment horizon is 10 years and about 30% when the horizon is 20 or 30 years. The corresponding figures for the position in the indexed bond are 100%, 130%, and 150%, respectively. This is a direct consequence of the dynamics of expected inflation, which loads more than the nominal short-term rate on the less persistent state variables driving the economy ($X_2$ and $X_3$).

The stock position for a conservative investor is very small, it is almost not sensitive to money illusion and, as expected, is increasing in the investment horizon. It is equal to zero when the latter equals 10 years, that is, the maturity of the bonds. The cash position decreases from 100% for short horizons to zero when the horizon is 10 years, when the portfolio is fully invested in bonds. The cash position keeps decreasing toward negative values, implying leveraged positions in the other assets, for horizons longer than 10 years.

While money illusion strongly tilts the portfolio from indexed to nominal bonds, the indexed bond position is zero only when the nominal bond maturity is equal to the investment horizon and money illusion is severe. Overall, given that nominal bonds for almost any maturity up to 30 years exist, we can confidently say that money illusion drives conservative investors away from the indexed bond market, as a conservative money-illusioned investor allocates all wealth into a nominal bond maturing at her investment horizon. Conversely, if indexed bonds were available for all maturities, conservative nonillusioned investors would invest only in indexed bonds.

*Intertemporal hedging demands.* The horizon effects, that is, the intertemporal hedging components of the optimal portfolio strategy in Figure 3, are substantial, and money illusion causes a shift from the indexed to the nominal bond market. Given that market prices of risk are constant so far, is this effect related to expected inflation or to the nominal short-term rate? In Figure 4, we show the projections of the intertemporal hedging components of each asset position onto the expected inflation and the nominal short-term rate, as well as the corresponding orthogonal components.

As can be noticed, the component of the intertemporal hedging demands projected onto expected inflation is substantial and very sensitive to the degree of money illusion, while the orthogonal component is far less sensitive. The mechanism at play is the following: investors take long positions in indexed bonds and short nominal bonds in order to hedge expected inflation. Money illusion tends to make these positions decrease in absolute value. The intertemporal hedging demand of the stock is mostly correlated with expected inflation and, as stock returns are positively correlated with the expected inflation $\pi$ (see Table 3, Panel B), the hedging demand is decreasing with $\alpha$. However, as the inflation-indexed bond has better inflation-hedging properties, the stock position is very small.

The two bottom rows of graphs in Figure 4 deliver a totally different picture, as the intertemporal hedging demands projected onto the nominal short-term rate is negligible, and the sensitivity to $\alpha$ is also very small. One may be surprised that, even for a money-illusioned investor, the projection of the hedging demand onto the nominal risk-free rate is small. This is due to the low correlation between the short-term rate and the returns of the 10-year nominal and real bonds, as documented in Table 3.

In short, when risk premia are constant, the intertemporal hedging demands for a conservative investor are linked to the rationale of hedging expected inflation, while the nominal interest rate plays a marginal role. Money illusion tends to reduce the demand for assets hedging future variations of expected inflation. When both types of bonds are available, they are used to perform most of the intertemporal hedging activity, while the stock plays a marginal role.

*Impact of money illusion on equity investments.* In order to further investigate the impact of money illusion on stock positions, we show in Figure 5 the portfolio strategy of nonillusioned and illusioned investors when the inflation-indexed bond is not available in the asset universe. The nominal bond position is increasing in the investment horizon, differing from Figure 3 where it flattens when indexed bonds are available,

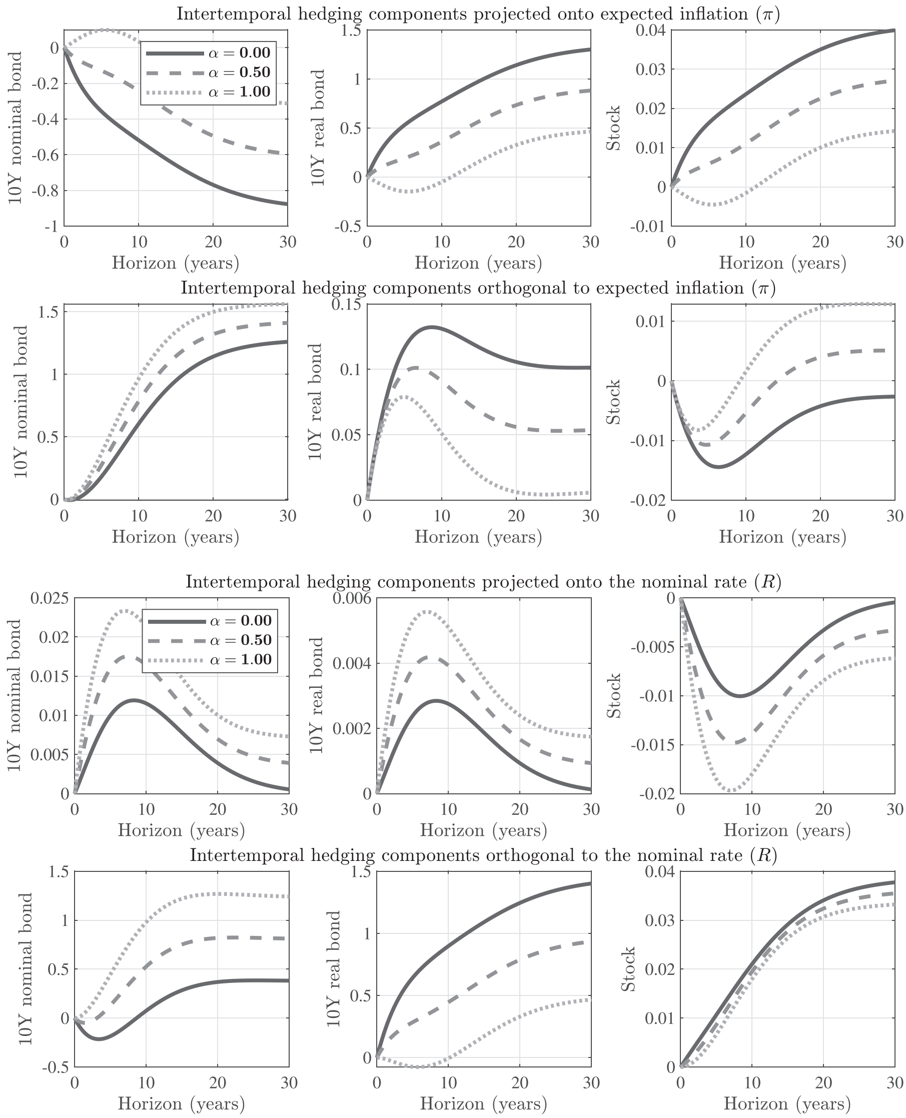


Fig 4. Intertemporal Hedging Components Projected onto (and Orthogonal to) Expected Inflation and the Nominal Short-Term Interest Rate for $\gamma \to \infty$ and Different Degrees of Money Illusion $\alpha$; Constant Risk Premia.

and is essentially insensitive to money illusion. The stock position increases with the horizon and is substantial for a nonillusioned investor (10% and 18%, for 10- and 30-year horizons, respectively), as the stock is used to hedge expected inflation in spite of the real bond. The stock position is instead substantially reduced when the degree of money illusion is higher.

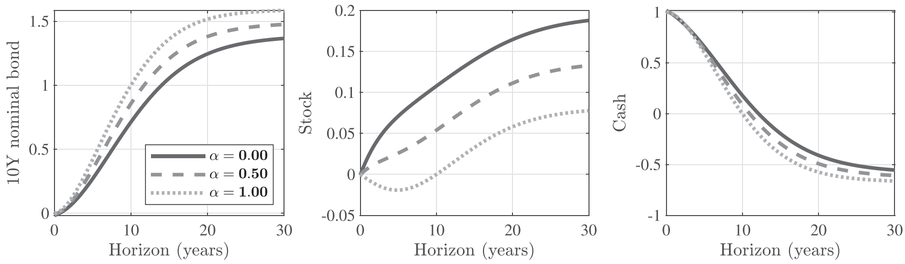


Fig 5. Optimal Portfolio Strategy for $\gamma \to \infty$ and Different Degrees of Money Illusion $\alpha$ when the 10-Year Inflation-Indexed Bond is Excluded from the Investable Universe; Constant Risk Premia.

In Online Appendix C, we analyze the dual case, more interesting for the understanding of the model than from a practical perspective, where the investor cannot access the nominal bond market. The analysis extends also to the case of a moderate investor, which we introduce further into this section.

*Synthesis.* For a conservative investor, the impact of money illusion is substantial, as it drives the investor out of the indexed bond market toward the nominal bond market. When inflation-indexed bonds are not available, money-illusion leads to a lower stock allocation.

### 3.2 *Moderate Investor*

Consider now a moderate investor, with a risk aversion $\gamma = 10$. For this investor, the risk/return trade-off offered by the investment opportunity set plays a role also through the speculative component, which is represented by the first term in (24). We want to assess the impact of money illusion in this context and see how the optimal portfolio strategy is similar and how it differs from the results obtained for the infinitely risk-averse investor. Furthermore, as risk aversion is finite, we are able to calculate annualized certainty equivalent returns of the strategies and, thus, to perform a welfare analysis.

*Portfolio strategy.* The portfolio strategy shown in Figure 6 mimics the patterns observed for the conservative investor. A moderate nonillusioned investor with a 10-year horizon invests 60% in the nominal bond, rather than 0% as the conservative investor. The position in the indexed bond is around 80%, rather than 100%. Despite these differences, money illusion affects the portfolio strategy similarly to the case of a conservative investor, as it entails a reduction of the optimal position in the real bond and an increase of the position in the nominal bond. Furthermore, the amount of this substitution effect is similar to that observed in Figure 3. As expected, the optimal stock position is higher relative to the case of a conservative investor, but the position is again almost insensitive to money illusion.

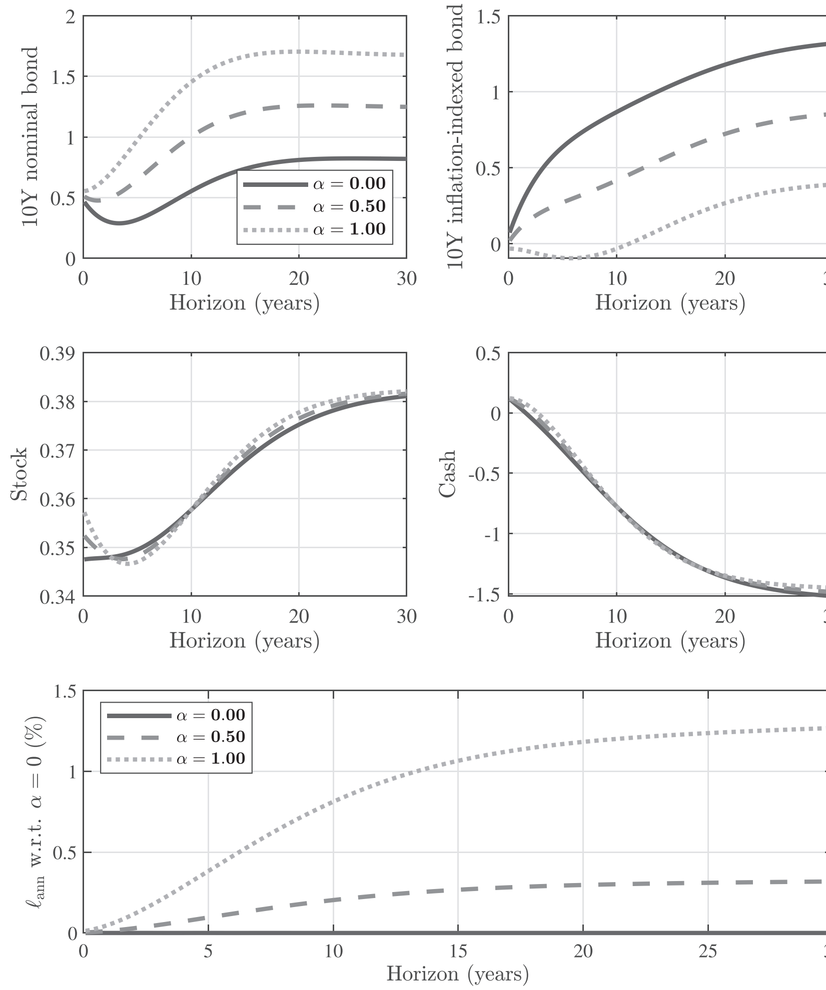


Fig 6. Optimal Portfolio Strategy for $\gamma = 10$ and Different Degrees of Money Illusion $\alpha$, as well as Utility Loss relative to a Nonillusioned Investor; Constant Risk Premia.

*Utility loss due to money illusion.* When money illusion is interpreted as an irrational behavior, the welfare cost it implies, as perceived by a rational nonillusioned agent, is particularly interesting. We evaluate it considering the portfolio strategy followed by an agent with a degree of money illusion $\alpha$ and calculate the expected utility perceived by a nonillusioned investor forced to implement this suboptimal strategy. The cost is expressed in terms of annualized certainty equivalent loss. The derivation of the value function for an investor following a suboptimal strategy is detailed in Online

Appendix B.3, while in Online Appendix B.4 we provide the expression for the loss.

For different degrees of money illusion $\alpha$, the opportunity cost $\ell_{\text{ann}}$ is shown in the bottom graph of Figure 6. As expected, the annualized loss increases with $\alpha$, but not linearly, as for a 10-year horizon and $\alpha = 0.5$ we observe an annualized loss of 0.25%, while for a totally illusioned agent ($\alpha = 1$) the loss is about 0.80% per annum. Furthermore, the annualized loss steeply increases with the investment horizon, up to around 10 years, and keeps increasing at a lower rate when the horizon is longer. For $\alpha = 1$ and a 30-year horizon, the annualized loss is about 1.25%.

Overall, it seems that, when all the investable assets we consider are available for dynamic trading, the opportunity cost of money illusion is substantial. Our estimate is in line with the empirical findings in Stephens and Tyran (2016), who observe that 10-year real portfolio returns are about 10 percentage points lower for Danish money-illusioned individuals.

*Perceived utility loss due to the unavailability of inflation-indexed bonds for different degrees of money illusion.* Figure 7 shows the optimal portfolio strategy when the investor has no access to inflation-indexed bonds. The effect of an increasing degree of money illusion on the portfolio weights is comparable to that already noted for the conservative investor, with little effect on the nominal bond position and a reduction in the stock position.

The most interesting result is shown, however, in the bottom panel of Figure 7, where the annualized certainty equivalent loss due to the exclusion of the inflation-indexed bond from the investable universe is shown. The opportunity cost of not having access to the real bond is increasing in the investment horizon and is substantial for a nonillusioned investor ($\alpha = 0$), being equal to 1.25% per annum for a 30-year horizon. This result is in line with the previous findings by Mkaouar, Prigent, and Abid (2017), who consider fully-rational nonillusioned investors. However, we also notice that the cost of removing the inflation-indexed bond is perceived as negligible by a totally money-illusioned investor ($\alpha = 1$).

The last observation is crucial, as it may represent a contributing factor to the low market demand for inflation-protected securities. Although the optimal portfolio weight associated with the inflation-indexed bond may not be exactly zero, because of the fact that an illusioned investor perceives a negligible loss for not investing in inflation-protected securities, her demand for these is likely to be low, as they can be effectively substituted by more traditional assets, such as nominal bonds and stocks.

*Synthesis.* It seems that money illusion significantly affects the optimal portfolio strategy of long-term investors, and in particular the positions taken in nominal and inflation-indexed bonds. The utility cost of money illusion, if evaluated from the point of view of a long-term nonillusioned investor, is significant. Although the optimal allocation of a money-illusioned investor also comprises an investment in inflation-indexed bonds, we show that by excluding inflation-indexed bonds from the investable universe, the utility loss perceived by a money-illusioned investor is very

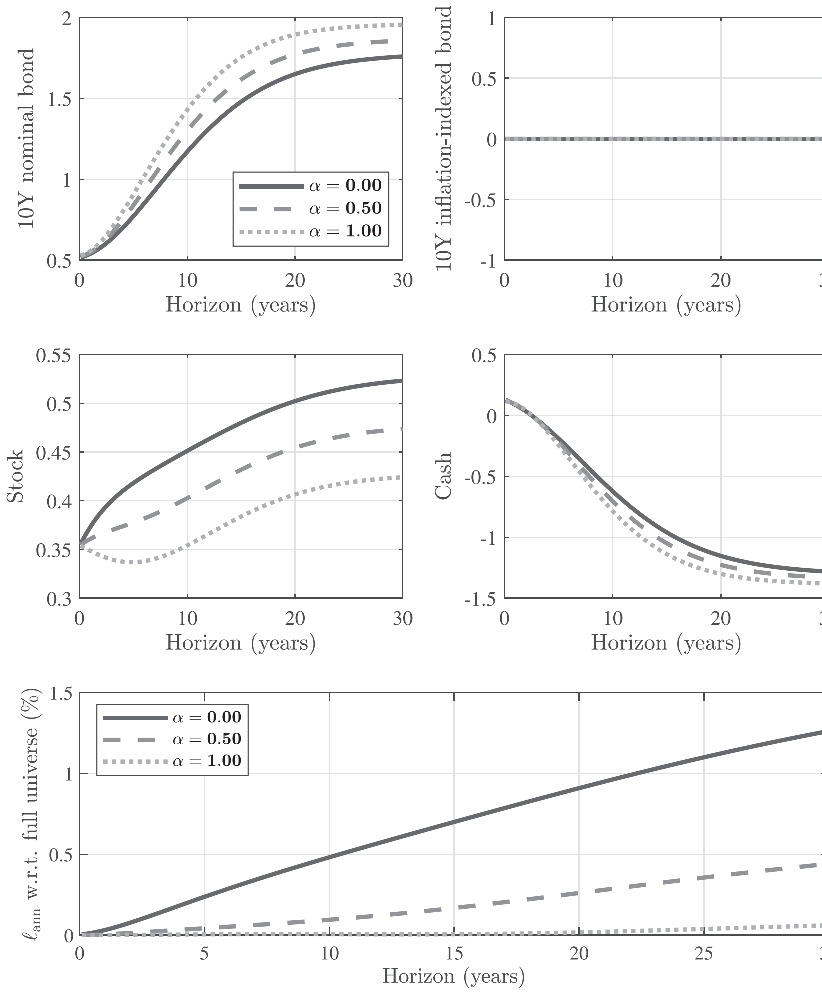


Fig 7. Optimal Portfolio Strategy for $\gamma = 10$ and Different Degrees of Money Illusion $\alpha$ when the 10-Year Inflation-Indexed is Excluded from the Investable Universe, as well as Utility Loss relative to the Case where the Inflation-Indexed Bond is Available; Constant Risk Premia.

small, as her perceived expected utility substituting real bonds with nominal bonds is almost unchanged.

## 4. ADDITIONAL EMPIRICAL FINDINGS: TIME-VARYING RISK PREMIA

In this section, we present additional empirical results that support the evidence obtained under constant risk premia. We evaluate the optimal portfolio strategy and *ex* *ante* welfare perceived by agents believing that risk premia are time-varying, who can be regarded as being rather sophisticated investors. We focus on the case of a moderate investor ($\gamma = 10$), considering the specification where risk premia are time-varying and their volatilities have been constrained, as specified in Section 2.3. The parameter estimates used are those in Table 1, Panel B. We relegate to Online Appendix E the analysis based on the specification with time-varying risk premia without any volatility constraint. This further analysis confirms the main conclusions drawn in this section, by providing qualitatively compatible results, but, as it could have been expected considering the evidence in Duffee (2011), Feldhütter et al. (2012), and Sarno, Schneider, and Wagner (2016), it overstates the optimal dynamic portfolio positions and the certainty equivalent returns.

*Portfolio strategy.* Figure 8 shows the optimal portfolio strategy when investors have access to the full investment universe. For short investment horizons, the optimal strategy is virtually identical to the case where risk premia are constant (Figure 6). When the horizon is increased beyond 5 years, the impact of time-varying risk premia seems to shift the optimal portfolio from nominal bonds to inflation-indexed bonds. For a 30-year horizon, the increase of the weight in the real bond is between 30% and 50%, corresponding to an approximately equivalent reduction of the weight in the nominal bond. This is true for any value of $\alpha$, which suggests that there is little interaction between money illusion and the intertemporal hedging demands due to the time variation of risk premia.

*Utility loss due to money illusion.* The welfare analysis in the graph at the bottom of Figure 8, showing the certainty equivalent loss attributable to money illusion, confirms that the welfare effect of money illusion is substantial. The loss is very similar in pattern and size to the case of constant risk premia. Indeed, considering a 10-year horizon, the annualized loss is about 1% for a totally illusioned investor with respect to a nonillusioned investor.

*Perceived utility loss due to the unavailability of inflation-indexed bonds for different degrees of money illusion.* We consider now the case where the inflation-indexed bond is removed from the investable universe. Figure 9 shows that the effect of money illusion on the optimal allocation is qualitatively similar to the case with constant risk premia (Figure 7). However, the effect of money illusion on the intertemporal hedging component associated with the nominal bond is more pronounced in this case, leading to a position for the illusioned investor which is higher by 30–40% for a horizon longer than 10 years relative to the nonillusioned investor. This difference corresponds to a different degree of leverage, as reflected by the cash position.

The evaluation of the opportunity cost, perceived by investors with different levels of $\alpha$, of excluding the real bond, is crucial to confirm that a money-illusioned investor does not perceive as useful the availability of inflation-protected securities. Indeed, in this situation, other than for inflation hedging, an investor who times the market also takes positions in the assets to hedge future variations in the risk premia. As can be noticed in Figure 9, the opportunity cost is substantial for the

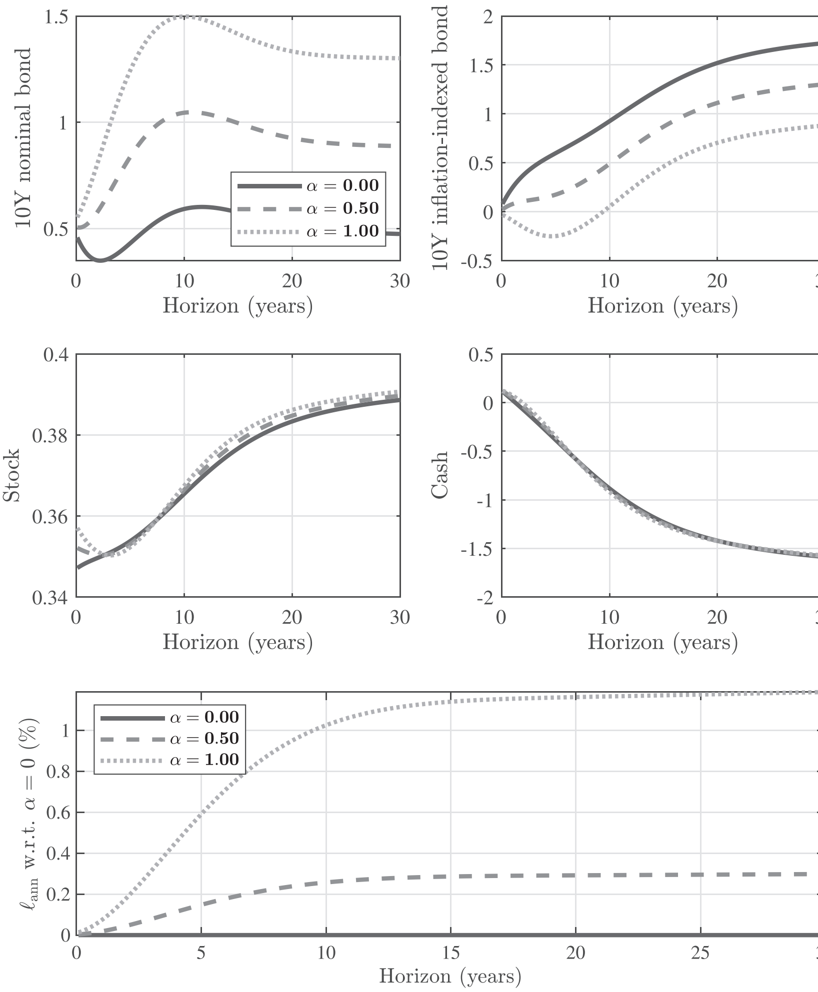


Fig 8. Optimal Portfolio Strategy for $\gamma = 10$ and Different Degrees of Money Illusion $\alpha$, as well as Utility Loss relative to a Nonillusioned Investor; Time-Varying Risk Premia with Volatility Constraints.

nonillusioned investor ($\alpha = 0$). The annualized loss increases steadily with the investment horizon, being equal to 0.5% per annum for a 10-year horizon, 1.1% for a 20-year horizon, and 1.6% for a 30-year horizon. However, considering a partially illusioned investor ($\alpha = 0.5$), the perceived loss is already significantly reduced, reaching a maximum of 0.9% for a 30-year horizon. A totally illusioned investor ($\alpha = 1$) perceives an even lower loss, which is roughly flat and equal to 0.25% per

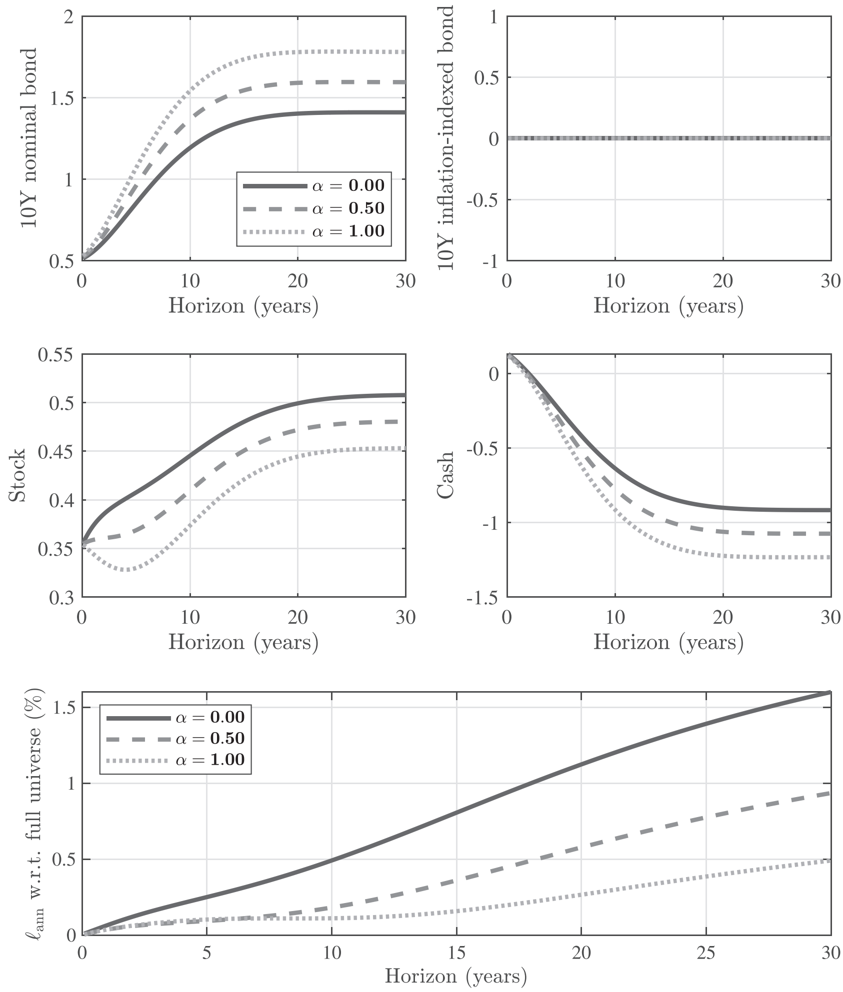


Fig 9. Optimal Portfolio Strategy for $\gamma = 10$ and Different Degrees of Money Illusion $\alpha$ when the 10-Year Inflation-Indexed Bond is Excluded from the Investable Universe, as well as Utility Loss relative to the Case where the Inflation-Indexed Bond is Available; Time-Varying Risk Premia with Volatility Constraints.

annum up to a 20-year horizon, and increases to a maximum of 0.5% for a 30-year horizon. [22]

22. The nonnegligible utility loss perceived by the totally money-illusioned investor for horizons shorter than 10 years is due to the fact that the investor cannot take short positions in the 10-year inflation-indexed bond, as implied by the optimal strategy in Figure 8, where the money-illusioned investor short sells the inflation-indexed bond (thus selling inflation protection) to finance long positions in other assets. This effect is of secondary importance, in the robustness checks we show that it reduces when the stock is excluded from the investable universe (Online Appendix G) and that it disappears when the investable universe is composed of 5-year bonds rather than of 10-year bonds (Online Appendix H).

In conclusion, our analysis confirms that a money-illusioned long-term investor, even when attempting to time the market by accounting for time-varying risk premia, perceives a significantly lower opportunity cost of not having access to inflation-indexed instruments relative to a nonillusioned investor. Among money-illusioned investors, even those following sophisticated dynamic strategies seem to be able to attain comparable expected utilities by substituting inflation-indexed bonds with more traditional assets, such as nominal bonds and stocks, and therefore have little incentive to enter the inflation-indexed bond market.

## 5. DISCUSSION AND ROBUSTNESS CHECKS

Our analysis shows that money illusion has significant consequences on the optimal allocation of a long-term investor, as well as on the perceived economic incentives to invest in inflation-indexed bonds. While the results shown in the paper refer to a situation where the state variables driving the economy are at their long-run means, in Online Appendix F we consider the time variation of the state variables throughout the available sample and show the time series of the optimal portfolio, as well as the perceived certainty equivalent losses from not investing in TIPS, for investors having different degrees of money illusion. While we observe strong time variations, all our key findings are maintained. In particular, despite significant time variations of the risk premia of the assets available for trade, a money-illusioned investor perceives a significantly lower economic incentive to enter the TIPS market than a nonillusioned investor for the whole time period considered.

There are other possible contributing factors to the low market demand for TIPS in the last decade. As pointed out by Gourio and Ngo (2020), the correlation between inflation and stock market returns turned from negative to slightly positive in the years following the 2008 financial crisis. Some investors could then rationally choose to be exposed to inflation risk to hedge against future bad states of the economy. Our model is totally compatible with such *rational* money-illusioned behavior, as the preferences that we employ could represent either an irrationally biased behavior or a fully-rational behavior. In addition, there is evidence for a low level of perceived inflation uncertainty in the decade after 2009 (e.g., Breach, D'Amico, and Orphanides 2020), which could have reduced the market interest in TIPS.

However, considering the market share of TIPS relative to the total Treasury market in Figure 1(a), we notice that the share is rather stable and around 7–10% since 2005. It actually tends to slightly increase since 2010 (together with the size of the TIPS market), when liquidity comes back to normal levels after the Great Recession, despite the fact that the inflation-stock returns correlation continues to be slightly positive and the perceived inflation risk rather low. We therefore believe that, while a rational money-illusioned behavior of some agents and a lower perceived inflation risk may contribute to the low demand for TIPS, these are not the key determinants.

We relegate to the Online Appendix further analyses and robustness checks. In Online Appendix D, we show that our baseline results are robust to shifts of the realized inflation risk premium, which is a quantity that previous studies have shown to be difficult to estimate. In Online Appendix G, we perform an asset allocation exercise excluding the stock index from the investable universe, verifying that all our baseline results are maintained, while in Online Appendix H we modify the maturity of the bonds included in the investable universe. In Online Appendix I, we consider levels of risk aversion different from the base case value ($\gamma = 10$), showing that all our results are qualitatively maintained. As the hedging motive is the key determinant of our findings, money illusion has stronger effects for higher levels of risk aversion and vice versa. In Online Appendix J, we check the robustness of our findings by reestimating the model using also survey-based long-run inflation forecasts. In Online Appendix K, we instead consider a different specification for the nominal and real term structure dynamics, using a four-factor no-arbitrage Nelson–Siegel-like model as in Christensen, Lopez, and Rudebusch (2010).

Finally, in Online Appendix L we repeat the analysis using an alternative data set obtained from the UK market and spanning the period 1985–2018. Other than because of the length of the sample, it is interesting to see whether our conclusions are robust to the choice of a market environment where the average inflation level and the inflation volatility are higher, as well as where the average inflation risk premium is negative. In this additional analysis, we find an even higher welfare cost of money illusion, qualitatively confirming all our conclusions.

## 6. CONCLUSION

Several authors have studied the effects of money illusion on financial markets. Modigliani and Cohn (1979) recognized money illusion as the source of major errors in the valuation of common stocks during periods of anticipated hyperinflation, concluding that these valuation mistakes were the main cause of a 50% undervaluation of U.S. stock value at the end of 1977. Their analysis was confirmed by Cohen, Polk, and Vuolteenaho (2005), who tested the effects of money illusion in a framework based on the capital asset pricing model, identifying an irrational increase of expected returns across all stocks, irrespective of the riskiness (beta) of the stock, during periods of hyperinflation. More recently, Schmeling and Schrimpf (2011) found that survey-based measures of expected inflation are able to predict aggregate stock returns, attributing this phenomenon to money illusion.

The present work shows that the mechanism at play in the aforementioned empirical investigations is also crucial for the decision making of a long-term investor. Our results confirm some of the findings in Stephens and Tyran (2016), who documented a tendency for money-illusioned investors to shift portfolios toward nominal assets.

In particular, an illusioned investor tends to reduce the allocation in inflation-indexed bonds, which intertemporally hedge expected inflation, in favor of nominal bonds. The effect on the stock allocation is marginal. We find the cost of money illusion to be around 1% per annum for a moderately risk-averse agent with an investment horizon of 10 years or longer. We also estimate that a nonillusioned investor who has no access to inflation-indexed bonds suffers from a utility loss of about 0.5–1.5% per annum for investment horizons of 10 to 30 years. Conversely, the perceived welfare loss of a money-illusioned investor is significantly lower. This finding identifies money illusion as potentially being at the origin of the scarce market demand for TIPS, as money-illusioned investors understate the utility loss entailed by substituting inflation-indexed bonds with nominal bonds.

Our results apply to individuals that irrationally suffer from money illusion, which has been shown to widely happen when the level of inflation is low, despite a possibly high inflation risk. Importantly, our analysis also applies to agents that rationally have preferences based on nominal returns. This is the case, for instance, for fund managers whose performance is benchmarked against nominal assets, or for pension funds which liabilities are not inflation-linked, as it is the case for most U.S. pension funds (Andonov, Bauer, and Cremers 2013).

From the technical standpoint, we contribute to the literature of dynamic asset allocation relying on affine term structure models, further investigating the empirical issues of essentially affine models that have recently been pointed out by Feldhütter et al. (2012) and Sarno, Schneider, and Wagner (2016). In particular, we complement the information set with survey-based forecasts and constrain the volatility of the risk premia in order to reduce the model overfitting, which typically leads to unrealistic optimal portfolio positions and largely overstates utility losses entailed by suboptimal strategies. While we provide out-of-sample forecasting evidence to support our approach, we leave for future research a formal analysis of these methodologies and an accurate fine tuning of the restrictions to be imposed in the estimation.

The analysis that we propose, at the cost of a significant increase of model complexity and estimation risk, as well as the need for a nontrivial numerical solution of the portfolio choice problem, could be extended to account for a time-varying degree of money illusion and a stochastic inflation volatility. Finally, our work may provide new insights on the relative mispricing of TIPS, inflation swaps, and nominal Treasury bonds (Fleckenstein, Longstaff, and Lustig 2014), by recognizing that the demand pressures for inflation protection either through TIPS or inflation swaps correspond to different categories of agents: the first often consisting of unsophisticated investors, who look for saving opportunities but might be money-illusioned, the second typically consisting of more sophisticated institutional investors, seeking for inflation-hedging and often funding-constrained. We model the demand of a price-taker operating in the first market, showing that money illusion generates significant distortions. A general equilibrium model accounting for heterogeneity in money illusion and funding constraints, as well as market segmentation, might contribute to the explanation of the inflation premium and the TIPS pricing puzzle.

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## LITERATURE CITED

Abrahams, Michael, Tobias Adrian, Richard K. Crump, Emanuel Moench, and Rui Yu. (2016) "Decomposing Real and Nominal Yield Curves." *Journal of Monetary Economics*, 84, 182–200.

Andonov, Aleksandar, Rob Bauer & Martijn Cremers (2013) "Pension Fund Asset Allocation and Liability Discount Rates: Camouflage and Reckless Risk Taking by US Public Plans?" Working Paper.

Andreasen, Martin M., Jens H.E. Christensen, and Simon Riddell. (2021) "The TIPS Liquidity Premium." *Review of Finance*, 25, 1639–75.

Barillas, Francisco (2011) "Can We Exploit Predictability in Bond Markets?" Working Paper.

Basak, Suleyman, and Hongjun Yan. (2010) "Equilibrium Asset Prices and Investor Behaviour in the Presence of Money Illusion." *Review of Economic Studies*, 77, 914–36.

Breach, Tomas, Stefania D'Amico, and Athanasios Orphanides. (2020) "The Term Structure and Inflation Uncertainty." *Journal of Financial Economics*, 138, 388–414.

Brennan, Michael J., and Yihong Xia. (2002) "Dynamic Asset Allocation under Inflation." *Journal of Finance*, 57, 1201–38.

Brunnermeier, Markus K., and Christian Julliard. (2008) "Money Illusion and Housing Frenzies." *Review of Financial Studies*, 21, 135–80.

Campbell, John Y., Yeung Lewis Chan, and Luis M. Viceira. (2003) "A Multivariate Model of Strategic Asset Allocation." *Journal of Financial Economics*, 67, 41–80.

Campbell, John Y., Robert J. Shiller, and Luis M. Viceira. (2009) "Understanding Inflation-Indexed Bond Markets." *Brookings Papers on Economic Activity*, Spring, 79–139.

Campbell, John Y., and Luis M. Viceira. (2001) "Who Should Buy Long-Term Bonds?" *American Economic Review*, 91, 99–127.

Cartea, Álvaro, Jonatan Saúl, and Juan Toro. (2012) "Optimal Portfolio Choice in Real Terms: Measuring the Benefits of TIPS." *Journal of Empirical Finance*, 19, 721–40.

Cavallo, Alberto, Guillermo Cruces, and Ricardo Perez-Truglia. (2017) "Inflation Expectations, Learning, and Supermarket Prices: Evidence from Survey Experiments." *American Economic Journal: Macroeconomics*, 9, 1–35.

Christensen, Jens H.E., and James M. Gillan. (2018) "Does Quantitative Easing Affect Market Liquidity?" Federal Reserve Bank of San Francisco Working Paper 2013-26.

Christensen, Jens H.E., Jose A. Lopez, and Glenn D. Rudebusch. (2010) "Inflation Expectations and Risk Premiums in an Arbitrage-Free Model of Nominal and Real Bond Yields." *Journal of Money, Credit and Banking*, 42, 143–78.

Cohen, Randolph B., Christopher Polk, and Tuomo Vuolteenaho. (2005) "Money Illusion in the Stock Market: The Modigliani-Cohn Hypothesis." *Quarterly Journal of Economics*, 120, 639–68.

D'Amico, Stefania, Don H. Kim, and Min Wei. (2018) "Tips from TIPS: The Informational Content of Treasury Inflation-Protected Security Prices." *Journal of Financial and Quantitative Analysis*, 53, 395–436.

David, Alexander, and Pietro Veronesi. (2013) "What Ties Return Volatilities to Price Valuations and Fundamentals?" *Journal of Political Economy*, 121, 682–746.

Duffee, Gregory R. (2002) "Term Premia and Interest Rate Forecasts in Affine Models." *Journal of Finance*, 57, 405–43.

Duffee, Gregory R. (2011) "Sharpe Ratios in Term Structure Models." Working papers, The Johns Hopkins University, Department of Economics.

Duffie, Darrell, and Rui Kan. (1996) "A Yield-Factor Model of Interest Rates." *Mathematical Finance*, 6, 379–406.

Dusansky, Richard, and Peter J. Kalman. (1974) "The Foundations of Money Illusion in a Neoclassical Micro-Monetary Model." *American Economic Review*, 64, 115–22.

Evans, Martin D.D. (1998) "Real Rates, Expected Inflation, and Inflation Risk Premia." *Journal of Finance*, 53, 187–218.

Fehr, Ernst, and Jean-Robert Tyran. (2001) "Does Money Illusion Matter?" *American Economic Review*, 91, 1239–62.

Fehr, Ernst, and Jean-Robert Tyran. (2007) "Money Illusion and Coordination Failure." *Games and Economic Behavior*, 58, 246–68.

Fehr, Ernst, and Jean-Robert Tyran. (2014) "Does Money Illusion Matter?: Reply." *American Economic Review*, 104, 1063–71.

Feldhütter, Peter, Linda Sandris Larsen, Claus Munk, and Anders B. Trolle. (2012) "Keep It Simple: Dynamic Bond Portfolios under Parameter Uncertainty." Working paper.

Fleckenstein, Matthias, Francis A. Longstaff, and Hanno Lustig. (2014) "The TIPS-Treasury Bond Puzzle." *Journal of Finance*, 69, 2151–97.

Fleming, Michael J., and Neel Krishnan. (2012) "The Microstructure of the TIPS Market." *Federal Reserve Bank of New York Economic Policy Review*, 18, 27–45.

Fleming, Michael J., and John Sporn. (2013) "Trading Activity and Price Transparency in the Inflation Swap Market." *Federal Reserve Bank of New York Economic Policy Review*, 19, 45–57.

Gabaix, Xavier. (2014) "A Sparsity-Based Model of Bounded Rationality." *Quarterly Journal of Economics*, 129, 1661–710.

García, Juan Ángel, and Adrian Van Rixel. (2007) "Inflation-Linked Bonds from a Central Bank Perspective." Documentos Ocasionales-Banco de España, 9–47.

Gourio, François, and Phuong Ngo. (2020) "Risk Premia at the ZLB: A Macroeconomic Interpretation." Working paper.

Gromb, Denis, and Dimitri Vayanos. (2018) "The Dynamics of Financially Constrained Arbitrage." *Journal of Finance*, 73, 1713–50.

Gürkaynak, Refet S., Brian Sack, and Jonathan H. Wright. (2007) "The US Treasury Yield Curve: 1961 to the Present." *Journal of Monetary Economics*, 54, 2291–304.

Gürkaynak, Refet S., Brian Sack, and Jonathan H. Wright. (2010) "The TIPS Yield Curve and Inflation Compensation." *American Economic Journal: Macroeconomics*, 2, 70–92.

He, Hua, and Neil D. Pearson. (1991) “Consumption and Portfolio Policies with Incomplete Markets and Short-Sale Constraints: The Infinite Dimensional Case.” *Journal of Economic Theory*, 54, 259–304.

Illeditsch, Philipp Karl. (2017) “Residual Inflation Risk.” *Management Science*, 64, 5289–5314.

Ireland, Peter N. (2015) “Monetary Policy, Bond Risk Premia, and The Economy.” *Journal of Monetary Economics*, 76, 124–40.

Joslin, Scott, Marcel Priebsch, and Kenneth J. Singleton. (2014) “Risk Premiums in Dynamic Term Structure Models with Unspanned Macro Risks.” *Journal of Finance*, 69, 1197–233.

Joslin, Scott, Kenneth J. Singleton, and Haoxiang Zhu. (2011) “A New Perspective on Gaussian Dynamic Term Structure Models.” *Review of Financial Studies*, 24, 926–70.

Suk Kim, Tong, and Edward Omberg. (1996) “Dynamic Nonmyopic Portfolio Behavior.” *Review of Financial Studies*, 9, 141–61.

Koijen, Ralph S.J., Theo E. Nijman, and Bas J.M. Werker. (2010) “When Can Life Cycle Investors Benefit from Time-Varying Bond Risk Premia?” *Review of Financial Studies*, 23, 741–80.

Kothari, S.P., and Jay Shanken. (2004) “Asset Allocation with Inflation-Protected Bonds.” *Financial Analysts Journal*, 60, 54–70.

Lioui, Abraham, and Andrea Tarelli. (2019) “Macroeconomic Environment, Money Demand and Portfolio Choice.” *European Journal of Operational Research*, 274, 357–74.

Luhan, Wolfgang J., and Johann Scharler. (2014) “Inflation Illusion and the Taylor Principle: An Experimental Study.” *Journal of Economic Dynamics and Control*, 45, 94–110.

Maio, Paulo F. (2018) “Does Inflation Explain Equity Risk Premia?” Working paper.

Marschak, Jacob. (1950) “The Rationale of the Demand for Money and of ‘Money Illusion’.” *Metroeconomica*, 2, 71–100.

Marschak, Jacob. (1974) “Money Illusion and Demand Analysis.” In *Economic Information, Decision, and Prediction*, pp. 206–21. Springer.

Martellini, Lionel, Vincent Milhau, and Andrea Tarelli. (2018) “Capital Structure Decisions and the Optimal Design of Corporate Market Debt Programs.” *Journal of Corporate Finance*, 49, 141–67.

Miao, Jianjun, and Danyang Xie. (2013) “Economic Growth under Money Illusion.” *Journal of Economic Dynamics and Control*, 37, 84–103.

Mkaouar, Farid, Jean-Luc Prigent, and Ilyes Abid. (2017) “Long-Term Investment with Stochastic Interest and Inflation Rates: The Need for Inflation-Indexed Bonds.” *Economic Modelling*, 67, 228–47.

Modigliani, Franco, and Richard A. Cohn. (1979) “Inflation, Rational Valuation and the Market.” *Financial Analysts Journal*, 35, 24–44.

Pflueger, Carolin E., and Luis M. Viceira. (2016) “Return Predictability in the Treasury Market: Real Rates, Inflation, and Liquidity.” In *Handbook of Fixed-Income Securities*, John Wiley & Sons, Inc., Hoboken, NJ, pp. 191–209.

Piazzesi, Monika, and Martin Schneider. (2008) “Inflation Illusion, Credit, and Asset Prices.” In *Asset Prices and Monetary Policy*, Chicago, IL: University of Chicago Press. pp. 147–89.

Risa, Stefano. (2001) “Nominal and Inflation Indexed Yields: Separating Expected Inflation and Inflation Risk Premia.” Working paper.

Sangvinatsos, Antonios, and Jessica A. Wachter. (2005) "Does the Failure of the Expectations Hypothesis Matter for Long-Term Investors?" *Journal of Finance*, 60, 179–230.

Sarno, Lucio, Paul Schneider, and Christian Wagner. (2016) "The Economic Value of Predicting Bond Risk Premia." *Journal of Empirical Finance*, 37, 247–67.

Schmeling, Maik, and Andreas Schrimpf. (2011) "Expected Inflation, Expected Stock Returns, and Money Illusion: What Can We Learn from Survey Expectations?" *European Economic Review*, 55, 702–19.

Shafir, Eldar, Peter Diamond, and Amos Tversky. (1997) "Money Illusion." *Quarterly Journal of Economics*, 112, 341–74.

Stephens, Thomas Alexander, and Jean-Robert Tyran. (2016) "Money Illusion and Household Finance." Working paper.

Tibshirani, Robert J., and Bradley Efron. (1993) "An Introduction to the Bootstrap." *Monographs on Statistics and Applied Probability*, 57, 1–436.

Wachter, Jessica A. (2003) "Risk Aversion and Allocation to Long-Term Bonds." *Journal of Economic Theory*, 112, 325–33.

## SUPPORTING INFORMATION

Additional supporting information may be found online in the Supporting Information section at the end of the article.

**Table A.1**: Parameter estimates when no constraints on the volatility of risk premia are imposed.

**Table A.2**: Historical and model-implied summary statistics when no constraints on the volatility of risk premia are imposed.

**Table A.3**: Correlations between asset returns and economic variables when no constraints on the volatility of risk premia are imposed.

**Table A.4**: Predictive ability of model specifications based on constant risk premia, as well as volatilityconstrained and unconstrained time-varying risk premia

**Table A.5**: Baseline model estimated observing also survey-based inflation expectations. Constant risk premia. Parameter estimates.

**Table A.6**: Baseline model estimated observing also survey-based inflation expectations. Constant risk premia. Historical and model-implied summary statistics.

**Table A.7**: Parameter estimates for the 4-factor affine no-arbitrage Nelson-Siegel term structure specification.

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**Table A.9**: UK-based analysis. Parameter estimates.

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**Table A.11**: UK-based analysis. Correlations between asset returns and economic variables.

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**Figure A.2**: Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$, when the 10-year nominal bond is excluded from the investable universe.

**Figure A.3**: Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$, when the 10-year nominal bond is excluded from the investable universe.

**Figure A.4**: Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$.

**Figure A.5**: Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$, when the 10-year inflation-indexed bond is excluded from the investable universe.

**Figure A.6**: Time series of relevant model-implied and observed variables. Time-varying risk premia (unconstrained volatility).

**Figure A.7**: Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$.

**Figure A.8**: Time series of in-sample optimal portfolio weights for a non-illusioned investor, obtained for different risk premia specifications.

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**Figure A.13**: Bond-only analysis. Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$, when the 10-year inflation-indexed bond is excluded from the investable universe.

**Figure A.14**: Bond-only analysis. Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$, when the 10-year inflation-indexed bond is excluded from the investable universe.

**Figure A.15**: Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$, when the investor can trade 5-year bonds.

**Figure A.16**: Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$, when the 5-year inflation-indexed bond is excluded from the investable universe.

**Figure A.17**: Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$, when the investor can trade 5-year bonds.

**Figure A.18**: Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$, when the 5-year inflation-indexed bond is excluded from the investable universe.

**Figure A.19**: Optimal portfolio strategy for $\gamma = 5$ and different degrees of money illusion $\alpha$.

**Figure A.20**: Optimal portfolio strategy for $\gamma = 5$ and different degrees of money illusion $\alpha$, when the 10-year inflation-indexed bond is excluded from the investable universe.

**Figure A.21**: Optimal portfolio strategy for $\gamma = 20$ and different degrees of money illusion $\alpha$. Utility loss with respect to a non-illusioned investor. Constant risk premia.

**Figure A.22**: Optimal portfolio strategy for $\gamma = 20$ and different degrees of money illusion $\alpha$, when the 10-year inflation-indexed bond is excluded from the investable universe.

**Figure A.23**: Optimal portfolio strategy for $\gamma = 5$ and different degrees of money illusion $\alpha$. Utility loss with respect to a non-illusioned investor.

**Figure A.24**: Optimal portfolio strategy for $\gamma = 5$ and different degrees of money illusion $\alpha$, when the 10-year inflation-indexed bond is excluded from the investable universe.

**Figure A.25**: Optimal portfolio strategy for $\gamma = 20$ and different degrees of money illusion $\alpha$.

**Figure A.26**: Optimal portfolio strategy for $\gamma = 20$ and different degrees of money illusion $\alpha$, when the 10-year inflation-indexed bond is excluded from the investable universe.

**Figure A.27**: Baseline model estimated observing also survey-based inflation expectations. Constant risk premia.

**Figure A.28**: Baseline model estimated observing also survey-based inflation expectations.

**Figure A.29**: Baseline model estimated observing also survey-based inflation expectations. Constant risk premia.

**Figure A.30**: 4-factor affine no-arbitrage Nelson-Siegel term structure specification with constant risk premia.

**Figure A.31**: 4-factor affine no-arbitrage Nelson-Siegel term structure specification with constant risk premia.

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**Figure A.33**: UK-based analysis. Time series of relevant model-implied and observed variables.

**Figure A.34**: UK-based analysis. Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$.

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**Figure A.36**: UK-based analysis. Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$, when the 10-year inflation-indexed bond is excluded from the investable universe.

**Figure A.37**: UK-based analysis. Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$, when the 10-year inflation-indexed bond is excluded from the investable universe.

# Money Illusion and TIPS Demand

# Online Appendix

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This Online Appendix is organized as follows. In Section A, we derive the pricing equations for nominal bonds (A.1), real bonds (A.2) and of the long-run expected stock return (A.3). The derivation of the optimal portfolio strategy is in Section B. In particular, we derive the optimal portfolio strategy for complete and incomplete markets (B.1), the optimal portfolio strategy for an infinitely risk-averse agent and constant market prices of risk (B.2), the welfare from suboptimal strategies (B.3) and the certainty equivalent utility loss (B.4). In Section C we address the dual case to that studied in the baseline analysis, calculating the optimal portfolio strategy and welfare losses when the nominal bond is not accessible. In Section D, we discuss the empirical findings obtained imposing a zero unexpected inflation risk premium. In Section E, we elaborate on the specification with time-varying risk premia where no volatility constraints are imposed, discussing the parameter estimates (E.1), comparing it with the other specifications in terms of out-of-sample predictive ability (E.2), and studying the portfolio strategy and welfare (E.3). In Section F we study the time series of the optimal in-sample portfolio positions and perceived losses when TIPS are not available. In Section G we perform an analysis based on an investable universe where only nominal and inflation-indexed bonds are available, while the stock is excluded. In Section H we study the optimal portfolio strategy and welfare effects considering maturities of the tradable bonds different from the baseline analysis. In Section I we instead consider different values of the risk aversion parameter. In Section J we study the effect of complementing the information set by survey-based long-run inflation expectations. In Section K we apply our methodology to a different market specification, where the nominal and real term structures are specified with a 4-factor no-arbitrage Nelson-Siegel-like model. Finally, in Section L, we perform an empirical analysis based on UK data.

## A Asset prices

### A.1 Nominal bond pricing

We conjecture that the nominal price of a zero-coupon nominal bond has the following functional form:

$$B\left(\mathbf{X}_t,\tau\right)=e^{-y^n(\mathbf{X}_t,\tau)\tau}=e^{A_0^B(\tau)+\mathbf{A}_1^B(\tau)\mathbf{X}_t},$$

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where $\tau$ stands for the time to maturity of the bond. From no-arbitrage arguments, the nominal price $B$ of a nominal bond satisfies the following PDE:

$$B_{\mathbf{X}}\mathbf{\Theta}\left(\bar{\mathbf{X}}-\mathbf{X}_t\right)+\frac{1}{2}\mathrm{tr}\left(B_{\mathbf{XX}}\mathbf{\Sigma}_{\mathbf{X}}'\mathbf{\Sigma}_{\mathbf{X}}\right)+B_t-R_tB=B_{\mathbf{X}}\mathbf{\Sigma}_{\mathbf{X}}'\Lambda_t,$$

where $B_{\cdot}$ stands for the partial derivative of $B$. Computing the derivatives:

$$\begin{aligned}
B_{\mathbf{X}} &= \mathbf{A}_1^B B\\
B_{\mathbf{XX}} &= \left(\mathbf{A}_1^B\right)'\mathbf{A}_1^B B\\
B_t &= -B_\tau=-\left(\frac{\partial}{\partial\tau}A_0^B\left(\tau\right)+\frac{\partial}{\partial\tau}\mathbf{A}_1^B\left(\tau\right)\mathbf{X}_t\right)B,
\end{aligned}$$

substituting into the PDE (A.1) for these derivatives, as well as the expressions for $R_t$ and $\mathbf{\Lambda}_t$ yields:

$$\begin{aligned}
\mathbf{A}_1^B\mathbf{\Sigma}_{\mathbf{X}}'\left(\mathbf{\Lambda}_0+\mathbf{\Lambda}_1\mathbf{X}_t\right) &= \mathbf{A}_1^B\mathbf{\Theta}\left(\bar{\mathbf{X}}-\mathbf{X}_t\right)+\frac{1}{2}\mathbf{A}_1^B\mathbf{\Sigma}_{\mathbf{X}}'\mathbf{\Sigma}_{\mathbf{X}}\left(\mathbf{A}_1^B\right)'-\left(\frac{\partial}{\partial\tau}A_0^B\left(\tau\right)+\frac{\partial}{\partial\tau}\mathbf{A}_1^B\left(\tau\right)\mathbf{X}_t\right)\\
&\quad-\left(R_0+\mathbf{R}_1'\mathbf{X}_t\right).
\end{aligned}$$

Matching the homogeneous term and the term in $\mathbf{X}_t$ yields:

$$\frac{\partial}{\partial\tau}A_0^B\left(\tau\right) = \mathbf{A}_1^B\left(\mathbf{\Theta}\bar{\mathbf{X}}-\mathbf{\Sigma}_{\mathbf{X}}'\mathbf{\Lambda}_0\right)+\frac{1}{2}\mathbf{A}_1^B\mathbf{\Sigma}_{\mathbf{X}}'\mathbf{\Sigma}_{\mathbf{X}}\left(\mathbf{A}_1^B\right)'-R_0, \tag{A.1}$$

$$\frac{\partial}{\partial\tau}\mathbf{A}_1^B\left(\tau\right) = -\mathbf{A}_1^B\left(\mathbf{\Theta}+\mathbf{\Sigma}_{\mathbf{X}}'\mathbf{\Lambda}_1\right)-\mathbf{R}_1'. \tag{A.2}$$

The nominal rate is given considering the limit for $\tau\to 0$:

$$R_t=R_0+\mathbf{R}_1'\mathbf{X}_t.$$

The dynamics of the price of a nominal zero-coupon bond with a time to maturity $\tau$ is:

$$\begin{aligned}
\frac{\mathrm{d}B\left(\mathbf{X}_t,\tau\right)}{B\left(\mathbf{X}_t,\tau\right)} &= \left[-\left(\frac{\partial}{\partial\tau}A_0^B\left(\tau\right)+\frac{\partial}{\partial\tau}\mathbf{A}_1^B\left(\tau\right)\mathbf{X}_t\right)+\mathbf{A}_1^B\mathbf{\Theta}\left(\bar{\mathbf{X}}-\mathbf{X}_t\right)+\frac{1}{2}\mathbf{A}_1^B\mathbf{\Sigma}_{\mathbf{X}}'\mathbf{\Sigma}_{\mathbf{X}}\left(\mathbf{A}_1^B\right)'\right]\mathrm{d}t\\
&\quad+\mathbf{A}_1^B\mathbf{\Sigma}_{\mathbf{X}}'\mathrm{d}\mathbf{z}_t.
\end{aligned}$$

Substituting for the expressions (A.1) and (A.2):

$$\begin{aligned}
\frac{\mathrm{d}B\left(\mathbf{X}_t,\tau\right)}{B\left(\mathbf{X}_t,\tau\right)} &= -\left[\begin{array}{c}\mathbf{A}_1^B\left(\mathbf{\Theta}\bar{\mathbf{X}}-\mathbf{\Sigma}_{\mathbf{X}}'\mathbf{\Lambda}_0\right)+\frac{1}{2}\mathbf{A}_1^B\mathbf{\Sigma}_{\mathbf{X}}'\mathbf{\Sigma}_{\mathbf{X}}\left(\mathbf{A}_1^B\right)'-R_0\\ -\mathbf{A}_1^B\left(\mathbf{\Theta}+\mathbf{\Sigma}_{\mathbf{X}}'\mathbf{\Lambda}_1\right)\mathbf{X}_t-\mathbf{R}_1'\mathbf{X}_t\end{array}\right]\mathrm{d}t\\
&\quad+\left[\mathbf{A}_1^B\mathbf{\Theta}\left(\bar{\mathbf{X}}-\mathbf{X}_t\right)+\frac{1}{2}\mathbf{A}_1^B\mathbf{\Sigma}_{\mathbf{X}}'\mathbf{\Sigma}_{\mathbf{X}}\left(\mathbf{A}_1^B\right)'\right]\mathrm{d}t\\
&\quad+\mathbf{A}_1^B\mathbf{\Sigma}_{\mathbf{X}}'\mathrm{d}\mathbf{z}_t.
\end{aligned}$$

We finally obtain:

$$\frac{\mathrm{d}B\left(\mathbf{X}_t,\tau\right)}{B\left(\mathbf{X}_t,\tau\right)} = \left[R_0+\mathbf{R}_1'\mathbf{X}_t+\mathbf{A}_1^B\mathbf{\Sigma}_{\mathbf{X}}'\left(\mathbf{\Lambda}_0+\mathbf{\Lambda}_1\mathbf{X}_t\right)\right]\mathrm{d}t+\mathbf{A}_1^B\mathbf{\Sigma}_{\mathbf{X}}'\mathrm{d}\mathbf{z}_t.$$

The time-varying risk premium is the instantaneous annualized expected excess return:

$$\frac{1}{\mathrm{d}t}\mathrm{E}_t\left[\frac{\mathrm{d}B\left(\mathbf{X}_t,\tau\right)}{B\left(\mathbf{X}_t,\tau\right)}\right]-R_t=\mathbf{A}_1^B\left(\tau\right)\mathbf{\Sigma}_{\mathbf{X}}'\left(\mathbf{\Lambda}_0+\mathbf{\Lambda}_1\mathbf{X}_t\right). \tag{A.3}$$

The model-implied expected average nominal yield for the maturity $\tau$ over a horizon $\tau_h$ is given by:

$$\frac{1}{\tau_h}\mathrm{E}_t\left[\frac{1}{\tau_h}\int_t^{t+\tau_h}y^n\left(\mathbf{X}_s,\tau\right)\mathrm{d}s\right]=\frac{A_0^{y^n}\left(\tau_h\right)}{\tau_h}+\frac{\mathbf{A}_1^{y^n}\left(\tau_h\right)}{\tau_h}\mathbf{X}_t, \tag{A.4}$$

where:

$$\begin{aligned}
A_0^{y^n}\left(\tau_h\right) &= \frac{A_0^B\left(\tau\right)}{\tau}\tau_h+\frac{\mathbf{A}_1^B\left(\tau\right)}{\tau}\int_0^{\tau_h}\left(I-e^{-\mathbf{\Theta}s}\right)\bar{\mathbf{X}}\mathrm{d}s,\\
\mathbf{A}_1^{y^n}\left(\tau_h\right) &= \frac{\mathbf{A}_1^B\left(\tau\right)}{\tau}\int_0^{\tau_h}e^{-\mathbf{\Theta}s}\mathrm{d}s.
\end{aligned}$$

## A.2 Real bond pricing

We conjecture that the nominal price of a zero-coupon inflation-indexed bond has the following functional form:

$$I\left(\mathbf{X}_t,P_t,\tau\right)=P_te^{-y^r\left(\mathbf{X}_t,\tau\right)\tau}=P_te^{A_0^I(\tau)+\mathbf{A}_1^I(\tau)\mathbf{X}_t}.$$

From no-arbitrage arguments, the nominal price $I$ of an inflation-indexed bond satisfies the following PDE:

$$\begin{aligned}
\left(I_{\mathbf{X}}\mathbf{\Sigma}_{\mathbf{X}}'+I_P\boldsymbol{\sigma}_P'P_t\right)\Lambda_t &= I_{\mathbf{X}}\mathbf{\Theta}\left(\bar{\mathbf{X}}-\mathbf{X}_t\right)+I_PP_t\pi_t+\frac{1}{2}\mathrm{tr}\left(I_{\mathbf{XX}}\mathbf{\Sigma}_{\mathbf{X}}'\mathbf{\Sigma}_{\mathbf{X}}\right)+I_{\mathbf{X}P}\mathbf{\Sigma}_{\mathbf{X}}'\boldsymbol{\sigma}_PP_t\\
&\quad+\frac{1}{2}I_{PP}\boldsymbol{\sigma}_P'\boldsymbol{\sigma}_PP_t^2+I_t-R_tI.
\end{aligned}$$

The derivatives can be computed explicitly as follows:

$$\begin{aligned}
I_{\mathbf{X}} &= \mathbf{A}_1^II,\\
I_P &= \frac{I}{P_t},\\
I_{\mathbf{XX}} &= \left(\mathbf{A}_1^I\right)'\mathbf{A}_1^II,\\
I_{\mathbf{X}P} &= \mathbf{A}_1^I\frac{I}{P_t},\\
I_{PP} &= 0,\\
I_t &= -I_\tau=-\left(\frac{\partial}{\partial\tau}A_0^I\left(\tau\right)+\frac{\partial}{\partial\tau}\mathbf{A}_1^I\left(\tau\right)\mathbf{X}_t\right)I.
\end{aligned}$$

Substituting into the PDE (A.5) the derivatives, as well as the expressions for $R_t$, $\pi_t$ and $\mathbf{\Lambda}_t$ yields:

$$\begin{aligned}
\left(\mathbf{A}_1^I\mathbf{\Sigma}_{\mathbf{X}}'+\boldsymbol{\sigma}_P'\right)\left(\mathbf{\Lambda}_0+\mathbf{\Lambda}_1\mathbf{X}_t\right) &= \mathbf{A}_1^I\mathbf{\Theta}\left(\bar{\mathbf{X}}-\mathbf{X}_t\right)+\left(\pi_0+\boldsymbol{\pi}_1'\mathbf{X}_t\right)+\frac{1}{2}\mathbf{A}_1^I\mathbf{\Sigma}_{\mathbf{X}}'\mathbf{\Sigma}_{\mathbf{X}}\left(\mathbf{A}_1^I\right)'\\
&\quad+\mathbf{A}_1^I\mathbf{\Sigma}_{\mathbf{X}}'\boldsymbol{\sigma}_P-\left(\frac{\partial}{\partial\tau}A_0^I\left(\tau\right)+\mathbf{X}_t\right)\\
&\quad-\left(R_0+\mathbf{R}_1'\mathbf{X}_t\right).
\end{aligned}$$

Matching the homogeneous term and the term in $\mathbf{X}_t$:

$$\begin{aligned}
\frac{\partial}{\partial\tau}A_0^I(\tau) &= \mathbf{A}_1^I\left(\boldsymbol{\Theta}\bar{\mathbf{X}}-\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Lambda}_0\right)+\mathbf{A}_1^I\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\sigma}_P+\frac{1}{2}\mathbf{A}_1^I\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}_{\mathbf{X}}\left(\mathbf{A}_1^I\right)'-R_0+\pi_0-\boldsymbol{\sigma}_P'\boldsymbol{\Lambda}_0, && \text{(A.5)}\\
\frac{\partial}{\partial\tau}\mathbf{A}_1^I(\tau) &= -\mathbf{A}_1^I\left(\boldsymbol{\Theta}+\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Lambda}_1\right)-\mathbf{R}_1'+\boldsymbol{\pi}_1'-\boldsymbol{\sigma}_P'\boldsymbol{\Lambda}_1. && \text{(A.6)}
\end{aligned}$$

The real rate is given considering the limit for $\tau\to 0$:

$$r_t=R_0-\pi_0+\boldsymbol{\sigma}_P'\boldsymbol{\Lambda}_0+\left(\mathbf{R}_1'-\boldsymbol{\pi}_1'+\boldsymbol{\sigma}_P'\boldsymbol{\Lambda}_1\right)\mathbf{X}_t=R_t-\underbrace{\left(\pi_t-\boldsymbol{\sigma}_P'\boldsymbol{\Lambda}_t\right)}_{\text{bei}_t^0}, \tag{A.7}$$

where the instantaneous break-even inflation rate, $\text{bei}_t^0$, is the difference between the instantaneous expected inflation, $\pi_t$, and the risk premium associated to the unexpected inflation, $\boldsymbol{\sigma}_P'\boldsymbol{\Lambda}_t$. The dynamics of the price of a zero-coupon real bond with time to maturity $\tau$ can be computed as:

$$\begin{aligned}
\frac{\mathrm{d}I\left(\mathbf{X}_t,P_t,\tau\right)}{I\left(\mathbf{X}_t,P_t,\tau\right)} &= \left[\begin{array}{c}-\left(\frac{\partial}{\partial\tau}A_0^I(\tau)+\frac{\partial}{\partial\tau}\mathbf{A}_1^I(\tau)\mathbf{X}_t\right)+\mathbf{A}_1^I\boldsymbol{\Theta}\left(\bar{\mathbf{X}}-\mathbf{X}_t\right)\\ +\left(\pi_0+\boldsymbol{\pi}_1'\mathbf{X}_t\right)+\frac{1}{2}\mathbf{A}_1^I\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}_{\mathbf{X}}\left(\mathbf{A}_1^I\right)'+\mathbf{A}_1^I\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\sigma}_P\end{array}\right]\mathrm{d}t\\
&\quad+\mathbf{A}_1^I\boldsymbol{\Sigma}_{\mathbf{X}}'\mathrm{d}\mathbf{z}_t+\boldsymbol{\sigma}_P'\mathrm{d}\mathbf{z}_t.
\end{aligned}$$

Substituting for the expressions (A.5) and (A.6):

$$\begin{aligned}
\frac{\mathrm{d}I\left(\mathbf{X}_t,P_t,\tau\right)}{I\left(\mathbf{X}_t,P_t,\tau\right)} &= -\left[\begin{array}{c}\mathbf{A}_1^I\left(\boldsymbol{\Theta}\bar{\mathbf{X}}-\boldsymbol{\Sigma}_{\mathbf{X}}'\left(\boldsymbol{\Lambda}_0-\boldsymbol{\sigma}_P\right)\right)+\frac{1}{2}\mathbf{A}_1^I\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}_{\mathbf{X}}\left(\mathbf{A}_1^I\right)'\\ -R_0+\pi_0-\boldsymbol{\sigma}_P'\boldsymbol{\Lambda}_0-\mathbf{A}_1^I\left(\boldsymbol{\Theta}+\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Lambda}_1\right)\mathbf{X}_t\\ -\mathbf{R}_1'\mathbf{X}_t+\boldsymbol{\pi}_1'\mathbf{X}_t-\boldsymbol{\sigma}_P'\boldsymbol{\Lambda}_1\mathbf{X}_t\end{array}\right]\mathrm{d}t\\
&\quad+\left[\begin{array}{c}\mathbf{A}_1^I\boldsymbol{\Theta}\left(\bar{\mathbf{X}}-\mathbf{X}_t\right)+\left(\pi_0+\boldsymbol{\pi}_1'\mathbf{X}_t\right)+\frac{1}{2}\mathbf{A}_1^I\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}_{\mathbf{X}}\left(\mathbf{A}_1^I\right)'\\ +\mathbf{A}_1^I\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\sigma}_P\end{array}\right]\mathrm{d}t\\
&\quad+\mathbf{A}_1^I\boldsymbol{\Sigma}_{\mathbf{X}}'\mathrm{d}\mathbf{z}_t+\boldsymbol{\sigma}_P'\mathrm{d}\mathbf{z}_t.
\end{aligned}$$

We finally obtain:

$$\begin{aligned}
\frac{\mathrm{d}I\left(\mathbf{X}_t,P_t,\tau\right)}{I\left(\mathbf{X}_t,P_t,\tau\right)} &= \left[R_0+\mathbf{R}_1'\mathbf{X}_t+\left(\mathbf{A}_1^I\boldsymbol{\Sigma}_{\mathbf{X}}'+\boldsymbol{\sigma}_P'\right)\left(\boldsymbol{\Lambda}_0+\boldsymbol{\Lambda}_1\mathbf{X}_t\right)\right]\mathrm{d}t\\
&\quad+\left(\mathbf{A}_1^I\boldsymbol{\Sigma}_{\mathbf{X}}'+\boldsymbol{\sigma}_P'\right)\mathrm{d}\mathbf{z}_t.
\end{aligned}$$

The time-varying risk premium is the instantaneous annualized expected excess return:

$$\frac{1}{\mathrm{d}t}\mathrm{E}_t\left[\frac{\mathrm{d}I\left(\mathbf{X}_t,P_t,\tau\right)}{I\left(\mathbf{X}_t,P_t,\tau\right)}\right]-R_t=\left(\mathbf{A}_1^I(\tau)\,\boldsymbol{\Sigma}_{\mathbf{X}}'+\boldsymbol{\sigma}_P'\right)\left(\boldsymbol{\Lambda}_0+\boldsymbol{\Lambda}_1\mathbf{X}_t\right) \tag{A.8}$$

The break-even inflation rate for a maturity $\tau$ is given by:

$$\text{bei}\left(\mathbf{X}_t,\tau\right)=y^n\left(\mathbf{X}_t,\tau\right)-y^r\left(\mathbf{X}_t,\tau\right)=\frac{A_0^I(\tau)-A_0^B(\tau)}{\tau}+\frac{\mathbf{A}_1^I(\tau)-\mathbf{A}_1^B(\tau)}{\tau}\mathbf{X}_t.$$

As in D'Amico et al. (2018), the model-implied expected inflation for a maturity $\tau$ is given by:

$$\frac{1}{\tau}\mathrm{E}_t\left[\log\frac{P_{t+\tau}}{P_t}\right] = \frac{A_0^P(\tau)}{\tau} + \frac{\mathbf{A}_1^P(\tau)}{\tau}\mathbf{X}_t, \tag{A.9}$$

where:

$$\begin{aligned} A_0^P(\tau) &= \left(\pi_0 - \frac{\boldsymbol{\sigma}_P'\boldsymbol{\sigma}_P}{2}\right)\tau + \boldsymbol{\pi}_1'\int_0^{\tau}\left(I - e^{-\boldsymbol{\Theta}s}\right)\bar{\mathbf{X}}\mathrm{d}s, \\ \mathbf{A}_1^P(\tau) &= \boldsymbol{\pi}_1'\int_0^{\tau}e^{-\boldsymbol{\Theta}s}\mathrm{d}s. \end{aligned}$$

### A.3 Long-run expected stock return

The model-implied expected annualized stock return over a horizon $\tau_h$ is given by:

$$\frac{1}{\tau_h}\mathrm{E}_t\left[\log\frac{S_{t+\tau_h}}{S_t}\right] = \frac{A_0^S(\tau_h)}{\tau_h} + \frac{\mathbf{A}_1^S(\tau_h)}{\tau_h}\mathbf{X}_t, \tag{A.10}$$

where:

$$\begin{aligned} A_0^S(\tau_h) &= \left(R_0 + \boldsymbol{\sigma}_S'\boldsymbol{\Lambda}_0 - \frac{\boldsymbol{\sigma}_S'\boldsymbol{\sigma}_S}{2}\right)\tau_h + \left(\mathbf{R}_1' + \boldsymbol{\sigma}_S'\boldsymbol{\Lambda}_1\right)\int_0^{\tau_h}\left(I - e^{-\boldsymbol{\Theta}s}\right)\bar{\mathbf{X}}\mathrm{d}s, \\ \mathbf{A}_1^S(\tau_h) &= \left(\mathbf{R}_1' + \boldsymbol{\sigma}_S'\boldsymbol{\Lambda}_1\right)\int_0^{\tau_h}e^{-\boldsymbol{\Theta}s}\mathrm{d}s. \end{aligned}$$

# B Portfolio strategy

## B.1 Optimal portfolio strategy in incomplete markets

The value function can be expressed as:

$$J(W_t, P_t, T-t) = \frac{W_t^{1-\gamma}P_t^{-(1-\alpha)(1-\gamma)}}{1-\gamma}F(\mathbf{X}_t, T-t) \tag{B.11}$$

With the martingale approach, it can be shown that:

$$W_t = l^{-\frac{1}{\gamma}}\Phi_t^{-\frac{1}{\gamma}}P_t^{-(1-\alpha)\frac{1-\gamma}{\gamma}}\mathrm{E}_t\left[\left(\frac{\Phi_T}{\Phi_t}\left(\frac{P_T}{P_t}\right)^{(1-\alpha)}\right)^{1-\frac{1}{\gamma}}\right], \tag{B.12}$$

where $\Phi_0 = 1$, $P_0 = 1$ and:

$$l^{-\frac{1}{\gamma}} = \frac{W_0}{\mathrm{E}_0\left[\left(\frac{\Phi_T}{\Phi_0}\left(\frac{P_T}{P_0}\right)^{(1-\alpha)}\right)^{1-\frac{1}{\gamma}}\right]}.$$

We rewrite time-$t$ wealth as follows:

$$W_t = W_0\Phi_t^{-\frac{1}{\gamma}}P_t^{-(1-\alpha)\frac{1-\gamma}{\gamma}}\frac{F(\mathbf{X}_t, T-t)}{F(\mathbf{X}_0, T)} \equiv G(\Phi_t, P_t, \mathbf{X}_t, T-t), \tag{B.13}$$

where:

$$F\left(\mathbf{X}_t, T-t\right) = \mathrm{E}_t\left[\left(\frac{\Phi_T}{\Phi_t}\left(\frac{P_T}{P_t}\right)^{(1-\alpha)}\right)^{1-\frac{1}{\gamma}}\right].$$

Computing the first-order condition of the equation of the value function (B.11) and solving by $W_t$, by comparison with (B.12), it immediately follows that $F\left(\mathbf{X}_t, T-t\right)$ is such that the value function can be expressed as follows:

$$J\left(W_t, P_t, t\right) = \frac{W_t^{1-\gamma} P_t^{-(1-\alpha)(1-\gamma)}}{1-\gamma}\left[F\left(\mathbf{X}_t, T-t\right)\right]^{\gamma}, \tag{B.14}$$

The function $F\left(\mathbf{X}_t, T-t\right)$ takes the form:

$$F\left(\mathbf{X}_t, T-t\right) = \exp\left\{\frac{1}{2}\mathbf{X}_t'\mathbf{B}_3\left(T-t\right)\mathbf{X}_t + \mathbf{B}_2\left(T-t\right)\mathbf{X}_t + B_1\left(T-t\right)\right\}. \tag{B.15}$$

Recalling that $G \equiv W_t$, the following no-arbitrage relation holds:

$$\mathcal{L}G + G_t - R_t G = \left(-G_\Phi \Phi_t \mathbf{\Lambda}_t' + G_P P_t \boldsymbol{\sigma}_P' + G_\mathbf{X}\mathbf{\Sigma}_\mathbf{X}'\right)\mathbf{\Lambda}_t, \tag{B.16}$$

where:

$$\begin{aligned}
\mathcal{L}G &= G_\Phi\left(-\Phi_t R_t\right) + G_P\left(P_t \pi_t\right) + G_\mathbf{X}\mathbf{\Theta}\left(\bar{\mathbf{X}} - \mathbf{X}_t\right) \\
&\quad + G_{\Phi\mathbf{X}}\mathbf{\Sigma}_\mathbf{X}'\left(-\Phi_t\mathbf{\Lambda}_t\right) + G_{P\mathbf{X}}\mathbf{\Sigma}_\mathbf{X}'\left(P_t\boldsymbol{\sigma}_P\right) + G_{\Phi P}\left(P_t\boldsymbol{\sigma}_P\right)'\left(-\Phi_t\mathbf{\Lambda}_t\right) \\
&\quad + \frac{1}{2}\left(G_{\Phi\Phi}\left(-\Phi_t\mathbf{\Lambda}_t\right)'\left(-\Phi_t\mathbf{\Lambda}_t\right) + G_{PP}\left(P_t\boldsymbol{\sigma}_P\right)'\left(P_t\boldsymbol{\sigma}_P\right) + \mathrm{tr}\left(G_{\mathbf{XX}}\mathbf{\Sigma}_\mathbf{X}'\mathbf{\Sigma}_\mathbf{X}\right)\right).
\end{aligned}$$

Substituting for the partial derivatives of $G$:

$$\begin{aligned}
\mathcal{L}G &= -\frac{1}{\gamma}\Phi_t^{-1}G\left(-\Phi_t R_t\right) - (1-\alpha)\frac{1-\gamma}{\gamma}P_t^{-1}G\left(P_t\pi_t\right) + \frac{F_\mathbf{X}}{F}G\mathbf{\Theta}\left(\bar{\mathbf{X}} - \mathbf{X}_t\right) \\
&\quad - \frac{1}{\gamma}\Phi_t^{-1}\frac{F_\mathbf{X}}{F}G\mathbf{\Sigma}_\mathbf{X}'\left(-\Phi_t\mathbf{\Lambda}_t\right) - (1-\alpha)\frac{1-\gamma}{\gamma}P_t^{-1}\frac{F_\mathbf{X}}{F}G\mathbf{\Sigma}_\mathbf{X}'\left(P_t\boldsymbol{\sigma}_P\right) \\
&\quad + (1-\alpha)\frac{1-\gamma}{\gamma^2}\Phi_t^{-1}P_t^{-1}G\left(P_t\boldsymbol{\sigma}_P\right)'\left(-\Phi_t\mathbf{\Lambda}_t\right) \\
&\quad + \frac{1}{2}\left(\begin{array}{c}
\frac{1}{\gamma}\left(1+\frac{1}{\gamma}\right)\Phi_t^{-2}G\left(-\Phi_t\mathbf{\Lambda}_t\right)'\left(-\Phi_t\mathbf{\Lambda}_t\right) \\
+(1-\alpha)\frac{1-\gamma}{\gamma}\left(1+(1-\alpha)\frac{1-\gamma}{\gamma}\right)P_t^{-2}G\left(P_t\boldsymbol{\sigma}_P\right)'\left(P_t\boldsymbol{\sigma}_P\right) \\
+\mathrm{tr}\left(\frac{F_{\mathbf{XX}}}{F}G\mathbf{\Sigma}_\mathbf{X}'\mathbf{\Sigma}_\mathbf{X}\right)
\end{array}\right).
\end{aligned}$$

Simplifying:

$$\begin{aligned}
\frac{\mathcal{L}G}{G} &= \frac{1}{\gamma}R_t - (1-\alpha)\frac{1-\gamma}{\gamma}\pi_t + \frac{F_\mathbf{X}}{F}\mathbf{\Theta}\left(\bar{\mathbf{X}} - \mathbf{X}_t\right) \\
&\quad + \frac{1}{\gamma}\frac{F_\mathbf{X}}{F}\mathbf{\Sigma}_\mathbf{X}'\mathbf{\Lambda}_t - (1-\alpha)\frac{1-\gamma}{\gamma}\frac{F_\mathbf{X}}{F}\mathbf{\Sigma}_\mathbf{X}'\boldsymbol{\sigma}_P - (1-\alpha)\frac{1-\gamma}{\gamma^2}\boldsymbol{\sigma}_P'\mathbf{\Lambda}_t \\
&\quad + \frac{1}{2}\left(\frac{1}{\gamma}\left(1+\frac{1}{\gamma}\right)\mathbf{\Lambda}_t'\mathbf{\Lambda}_t + (1-\alpha)\frac{1-\gamma}{\gamma}\left(1+(1-\alpha)\frac{1-\gamma}{\gamma}\right)\boldsymbol{\sigma}_P'\boldsymbol{\sigma}_P + \mathrm{tr}\left(\frac{F_{\mathbf{XX}}}{F}\mathbf{\Sigma}_\mathbf{X}'\mathbf{\Sigma}_\mathbf{X}\right)\right).
\end{aligned}$$

Substituting into (B.16):

$$\begin{aligned}
\left(\frac{1}{\gamma}\mathbf{\Lambda}_t' - (1-\alpha)\frac{1-\gamma}{\gamma}\boldsymbol{\sigma}_P' + \frac{F_{\mathbf{X}}}{F}\mathbf{\Sigma}_{\mathbf{X}}'\right)\mathbf{\Lambda}_t \;=\;& \frac{1}{\gamma}R_t - (1-\alpha)\frac{1-\gamma}{\gamma}\pi_t + \frac{F_{\mathbf{X}}}{F}\mathbf{\Theta}\left(\bar{\mathbf{X}} - \mathbf{X}_t\right) \\
&+\frac{1}{\gamma}\frac{F_{\mathbf{X}}}{F}\mathbf{\Sigma}_{\mathbf{X}}'\mathbf{\Lambda}_t - (1-\alpha)\frac{1-\gamma}{\gamma}\frac{F_{\mathbf{X}}}{F}\mathbf{\Sigma}_{\mathbf{X}}'\boldsymbol{\sigma}_P \\
&-(1-\alpha)\frac{1-\gamma}{\gamma^2}\boldsymbol{\sigma}_P'\mathbf{\Lambda}_t + \frac{1}{2\gamma}\left(1+\frac{1}{\gamma}\right)\mathbf{\Lambda}_t'\mathbf{\Lambda}_t \\
&+(1-\alpha)\frac{1-\gamma}{2\gamma}\left(1+(1-\alpha)\frac{1-\gamma}{\gamma}\right)\boldsymbol{\sigma}_P'\boldsymbol{\sigma}_P \\
&+\frac{1}{2}\mathrm{tr}\left(\frac{F_{\mathbf{XX}}}{F}\mathbf{\Sigma}_{\mathbf{X}}'\mathbf{\Sigma}_{\mathbf{X}}\right) + \frac{G_t}{G} - R_t.
\end{aligned}$$

Rearranging and collecting terms:

$$\begin{aligned}
0 \;=\;& \left(\frac{1}{\gamma}-1\right)R_t - (1-\alpha)\frac{1-\gamma}{\gamma}\pi_t + \frac{F_{\mathbf{X}}}{F}\mathbf{\Theta}\left(\bar{\mathbf{X}} - \mathbf{X}_t\right) \\
&+\left(\frac{1}{\gamma}-1\right)\frac{F_{\mathbf{X}}}{F}\mathbf{\Sigma}_{\mathbf{X}}'\mathbf{\Lambda}_t - (1-\alpha)\frac{1-\gamma}{\gamma}\frac{F_{\mathbf{X}}}{F}\mathbf{\Sigma}_{\mathbf{X}}'\boldsymbol{\sigma}_P \\
&+\left(1-\frac{1}{\gamma}\right)(1-\alpha)\frac{1-\gamma}{\gamma}\boldsymbol{\sigma}_P'\mathbf{\Lambda}_t \\
&+\frac{1}{2\gamma}\left(\frac{1}{\gamma}-1\right)\mathbf{\Lambda}_t'\mathbf{\Lambda}_t + (1-\alpha)\frac{1-\gamma}{2\gamma}\left(1+(1-\alpha)\frac{1-\gamma}{\gamma}\right)\boldsymbol{\sigma}_P'\boldsymbol{\sigma}_P \\
&+\frac{1}{2}\mathrm{tr}\left(\frac{F_{\mathbf{XX}}}{F}\mathbf{\Sigma}_{\mathbf{X}}'\mathbf{\Sigma}_{\mathbf{X}}\right) + \frac{F_t}{F}.
\end{aligned}$$

It is at this point necessary to elaborate on market completeness. The optimal portfolio strategy is obtained equaling the diffusion terms of the dynamics of (log) wealth, written either as in (B.13) or as a linear combination of the dynamics of the underlying assets:

$$\mathbf{\Sigma}\boldsymbol{\omega} = \frac{1}{\gamma}\mathbf{\Lambda}_t - (1-\alpha)\frac{1-\gamma}{\gamma}\boldsymbol{\sigma}_P + \mathbf{\Sigma}_{\mathbf{X}}\frac{(F_{\mathbf{X}})'}{F}. \tag{B.17}$$

This $n$-dimensional equation imposes that the $N$ non-redundant traded assets, for which the volatility vectors are at the l.h.s., span the dynamics of optimal wealth at the r.h.s.. In the case of complete markets this happens without issues, as $\mathbf{\Sigma}$ is invertible. If instead markets are incomplete, we follow Sangvinatsos and Wachter (2005) and impose some further restrictions to make sure that the r.h.s. is completely spanned by the traded assets. We decompose the r.h.s. of the equation into a contribution spanned by the traded assets and a contribution orthogonal to the asset space. In order to decompose the vector of market prices of risk, we write $\mathbf{\Lambda}_t = \mathbf{\Lambda}_t^* + \boldsymbol{\nu}_t^*$, where $\mathbf{\Lambda}_t^*$ belongs to the column space of the volatility matrix of the traded assets and $\boldsymbol{\nu}_t^*$ belongs to the null space. We pre-multiply (B.17) by $\mathbf{I} - \mathbf{\Sigma}\left(\mathbf{\Sigma}'\mathbf{\Sigma}\right)^{-1}\mathbf{\Sigma}'$:

$$\begin{aligned}
\underbrace{\left(\mathbf{I} - \mathbf{\Sigma}\left(\mathbf{\Sigma}'\mathbf{\Sigma}\right)^{-1}\mathbf{\Sigma}'\right)\mathbf{\Sigma}\boldsymbol{\omega}}_{0} \;=\;& \frac{1}{\gamma}\left(\mathbf{I} - \mathbf{\Sigma}\left(\mathbf{\Sigma}'\mathbf{\Sigma}\right)^{-1}\mathbf{\Sigma}'\right)\left(\mathbf{\Lambda}_t^* + \boldsymbol{\nu}_t^*\right) \\
&-(1-\alpha)\frac{1-\gamma}{\gamma}\left(\mathbf{I} - \mathbf{\Sigma}\left(\mathbf{\Sigma}'\mathbf{\Sigma}\right)^{-1}\mathbf{\Sigma}'\right)\boldsymbol{\sigma}_P
\end{aligned}$$

$$+\left(\mathbf{I}-\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\right)\boldsymbol{\Sigma}_{\mathbf{X}}\frac{\left(F_{\mathbf{X}}\right)'}{F}.$$

This condition will affect the function $F$, which we still have to determine, and therefore the dynamics of optimal wealth and the optimal portfolio weights. As $\left(\mathbf{I}-\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\right)\boldsymbol{\Lambda}_t^*=\mathbf{0}$, then:

$$\boldsymbol{\nu}_t^*=(1-\alpha)(1-\gamma)\left(\mathbf{I}-\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\right)\boldsymbol{\sigma}_P-\gamma\left(\mathbf{I}-\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\right)\boldsymbol{\Sigma}_{\mathbf{X}}\frac{\left(F_{\mathbf{X}}\right)'}{F}.$$

Pre-multiplying instead (B.17) by $\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'$:

$$\begin{aligned}\underbrace{\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\boldsymbol{\omega}}_{\boldsymbol{\Sigma}\boldsymbol{\omega}} &= \frac{1}{\gamma}\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\left(\boldsymbol{\Lambda}_t^*+\boldsymbol{\nu}_t^*\right)-(1-\alpha)\frac{1-\gamma}{\gamma}\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\sigma}_P \\ &\quad+\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Sigma}_{\mathbf{X}}\frac{\left(F_{\mathbf{X}}\right)'}{F}.\end{aligned}$$

As $\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\nu}_t^*=\mathbf{0}$, by pre-multiplying again by $\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'$, we obtain the expression for the optimal portfolio weights (21).

We employ the following notations:

$$\boldsymbol{\Lambda}_t^*=\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Lambda}_t=\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\left(\boldsymbol{\Lambda}_0+\boldsymbol{\Lambda}_1\mathbf{X}_t\right)=\boldsymbol{\Lambda}_0^*+\boldsymbol{\Lambda}_1^*\mathbf{X}_t,$$

where:

$$\begin{aligned}\boldsymbol{\Lambda}_0^* &= \boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Lambda}_0,\\ \boldsymbol{\Lambda}_1^* &= \boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Lambda}_1,\end{aligned}$$

In the remaining, we also employ the following notations:

$$\begin{aligned}\boldsymbol{\sigma}_P^{\perp} &= \left(\mathbf{I}-\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\right)\boldsymbol{\sigma}_P,\\ \boldsymbol{\Sigma}_{\mathbf{X}}^{\perp} &= \left(\mathbf{I}-\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\right)\boldsymbol{\Sigma}_{\mathbf{X}}.\end{aligned}$$

The market prices of risk can therefore be decomposed as:

$$\boldsymbol{\Lambda}_t=\boldsymbol{\Lambda}_t^*+\boldsymbol{\nu}_t^*=\boldsymbol{\Lambda}_0^*+\boldsymbol{\Lambda}_1^*\mathbf{X}_t+(1-\alpha)(1-\gamma)\boldsymbol{\sigma}_P^{\perp}-\gamma\boldsymbol{\Sigma}_{\mathbf{X}}^{\perp}\frac{\left(F_{\mathbf{X}}\right)'}{F}.$$

Setting $\tau=T-t$, we guess the functional form for $F\left(\mathbf{X}_t,\tau\right)$:

$$F\left(\mathbf{X}_t,\tau\right)=\exp\left\{\frac{1}{2}\mathbf{X}_t'\mathbf{B}_3(\tau)\mathbf{X}_t+\mathbf{B}_2(\tau)\mathbf{X}_t+B_1(\tau)\right\},$$

and compute the derivatives:

$$F_{\mathbf{X}} = \left(\frac{1}{2}\mathbf{X}_t'\left(\mathbf{B}_3(\tau)+\mathbf{B}_3'(\tau)\right)+\mathbf{B}_2(\tau)\right)F=\left(\mathbf{X}_t'\tilde{\mathbf{B}}_3(\tau)+\mathbf{B}_2(\tau)\right)F,$$

$$
\begin{aligned}
F_{\mathbf{XX}} &= \left(\frac{1}{2}\mathbf{X}_t'\left(\mathbf{B}_3(\tau)+\mathbf{B}_3'(\tau)\right)+\mathbf{B}_2(\tau)\right)'\left(\frac{1}{2}\mathbf{X}_t'\left(\mathbf{B}_3(\tau)+\mathbf{B}_3'(\tau)\right)+\mathbf{B}_2(\tau)\right)F \\
&\quad +\frac{1}{2}\left(\mathbf{B}_3(\tau)+\mathbf{B}_3'(\tau)\right)F \\
&= \frac{1}{4}\left(\mathbf{B}_3(\tau)+\mathbf{B}_3'(\tau)\right)'\mathbf{X}_t\mathbf{X}_t'\left(\mathbf{B}_3(\tau)+\mathbf{B}_3'(\tau)\right)F+\frac{1}{2}\mathbf{B}_2'(\tau)\mathbf{X}_t'\left(\mathbf{B}_3(\tau)+\mathbf{B}_3'(\tau)\right)F \\
&\quad +\frac{1}{2}\left(\mathbf{B}_3(\tau)+\mathbf{B}_3'(\tau)\right)'\mathbf{X}_t\mathbf{B}_2(\tau)F+\mathbf{B}_2'(\tau)\mathbf{B}_2(\tau)F+\frac{1}{2}\left(\mathbf{B}_3(\tau)+\mathbf{B}_3'(\tau)\right)F \\
&= \tilde{\mathbf{B}}_3(\tau)\mathbf{X}_t\mathbf{X}_t'\tilde{\mathbf{B}}_3(\tau)F+\mathbf{B}_2'(\tau)\mathbf{X}_t'\tilde{\mathbf{B}}_3(\tau)F+\tilde{\mathbf{B}}_3(\tau)\mathbf{X}_t\mathbf{B}_2(\tau)F \\
&\quad +\mathbf{B}_2'(\tau)\mathbf{B}_2(\tau)F+\tilde{\mathbf{B}}_3(\tau)F, \\
F_t &= -\left(\frac{1}{2}\mathbf{X}_t'\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{B}_3(\tau)\mathbf{X}_t+\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{B}_2(\tau)\mathbf{X}_t+\frac{\mathrm{d}}{\mathrm{d}\tau}B_1(\tau)\right)F,
\end{aligned}
$$

where $\tilde{\mathbf{B}}_3(\tau)=\frac{\mathbf{B}_3(\tau)+\mathbf{B}_3'(\tau)}{2}$. The market prices of risk can be rewritten as:

$$
\begin{aligned}
\boldsymbol{\Lambda}_t &= \boldsymbol{\Lambda}_t^*+\boldsymbol{\nu}_t^* \\
&= \boldsymbol{\Lambda}_0^*+\boldsymbol{\Lambda}_1^*\mathbf{X}_t+(1-\alpha)(1-\gamma)\boldsymbol{\sigma}_P^{\perp}-\gamma\boldsymbol{\Sigma}_{\mathbf{X}}^{\perp}\left(\tilde{\mathbf{B}}_3(\tau)\mathbf{X}_t+\mathbf{B}_2'(\tau)\right) \\
&= \boldsymbol{\Lambda}_0^*+(1-\alpha)(1-\gamma)\boldsymbol{\sigma}_P^{\perp}-\gamma\boldsymbol{\Sigma}_{\mathbf{X}}^{\perp}\mathbf{B}_2'(\tau)+\left(\boldsymbol{\Lambda}_1^*-\gamma\boldsymbol{\Sigma}_{\mathbf{X}}^{\perp}\tilde{\mathbf{B}}_3(\tau)\right)\mathbf{X}_t \\
&= \tilde{\boldsymbol{\Lambda}}_0^*+\tilde{\boldsymbol{\Lambda}}_1^*\mathbf{X}_t,
\end{aligned}
$$

where $\tilde{\boldsymbol{\Lambda}}_0^*=\boldsymbol{\Lambda}_0^*+(1-\alpha)(1-\gamma)\boldsymbol{\sigma}_P^{\perp}-\gamma\boldsymbol{\Sigma}_{\mathbf{X}}^{\perp}\mathbf{B}_2'(\tau)$ and $\tilde{\boldsymbol{\Lambda}}_1^*=\boldsymbol{\Lambda}_1^*-\gamma\boldsymbol{\Sigma}_{\mathbf{X}}^{\perp}\tilde{\mathbf{B}}_3(\tau)$. Substituting in the PDE for the expressions of $R_t$, $\pi_t$ and $\boldsymbol{\Lambda}_t$, as well as for the derivatives of $F$:

$$
\begin{aligned}
0 &= \left(\frac{1}{\gamma}-1\right)\left(R_0+\mathbf{R}_1'\mathbf{X}_t\right)-(1-\alpha)\frac{1-\gamma}{\gamma}\left(\pi_0+\boldsymbol{\pi}_1'\mathbf{X}_t\right)+\left(\mathbf{X}_t'\tilde{\mathbf{B}}_3(\tau)+\mathbf{B}_2(\tau)\right)\boldsymbol{\Theta}\left(\bar{\mathbf{X}}-\mathbf{X}_t\right) \\
&\quad +\left(\frac{1}{\gamma}-1\right)\left(\mathbf{X}_t'\tilde{\mathbf{B}}_3(\tau)+\mathbf{B}_2(\tau)\right)\boldsymbol{\Sigma}_{\mathbf{X}}'\left(\tilde{\boldsymbol{\Lambda}}_0^*+\tilde{\boldsymbol{\Lambda}}_1^*\mathbf{X}_t\right) \\
&\quad -(1-\alpha)\frac{1-\gamma}{\gamma}\left(\mathbf{X}_t'\tilde{\mathbf{B}}_3(\tau)+\mathbf{B}_2(\tau)\right)\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\sigma}_P+\left(1-\frac{1}{\gamma}\right)(1-\alpha)\frac{1-\gamma}{\gamma}\boldsymbol{\sigma}_P'\left(\tilde{\boldsymbol{\Lambda}}_0^*+\tilde{\boldsymbol{\Lambda}}_1^*\mathbf{X}_t\right) \\
&\quad +\frac{1}{2\gamma}\left(\frac{1}{\gamma}-1\right)\left(\tilde{\boldsymbol{\Lambda}}_0^*+\tilde{\boldsymbol{\Lambda}}_1^*\mathbf{X}_t\right)'\left(\tilde{\boldsymbol{\Lambda}}_0^*+\tilde{\boldsymbol{\Lambda}}_1^*\mathbf{X}_t\right)+(1-\alpha)\frac{1-\gamma}{2\gamma}\left(1+(1-\alpha)\frac{1-\gamma}{\gamma}\right)\boldsymbol{\sigma}_P'\boldsymbol{\sigma}_P \\
&\quad +\frac{1}{2}\mathrm{tr}\left(\left(\tilde{\mathbf{B}}_3(\tau)\mathbf{X}_t\mathbf{X}_t'\tilde{\mathbf{B}}_3(\tau)+\mathbf{B}_2'(\tau)\mathbf{X}_t'\tilde{\mathbf{B}}_3(\tau)+\tilde{\mathbf{B}}_3(\tau)\mathbf{X}_t\mathbf{B}_2(\tau)+\mathbf{B}_2'(\tau)\mathbf{B}_2(\tau)+\tilde{\mathbf{B}}_3(\tau)\right)\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}_{\mathbf{X}}\right) \\
&\quad -\left(\frac{1}{2}\mathbf{X}_t'\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{B}_3(\tau)\mathbf{X}_t+\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{B}_2(\tau)\mathbf{X}_t+\frac{\mathrm{d}}{\mathrm{d}\tau}B_1(\tau)\right).
\end{aligned}
$$

Isolating the term in $\mathbf{X}_t'\ldots\mathbf{X}_t$ and solving for $\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{B}_3(\tau)$:

$$
\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{B}_3(\tau) = -2\tilde{\mathbf{B}}_3(\tau)\boldsymbol{\Theta}+2\left(\frac{1}{\gamma}-1\right)\tilde{\mathbf{B}}_3(\tau)\boldsymbol{\Sigma}_{\mathbf{X}}'\tilde{\boldsymbol{\Lambda}}_1^*+\frac{1}{\gamma}\left(\frac{1}{\gamma}-1\right)\left(\tilde{\boldsymbol{\Lambda}}_1^*\right)'\tilde{\boldsymbol{\Lambda}}_1^*+\tilde{\mathbf{B}}_3(\tau)\boldsymbol{\Sigma}_X'\boldsymbol{\Sigma}_{\mathbf{X}}\tilde{\mathbf{B}}_3.
$$

Isolating the term in $\mathbf{X}_t$ and solving for $\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{B}_2(\tau)$:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{B}_2(\tau) &= \left(\frac{1}{\gamma}-1\right)\mathbf{R}_1'-(1-\alpha)\frac{1-\gamma}{\gamma}\boldsymbol{\pi}_1'+\bar{\mathbf{X}}'\boldsymbol{\Theta}'\tilde{\mathbf{B}}_3(\tau)-\mathbf{B}_2(\tau)\boldsymbol{\Theta} \\
&\quad +\left(\frac{1}{\gamma}-1\right)\left(\tilde{\boldsymbol{\Lambda}}_0^*\right)'\boldsymbol{\Sigma}_{\mathbf{X}}\tilde{\mathbf{B}}_3(\tau)+\left(\frac{1}{\gamma}-1\right)\mathbf{B}_2(\tau)\boldsymbol{\Sigma}_{\mathbf{X}}'\tilde{\boldsymbol{\Lambda}}_1^*
\end{aligned}
$$

$$
\begin{aligned}
&-\left(1-\alpha\right)\frac{1-\gamma}{\gamma}\boldsymbol{\sigma}_P'\boldsymbol{\Sigma}_{\mathbf{X}}\tilde{\mathbf{B}}_3\left(\tau\right)-\left(1-\alpha\right)\left(\frac{\gamma-1}{\gamma}\right)^2\boldsymbol{\sigma}_P'\tilde{\boldsymbol{\Lambda}}_1^*\\
&+\frac{1}{\gamma}\left(\frac{1}{\gamma}-1\right)\left(\tilde{\boldsymbol{\Lambda}}_0^*\right)'\tilde{\boldsymbol{\Lambda}}_1^*+\mathbf{B}_2\left(\tau\right)\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}_{\mathbf{X}}\tilde{\mathbf{B}}_3\left(\tau\right).
\end{aligned}
$$

Isolating the homogeneous term and solving for $\frac{\mathrm{d}}{\mathrm{d}\tau}B_1\left(\tau\right)$:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}B_1\left(\tau\right) \quad = \quad & \left(\frac{1}{\gamma}-1\right)R_0-\left(1-\alpha\right)\frac{1-\gamma}{\gamma}\pi_0+\mathbf{B}_2\left(\tau\right)\boldsymbol{\Theta}\bar{\mathbf{X}}+\left(\frac{1}{\gamma}-1\right)\mathbf{B}_2\left(\tau\right)\boldsymbol{\Sigma}_{\mathbf{X}}'\tilde{\boldsymbol{\Lambda}}_0^*\\
&-\left(1-\alpha\right)\frac{1-\gamma}{\gamma}\mathbf{B}_2\left(\tau\right)\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\sigma}_P-\left(1-\alpha\right)\left(\frac{\gamma-1}{\gamma}\right)^2\boldsymbol{\sigma}_P'\tilde{\boldsymbol{\Lambda}}_0^*\\
&+\frac{1}{2\gamma}\left(\frac{1}{\gamma}-1\right)\left(\tilde{\boldsymbol{\Lambda}}_0^*\right)'\tilde{\boldsymbol{\Lambda}}_0^*+\left(1-\alpha\right)\frac{1-\gamma}{2\gamma}\left(1+\left(1-\alpha\right)\frac{1-\gamma}{\gamma}\right)\boldsymbol{\sigma}_P'\boldsymbol{\sigma}_P\\
&+\frac{1}{2}\mathbf{B}_2\left(\tau\right)\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}_{\mathbf{X}}\mathbf{B}_2'\left(\tau\right)+\frac{1}{2}\mathrm{tr}\left(\tilde{\mathbf{B}}_3\left(\tau\right)\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}_{\mathbf{X}}\right).
\end{aligned}
$$

In order to write the optimal portfolio weights, recall that $\frac{F_{\mathbf{X}}}{F}=\mathbf{X}_t'\tilde{\mathbf{B}}_3\left(\tau\right)+\mathbf{B}_2\left(\tau\right)$. Then, substituting into (21):

$$
\begin{aligned}
\boldsymbol{\omega}_t \quad = \quad & \frac{1}{\gamma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Lambda}_t-\left(1-\alpha\right)\frac{1-\gamma}{\gamma}\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\sigma}_P\\
&+\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Sigma}_{\mathbf{X}}\left(\tilde{\mathbf{B}}_3\left(\tau\right)\mathbf{X}_t+\mathbf{B}_2'\left(\tau\right)\right).
\end{aligned}
\tag{B.18}
$$

## B.2 Optimal portfolio strategy for an infinitely risk-averse agent and constant market prices of risk

For $\gamma\to\infty$, in the case of constant market prices of risk ($\boldsymbol{\Lambda}_1=\mathbf{0}$), it is possible to write an explicit solution to the problem. Indeed, in this case $\mathbf{B}_3(\tau)\to\mathbf{0}$ and the optimal portfolio strategy becomes independent of the current state $\mathbf{X}_t$:

$$
\boldsymbol{\omega}_t=\left(1-\alpha\right)\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\sigma}_P+\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Sigma}_{\mathbf{X}}\mathbf{B}_2'\left(\tau\right),
\tag{B.19}
$$

where:

$$
\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{B}_2\left(\tau\right)=-\mathbf{R}_1'+\left(1-\alpha\right)\boldsymbol{\pi}_1'-\mathbf{B}_2\left(\tau\right)\boldsymbol{\Theta},
\tag{B.20}
$$

for which the solution is:

$$
\mathbf{B}_2\left(\tau\right)=\left(\mathbf{R}_1'-\left(1-\alpha\right)\boldsymbol{\pi}_1'\right)\left(\mathbf{e}^{-\boldsymbol{\Theta}\tau}-\mathbf{I}\right)\boldsymbol{\Theta}^{-1}.
\tag{B.21}
$$

We can then rewrite the optimal portfolio strategy as:

$$
\begin{aligned}
\boldsymbol{\omega}_t=&\left(1-\alpha\right)\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\sigma}_P+\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Sigma}_{\mathbf{X}}\left[\mathbf{R}_1'\left(\mathbf{e}^{-\boldsymbol{\Theta}\tau}-\mathbf{I}\right)\boldsymbol{\Theta}^{-1}\right]'\\
&-\left(1-\alpha\right)\left(\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\right)^{-1}\boldsymbol{\Sigma}'\boldsymbol{\Sigma}_{\mathbf{X}}\left[\boldsymbol{\pi}_1'\left(\mathbf{e}^{-\boldsymbol{\Theta}\tau}-\mathbf{I}\right)\boldsymbol{\Theta}^{-1}\right]'.
\end{aligned}
\tag{B.22}
$$

Recall that, for constant risk premia, the coefficient $\mathbf{A}_1^B$, used for nominal bond pricing, satisfies the following relation:

$$\frac{\partial}{\partial \tau}\mathbf{A}_1^B(\tau) = -\mathbf{A}_1^B(\tau)\,\mathbf{\Theta} - \mathbf{R}_1', \tag{B.23}$$

Therefore, the explicit solution for $\mathbf{A}_1^B$ is:

$$\mathbf{A}_1^B(\tau) = \mathbf{R}_1'\left(\mathbf{e}^{-\mathbf{\Theta}\tau} - \mathbf{I}\right)\mathbf{\Theta}^{-1}. \tag{B.24}$$

The same applies to the coefficient $\mathbf{A}_1^I$, used for real bond pricing:

$$\frac{\partial}{\partial \tau}\mathbf{A}_1^I(\tau) = -\mathbf{A}_1^I(\tau)\,\mathbf{\Theta} - \left(\mathbf{R}_1' - \boldsymbol{\pi}_1'\right), \tag{B.25}$$

for which the solution is:

$$\mathbf{A}_1^I(\tau) = \left(\mathbf{R}_1' - \boldsymbol{\pi}_1'\right)\left(\mathbf{e}^{-\mathbf{\Theta}\tau} - \mathbf{I}\right)\mathbf{\Theta}^{-1}. \tag{B.26}$$

Therefore:

$$\boldsymbol{\omega}_t = (1-\alpha)\left(\mathbf{\Sigma}'\mathbf{\Sigma}\right)^{-1}\mathbf{\Sigma}'\left(\mathbf{\Sigma_X}\mathbf{A}_1^I(\tau) + \boldsymbol{\sigma}_P\right) + \alpha\left(\mathbf{\Sigma}'\mathbf{\Sigma}\right)^{-1}\mathbf{\Sigma}'\mathbf{\Sigma_X}\mathbf{A}_1^B(\tau), \tag{B.27}$$

where the first term, with weight $1-\alpha$, replicates an inflation-indexed bond with time-to-maturity $\tau$, and the second term, with weight $\alpha$, replicates a nominal bond with time-to-maturity $\tau$.

## B.3 Utility from suboptimal strategies

Given a certain degree of money illusion $\alpha$, the optimal portfolio strategy is given by (B.18). We are interested in assessing the welfare of an investor, following a certain portfolio strategy, which is suboptimal with respect to her degree of money illusion $\hat{\alpha}$. In particular, we are interested in assessing the utility loss sustained by a non-illusioned agent ($\hat{\alpha} = 0$), in the case where she follows the same portfolio strategy as an illusioned agent ($\alpha > 0$). The portfolio weights, calculated (sub-optimally) for $\alpha > 0$, take the form:

$$\begin{aligned}\hat{\boldsymbol{\omega}}_t(\tau) &= \frac{1}{\gamma}\left(\mathbf{\Sigma}'\mathbf{\Sigma}\right)^{-1}\mathbf{\Sigma}'\left(\mathbf{\Lambda}_0 + \mathbf{\Lambda}_1\mathbf{X}_t\right) + (1-\alpha)\left(1-\frac{1}{\gamma}\right)\left(\mathbf{\Sigma}'\mathbf{\Sigma}\right)^{-1}\mathbf{\Sigma}'\boldsymbol{\sigma}_P \\ &\quad + \left(\mathbf{\Sigma}'\mathbf{\Sigma}\right)^{-1}\mathbf{\Sigma}'\mathbf{\Sigma_X}\left(\tilde{\mathbf{B}}_3(\tau)\,\mathbf{X}_t + \mathbf{B}_2'(\tau)\right) \\ &\equiv \hat{\boldsymbol{\omega}}_0(\tau) + \hat{\boldsymbol{\omega}}_1(\tau)\,\mathbf{X}_t,\end{aligned} \tag{B.28}$$

where $\tau = T - t$ and:

$$\begin{aligned}\hat{\boldsymbol{\omega}}_0(\tau) &= \frac{1}{\gamma}\left(\mathbf{\Sigma}'\mathbf{\Sigma}\right)^{-1}\mathbf{\Sigma}'\mathbf{\Lambda}_0 + (1-\alpha)\left(1-\frac{1}{\gamma}\right)\left(\mathbf{\Sigma}'\mathbf{\Sigma}\right)^{-1}\mathbf{\Sigma}'\boldsymbol{\sigma}_P + \left(\mathbf{\Sigma}'\mathbf{\Sigma}\right)^{-1}\mathbf{\Sigma}'\mathbf{\Sigma_X}\mathbf{B}_2'(\tau), \\ \hat{\boldsymbol{\omega}}_1(\tau) &= \frac{1}{\gamma}\left(\mathbf{\Sigma}'\mathbf{\Sigma}\right)^{-1}\mathbf{\Sigma}'\mathbf{\Lambda}_1 + \left(\mathbf{\Sigma}'\mathbf{\Sigma}\right)^{-1}\mathbf{\Sigma}'\mathbf{\Sigma_X}\tilde{\mathbf{B}}_3(\tau).\end{aligned}$$

The expected utility over terminal wealth (evaluated for a degree of money illusion $\hat{\alpha}$), $\hat{J}(\tau, W_t, \mathbf{X}_t, P_t)$, is a martingale, as it represent an expectation of future utility. Thus, it satisfies the following PDE:

$$\frac{\partial}{\partial t}\hat{J} + \mathcal{L}\hat{J} = 0,$$

where $\mathcal{L}\hat{J}$ is the following differential operator:

$$\begin{aligned}
\mathcal{L}\hat{J} &= \hat{J}_W W_t \left(R_t + \hat{\boldsymbol{\omega}}_t'(\tau)\boldsymbol{\Sigma}'\boldsymbol{\Lambda}_t\right) + \hat{J}_{\mathbf{X}}\boldsymbol{\Theta}\left(\bar{\mathbf{X}} - \mathbf{X}_t\right) + \hat{J}_P P_t \pi_t \\
&\quad + \hat{J}_{\mathbf{X}W} W_t \boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}\hat{\boldsymbol{\omega}}_t(\tau) + \hat{J}_{WP} W_t P_t \hat{\boldsymbol{\omega}}_t'(\tau)\boldsymbol{\Sigma}'\boldsymbol{\sigma}_P + \hat{J}_{\mathbf{X}P} P_t \boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\sigma}_P \\
&\quad + \frac{1}{2}\hat{J}_{WW} W_t^2 \hat{\boldsymbol{\omega}}_t'(\tau)\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\hat{\boldsymbol{\omega}}_t(\tau) + \frac{1}{2}\mathrm{tr}\left(\hat{J}_{\mathbf{X}\mathbf{X}}\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}_{\mathbf{X}}\right) + \frac{1}{2}\hat{J}_{PP} P_t^2 \boldsymbol{\sigma}_P'\boldsymbol{\sigma}_P.
\end{aligned}$$

We guess the following functional form for $\hat{J}(\tau, W_t, \mathbf{X}_t, P_t)$:

$$\hat{J}(\tau, W_t, \mathbf{X}_t, P_t) = \frac{W_t^{1-\gamma} P_t^{-(1-\hat{\alpha})(1-\gamma)}}{1-\gamma} H(\mathbf{X}_t, \tau).$$

Substituting into the PDE and simplifying:

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}H + (1-\gamma)\left(R_t + \hat{\boldsymbol{\omega}}_t'(\tau)\boldsymbol{\Sigma}'\boldsymbol{\Lambda}_t\right)H + H_{\mathbf{X}}\boldsymbol{\Theta}\left(\bar{\mathbf{X}} - \mathbf{X}_t\right) - (1-\hat{\alpha})(1-\gamma)\pi_t H \\
&\quad + (1-\gamma)H_{\mathbf{X}}\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}\hat{\boldsymbol{\omega}}_t(\tau) - (1-\hat{\alpha})(1-\gamma)^2\hat{\boldsymbol{\omega}}_t'(\tau)\boldsymbol{\Sigma}'\boldsymbol{\sigma}_P H - (1-\hat{\alpha})(1-\gamma)H_{\mathbf{X}}\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\sigma}_P \\
&\quad - \frac{1}{2}\gamma(1-\gamma)\hat{\boldsymbol{\omega}}_t'(\tau)\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\hat{\boldsymbol{\omega}}_t(\tau)H + \frac{1}{2}\mathrm{tr}\left(H_{\mathbf{X}\mathbf{X}}\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}_{\mathbf{X}}\right) \\
&\quad + \frac{1}{2}(1-\hat{\alpha})(1-\gamma)(2-\hat{\alpha}-\gamma+\hat{\alpha}\gamma)\boldsymbol{\sigma}_P'\boldsymbol{\sigma}_P H.
\end{aligned}$$

We make a guess on the functional form of $H$:

$$H(\mathbf{X}_t, \tau) = \exp\left\{\frac{1}{2}\mathbf{X}_t'\hat{\mathbf{B}}_3(\tau)\mathbf{X}_t + \hat{\mathbf{B}}_2(\tau)\mathbf{X}_t + \hat{B}_1(\tau)\right\}.$$

The partial derivatives are given by:

$$\begin{aligned}
H_{\mathbf{X}} &= \left(\frac{1}{2}\mathbf{X}_t'\left(\hat{\mathbf{B}}_3(\tau) + \hat{\mathbf{B}}_3'(\tau)\right) + \hat{\mathbf{B}}_2(\tau)\right)H \\
&= \left(\mathbf{X}_t'\tilde{\hat{\mathbf{B}}}_3(\tau) + \hat{\mathbf{B}}_2(\tau)\right)H \\
H_{\mathbf{X}\mathbf{X}} &= \frac{1}{4}\left(\hat{\mathbf{B}}_3(\tau) + \hat{\mathbf{B}}_3'(\tau)\right)'\mathbf{X}_t\mathbf{X}_t'\left(\hat{\mathbf{B}}_3(\tau) + \hat{\mathbf{B}}_3'(\tau)\right)H + \frac{1}{2}\hat{\mathbf{B}}_2'(\tau)\mathbf{X}_t'\left(\hat{\mathbf{B}}_3(\tau) + \hat{\mathbf{B}}_3'(\tau)\right)H \\
&\quad + \frac{1}{2}\left(\hat{\mathbf{B}}_3(\tau) + \hat{\mathbf{B}}_3'(\tau)\right)'\mathbf{X}_t\hat{\mathbf{B}}_2(\tau)H + \hat{\mathbf{B}}_2'(\tau)\hat{\mathbf{B}}_2(\tau)H + \frac{1}{2}\left(\hat{\mathbf{B}}_3(\tau) + \hat{\mathbf{B}}_3'(\tau)\right)H \\
&= \left(\tilde{\hat{\mathbf{B}}}_3(\tau)\mathbf{X}_t\mathbf{X}_t'\tilde{\hat{\mathbf{B}}}_3(\tau) + \hat{\mathbf{B}}_2'(\tau)\mathbf{X}_t'\tilde{\hat{\mathbf{B}}}_3(\tau) + \tilde{\hat{\mathbf{B}}}_3(\tau)\mathbf{X}_t\hat{\mathbf{B}}_2(\tau) + \hat{\mathbf{B}}_2'(\tau)\hat{\mathbf{B}}_2(\tau) + \tilde{\hat{\mathbf{B}}}_3(\tau)\right)H \\
\frac{\partial}{\partial t}H &= -\left(\frac{1}{2}\mathbf{X}_t'\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{\mathbf{B}}_3(\tau)\mathbf{X}_t + \frac{\mathrm{d}}{\mathrm{d}\tau}\hat{\mathbf{B}}_2(\tau)\mathbf{X}_t + \frac{\mathrm{d}}{\mathrm{d}\tau}\hat{B}_1(\tau)\right)H,
\end{aligned}$$

where $\tilde{\hat{\mathbf{B}}}_3(\tau) = \frac{\hat{\mathbf{B}}_3(\tau)+\hat{\mathbf{B}}_3'(\tau)}{2}$. Substituting into the PDE the partial derivatives, as well as the quantities $R_t$, $\pi_t$, $\boldsymbol{\Lambda}_t$ (note that $\boldsymbol{\Sigma}'\boldsymbol{\Lambda}_t = \boldsymbol{\Sigma}'\boldsymbol{\Lambda}_t^*$) and $\hat{\boldsymbol{\omega}}_t(\tau)$:

$$\begin{aligned}
0 &= -\left(\frac{1}{2}\mathbf{X}_t'\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{\mathbf{B}}_3(\tau)\mathbf{X}_t + \frac{\mathrm{d}}{\mathrm{d}\tau}\hat{\mathbf{B}}_2(\tau)\mathbf{X}_t + \frac{\mathrm{d}}{\mathrm{d}\tau}\hat{B}_1(\tau)\right) \\
&\quad (1-\gamma)\left(R_0 + \mathbf{R}_1'\mathbf{X}_t + \left(\hat{\boldsymbol{\omega}}_0(\tau) + \hat{\boldsymbol{\omega}}_1(\tau)\mathbf{X}_t\right)'\boldsymbol{\Sigma}'\left(\boldsymbol{\Lambda}_0 + \boldsymbol{\Lambda}_1\mathbf{X}_t\right)\right) \\
&\quad + \left(\mathbf{X}_t'\tilde{\hat{\mathbf{B}}}_3(\tau) + \hat{\mathbf{B}}_2(\tau)\right)\boldsymbol{\Theta}\left(\bar{\mathbf{X}} - \mathbf{X}_t\right)
\end{aligned}$$

$$
\begin{aligned}
&-\left(1-\hat{\alpha}\right)\left(1-\gamma\right)\left(\pi_0+\boldsymbol{\pi}_1'\mathbf{X}_t\right)\\
&+\left(1-\gamma\right)\left(\mathbf{X}_t'\tilde{\hat{\mathbf{B}}}_3\left(\tau\right)+\hat{\mathbf{B}}_2\left(\tau\right)\right)\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}\left(\hat{\boldsymbol{\omega}}_0\left(\tau\right)+\hat{\boldsymbol{\omega}}_1\left(\tau\right)\mathbf{X}_t\right)\\
&-\left(1-\hat{\alpha}\right)\left(1-\gamma\right)^2\left(\hat{\boldsymbol{\omega}}_0\left(\tau\right)+\hat{\boldsymbol{\omega}}_1\left(\tau\right)\mathbf{X}_t\right)'\boldsymbol{\Sigma}'\boldsymbol{\sigma}_P\\
&-\left(1-\hat{\alpha}\right)\left(1-\gamma\right)\left(\mathbf{X}_t'\tilde{\hat{\mathbf{B}}}_3\left(\tau\right)+\hat{\mathbf{B}}_2\left(\tau\right)\right)\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\sigma}_P\\
&-\frac{1}{2}\gamma\left(1-\gamma\right)\left(\hat{\boldsymbol{\omega}}_0\left(\tau\right)+\hat{\boldsymbol{\omega}}_1\left(\tau\right)\mathbf{X}_t\right)'\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\left(\hat{\boldsymbol{\omega}}_0\left(\tau\right)+\hat{\boldsymbol{\omega}}_1\left(\tau\right)\mathbf{X}_t\right)\\
&+\frac{1}{2}\mathrm{tr}\left(\left(\tilde{\hat{\mathbf{B}}}_3\left(\tau\right)\mathbf{X}_t\mathbf{X}_t'\tilde{\hat{\mathbf{B}}}_3\left(\tau\right)+\hat{\mathbf{B}}_2'\left(\tau\right)\mathbf{X}_t'\tilde{\hat{\mathbf{B}}}_3\left(\tau\right)+\tilde{\hat{\mathbf{B}}}_3\left(\tau\right)\mathbf{X}_t\hat{\mathbf{B}}_2\left(\tau\right)+\hat{\mathbf{B}}_2'\left(\tau\right)\hat{\mathbf{B}}_2\left(\tau\right)+\tilde{\hat{\mathbf{B}}}_3\left(\tau\right)\right)\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}_{\mathbf{X}}\right)\\
&+\frac{1}{2}\left(1-\hat{\alpha}\right)\left(1-\gamma\right)\left(2-\hat{\alpha}-\gamma+\hat{\alpha}\gamma\right)\boldsymbol{\sigma}_P'\boldsymbol{\sigma}_P.
\end{aligned}
$$

Isolating the terms in $\mathbf{X}_t' \ldots \mathbf{X}_t$ and solving for $\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{\mathbf{B}}_3\left(\tau\right)$ yields:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{\mathbf{B}}_3\left(\tau\right) \quad = \quad & 2\left(1-\gamma\right)\hat{\boldsymbol{\omega}}_1'\left(\tau\right)\boldsymbol{\Sigma}'\boldsymbol{\Lambda}_1-2\tilde{\hat{\mathbf{B}}}_3\left(\tau\right)\boldsymbol{\Theta}+2\left(1-\gamma\right)\tilde{\hat{\mathbf{B}}}_3\left(\tau\right)\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}\hat{\boldsymbol{\omega}}_1\left(\tau\right)\\
&-\gamma\left(1-\gamma\right)\hat{\boldsymbol{\omega}}_1'\left(\tau\right)\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\hat{\boldsymbol{\omega}}_1\left(\tau\right)+\tilde{\hat{\mathbf{B}}}_3\left(\tau\right)\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}_{\mathbf{X}}\tilde{\hat{\mathbf{B}}}_3\left(\tau\right).
\end{aligned}
$$

Isolating the terms in $\mathbf{X}_t$ and solving for $\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{\mathbf{B}}_2\left(\tau\right)$:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{\mathbf{B}}_2\left(\tau\right) \quad = \quad & \left(1-\gamma\right)\left(\mathbf{R}_1'+\hat{\boldsymbol{\omega}}_0'\left(\tau\right)\boldsymbol{\Sigma}'\boldsymbol{\Lambda}_1+\boldsymbol{\Lambda}_0'\boldsymbol{\Sigma}\hat{\boldsymbol{\omega}}_1\left(\tau\right)\right)\\
&+\bar{\mathbf{X}}'\boldsymbol{\Theta}'\tilde{\hat{\mathbf{B}}}_3\left(\tau\right)-\hat{\mathbf{B}}_2\left(\tau\right)\boldsymbol{\Theta}-\left(1-\hat{\alpha}\right)\left(1-\gamma\right)\boldsymbol{\pi}_1'\\
&+\left(1-\gamma\right)\hat{\boldsymbol{\omega}}_0'\left(\tau\right)\boldsymbol{\Sigma}'\boldsymbol{\Sigma}_{\mathbf{X}}\tilde{\hat{\mathbf{B}}}_3\left(\tau\right)+\left(1-\gamma\right)\hat{\mathbf{B}}_2\left(\tau\right)\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}\hat{\boldsymbol{\omega}}_1\left(\tau\right)\\
&-\left(1-\hat{\alpha}\right)\left(1-\gamma\right)^2\boldsymbol{\sigma}_P'\boldsymbol{\Sigma}\hat{\boldsymbol{\omega}}_1\left(\tau\right)-\left(1-\hat{\alpha}\right)\left(1-\gamma\right)\boldsymbol{\sigma}_P'\boldsymbol{\Sigma}_{\mathbf{X}}\tilde{\hat{\mathbf{B}}}_3\left(\tau\right)\\
&-\gamma\left(1-\gamma\right)\hat{\boldsymbol{\omega}}_0'\left(\tau\right)\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\hat{\boldsymbol{\omega}}_1\left(\tau\right)+\hat{\mathbf{B}}_2\left(\tau\right)\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}_{\mathbf{X}}\tilde{\hat{\mathbf{B}}}_3\left(\tau\right).
\end{aligned}
$$

Isolating the homogeneous terms and solving for $\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{B}_1\left(\tau\right)$:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{B}_1\left(\tau\right) \quad = \quad & \left(1-\gamma\right)\left(R_0+\hat{\boldsymbol{\omega}}_0'\left(\tau\right)\boldsymbol{\Sigma}'\boldsymbol{\Lambda}_0\right)+\hat{\mathbf{B}}_2\left(\tau\right)\boldsymbol{\Theta}\bar{\mathbf{X}}-\left(1-\hat{\alpha}\right)\left(1-\gamma\right)\pi_0\\
&+\left(1-\gamma\right)\hat{\mathbf{B}}_2\left(\tau\right)\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}\hat{\boldsymbol{\omega}}_0\left(\tau\right)-\left(1-\hat{\alpha}\right)\left(1-\gamma\right)^2\hat{\boldsymbol{\omega}}_0'\left(\tau\right)\boldsymbol{\Sigma}'\boldsymbol{\sigma}_P\\
&-\left(1-\hat{\alpha}\right)\left(1-\gamma\right)\hat{\mathbf{B}}_2\left(\tau\right)\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\sigma}_P-\frac{1}{2}\gamma\left(1-\gamma\right)\hat{\boldsymbol{\omega}}_0'\left(\tau\right)\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\hat{\boldsymbol{\omega}}_0\left(\tau\right)\\
&+\frac{1}{2}\hat{\mathbf{B}}_2\left(\tau\right)\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}_{\mathbf{X}}\hat{\mathbf{B}}_2'\left(\tau\right)+\frac{1}{2}\mathrm{tr}\left(\tilde{\hat{\mathbf{B}}}_3\left(\tau\right)\boldsymbol{\Sigma}_{\mathbf{X}}'\boldsymbol{\Sigma}_{\mathbf{X}}\right)\\
&+\frac{1}{2}\left(1-\hat{\alpha}\right)\left(1-\gamma\right)\left(2-\hat{\alpha}-\gamma+\hat{\alpha}\gamma\right)\boldsymbol{\sigma}_P'\boldsymbol{\sigma}_P.
\end{aligned}
$$

## B.4 Certainty equivalent utility loss

The certainty equivalent utility loss $\ell$, incurred by following a suboptimal strategy, can be measured in terms of fraction of initial wealth. It can be obtained by solving the following problem:

$$
\hat{J}\left(T,W_0,\mathbf{X}_0,P_0\right)=J\left(T,W_0\left(1-\ell\right),\mathbf{X}_0,P_0\right).
$$

Developing the r.h.s.:

$$\hat{J}(T, W_0, \mathbf{X}_0, P_0) = \frac{W_0^{1-\gamma}(1-\ell)^{1-\gamma} P_0^{-(1-\alpha)(1-\gamma)}}{1-\gamma} [F(\mathbf{X}_0, T)]^{\gamma} = (1-\ell)^{1-\gamma} J(T, W_0, \mathbf{X}_0, P_0),$$

and, finally, the initial certainty equivalent loss can be expressed as:

$$\ell = 1 - \left( \frac{\hat{J}(T, W_0, \mathbf{X}_0, P_0)}{J(T, W_0, \mathbf{X}_0, P_0)} \right)^{\frac{1}{1-\gamma}}.$$

It is useful to calculate an annualized certainty equivalent loss, which, for an investment horizon equal to $\tau$, we evaluate as:

$$\ell_{\text{ann}} = 1 - \left( \frac{\hat{J}(T, W_0, \mathbf{X}_0, P_0)}{J(T, W_0, \mathbf{X}_0, P_0)} \right)^{\frac{1}{(1-\gamma)\tau}} = 1 - (1-\ell)^{\frac{1}{\tau}}.$$

# C Portfolio strategy and welfare losses when the nominal bond is not accessible

Referring to the baseline specification with constant risk premia, Figure A.1 shows the optimal portfolio strategy of a conservative investor with no access to nominal bonds. This situation is highly hypothetical, given the current enormous size of nominal bond markets, but it complements the intuition obtained in Figure 5, where the investor can not trade the inflation-indexed bond. When only the indexed bond is available, its position is monotonically increasing in the investment horizon and decreases with the degree of money illusion, due to a reduced benefit associated to inflation hedging. Furthermore, this long position is partially offset by a negative stock position when the degree of money illusion is high, while the stock position is virtually zero for a non-illusioned investor, who fully benefits from the inflation-hedging properties of the real bond.

Referring instead to the analysis in Section 4.2, Figure A.2 shows the optimal portfolio strategy and certainty equivalent loss of a moderate investor who can not invest in nominal bonds. The results are complementary to those obtained in Figure 7: money illusion reduces the allocations in both the inflation-indexed bond and the stock market. In terms of opportunity cost of removing nominal bonds from the investable universe, a money-illusioned investor is severely hurt, while a non-illusioned investor, who tends to favor the allocation into inflation-indexed bonds (Figure 6), is significantly less affected.

Figure A.3 shows a similar analysis based on the specification with time-varying risk premia, for which the main results are in Section 5. When the nominal bond is excluded from the investable universe, the highest utility loss is sustained by the money-illusioned investor (about 2% per annum for a 10- to 30-year horizon), while a non-illusioned investor perceives a loss equal to about 0.4% per annum.

# D Empirical findings imposing a zero unexpected inflation risk premium

The estimation of the inflation risk premium is an object of debate in the literature concerning inflation-indexed securities. In our setting, in particular, the unexpected inflation risk premium is equal to $\sigma_P' \mathbf{\Lambda}_t$ and determines the difference between the instantaneous expected inflation $\pi_t$ and the instantaneous break-even inflation:

$$\text{bei}_t = R_t - r_t = \pi_t - \sigma_P' \mathbf{\Lambda}_t. \tag{D.29}$$

We are interested in assessing whether our empirical findings in terms of portfolio strategy and welfare are robust with respect to the estimate obtained for the unexpected inflation risk premium. The base case estimate of this quantity, for the setting with constant risk premia, is non-negligible and equal to $\sigma_P' \tilde{\mathbf{\Lambda}} = 1.17\%$. We decided to repeat our analysis imposing this quantity to be equal to zero. In order to do so, we needed to determine a modified vector of constant market prices of risk, $\tilde{\mathbf{\Lambda}}_0$, so that $\sigma_P' \mathbf{\Lambda}_0 = 0$. $\tilde{\mathbf{\Lambda}}$ is a $5 \times 1$ vector. Therefore, we needed to impose four additional constraints to pin it down. We chose to impose that the risk premia on the 2- and 10-year nominal bonds, as well as that of the 10-year inflation-indexed bond and the stock, are unchanged with respect to their base case values. This translates into the following conditions:

$$\begin{bmatrix} \mathbf{A}_1^B(2)\,\mathbf{\Sigma}_X' \\ \mathbf{A}_1^B(10)\,\mathbf{\Sigma}_X' \\ \mathbf{A}_1^I(10)\,\mathbf{\Sigma}_X' + \boldsymbol{\sigma}_P' \\ \boldsymbol{\sigma}_P' \\ \boldsymbol{\sigma}_S' \end{bmatrix} \tilde{\mathbf{\Lambda}}_0 = \begin{bmatrix} \mathbf{A}_1^B(2)\,\mathbf{\Sigma}_X' \mathbf{\Lambda}_0 \\ \mathbf{A}_1^B(10)\,\mathbf{\Sigma}_X' \mathbf{\Lambda}_0 \\ \left(\mathbf{A}_1^I(10)\,\mathbf{\Sigma}_X' + \boldsymbol{\sigma}_P'\right) \mathbf{\Lambda}_0 \\ 0 \\ \boldsymbol{\sigma}_S' \mathbf{\Lambda}_0 \end{bmatrix}.$$

Figure A.4 shows the optimal portfolio strategy for different degrees of money illusion, as well as the certainty equivalent annualized loss attributable to money illusion. The results are virtually identical to those obtained for the base case estimate of $\mathbf{\Lambda}_0$ (Figure 6).

Figure A.5 instead shows the optimal portfolio positions when the inflation-indexed bond is removed from the investable universe, as well as the perceived loss by the investors with different values of $\alpha$. The results are, again, very similar to those shown in Figure 7.

Overall, it seems that the conclusions we draw in Section 4 are not sensitive to the estimate of the unexpected inflation risk premium.

# E Empirical findings with unconstrained volatility of time-varying risk premia

In this section, we report the empirical results obtained when we do not impose any restrictions on the volatility of the risk premia in the estimation. After presenting the parameter estimates for the unconstrained model, we perform an out-of-sample prediction exercise, comparing the different model specifications in terms of predictive ability. We then report the portfolio choice implications of letting the risk premia unconstrained. The results highlight the necessity of robustifying the estimation of the model based on unconstrained time-varying risk premia, which we do in the baseline analysis.

## E.1 Estimation

We consider a specification with time-varying risk premia, initially with no restrictions imposed on the matrix $\mathbf{\Lambda}_1$. As in Christensen et al. (2010), we then iterate the estimation procedure, by progressively imposing a zero restriction on the element of $\mathbf{\Lambda}_1$ with the lowest $t$-statistics. We stop when all the elements of $\mathbf{\Lambda}_1$ have a $t$-stat higher than 2. The parameter estimates are listed in Table A.1. The standard deviation of the observation errors of the 10-year average yield forecast, $\sigma_\epsilon^{BF}$, is 1.00%, while that for the 10-year stock return forecast, $\sigma_\epsilon^{SF}$, is 0.50%. While we could expect a more flexible model to better fit the data in the sample, the improvement relative to the constrained models is negligible.

The summary statistics and characteristics of the distributions of the most relevant economic and financial quantities, both historical and as implied by the estimated parameters, are listed in Table A.2. It is very interesting to note how the model-implied volatility of expected inflation is affected by considering time-varying risk premia with no restriction on their volatility. In addition, the estimated risk premia seem to be extremely (and unrealistically) volatile in this case.

Table A.3 shows the correlations both from the historical distribution and as implied by the estimated parameters (for both variable and constant risk premia).

Figure A.6 shows the time series of the model-implied macroeconomic variables, the risk premia and the maximum achievable Sharpe ratio (considering a 10-year nominal bond, a 10-year inflation-indexed bond and the stock index). When compared to Figure 2b, it is noticeable how the volatility of the expected inflation (which is unobservable) is higher in this setting. The annualized 10-year nominal bond risk premium, which is stabilized by the observation of the survey-based expected average 10-year nominal yield, ranges from 0% to 10%, while the 10-year real risk premium, for which we do not observe survey data, is way more volatile, ranging from about $-7\%$ to about 32%. The equity premium, for which we also observe survey expectations, ranges between 0% and 10%. The time series of mean-variance portfolio weights clearly shows the limits of this setting for portfolio choice: for a myopic nominal investor with $\gamma = 10$, the portfolio weights of the bonds range from about $-500\%$ to 1000%. The ex-ante Sharpe ratio (in annual terms) achievable with the mean-variance portfolio is on average higher than 1, with unreasonably high peaks around 5.

## E.2 Out-of-sample predictive ability

In this section, we assess the out-of-sample predictive ability of the three model specifications, i.e., based on constant risk premia (CRP), constrained time-varying risk premia (CTVRP), and unconstrained time-varying risk premia (UTVRP). To do so, we recursively estimate the parameters over an expanding window. The initial window spans the period December 1999 through December 2005. The window is then expanded by 1-month steps until the end of the available sample (December 2019) is reached. For each model specification and estimation window, we use the parameter estimates and the end-of-period values of the state variables to generate 1-, 12-, 36-, and 60-month-ahead predictions of the nominal and real yield variations, as well as of equity logarithmic returns. We then calculate the prediction errors as the differences between predictions and the corresponding realized values. Finally, from the time series of prediction errors, we evaluate the root-mean-squared errors (RMSE) and report them in Table A.4.

The specification based on constant risk premia is the best performer (lowest RMSE) for the prediction of nominal and real yield variations almost across the board, as long as the predictive horizon is sufficiently long (12 months or longer), while the second best is represented by the specification with volatility-constrained time-varying risk premia. A few exceptions to this conclusion are observed for a 1-month predictive horizon, where the specification based on volatility-constrained time-varying risk premia better forecasts yields for maturities equal to 5 years or above. For such predictive horizon, however, the differences in RMSE across models are rather small. Finally, focusing on stock return predictability, the relative forecasting performance of the three specifications depends on the predictive horizon. For a short predictive horizon (1 month), the best model is CRP, followed by CTVRP. For the other horizons, the specifications with time-varying risk premia dominate showing rather similar performances, with CTVRP being the best performer for a 12-month horizon and UTVRP for 36- and 60-month horizons. Overall, considering that our analysis is mainly focused on long-horizon bond allocation, the above results support our choice of considering the setup based on constant risk premia as baseline and the constrained-volatility model as an alternative specification.

### E.3 Portfolio strategy and welfare

**Portfolio strategy** Figure A.7 shows the optimal portfolio strategy when investors have access to the full investment universe and the state variables are at their long-run means. For short investment horizons, the optimal strategy is similar to the case where risk premia are constant (Figure 6) or time-varying with a constrained volatility (Figure 8). When the horizon is increased, the impact of the time-varying risk premia with unconstrained volatility significantly affects the portfolio strategy. However, the qualitative results at the long-run mean of the state of the economy are similar to the other specifications, with an increase of the nominal bond position and a decrease of the real bond position with $\alpha$, as well as positions in the stock and cash, which are rather insensitive to $\alpha$.

**Utility loss due to money illusion** The welfare analysis in the graph at the bottom, showing the certainty equivalent loss attributable to money illusion, confirms that the welfare effect of money illusion is substantial. The pattern of the loss is very similar to the case of constant risk premia, although the size of the relative loss seems to be lower than in the cases of constant or volatility-constrained risk premia. For example, the annualized loss is about 0.85% for a totally illusioned investor with a 30-year horizon with respect to a non-illusioned investor (rather than 1.2% for the case with volatility-constrained risk premia).

Figure A.7, however, shows relative losses, which somehow hide the actual value of certainty equivalent returns. In-sample certainty equivalent returns are, in this specification, unrealistically high. For example, considering the state variables at their long-run means, a non-illusioned agent with a 30-year horizon has a certainty equivalent real annualized return equal to 2.83% when risk premia are constant, and equal to 3.28% when risk premia are time-varying with a constrained volatility. The in-sample certainty equivalent real annualized return is instead unrealistically high and equal to 12.38% when risk premia are time-varying with an unconstrained volatility.

This figure is due to the fact that optimal portfolio positions, as soon as the state of the economy departs from the long-run mean, quickly assume unrealistically high values. To point out this issue, on the left graphs in Figure A.8 we show the time series of in-sample portfolio weights, considering a non-illusioned investor ($\alpha = 0$) with a 10-year horizon, for the three specifications of risk premia. The weights are calculated for each date considering the time series of the state variables $\mathbf{X}_t$ and the estimated model parameters. As can be noticed, the positions, which are expressed in fractions of total wealth, are constant for the case of constant risk premia, time-varying within a realistically implementable range (from −25% to about 200%) for the case of volatility-constrained risk premia, and time-varying within an unrealistically wide range (from −1500% to 2500%) for the case of unconstrained risk premia. In particular, the in-sample overfitting of bond risk premia causes incredibly large and offsetting positions between the nominal bond and the real bond, which correspond to quasi-arbitrage opportunities. The extremely large in-sample certainty equivalents, shown in the graphs on the right, reach a peak of almost 40% per annum and are a consequence of this same issue.

**Perceived utility loss due to the unavailability of inflation-indexed bonds for different degrees of money illusion** We consider now the case where one of the bonds is removed from the investable universe. In particular, Figure A.9 shows the case where the inflation-indexed bond is unavailable. In terms of portfolio positions, qualitatively, the effect of money illusion on the optimal allocation is similar to the case with volatility-constrained risk premia (Figure 9), although in this case the effect of money illusion on the portfolio positions seems to be less pronounced.

We then evaluate the opportunity cost, perceived by investors with different levels of $\alpha$, of excluding the real bond. As can be noticed in Figure A.9, the opportunity cost of excluding the inflation-indexed bond, although slightly higher for the non-illusioned investor, is huge for any level of $\alpha$. In particular, the annualized loss reaches a level of about 9% per annum for a 30-year horizon. This is a consequence of the fact that, once a bond is excluded from the investable universe, it is not possible for the investor to exploit the quasi-arbitrage opportunities arising from the in-sample over-fitted bond risk premia. This result is consistent with the empirical findings of other studies relying on unconstrained essentially affine term structure models, such as the huge utility losses incurred when following suboptimal strategies, as documented by Sangvinatsos and Wachter (2005) and Barillas (2011), or the extremely high sensitivity to parameter uncertainty, as documented by Feldhütter et al. (2012).

# F Time variation of optimal portfolio positions and perceived losses when the TIPS market is not accessible

For the two main specifications, assuming respectively constant risk premia and volatility-constrained time-varying risk premia, we plot in the top graphs of Figure A.10 the time series of the in-sample optimal portfolio weights for a non-money-illusioned investor ($\alpha = 0$) and a totally money-illusioned investor ($\alpha = 1$). The positions are calculated according to the current value of the state variables $\mathbf{X}_t$ and considering a 30-year investment horizon.

As expected, for an investor considering constant risk premia, the positions are constant throughout the sample. For an investor considering time-varying risk premia, the positions in the two bonds are negatively correlated, e.g. an increase in the optimal position in the nominal bond tends to correspond to a decrease in the position in the real bond. Unsurprisingly, for a non-money-illusioned investor the optimal position in the inflation-indexed bond is always higher than the optimal position in the nominal bond. Conversely, a totally money-illusioned investor would invest more in nominal bonds most of the time, but there are periods where the optimal investment in the inflation-indexed bond is higher than that of the nominal bond. It turns out that these are periods when the expected inflation is rather higher than the break-even inflation. As can be noticed, the stock position is almost unaffected by money illusion.

The bottom graphs of Figure A.10 show the certainty equivalent annualized losses, as perceived by an investor with a 30-year horizon starting at a given date, when the TIPS is removed from the investable universe. The solid lines refer to a non-money-illusioned investor, while the dashed lines refer to a totally money-illusioned investor. For the analysis based on constant risk premia, the perceived losses are constant: 1.3% per annum for a non-money-illusioned investor and nearly zero for a money-illusioned investor. For an investor exploiting the time variation of the risk premia, the perceived losses are slightly higher time varying. They tend to decrease throughout the sample, being comprised between 1.4% and 2% per annum for a non-money-illusioned investor and between 0.3% and 0.7% per annum for a money-illusioned investor. The losses tend to be higher when the optimal position in the inflation-indexed bond (when it is accessible) is higher. Interestingly, the difference between the losses perceived by the two investors is similar to the difference estimated under constant risk premia. If compared to a non-money-illusioned investor, a money-illusioned investor perceives a substantially lower economic incentive to invest in the TIPS market at all times.

# G Empirical findings with a bond-only investable universe

In this section, we check that our main empirical findings are mostly driven by the characteristics of the nominal and inflation-indexed bonds, while they are affected to a lesser extent by the possibility of investing in the stock index that we have included in our baseline analysis. To do so, using U.S. data, we repeat our analysis excluding the stock from the investable universe.

For what concerns the effects of money illusion on the optimal portfolio strategy and the certainty equivalent, the results respectively in Figures A.11 (constant risk premia) and A.12 (time-varying risk premia), if compared to those in Figures 6 and 8, show totally comparable portfolio shifts and utility losses entailed by money illusion.

For what concerns the optimal strategy in the absence of the inflation-indexed bond, Figure A.13 refers to the specification with constant risk premia and shows similar effects to those in Figure 7, with slightly smaller differences between bond allocations corresponding to agents with different degrees of money illusion. This stems from the fact that the stock can not be used as a substitute for the missing inflation-indexed bond in order to hedge inflation. Also the utility losses entailed by the removal of the inflation-indexed bond from the investable universe follow very similar patterns to those in the baseline analysis, although slightly larger welfare effects are observed in the case where the stock index is not accessible.

It is interesting to notice that, for the specification with time-varying risk premia, the optimal strategy for a short-horizon money-illusioned investor in Figure A.12 does not entail short positions, which are present in the baseline analysis in Figure 8. The welfare losses perceived for short horizons by the money-illusioned investor (Figure A.14) are thus even smaller than in the baseline analysis (Figure

9).

# H Empirical findings for different maturities of the investable bonds

In the baseline analysis we consider an investable universe composed of a 10-year nominal bond, a 10-year inflation-indexed bond, a stock index and a nominal money market account. In this Appendix we want to show that our results are not driven by the particular choice of the maturity of the bonds. We run once more the analysis by considering a 5-year nominal bond and a 5-year inflation-indexed bond, in spite of the 10-year bonds. We consider both the specifications with constant and time-varying risk premia.

## H.1 Constant risk premia

Figure A.15 shows that the optimal portfolio weights in the 5-year bonds and the stock index follow very similar patterns to those obtained in the baseline analysis (Figure 6), the most important difference being that the allocations for all bonds are larger. This is due to the fact that the risk exposures of these bonds, mainly to nominal and real rate risks, are lower than those of the 10-year bonds, making it necessary to increase the portfolio positions in order to achieve similar risk exposures. This makes bond positions, in particular for the 5-year inflation-indexed bond, to be always positive. Money illusion entails a significant portfolio shift from inflation-indexed to nominal bonds, especially for long investment horizons. The welfare losses entailed by money illusion are very similar to those obtained in the baseline analysis.

Figure A.16 shows the optimal allocation when the 5-year inflation-indexed bond is removed from the investable universe. The positions are again qualitatively similar to those in Figure 7, with higher positions in the nominal bond. The welfare losses perceived by investors with different degrees of money illusion are higher than those in the baseline analysis, but the patterns are very similar, with a markedly higher perceived certainty equivalent annualized loss suffered by the non-illusioned investor (about 2.2% p.a. for a 30-year horizon) with respect to the loss perceived by the totally money-illusioned investor (about 0.4% p.a. for a 30-year horizon).

## H.2 Time-varying risk premia

The analysis performed considering time-varying risk premia leads to similar results as for the case of constant risk premia. The optimal positions in the 5-year nominal and inflation-indexed bonds in Figure A.17 have similar patterns to those in Figure 8 for 10-year bonds, but the positions are larger and, importantly, the position of the inflation-indexed bond is always positive. This is mainly due to the optimal myopic allocation, which non-trivially depends on the instantaneous expected returns, volatilities and correlations of the assets available for trading. The welfare loss suffered by a non-illusioned investor, when forced to follow the portfolio strategy of the money-illusioned investor, is about 0.85% p.a. for a 30-year horizon (it is 1.2% p.a. in the baseline analysis).

When the inflation-indexed bond is excluded from the investable universe, as can be seen in Figure A.18, the perceived utility loss is higher than in the baseline analysis (Figure 9) for all investors, and also the differences in perceived utility losses between investors with different degrees of money illusion are higher in this case for any investment horizon. For example, the difference in perceived loss between the non-illusioned investor and the totally money-illusoned investor is more than 1.5% p.a. for a 30-year horizon, as opposed to slightly more than 1% in the baseline analysis. Furthermore, considering short investment horizons (below 10 years), the effect noticeable in Figure 9, where the perceived utility loss is similar between investors with $\alpha = 0$ and $\alpha = 1$, is in this case absent because of the fact that the optimal strategy, when the 5-year inflation-indexed bond is available (Figure A.17), does not require short positions in the inflation-indexed bond for the totally money-illusioned investor, while in the baseline analysis (Figure 8) the optimal strategy requires a short position in the 10-year inflation-indexed bond for short investment horizons, which allows to finance long positions in other risky assets. When the money-illusioned investor is prevented from investing in the 10-year bond, she suffers from the inability to take this short position, and hence a non-negligible perceived utility loss, which in this case is not due to the impossibility to hedge inflation, but to the limited ability to short sell an asset, even if this entails increasing inflation exposure, to finance other long positions. This effect is absent in the case where the investable universe includes 5-year bonds.

# I Empirical findings for levels of risk aversion different from $\gamma = 10$

In this Appendix we consider levels of risk aversion different from the base case value $\gamma = 10$. In particular, with respect to the baseline analysis, we consider a more risk-tolerant investor ($\gamma = 5$) and a more risk-averse investor ($\gamma = 20$).

## I.1 Constant risk premia

Figure A.19 shows that, when the risk aversion is lower than in the baseline analysis (Figure 6), the optimal portfolio positions follow similar patterns, although positions are typically larger. For what concerns the welfare implications of money illusion, these are qualitatively similar to the base case analysis, while they are quantitatively smaller, as the certainty equivalent annualized losses are about halved.

When the inflation-indexed bond is removed from the investable universe (Figure A.20), the results are qualitatively similar to those obtained for $\gamma = 10$ (Figure 7), although the sizes of the perceived losses are about halved.

Figure A.21 shows instead the optimal portfolio strategy for more conservative investors ($\gamma = 20$). The patterns of the positions are again similar to those obtained in the baseline analysis (Figure 6), although the sizes of the positions in the risky assets are lower, as it was expected. The certainty equivalent welfare losses entailed by money illusion are about twice as large as those in the baseline analysis for $\gamma = 10$.

When the inflation-indexed bond is removed from the investable universe (Figure A.22), the results are qualitatively similar to those obtained for $\gamma = 10$ (Figure 7), although the sizes of the perceived losses are about twice as large.

### I.2 Time-varying risk premia

We consider now the model specification with time-varying risk premia. Figure A.23 shows the optimal strategy for investors with a risk aversion of 5, where the positions follow very similar patterns to those in Figure 8 but are quantitatively larger, entailing also higher levels of leverage. The welfare losses suffered by a non-illusioned investor, forced to follow the strategy of a money-illusioned investor are lower than in the base case.

Figure A.24 shows the optimal strategy and welfare losses suffered by investors with different degrees of money illusion when the inflation-indexed bond is removed from the investable universe. Also in this case, similar conclusions to the baseline results in Figure 9 can be drawn, although the differential between the welfare losses perceived by the non-illusioned and the money-illusioned investors are lower.

When the level of risk aversion is increased, the results are again qualitatively similar to those obtained in the baseline analysis, as can be noticed in Figures A.25 and A.26, although the welfare effects entailed by money illusion and the perceived utility losses entailed by not allowing non-illusioned investors to access inflation-indexed bonds are significantly larger, due to the higher importance of the hedging motive.

The results in this Appendix confirm that money illusion significantly affects inflation risk hedging, while it has a lower impact in terms of speculative decisions. Investors with a high level of risk aversion are subject to stronger welfare effects when money illusion is at play, entailing also greater differential welfare incentives to non-illusioned investors, as opposed to money-illusioned investors, to include inflation-indexed bonds in their portfolio.

## J Empirical findings estimating the baseline model observing also survey-based inflation expectations

In this Section, as it has been done in the works by D'Amico et al. (2018) and Breach et al. (2020), we expand the information set used to estimate the model by also observing time series of inflation expectations. In particular, we consider the 10-year CPI inflation rate forecast from the Survey of Professional Forecasters (`CPI10` median responses).

Under our model specification, the expected inflation, i.e., the expected logarithmic increment of the price level, over a time interval $\tau$, is given in (A.9). The observed forecast of the 10-year inflation rate is assumed to be equal to the model-implied value plus an observation error with standard deviation $\sigma_\epsilon^{IF}$.

### J.1 Estimation

The model can be estimated using the same methodology described in Section 3 for the baseline specification. Table A.5 shows the parameter estimates and their asymptotic standard errors. Most estimates are statistically significant, including the market prices of risk in $\mathbf{\Lambda}_0$, with the exception of the elements associated to the level factors respectively of the nominal and real yield curves (first and fourth elements of $\mathbf{\Lambda}_0$). The standard deviation of the observation errors of the 10-year inflation rate forecast $\sigma_\epsilon^{IF}$ is 0.31%, while the standard deviation of the observation errors of the expected average over 10 years of the 10-year nominal yield, $\sigma_\epsilon^{BF}$, is 1.05% and that for the 10-year expected stock market return $\sigma_\epsilon^{SF}$ is 0.85%.

Table A.6 reports the annualized mean values and the volatilities, both historical and model-implied, of bond yields, realized inflation and realized equity returns. The estimate of the average expected inflation is about 0.25% higher than that obtained for the baseline analysis in Table 2a, corresponding to a slightly higher estimate for the unexpected inflation risk premium and real bond risk premia.

Figure A.27 shows the time series of the model-implied macroeconomic variables, the risk premia and the maximum achievable Sharpe ratio (considering a 10-year nominal bond, a 10-year inflation-indexed bond and the stock index).

### J.2 Portfolio strategy and welfare

**Portfolio strategy** Figure A.28 shows the optimal portfolio strategy and certainty equivalent annualized losses entailed by money illusion. Despite a slightly higher optimal position in the inflation-indexed bond, due to the higher inflation risk premium, the results are very similar to those obtained in the baseline analysis (Figure 6).

**Perceived utility loss due to the unavailability of inflation-indexed bonds for different degrees of money illusion** We consider in Figure A.29 the case where the inflation-indexed bond is removed from the investable universe. Also in this case, the results in terms of portfolio positions and economic incentives to invest in TIPS are very similar to those obtained in the baseline analysis (Figure 7).

## K Empirical findings under a 4-factor affine no-arbitrage Nelson-Siegel term structure specification

### K.1 Model specification

In this section, we implement a 4-factor affine no-arbitrage Nelson-Siegel term structure model similar to the one proposed by Christensen et al. (2010), checking whether our empirical results are robust to the choice of the affine model chosen. In this specification, there are two separate level factors, $L_t^R$ and $L_t^r$, respectively for the nominal and the real yield curves, while a slope factor $S_t^R$ and a curvature factor $C_t^R$ are common to the two curves. The vector of state variables is $\mathbf{X}_t = \begin{bmatrix} L_t^R & S_t^R & C_t^R & L_t^r \end{bmatrix}'$ and its $\mathbb{Q}$-dynamics is:

$$\mathrm{d}\mathbf{X}_t = -\mathbf{\Theta}^{\mathbb{Q}}\mathbf{X}_t\mathrm{d}t + \mathbf{\Sigma}_{\mathbf{X}}'\mathrm{d}\mathbf{z}_t^{\mathbb{Q}}. \tag{K.30}$$

The $4 \times 4$ mean-reversion matrix $\mathbf{\Theta}^{\mathbb{Q}}$ is restricted to depend only on one parameter, $\lambda$

$$\mathbf{\Theta}^{\mathbb{Q}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda & -\lambda & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{K.31}$$

Note that, as in Christensen et al. (2010), the level factors are unit-root processes under the $\mathbb{Q}$-measure. As in our baseline 3-factor model specification, we adopt a generic upper-triangular definition for the volatility matrix $\mathbf{\Sigma_X}$, which allows the innovations in the factors to be correlated between them. This entails that, given the vector of market prices of risk $\mathbf{\Lambda}_t = \mathbf{\Lambda}_0 + \mathbf{\Lambda}_1 \mathbf{X}_t$, the $\mathbb{P}$-dynamics of $\mathbf{X}_t$ is:

$$\mathrm{d}\mathbf{X}_t = \left(\mathbf{\Sigma}_\mathbf{X}' \mathbf{\Lambda}_0 - \mathbf{\Theta}\mathbf{X}_t\right) \mathrm{d}t + \mathbf{\Sigma}_\mathbf{X}' \mathrm{d}\mathbf{z}_t, \tag{K.32}$$

where $\mathbf{\Theta} = \mathbf{\Theta}^{\mathbb{Q}} - \mathbf{\Sigma}_\mathbf{X}' \mathbf{\Lambda}_1$.[1]

The instantaneous nominal and real interest rates are respectively defined as:

$$R_t = L_t^R + S_t^R, \qquad r_t = L_t^r + \alpha^r S_t^R. \tag{K.33}$$

Using the same notations introduced in Section 2.1, the following relations hold:

$$R_0 = 0, \qquad \pi_0^{\mathbb{Q}} = 0, \qquad \pi_0 = \pi_0^{\mathbb{Q}} + \boldsymbol{\sigma}_P' \mathbf{\Lambda}_0,$$

$$\mathbf{R}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{\pi}_1^{\mathbb{Q}} = \begin{bmatrix} 1 \\ (1-\alpha^r) \\ 0 \\ -1 \end{bmatrix}, \qquad \boldsymbol{\pi}_1 = \boldsymbol{\pi}_1^{\mathbb{Q}} + \boldsymbol{\sigma}_P' \mathbf{\Lambda}_1.$$

Under this framework, as in the baseline specification, bond nominal and real yields can be written as affine functions of the state variables. In this case the loadings are the same as in the reduced-form yield curve model by Nelson et al. (1987):

$$\begin{aligned} y^n \left(\mathbf{X}_t, \tau\right) &= -\frac{A_0^B(\tau)}{\tau} - \frac{\mathbf{A}_1^B(\tau)}{\tau}\mathbf{X}_t, \\ y^r \left(\mathbf{X}_t, \tau\right) &= -\frac{A_0^I(\tau)}{\tau} - \frac{\mathbf{A}_1^I(\tau)}{\tau}\mathbf{X}_t, \end{aligned}$$

where:

$$\begin{aligned} \mathbf{A}_1^B(\tau) &= -\tau \begin{bmatrix} 1 & \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) & \left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) & 0 \end{bmatrix}, \\ \mathbf{A}_1^I(\tau) &= -\tau \begin{bmatrix} 0 & \alpha^r\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) & \alpha^r\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) & 1 \end{bmatrix}, \end{aligned}$$

and $A_0^B(\tau)$ and $A_0^I(\tau)$ can be obtained integrating the following ODEs:

$$\begin{aligned} \frac{\partial}{\partial\tau} A_0^B(\tau) &= \frac{1}{2}\mathbf{A}_1^B(\tau)\mathbf{\Sigma}_\mathbf{X}'\mathbf{\Sigma}_\mathbf{X}\left(\mathbf{A}_1^B(\tau)\right)' - R_0, \\ \frac{\partial}{\partial\tau} A_0^I(\tau) &= \mathbf{A}_1^I(\tau)\mathbf{\Sigma}_\mathbf{X}'\boldsymbol{\sigma}_P + \frac{1}{2}\mathbf{A}_1^I(\tau)\mathbf{\Sigma}_\mathbf{X}'\mathbf{\Sigma}_\mathbf{X}\left(\mathbf{A}_1^I(\tau)\right)' - R_0 + \pi_0 - \boldsymbol{\sigma}_P'\mathbf{\Lambda}_0. \end{aligned}$$

[1] Note that, as under this specification the matrix $\mathbf{\Theta}^{\mathbb{Q}}$ is singular, when the market prices of risk are constant ($\mathbf{\Lambda}_1 = \mathbf{0}$), it is not possible to find a vector $\bar{\mathbf{X}}$ such that $\mathbf{\Theta}\bar{\mathbf{X}} = \mathbf{\Sigma}_\mathbf{X}'\mathbf{\Lambda}_0$. This is the reason why we prefer to write the $\mathbb{P}$-dynamics of $\mathbf{X}_t$ as in (K.32) rather than as in (4). All the equations used to determine the bond prices and the optimal asset allocation in the baseline specification still hold, with the only attention that whenever a term $\mathbf{\Theta}\bar{\mathbf{X}}$ appears, it must be replaced by $\mathbf{\Sigma}_\mathbf{X}'\mathbf{\Lambda}_0$.

### K.2 Estimation

The model can be estimated using the same methodology described in Section 3 for the baseline specification. Table A.7 shows the parameter estimates and their standard errors. Most estimates are statistically significant, including the market prices of risk in $\mathbf{\Lambda}_0$, with the exception of the elements associated to the level factors respectively of the nominal and real yield curves (first and fourth elements of $\mathbf{\Lambda}_0$).

Table A.8 reports the annualized mean values and the volatilities, both historical and model-implied, of bond yields, realized inflation and realized equity returns.

### K.3 Portfolio strategy and welfare

**Portfolio strategy** Figure A.31 shows the optimal portfolio strategy obtained when the yield curves follow the 4-factor no-arbitrage Nelson-Siegel specification. As a consequence of the fact that two state variables follow unit-root processes, it turns out that in some occasions the optimal portfolio positions tend to diverge when the investment horizon becomes longer. Despite this fact, which leads to larger positions than in Figure 6 for long investment horizons, the baseline results are confirmed: money illusion entails a significant shift of the long-term optimal portfolio weights from inflation-indexed to nominal bonds. The effect on the stock allocation is also slightly more sizable than in the baseline analysis for horizons longer than 15 years. From the point of view of a long-term non-illusioned agent, the utility cost of money illusion is substantial, ranging from about 0.8% p.a. for a 10-year horizon to about 5% p.a. for a 30-year horizon.

**Perceived utility loss due to the unavailability of inflation-indexed bonds for different degrees of money illusion** We consider in Figure A.32 the case where the inflation-indexed bond is removed from the investable universe. In terms of portfolio positions, the effect of money illusion on the optimal allocation is qualitatively similar to that obtained for the baseline specification (Figure 7), but in this case the substitution effect between the stock and the nominal bond for an increasing $\alpha$ is stronger. The perceived utility loss for not accessing inflation-indexed bonds is again very small for a fully money-illusioned investor, while it is significantly higher for a partially money-illusioned or a non-illusioned agent.

## L Empirical findings on UK data

We report in this section the empirical results obtained using UK data from December 1985 to July 2018. It is an interesting analysis, because the sample period is almost twice as long as the one available for the U.S. market, as well as for the fact that, due to the higher development of the market of inflation-indexed instruments, we expect a smaller inflation risk premium in the UK market. Eventually, performing the analysis on a different dataset represents also a useful robustness check for our baseline results.

## L.1 Estimation

For this analysis, we use monthly data from December 1985 to July 2018 for 1-year, 2-year, 3-year, 5-year, 7-year and 10-year nominal bond yields, as well as 5-year, 7-year and 10-year real bond yields, available on the website of the Bank of England. Principal and coupon payments of inflation-indexed gilts are adjusted according to the Retail Price Index (RPI), which time series is also available from the Bank of England and that we use as price index. For the stock market, we use total returns over the FTSE All-Share Index.

Table A.9 shows the parameter estimates for both the specifications with constant and time-varying risk premia, which we also constrain in volatility as we did for the estimation performed over U.S. data. Table A.10 reports instead the means and volatilities of relevant historical and model-implied variables. These can be directly compared with the parameter estimates and statistics obtained from U.S. data and reported in Tables 1 and 2.

Focusing on Table A.10a, as can be noticed, the average level of nominal and real yields is higher than for the U.S. sample. The same applies to the volatility of nominal yields, while real yields seem to be slightly less volatile in the UK sample. The average realized inflation is also way higher in the UK dataset (3.34%) than in the U.S. dataset (2.17%), and the realized inflation volatility is also higher (1.46% vs 1.07%). Estimated nominal and real bond risk premia are lower for the UK data. Furthermore, the realized inflation risk premium is slightly negative for the UK sample ($-0.07$%), while it is positive in the U.S. sample (1.18%).

As far as the correlations are concerned, comparing Table A.11 with Table 3, it can be noticed that equity returns are positively correlated with nominal and real bond returns in the UK dataset, while in the U.S. dataset equity returns are negatively correlated with nominal bond returns and positively correlated with real bond returns. Another difference is the estimated correlation of the real rate $r$ with the expected inflation $\pi$, which estimate is negative in the UK sample, but not as strongly negative ($-0.91$) as the estimate obtained from the U.S. sample.

Finally, Figure A.33 shows the time series of the model-implied macroeconomic variables, the risk premia and the maximum achievable Sharpe ratio (considering a 10-year nominal bond, a 10-year inflation-indexed bond and the stock index). As can be noticed, in the time-varying setting, the realized inflation risk premiums shows an increasing pattern, being negative at the beginning of the sample and positive at the end of the sample.

## L.2 Portfolio strategy and welfare

**Portfolio strategy** Figure A.34 shows the optimal portfolio strategy for the specification based on constant risk premia. Although the patterns are qualitatively similar to those shown for the U.S. market in Figure 6, in this case, for investment horizons longer than 10 years, a money-illusioned investor would take a sizable short position the inflation-indexed bond. This is probably due to the fact that the realized inflation premium is slightly negative. In the long-run, a money-illusioned investor prefers to even have a negative exposure to inflation in order to benefit from this premium. The utility cost of money illusion is even more sizable than for the analysis based on U.S. data, probably due to the fact that realized inflation is more volatile in the UK dataset.

The results obtained under time-varying risk premia, which we show in Figure A.35 are very similar to those obtained for the U.S. market (Figure 8), again with a higher welfare effect entailed by money illusion in the analysis based on UK data.

**Perceived utility loss due to the unavailability of inflation-indexed bonds for different degrees of money illusion** We consider now the case where one of the bonds is removed from the investable universe. For the specification with constant risk premia, Figure A.36 shows the case where the inflation-indexed bond is unavailable. In terms of portfolio positions, the effect of money illusion on the optimal allocation is similar to that obtained for the U.S. dataset (Figure 7), although in the case based on UK data, being the money-illusioned investor unable to short the inflation-indexed bond for long horizons, and thus not being able to benefit from selling the negative inflation premium, also the money-illusioned investor perceives a considerable utility loss for a very long investment horizon (more than 20 years). For horizons shorter than 20 years, however, the utility loss perceived by a money-illusioned agent is nearly zero, as in our baseline analysis.

Finally, the analysis in Figure A.37 refers to the specification with time-varying risk premia and shows results that are totally comparable to those based on U.S. estimates (Figure 9).

## References


Barillas, F. (2011). Can we exploit predictability in bond markets? Working paper.

Breach, T., S. D'Amico, and A. Orphanides (2020). The term structure and inflation uncertainty. *Journal of Financial Economics 138*(2), 388–414.

Christensen, J. H., J. A. Lopez, and G. D. Rudebusch (2010). Inflation expectations and risk premiums in an arbitrage-free model of nominal and real bond yields. *Journal of Money, Credit and Banking 42*(s1), 143–178.

D'Amico, S., D. H. Kim, and M. Wei (2018). Tips from TIPS: The informational content of Treasury Inflation-Protected Security prices. *Journal of Financial and Quantitative Analysis 53*(1), 395–436.

Feldhütter, P., L. S. Larsen, C. Munk, and A. B. Trolle (2012). Keep it simple: Dynamic bond portfolios under parameter uncertainty. Working paper.

Nelson, C. R., A. F. Siegel, et al. (1987). Parsimonious modeling of yield curves. *The Journal of Business 60*(4), 473–489.

Sangvinatsos, A. and J. A. Wachter (2005). Does the failure of the expectations hypothesis matter for long-term investors? *The Journal of Finance 60*(1), 179–230.

**Table A.1:** Parameter estimates when no constraints on the volatility of risk premia are imposed.

The table shows the maximum-likelihood estimates of the model parameters. The values in parentheses are the asymptotic standard errors of the estimates. Starting from an unrestricted matrix $\mathbf{\Lambda}_1$, the estimation is iterated by progressively imposing a zero restriction on the element of $\mathbf{\Lambda}_1$ with the lowest $t$-statistics. The results reported in the table correspond to the estimate obtained when all the elements of $\mathbf{\Lambda}_1$ have a $t$-stat higher than 2. The sample period runs from January 1999 until December 2019.

| $R_0$ | $\mathbf{R}_1$ | $\pi_0$ | $\boldsymbol{\pi}_1$ | $\mathbf{\Theta}$ | | | $\sigma_\epsilon^B$ | $\sigma_\epsilon^I$ |
|---|---|---|---|---|---|---|---|---|
| 0.0178 | 0.3596 | 0.0210 | 0.1067 | 0.1130 | 0.2531 | 2.7564 | 0.0013 | 0.0007 |
| (0.0001) | (0.0012) | (0.0018) | (0.0383) | (0.0470) | (0.1824) | (0.5509) | (0.0000) | (0.0000) |
| | −0.4297 | | −0.3291 | 0.0349 | 0.2145 | −0.0779 | | |
| | (0.0045) | | (0.1370) | (0.0325) | (0.1270) | (0.3750) | | |
| | 0.2132 | | −1.7429 | 0.0045 | 0.2672 | 1.2497 | | |
| | (0.0115) | | (0.3672) | (0.0305) | (0.1075) | (0.3020) | | |
| $\mathbf{\Lambda}_0$ | $\mathbf{\Lambda}_1$ | | | $\mathbf{\Sigma}_X$ | | | $\boldsymbol{\sigma}_P$ | $\boldsymbol{\sigma}_S$ |
| −0.6673 | −2.1701 | −42.8899 | −101.5027 | 0.0196 | 0.0068 | −0.0035 | 0.0002 | 0.0325 |
| (0.0302) | (2.3943) | (9.5134) | (28.5585) | (0.0009) | (0.0008) | (0.0006) | (0.0006) | (0.0094) |
| 0.5057 | −0.8670 | 26.4088 | 55.9389 | 0 | 0.0110 | 0.0020 | −0.0005 | −0.0269 |
| (0.0481) | (3.7212) | (14.0668) | (37.7858) | | (0.0005) | (0.0006) | (0.0006) | (0.0094) |
| −0.1620 | −2.3824 | −43.9248 | −186.2893 | 0 | 0 | 0.0084 | −0.0039 | −0.0491 |
| (0.0539) | (3.8272) | (14.0364) | (36.6662) | | | (0.0004) | (0.0005) | (0.0088) |
| 1.1834 | 5.9460 | 24.0838 | −172.9475 | 0 | 0 | 0 | 0.0080 | −0.0188 |
| (0.2389) | (4.3214) | (16.3750) | (43.1108) | | | | (0.0004) | (0.0086) |
| 0.8215 | −0.8405 | 4.7544 | −76.7747 | 0 | 0 | 0 | 0 | 0.1343 |
| (0.1055) | (2.1410) | (8.9725) | (29.0113) | | | | | (0.0060) |

**Table A.2:** Historical and model-implied summary statistics when no constraints on the volatility of risk premia are imposed.

The table shows annualized historical and model-implied means and volatilities of bond yields, equity log returns, realized inflation and their model-implied means and volatilities. The table also shows the model-implied means and volatilities of the bond risk premia, the equity risk premium, the nominal risk-free rate and the expected inflation.

| Time series | Mean value | | Volatility | |
|---|---|---|---|---|
| | Estimation | Data | Estimation | Data |
| 3M nominal yield | 1.84% | 1.81% | 0.57% | 0.61% |
| 6M nominal yield | 1.90% | 1.91% | 0.56% | 0.59% |
| 1Y nominal yield | 2.01% | 2.05% | 0.57% | 0.69% |
| 2Y nominal yield | 2.25% | 2.25% | 0.62% | 0.80% |
| 3Y nominal yield | 2.49% | 2.47% | 0.69% | 0.86% |
| 5Y nominal yield | 2.92% | 2.91% | 0.83% | 0.90% |
| 7Y nominal yield | 3.28% | 3.28% | 0.93% | 0.91% |
| 10Y nominal yield | 3.70% | 3.70% | 0.97% | 0.91% |
| 5Y real yield | 1.10% | 1.09% | 0.85% | 0.91% |
| 7Y real yield | 1.31% | 1.33% | 0.83% | 0.81% |
| 10Y real yield | 1.58% | 1.57% | 0.81% | 0.72% |
| Log realized inflation | 2.09% | 2.15% | 0.90% | 0.99% |
| Equity log returns | 6.72% | 6.91% | 15.03% | 15.10% |
| 3M nominal risk premium | 0.12% | | 0.23% | |
| 6M nominal risk premium | 0.24% | | 0.43% | |
| 1Y nominal risk premium | 0.48% | | 0.77% | |
| 2Y nominal risk premium | 0.96% | | 1.25% | |
| 3Y nominal risk premium | 1.42% | | 1.52% | |
| 5Y nominal risk premium | 2.23% | | 1.71% | |
| 7Y nominal risk premium | 2.88% | | 1.72% | |
| 10Y nominal risk premium | 3.54% | | 1.81% | |
| 5Y real risk premium | 2.04% | | 5.23% | |
| 7Y real risk premium | 2.50% | | 6.06% | |
| 10Y real risk premium | 3.06% | | 6.68% | |
| Unexpected inflation risk premium | 0.97% | | 0.87% | |
| Equity risk premium | 6.07% | | 2.44% | |
| Nominal risk-free rate | 1.78% | | 0.58% | |
| Expected inflation | 2.10% | | 1.73% | |

**Table A.3:** Correlations between asset returns and economic variables when no constraints on the volatility of risk premia are imposed.

Panel (a) shows unconditional correlations between nominal and real bond returns, stock returns and realized inflation, calculated from the monthly time series. Panel (b) reports one-month conditional pairwise correlations between nominal and real bond returns, stock returns, realized inflation, nominal interest rate, expected inflation and real interest rate.

**(a)** Data

| | 3M nom | 1Y nom | 2Y nom | 5Y nom | 10Y nom | 5Y real | 10Y real | Equity | CPI |
|---|---|---|---|---|---|---|---|---|---|
| 3M nom | 1.000 | | | | | | | | |
| 1Y nom | 0.662 | 1.000 | | | | | | | |
| 2Y nom | 0.472 | 0.923 | 1.000 | | | | | | |
| 5Y nom | 0.263 | 0.713 | 0.893 | 1.000 | | | | | |
| 10Y nom | 0.122 | 0.513 | 0.685 | 0.911 | 1.000 | | | | |
| 5Y real | 0.018 | 0.327 | 0.420 | 0.504 | 0.478 | 1.000 | | | |
| 10Y real | 0.016 | 0.341 | 0.453 | 0.628 | 0.692 | 0.901 | 1.000 | | |
| Equity | −0.142 | −0.286 | −0.351 | −0.327 | −0.261 | 0.040 | 0.018 | 1.000 | |
| CPI | −0.106 | −0.144 | −0.142 | −0.197 | −0.237 | 0.344 | 0.140 | 0.073 | 1.000 |

**(b)** Time-varying risk premia without volatility restrictions

| | 3M nom | 1Y nom | 2Y nom | 5Y nom | 10Y nom | 5Y real | 10Y real | Equity | CPI | $R$ | $\pi$ | $r$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3M nom | 1.000 | | | | | | | | | | | |
| 1Y nom | 0.957 | 1.000 | | | | | | | | | | |
| 2Y nom | 0.834 | 0.956 | 1.000 | | | | | | | | | |
| 5Y nom | 0.509 | 0.728 | 0.897 | 1.000 | | | | | | | | |
| 10Y nom | 0.282 | 0.534 | 0.756 | 0.966 | 1.000 | | | | | | | |
| 5Y real | 0.153 | 0.263 | 0.366 | 0.499 | 0.582 | 1.000 | | | | | | |
| 10Y real | 0.135 | 0.322 | 0.499 | 0.711 | 0.805 | 0.945 | 1.000 | | | | | |
| Equity | −0.215 | −0.278 | −0.317 | −0.308 | −0.256 | 0.113 | −0.005 | 1.000 | | | | |
| CPI | −0.009 | −0.086 | −0.154 | −0.212 | −0.197 | 0.254 | 0.103 | 0.081 | 1.000 | | | |
| $R$ | −0.980 | −0.913 | −0.757 | −0.397 | −0.167 | −0.157 | −0.091 | 0.162 | −0.055 | 1.000 | | |
| $\pi$ | −0.374 | −0.506 | −0.593 | −0.588 | −0.480 | 0.407 | 0.129 | 0.429 | 0.466 | 0.252 | 1.000 | |
| $r$ | −0.027 | 0.004 | 0.027 | 0.012 | −0.062 | −0.846 | −0.632 | −0.301 | −0.444 | 0.110 | −0.807 | 1.000 |

**Table A.4:** Predictive ability of model specifications based on constant risk premia, as well as volatility-constrained and unconstrained time-varying risk premia.

The table reports 1-, 12-, 36-, and 60-month-ahead out-of-sample prediction root-mean-squared errors (RMSE, expressed as percentages) of nominal and real yield variations, as well as of logarithmic equity returns. The model specifications considered are based on constant risk premia (CRP), volatility-constrained time-varying risk premia (CTVRP), and unconstrained time-varying risk premia (UTVRP). The estimation is performed over an expanding window, considering an initial sample starting in December 1999 and ending in December 2005, which is then recursively expanded by monthly steps through December 2019. The prediction is performed over an out-of-sample period starting in January 2006, considering the estimated parameters and the values of the state variables at the end of the most recent estimation window. For each instrument and predictive horizon, the RMSE of the best performing model is represented in **bold underlined**, while the RMSE of the second best is underlined.

| | 1-month prediction | | | 12-month prediction | | | 36-month prediction | | | 60-month prediction | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | CRP | CTVRP | UTVRP | CRP | CTVRP | UTVRP | CRP | CTVRP | UTVRP | CRP | CTVRP | UTVRP |
| 3M nominal yield | **0.1527** | 0.1538 | 0.1687 | **0.9923** | 1.0510 | 1.1466 | **1.8008** | 1.8196 | 2.3401 | **1.9736** | 2.0410 | 4.8409 |
| 6M nominal yield | **0.1369** | 0.1408 | 0.1539 | **0.9632** | 1.0126 | 1.1218 | **1.8351** | 1.8512 | 2.3627 | **2.0777** | 2.1454 | 4.8633 |
| 1Y nominal yield | **0.1689** | 0.1721 | 0.1799 | **0.9204** | 0.9538 | 1.0799 | **1.7833** | 1.7910 | 2.2692 | **2.1029** | 2.1658 | 4.6899 |
| 2Y nominal yield | **0.1958** | 0.1971 | 0.2035 | **0.8740** | 0.8878 | 1.0314 | **1.7157** | 1.7267 | 2.0936 | **2.1130** | 2.1801 | 4.1058 |
| 3Y nominal yield | **0.2145** | 0.2149 | 0.2215 | **0.8391** | 0.8423 | 0.9859 | **1.6472** | 1.6688 | 1.9260 | **2.0809** | 2.1564 | 3.4695 |
| 5Y nominal yield | 0.2342 | **0.2339** | 0.2412 | **0.8039** | 0.8040 | 0.9254 | **1.5340** | 1.5844 | 1.7045 | **2.0115** | 2.1126 | 2.6025 |
| 7Y nominal yield | 0.2456 | **0.2452** | 0.2532 | **0.7893** | 0.7953 | 0.8925 | **1.4458** | 1.5238 | 1.5949 | **1.9574** | 2.0845 | 2.2545 |
| 10Y nominal yield | 0.2548 | **0.2548** | 0.2630 | **0.7688** | 0.7815 | 0.8537 | **1.3409** | 1.4421 | 1.4979 | **1.8899** | 2.0352 | 2.0989 |
| 5Y real yield | 0.2707 | **0.2687** | 0.2725 | 0.8459 | **0.8434** | 0.9244 | **1.4792** | 1.5646 | 1.6929 | **1.8962** | 2.0644 | 2.1907 |
| 7Y real yield | 0.2446 | **0.2436** | 0.2488 | **0.7838** | 0.7926 | 0.8639 | **1.3823** | 1.4738 | 1.5889 | **1.7837** | 1.9576 | 2.0847 |
| 10Y real yield | 0.2199 | **0.2196** | 0.2251 | **0.7079** | 0.7258 | 0.7847 | **1.2257** | 1.3305 | 1.4341 | **1.6139** | 1.7948 | 1.9712 |
| Equity log return | **4.2484** | 4.2524 | 4.2753 | 17.5899 | **17.3140** | 17.3681 | 31.1114 | 28.1476 | **28.0934** | 38.6718 | 33.0592 | **32.9554** |

**Table A.5:** Baseline model estimated observing also survey-based inflation expectations. Constant risk premia. Parameter estimates.

The table shows the maximum-likelihood parameters estimates when the information set includes also the 10-year CPI inflation rate forecast from the Survey of Professional Forecasters. Risk premia are constant. The values in parentheses are the asymptotic standard errors of the estimates. The sample period runs from January 1999 until December 2019.

| $R_0$ | $\mathbf{R}_1$ | $\pi_0$ | $\boldsymbol{\pi}_1$ | $\boldsymbol{\Theta}$ | | | $\sigma_\epsilon^B$ | $\sigma_\epsilon^I$ |
|---|---|---|---|---|---|---|---|---|
| 0.0178 | 0.3584 | 0.0237 | 0.0576 | 0.0661 | −0.5678 | 0.8392 | 0.0013 | 0.0007 |
| (0.0001) | (0.0012) | (0.0002) | (0.0045) | (0.0027) | (0.0125) | (0.0300) | (0.0000) | (0.0000) |
| | −0.4260 | | −0.7630 | 0.0132 | 0.2171 | −0.1757 | | |
| | (0.0046) | | (0.0296) | (0.0017) | (0.0074) | (0.0179) | | |
| | 0.2306 | | −1.2423 | −0.0131 | 0.1294 | 0.2429 | | |
| | (0.0117) | | (0.0626) | (0.0015) | (0.0088) | (0.0201) | | |
| $\boldsymbol{\Lambda}_0$ | $\boldsymbol{\Lambda}_1$ | | | $\boldsymbol{\Sigma}_X$ | | | $\boldsymbol{\sigma}_P$ | $\boldsymbol{\sigma}_S$ |
| −0.6309 | 0 | 0 | 0 | 0.0202 | 0.0063 | −0.0033 | 0.0002 | 0.0365 |
| (0.0285) | | | | (0.0009) | (0.0008) | (0.0005) | (0.0006) | (0.0093) |
| 0.4448 | 0 | 0 | 0 | 0 | 0.0113 | 0.0017 | −0.0003 | −0.0269 |
| (0.0442) | | | | | (0.0005) | (0.0005) | (0.0006) | (0.0093) |
| −0.0527 | 0 | 0 | 0 | 0 | 0 | 0.0083 | −0.0038 | −0.0462 |
| (0.0489) | | | | | | (0.0004) | (0.0006) | (0.0087) |
| 1.6614 | 0 | 0 | 0 | 0 | 0 | 0 | 0.0084 | −0.0189 |
| (0.1075) | | | | | | | (0.0004) | (0.0086) |
| 0.9249 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.1349 |
| (0.1238) | | | | | | | | (0.0060) |

**Table A.6:** Baseline model estimated observing also survey-based inflation expectations. Constant risk premia. Historical and model-implied summary statistics.

The table shows annualized historical and model-implied means and volatilities of bond yields, equity log returns, realized inflation and their model-implied means and volatilities. The table also shows the model-implied means and volatilities of the bond risk premia, the equity risk premium, the nominal risk-free rate and the expected inflation.

| Time series | Mean value | | Volatility | |
|---|---|---|---|---|
| | Estimation | Data | Estimation | Data |
| 3M nominal yield | 1.84% | 1.81% | 0.60% | 0.61% |
| 6M nominal yield | 1.90% | 1.91% | 0.60% | 0.59% |
| 1Y nominal yield | 2.01% | 2.05% | 0.60% | 0.69% |
| 2Y nominal yield | 2.25% | 2.25% | 0.65% | 0.80% |
| 3Y nominal yield | 2.48% | 2.47% | 0.72% | 0.86% |
| 5Y nominal yield | 2.92% | 2.91% | 0.84% | 0.90% |
| 7Y nominal yield | 3.29% | 3.28% | 0.91% | 0.91% |
| 10Y nominal yield | 3.70% | 3.70% | 0.93% | 0.91% |
| 5Y real yield | 1.10% | 1.09% | 0.85% | 0.91% |
| 7Y real yield | 1.30% | 1.33% | 0.82% | 0.81% |
| 10Y real yield | 1.58% | 1.57% | 0.81% | 0.72% |
| Log realized inflation | 2.37% | 2.15% | 0.92% | 0.99% |
| Equity log returns | 6.72% | 6.91% | 15.08% | 15.10% |
| 3M nominal risk premium | 0.11% | | 0.00% | |
| 6M nominal risk premium | 0.23% | | 0.00% | |
| 1Y nominal risk premium | 0.47% | | 0.00% | |
| 2Y nominal risk premium | 0.95% | | 0.00% | |
| 3Y nominal risk premium | 1.42% | | 0.00% | |
| 5Y nominal risk premium | 2.24% | | 0.00% | |
| 7Y nominal risk premium | 2.86% | | 0.00% | |
| 10Y nominal risk premium | 3.44% | | 0.00% | |
| 5Y real risk premium | 2.29% | | 0.00% | |
| 7Y real risk premium | 2.77% | | 0.00% | |
| 10Y real risk premium | 3.33% | | 0.00% | |
| Unexpected inflation risk premium | 1.39% | | 0.00% | |
| Equity risk premium | 6.08% | | 0.00% | |
| Nominal risk-free rate | 1.78% | | 0.61% | |
| Expected inflation | 2.37% | | 1.48% | |

**Table A.7:** Parameter estimates for the 4-factor affine no-arbitrage Nelson-Siegel term structure specification.

The table shows the maximum-likelihood parameters estimates for the model described in Section K.1. The values in parentheses are the asymptotic standard errors of the estimates. The sample period runs from January 1999 until December 2019.

| $R_0$ | $\mathbf{R}_1$ | $\pi_0$ | $\boldsymbol{\pi}_1$ | $\boldsymbol{\Theta}^{\mathbb{Q}}$ | | | | $\alpha^r$ | $\sigma_\epsilon^B$ | $\sigma_\epsilon^I$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0.0076 | 1 | 0 | 0 | 0 | 0 | 0.6206 | 0.0008 | 0.0011 |
| | | (0.0021) | | | | | | (0.0115) | (0.0000) | (0.0000) |
| | 1 | | 0.3794 | 0 | 0.4161 | −0.4161 | 0 | | | |
| | | | (0.0115) | | (0.0090) | (0.0090) | | | | |
| | 0 | | 0 | 0 | 0 | 0.4161 | 0 | | | |
| | | | | | | (0.0090) | | | | |
| | 0 | | −1 | 0 | 0 | 0 | 0 | | | |

| $\boldsymbol{\Lambda}_0$ | $\boldsymbol{\Lambda}_1$ | | | | $\boldsymbol{\Sigma}_X$ | | | | $\boldsymbol{\sigma}_P$ | $\boldsymbol{\sigma}_S$ |
|---|---|---|---|---|---|---|---|---|---|---|
| −0.1514 | 0 | 0 | 0 | 0 | 0.0096 | −0.0100 | −0.0143 | 0.0061 | 0.0026 | −0.0075 |
| (0.2198) | | | | | (0.0006) | (0.0007) | (0.0028) | (0.0006) | (0.0007) | (0.0119) |
| −0.7133 | 0 | 0 | 0 | 0 | 0 | 0.0060 | −0.0072 | −0.0004 | 0.0007 | 0.0138 |
| (0.2242) | | | | | | (0.0003) | (0.0020) | (0.0004) | (0.0006) | (0.0107) |
| −0.6645 | 0 | 0 | 0 | 0 | 0 | 0 | 0.0277 | −0.0007 | 0.0013 | 0.0542 |
| (0.2259) | | | | | | | (0.0013) | (0.0003) | (0.0006) | (0.0093) |
| −0.2887 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.0053 | −0.0033 | −0.0433 |
| (0.2277) | | | | | | | | (0.0002) | (0.0006) | (0.0093) |
| 1.0305 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.0082 | −0.0176 |
| (0.2362) | | | | | | | | | (0.0004) | (0.0091) |
| 0.8434 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.1318 |
| (0.2437) | | | | | | | | | | (0.0062) |

**Table A.8:** 4-factor affine no-arbitrage Nelson-Siegel term structure specification. Historical and model-implied summary statistics.

The table shows annualized historical and model-implied means and volatilities of bond yields, equity log returns, realized inflation and their model-implied means and volatilities. The table also shows the model-implied means and volatilities of the bond risk premia, the equity risk premium, the nominal risk-free rate and the expected inflation.

| Time series | Mean value | | Volatility | |
|---|---|---|---|---|
| | Estimation | Data | Estimation | Data |
| 3M nominal yield | 1.85% | 1.81% | 0.56% | 0.61% |
| 6M nominal yield | 1.90% | 1.91% | 0.54% | 0.59% |
| 1Y nominal yield | 2.00% | 2.05% | 0.58% | 0.69% |
| 2Y nominal yield | 2.24% | 2.25% | 0.72% | 0.80% |
| 3Y nominal yield | 2.48% | 2.47% | 0.80% | 0.86% |
| 5Y nominal yield | 2.94% | 2.91% | 0.83% | 0.90% |
| 7Y nominal yield | 3.30% | 3.28% | 0.79% | 0.91% |
| 10Y nominal yield | 3.68% | 3.70% | 0.74% | 0.91% |
| 5Y real yield | 1.11% | 1.09% | 0.69% | 0.91% |
| 7Y real yield | 1.33% | 1.33% | 0.68% | 0.81% |
| 10Y real yield | 1.55% | 1.57% | 0.67% | 0.72% |
| Log realized inflation | 2.15% | 2.15% | 0.93% | 0.99% |
| Equity log returns | 6.76% | 6.91% | 15.08% | 15.10% |
| 3M nominal risk premium | 0.12% | | 0.00% | |
| 6M nominal risk premium | 0.25% | | 0.00% | |
| 1Y nominal risk premium | 0.55% | | 0.00% | |
| 2Y nominal risk premium | 1.21% | | 0.00% | |
| 3Y nominal risk premium | 1.86% | | 0.00% | |
| 5Y nominal risk premium | 2.96% | | 0.00% | |
| 7Y nominal risk premium | 3.76% | | 0.00% | |
| 10Y nominal risk premium | 4.57% | | 0.00% | |
| 5Y real risk premium | 3.00% | | 0.00% | |
| 7Y real risk premium | 3.66% | | 0.00% | |
| 10Y real risk premium | 4.41% | | 0.00% | |
| Realized inflation risk premium | 0.76% | | 0.00% | |
| Equity risk premium | 6.09% | | 0.00% | |
| Nominal risk-free rate | 1.81% | | 0.60% | |
| Expected inflation | 2.16% | | 0.60% | |

**Table A.9:** UK-based analysis. Parameter estimates.

The table shows the maximum-likelihood estimates of the model parameters. The sample period runs from December 1985 to July 2018. Panel (a) shows the parameter estimates obtained for the specification with constant risk premia ($\mathbf{\Lambda}_1 = \mathbf{0}$). The values in parentheses are the asymptotic standard errors of the estimates. Panel (b) shows the parameter estimates obtained for the specification with time-varying risk premia and restrictions on the maximum volatility of bond, realized inflation and stock risk premia. The volatility of the risk premia for the 3- and 10-year nominal bonds, and for the 7-year inflation-indexed bond, are imposed to be lower than 1%. The volatility of the realized inflation risk premium is not higher than 0.5% per annum and the volatility of the equity premium is not higher than 1% per annum. The values in parentheses are the standard errors of the estimates.

**(a)** Constant risk premia

| $R_0$ | $\mathbf{R}_1$ | $\pi_0$ | $\boldsymbol{\pi}_1$ | $\mathbf{\Theta}$ | | | $\sigma_\epsilon^B$ | $\sigma_\epsilon^I$ |
|---|---|---|---|---|---|---|---|---|
| 0.0482 | 0.4425 | 0.0337 | 0.2147 | 0.0661 | −0.5287 | −0.9337 | 0.0009 | 0.0009 |
| (0.0001) | (0.0009) | (0.0025) | (0.0022) | (0.0011) | (0.0091) | (0.0114) | (0.0000) | (0.0000) |
| | −0.6427 | | −1.0922 | −0.0189 | 0.3049 | 0.1013 | | |
| | (0.0078) | | (0.0265) | (0.0008) | (0.0088) | (0.0096) | | |
| | −0.7436 | | 0.1934 | −0.0039 | 0.1885 | 0.1955 | | |
| | (0.0099) | | (0.0272) | (0.0008) | (0.0064) | (0.0081) | | |
| $\mathbf{\Lambda}_0$ | $\mathbf{\Lambda}_1$ | | | $\mathbf{\Sigma}_X$ | | | $\boldsymbol{\sigma}_P$ | $\boldsymbol{\sigma}_S$ |
| −0.2150 | 0 | 0 | 0 | 0.0268 | −0.0007 | 0.0007 | 0.0005 | −0.0242 |
| (0.0074) | | | | (0.0010) | (0.0006) | (0.0004) | (0.0007) | (0.0078) |
| −0.0173 | 0 | 0 | 0 | 0 | 0.0116 | −0.0011 | −0.0003 | −0.0224 |
| (0.0100) | | | | | (0.0004) | (0.0004) | (0.0007) | (0.0077) |
| 0.0401 | 0 | 0 | 0 | 0 | 0 | 0.0083 | 0.0009 | −0.0079 |
| (0.0113) | | | | | | (0.0003) | (0.0007) | (0.0077) |
| −0.0477 | 0 | 0 | 0 | 0 | 0 | 0 | 0.0140 | 0.0015 |
| (0.1770) | | | | | | | (0.0005) | (0.0076) |
| 0.2868 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.1495 |
| (0.1768) | | | | | | | | (0.0053) |

**(b)** Time-varying risk premia with volatility restrictions

| $R_0$ | $\mathbf{R}_1$ | $\pi_0$ | $\boldsymbol{\pi}_1$ | $\mathbf{\Theta}$ | | | $\sigma_\epsilon^B$ | $\sigma_\epsilon^I$ |
|---|---|---|---|---|---|---|---|---|
| 0.0482 | 0.4425 | 0.0338 | 0.1078 | 0.0674 | 0.1204 | −0.8229 | 0.0009 | 0.0009 |
| (0.0002) | (0.0027) | (0.0049) | (0.0749) | (0.0591) | (0.1534) | (0.2341) | (0.0000) | (0.0000) |
| | −0.6442 | | −1.1201 | −0.0297 | 0.1696 | 0.0712 | | |
| | (0.0082) | | (0.1902) | (0.0339) | (0.0944) | (0.1613) | | |
| | −0.7438 | | 0.3843 | 0.0054 | −0.0939 | 0.3181 | | |
| | (0.0161) | | (0.2683) | (0.0277) | (0.0778) | (0.1237) | | |
| $\mathbf{\Lambda}_0$ | $\mathbf{\Lambda}_1$ | | | $\mathbf{\Sigma}_X$ | | | $\boldsymbol{\sigma}_P$ | $\boldsymbol{\sigma}_S$ |
| −0.2172 | −0.0513 | −24.5926 | −4.1829 | 0.0265 | −0.0005 | 0.0009 | 0.0007 | −0.0242 |
| (0.0135) | (2.1745) | (5.7366) | (8.8493) | (0.0011) | (0.0007) | (0.0006) | (0.0008) | (0.0151) |
| −0.0143 | 0.9121 | 10.7036 | 2.5268 | 0 | 0.0116 | −0.0012 | −0.0002 | −0.0218 |
| (0.0218) | (2.9410) | (8.0468) | (13.4909) | | (0.0004) | (0.0004) | (0.0009) | (0.0102) |
| 0.0452 | −0.9774 | 38.3922 | −14.0737 | 0 | 0 | 0.0083 | 0.0007 | −0.0075 |
| (0.0303) | (3.7849) | (10.5009) | (16.8228) | | | (0.0004) | (0.0009) | (0.0085) |
| −0.0410 | −7.7586 | −2.3811 | 15.2210 | 0 | 0 | 0 | 0.0137 | 0.0001 |
| (0.3555) | (5.5391) | (13.9290) | (19.1601) | | | | (0.0006) | (0.0094) |
| 0.2870 | −1.2047 | −1.4300 | 5.9979 | 0 | 0 | 0 | 0 | 0.1493 |
| (0.2165) | (1.5776) | (4.2625) | (5.1218) | | | | | (0.0093) |

**Table A.10:** UK-based analysis. Historical and model-implied summary statistics.

The table shows annualized historical and model-implied means and volatilities of bond yields, equity log returns, realized inflation and their model-implied means and volatilities. The table also shows the model-implied means and volatilities of the bond risk premia, the equity risk premium, the nominal risk-free rate and the expected inflation.

**(a)** Constant risk premia

| Time series | Mean value | | Volatility | |
|---|---|---|---|---|
| | Estimation | Data | Estimation | Data |
| 1Y nominal yield | 4.94% | 4.96% | 1.27% | 1.18% |
| 2Y nominal yield | 5.05% | 5.03% | 1.14% | 1.15% |
| 3Y nominal yield | 5.16% | 5.14% | 1.07% | 1.12% |
| 5Y nominal yield | 5.33% | 5.34% | 1.03% | 1.07% |
| 7Y nominal yield | 5.48% | 5.49% | 1.03% | 1.03% |
| 10Y nominal yield | 5.63% | 5.62% | 1.05% | 0.97% |
| 5Y real yield | 1.68% | 1.67% | 0.89% | 0.91% |
| 7Y real yield | 1.77% | 1.79% | 0.81% | 0.79% |
| 10Y real yield | 1.90% | 1.89% | 0.72% | 0.68% |
| Log realized inflation | 3.36% | 3.34% | 1.41% | 1.46% |
| Equity log returns | 8.45% | 9.28% | 15.33% | 15.31% |
| 1Y nominal risk premium | 0.25% | | 0.00% | |
| 2Y nominal risk premium | 0.48% | | 0.00% | |
| 3Y nominal risk premium | 0.68% | | 0.00% | |
| 5Y nominal risk premium | 1.05% | | 0.00% | |
| 7Y nominal risk premium | 1.37% | | 0.00% | |
| 10Y nominal risk premium | 1.77% | | 0.00% | |
| 5Y real risk premium | 0.59% | | 0.00% | |
| 7Y real risk premium | 0.81% | | 0.00% | |
| 10Y real risk premium | 1.11% | | 0.00% | |
| Realized inflation risk premium | −0.07% | | 0.00% | |
| Equity risk premium | 4.81% | | 0.00% | |
| Nominal risk-free rate | 4.82% | | 1.49% | |
| Expected inflation | 3.37% | | 1.46% | |

**(b)** Constrained time-varying risk premia

| Time series | Mean value | | Volatility | |
|---|---|---|---|---|
| | Estimation | Data | Estimation | Data |
| 1Y nominal yield | 4.94% | 4.96% | 1.24% | 1.18% |
| 2Y nominal yield | 5.05% | 5.03% | 1.11% | 1.15% |
| 3Y nominal yield | 5.16% | 5.14% | 1.05% | 1.12% |
| 5Y nominal yield | 5.33% | 5.34% | 1.02% | 1.07% |
| 7Y nominal yield | 5.48% | 5.49% | 1.03% | 1.03% |
| 10Y nominal yield | 5.63% | 5.62% | 1.05% | 0.97% |
| 5Y real yield | 1.68% | 1.67% | 0.89% | 0.91% |
| 7Y real yield | 1.77% | 1.79% | 0.81% | 0.79% |
| 10Y real yield | 1.90% | 1.89% | 0.72% | 0.68% |
| Log realized inflation | 3.37% | 3.34% | 1.38% | 1.46% |
| Equity log returns | 8.45% | 9.28% | 15.30% | 15.31% |
| 1Y nominal risk premium | 0.25% | | 0.53% | |
| 2Y nominal risk premium | 0.47% | | 0.84% | |
| 3Y nominal risk premium | 0.68% | | 1.00% | |
| 5Y nominal risk premium | 1.05% | | 1.05% | |
| 7Y nominal risk premium | 1.36% | | 0.97% | |
| 10Y nominal risk premium | 1.77% | | 1.00% | |
| 5Y real risk premium | 0.59% | | 0.97% | |
| 7Y real risk premium | 0.81% | | 1.00% | |
| 10Y real risk premium | 1.11% | | 0.90% | |
| Realized inflation risk premium | −0.07% | | 0.32% | |
| Equity risk premium | 4.81% | | 1.00% | |
| Nominal risk-free rate | 4.82% | | 1.45% | |
| Expected inflation | 3.38% | | 1.43% | |

**Table A.11:** UK-based analysis. Correlations between asset returns and economic variables.

Panel (a) shows the unconditional correlations between nominal and real bond returns, stock returns and realized inflation, calculated from the monthly time series. Panels (b) and (c) report the one-month conditional pairwise correlations between nominal and real bond returns, stock returns, realized inflation, nominal interest rate, expected inflation and real interest rate.

**(a)** Data

| | 1Y nom | 2Y nom | 5Y nom | 10Y nom | 5Y real | 10Y real | Equity | CPI |
|---|---|---|---|---|---|---|---|---|
| 1Y nom | 1.000 | | | | | | | |
| 2Y nom | 0.954 | 1.000 | | | | | | |
| 5Y nom | 0.806 | 0.920 | 1.000 | | | | | |
| 10Y nom | 0.610 | 0.731 | 0.920 | 1.000 | | | | |
| 5Y real | 0.306 | 0.405 | 0.418 | 0.317 | 1.000 | | | |
| 10Y real | 0.326 | 0.435 | 0.516 | 0.485 | 0.911 | 1.000 | | |
| Equity | 0.071 | 0.091 | 0.144 | 0.147 | 0.167 | 0.197 | 1.000 | |
| CPI | −0.062 | −0.072 | −0.087 | −0.102 | 0.255 | 0.184 | 0.006 | 1.000 |

**(b)** Constant risk premia

| | 1Y nom | 2Y nom | 5Y nom | 10Y nom | 5Y real | 10Y real | Equity | CPI | $R$ | $\pi$ | $r$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1Y nom | 1.000 | | | | | | | | | | |
| 2Y nom | 0.983 | 1.000 | | | | | | | | | |
| 5Y nom | 0.837 | 0.921 | 1.000 | | | | | | | | |
| 10Y nom | 0.651 | 0.766 | 0.948 | 1.000 | | | | | | | |
| 5Y real | 0.346 | 0.427 | 0.475 | 0.313 | 1.000 | | | | | | |
| 10Y real | 0.395 | 0.519 | 0.667 | 0.584 | 0.942 | 1.000 | | | | | |
| Equity | 0.074 | 0.107 | 0.157 | 0.154 | 0.185 | 0.214 | 1.000 | | | | |
| CPI | −0.050 | −0.057 | −0.071 | −0.086 | 0.031 | −0.001 | 0.006 | 1.000 | | | |
| $R$ | −0.979 | −0.928 | −0.717 | −0.509 | −0.229 | −0.234 | −0.034 | 0.044 | 1.000 | | |
| $\pi$ | −0.662 | −0.610 | −0.496 | −0.489 | 0.455 | 0.310 | 0.055 | 0.084 | 0.710 | 1.000 | |
| $r$ | −0.438 | −0.438 | −0.306 | −0.039 | −0.894 | −0.713 | −0.116 | −0.051 | 0.403 | −0.359 | 1.000 |

**(c)** Constrained time-varying risk premia

| | 1Y nom | 2Y nom | 5Y nom | 10Y nom | 5Y real | 10Y real | Equity | CPI | $R$ | $\pi$ | $r$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1Y nom | 1.000 | | | | | | | | | | |
| 2Y nom | 0.983 | 1.000 | | | | | | | | | |
| 5Y nom | 0.834 | 0.920 | 1.000 | | | | | | | | |
| 10Y nom | 0.651 | 0.768 | 0.949 | 1.000 | | | | | | | |
| 5Y real | 0.333 | 0.418 | 0.471 | 0.313 | 1.000 | | | | | | |
| 10Y real | 0.385 | 0.513 | 0.666 | 0.585 | 0.942 | 1.000 | | | | | |
| Equity | 0.074 | 0.107 | 0.156 | 0.152 | 0.185 | 0.213 | 1.000 | | | | |
| CPI | −0.055 | −0.062 | −0.076 | −0.088 | 0.023 | −0.008 | −0.004 | 1.000 | | | |
| $R$ | −0.981 | −0.929 | −0.716 | −0.509 | −0.227 | −0.235 | −0.037 | 0.047 | 1.000 | | |
| $\pi$ | −0.491 | −0.434 | −0.341 | −0.388 | 0.635 | 0.474 | 0.084 | 0.080 | 0.538 | 1.000 | |
| $r$ | −0.409 | −0.413 | −0.289 | −0.029 | −0.897 | −0.714 | −0.119 | −0.045 | 0.385 | −0.569 | 1.000 |

**Figure A.1:** Optimal portfolio strategy for $\gamma \to \infty$ and different degrees of money illusion $\alpha$, when the 10-year nominal bond is excluded from the investable universe. Constant risk premia.

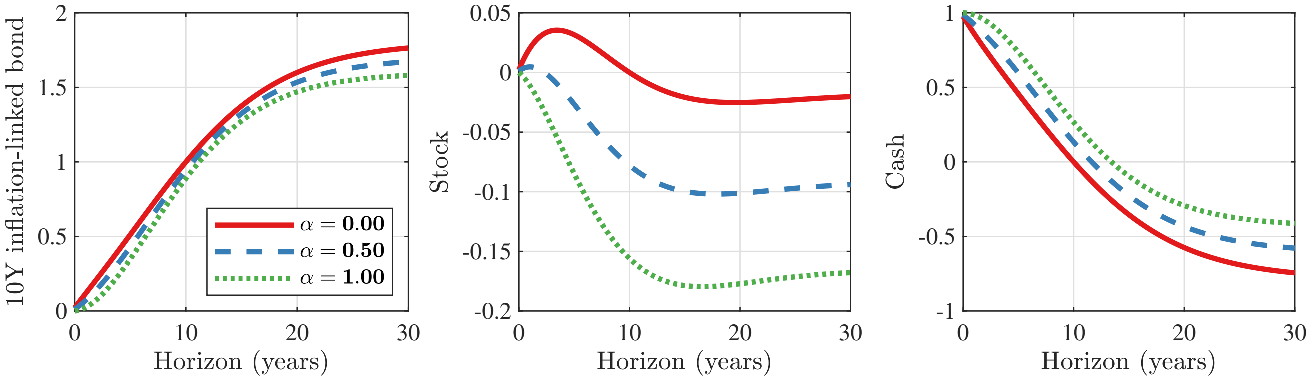


**Figure A.2:** Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$, when the 10-year nominal bond is excluded from the investable universe. Utility loss with respect to the case where the nominal bond is available. Constant risk premia.

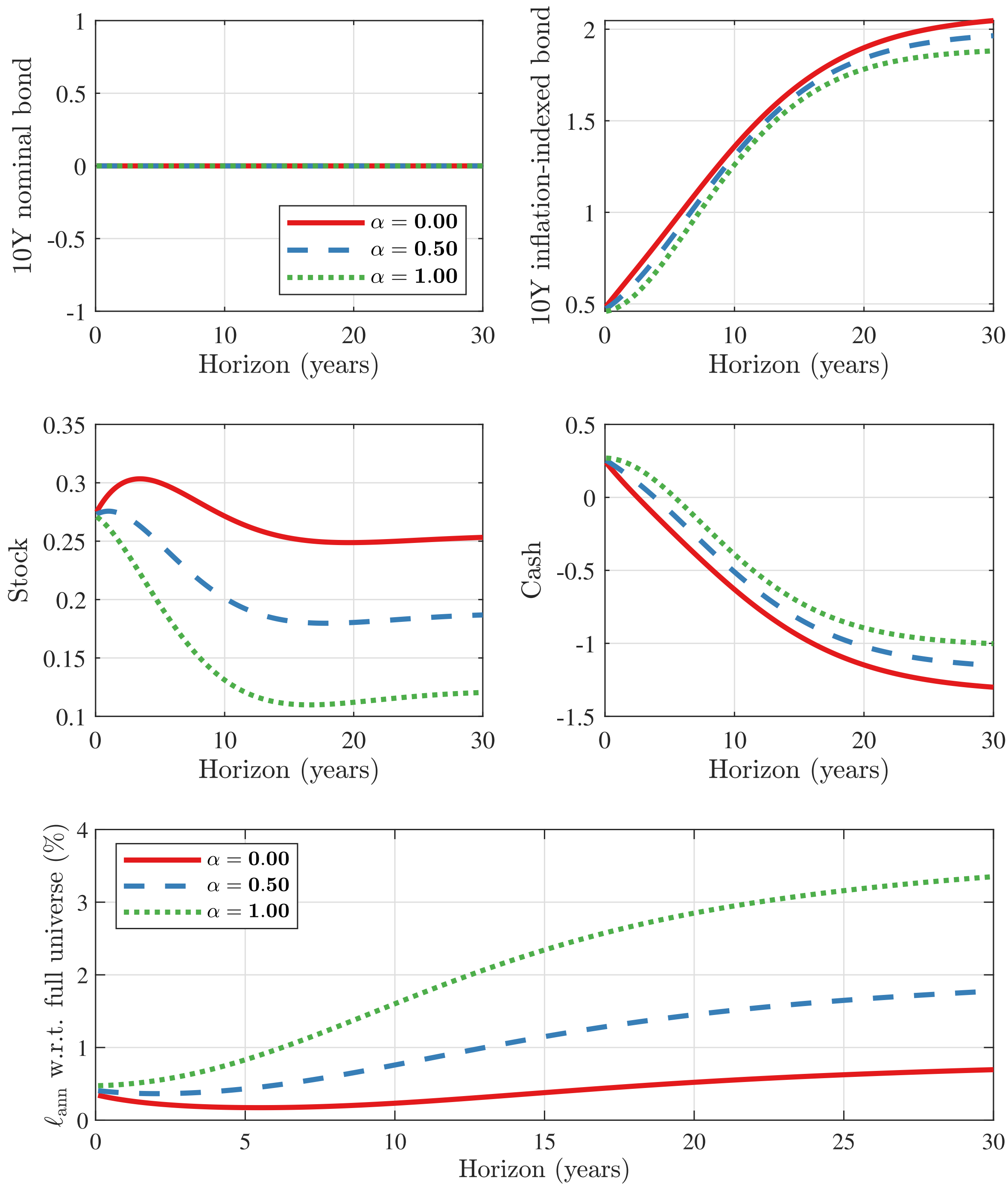


**Figure A.3:** Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$, when the 10-year nominal bond is excluded from the investable universe. Utility loss with respect to the case where the nominal bond is available. Time-varying risk premia (with volatility constraints).

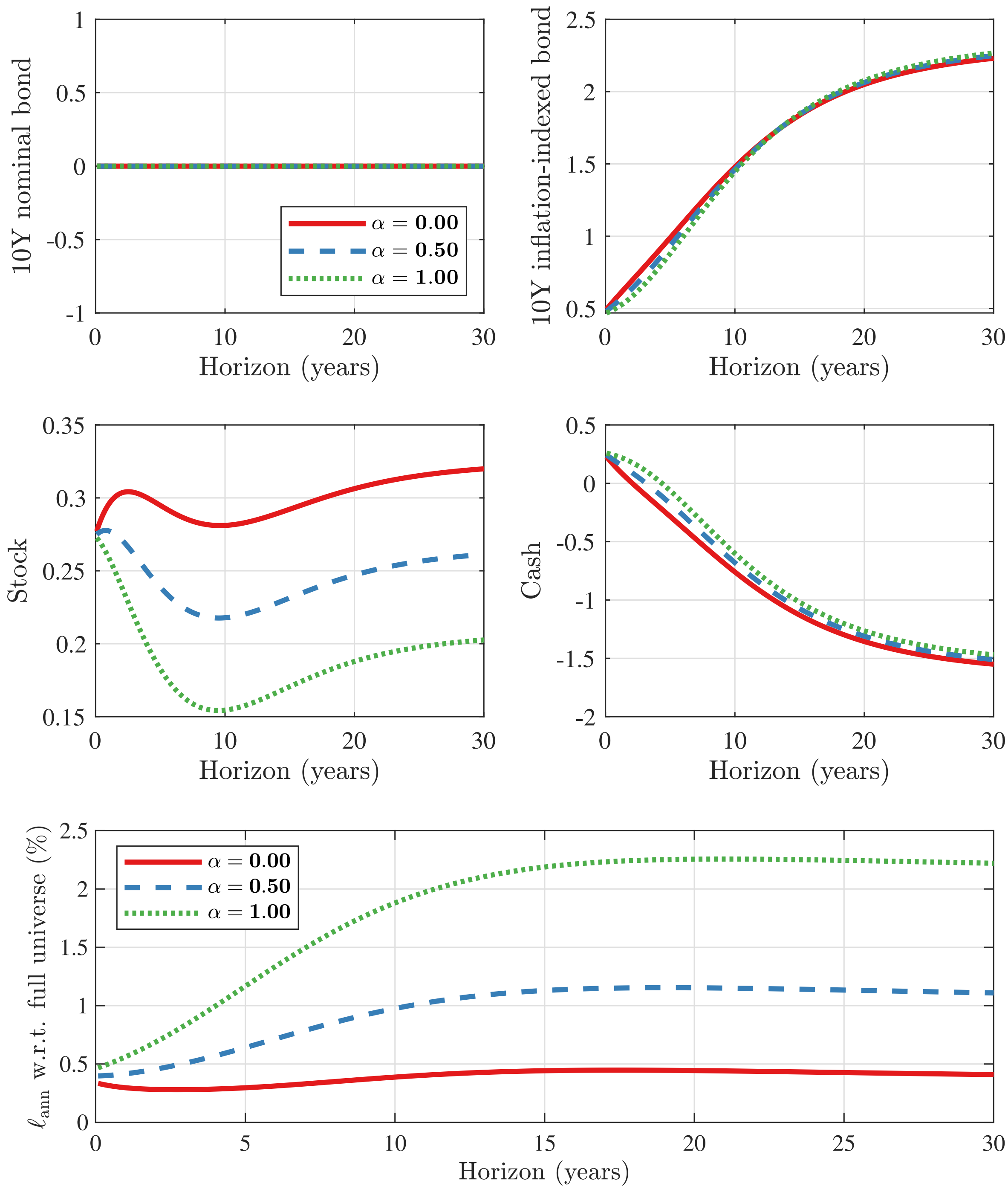


**Figure A.4:** Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$. Utility loss with respect to a non-illusioned investor. Constant risk premia and zero realized inflation risk premium.

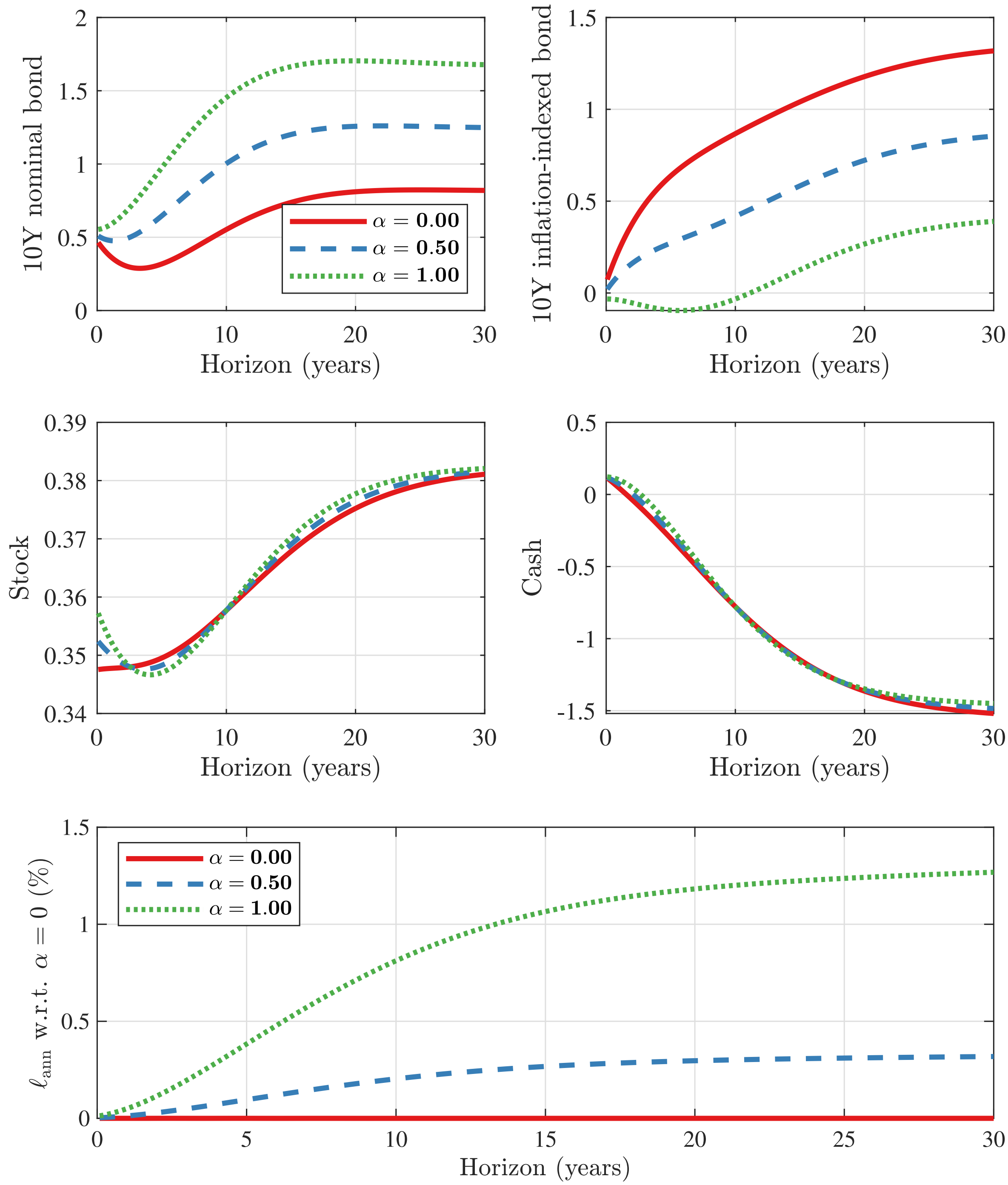


**Figure A.5:** Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$, when the 10-year inflation-indexed bond is excluded from the investable universe. Utility loss with respect to the case where the inflation-indexed bond is available. Constant risk premia and zero realized inflation risk premium.

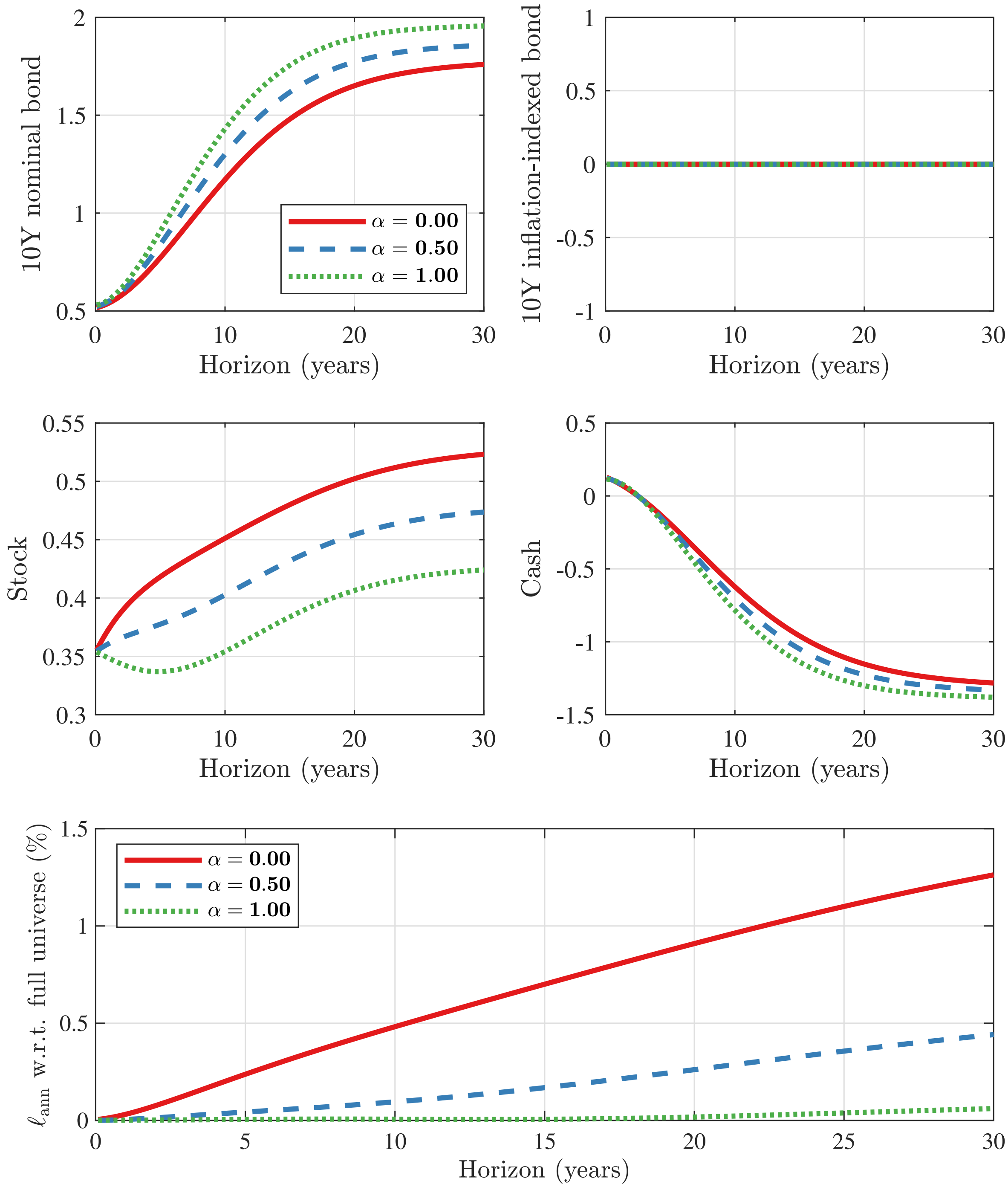


**Figure A.6:** Time series of relevant model-implied and observed variables. Time-varying risk premia (unconstrained volatility).

The top graphs show the model-implied instantaneous nominal rate, real rate and break-even inflation. The second graphs from the top show the model-implied 10-year nominal yield, real yield, break-even inflation, expected inflation and the 10-year CPI inflation rate forecast from the Survey of Professional Forecasters. The third graphs from the top show the model-implied risk premia of the risky assets and of the unexpected inflation. The bottom graphs show the portfolio allocation of a nominal mean-variance investor ($\gamma = 10$) and the corresponding maximum ex-ante Sharpe ratio (considering a 10-year nominal bond, a 10-year inflation-indexed bond and the stock index).

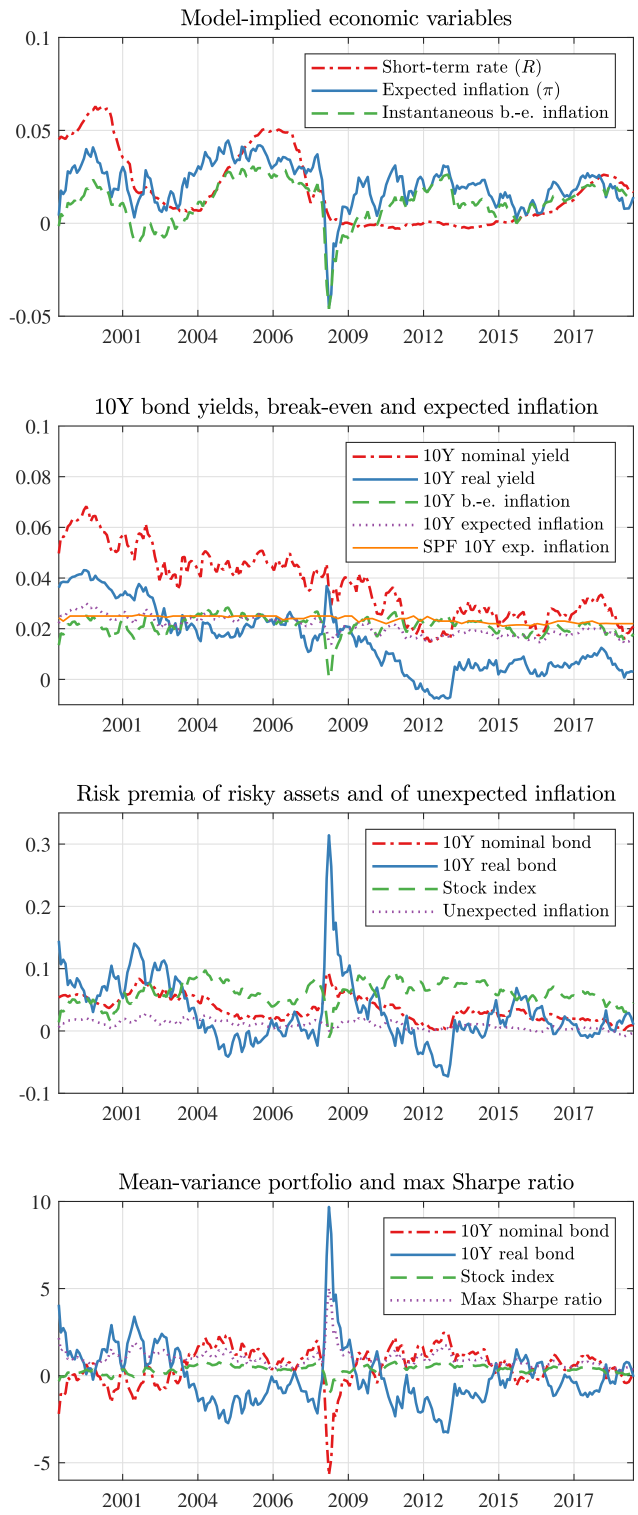


**Figure A.7:** Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$. Utility loss with respect to a non-illusioned investor. Time-varying risk premia (unconstrained volatility).

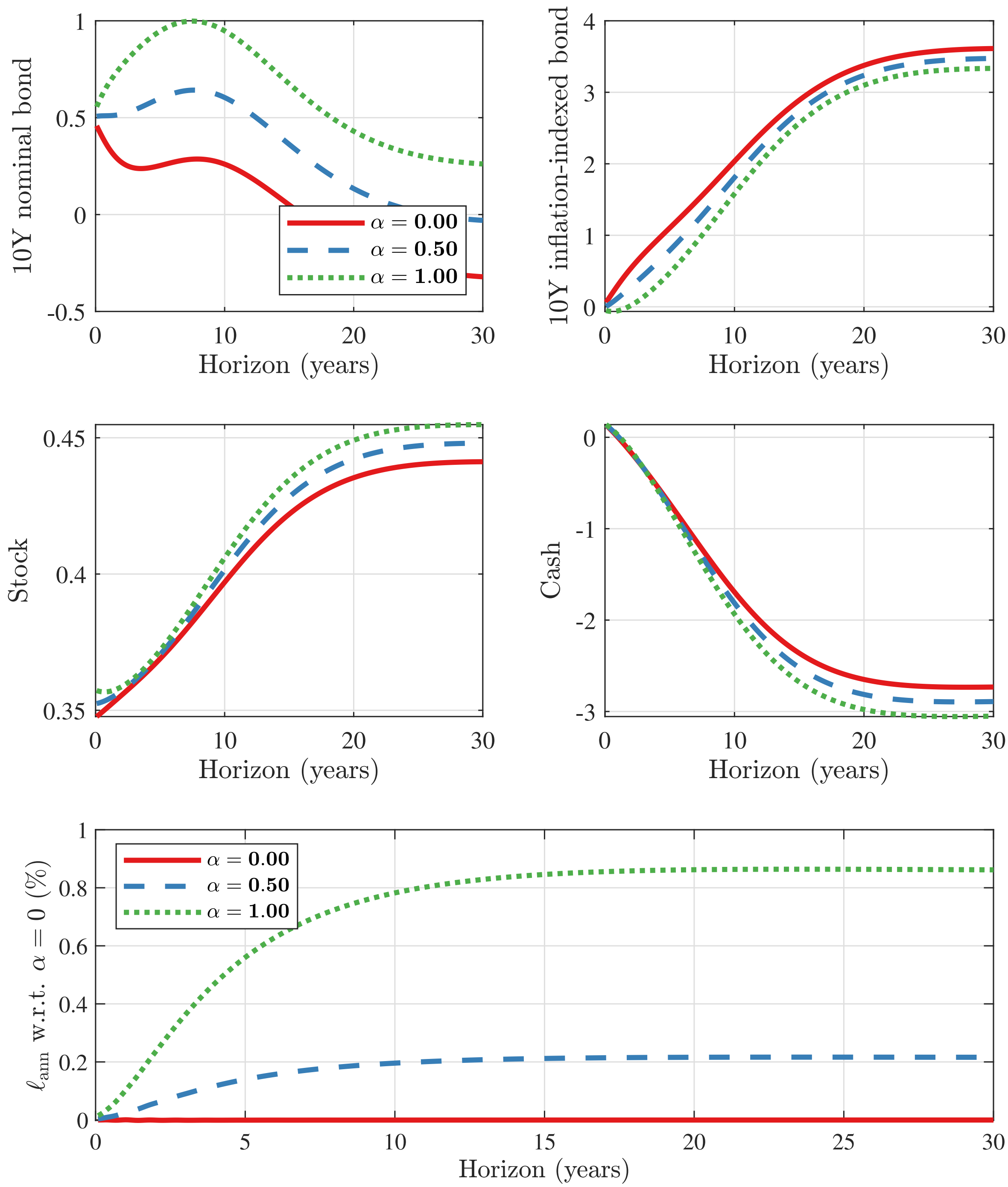
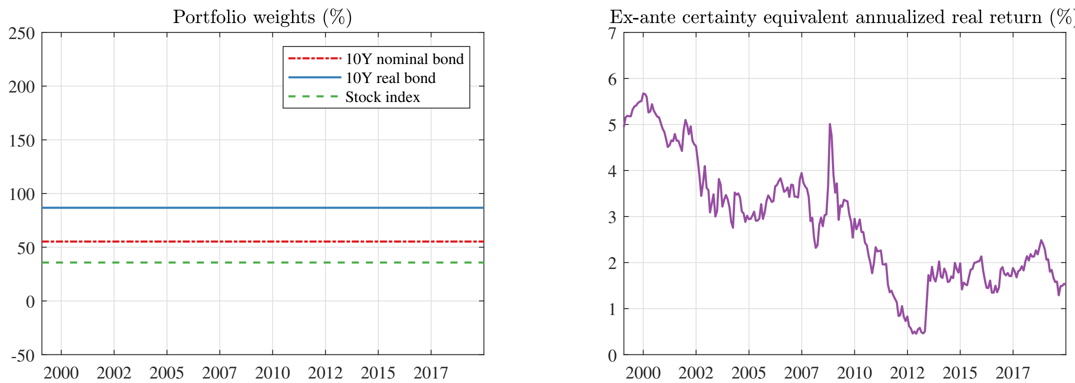


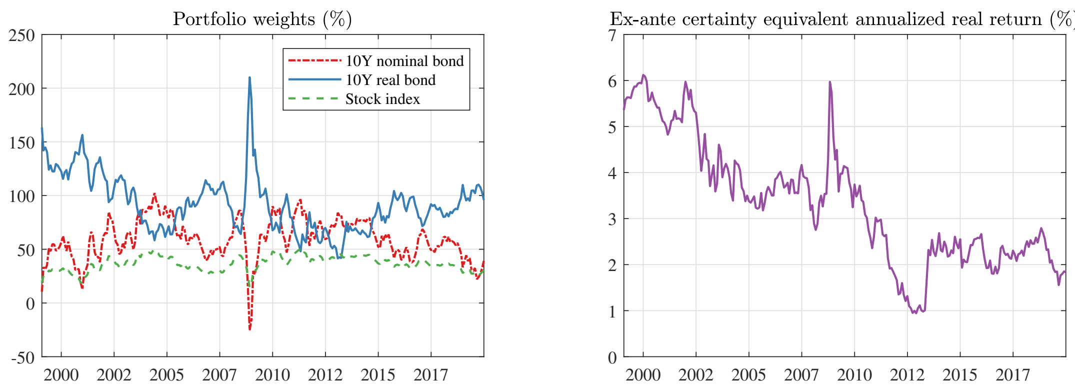
**Figure A.8:** Time series of in-sample optimal portfolio weights for a non-illusioned investor, obtained for different risk premia specifications.

The graphs on the left show the optimal in-sample portfolio weights, calculated for each date considering the time series of the state variables $\mathbf{X}_t$ and the estimated model parameters. The graphs on the right show the ex-ante certainty equivalent annualized real return for a strategy starting at the current date. The investor is not money-illusioned, i.e., $\alpha = 0$, and the investment horizon is 10-year long. Panel (a) refers to the model specification with constant risk premia, using the estimates in Table 1a. Panel (b) refers to the model specification with time-varying risk premia constrained in volatility, using the estimates in Table 1b. Panel (c) refers to the model specification with time-varying risk premia with no volatility constraints, using the estimates in Table A.1. All values are expressed as percentages.

**(a)** Constant risk premia




**(b)** Time-varying risk premia (constrained volatility)




**(c)** Time-varying risk premia (unconstrained volatility)

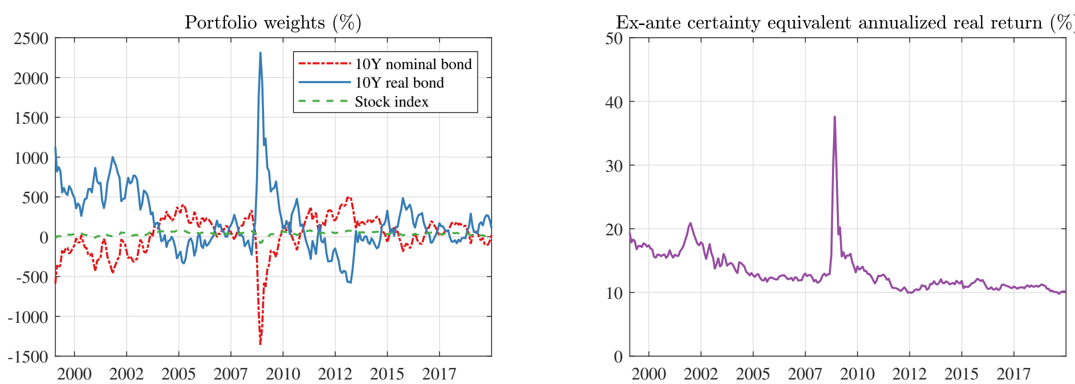


**Figure A.9:** Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$, when the 10-year inflation-indexed bond is excluded from the investable universe. Utility loss with respect to the case where the inflation-indexed bond is available. Time-varying risk premia (unconstrained volatility).

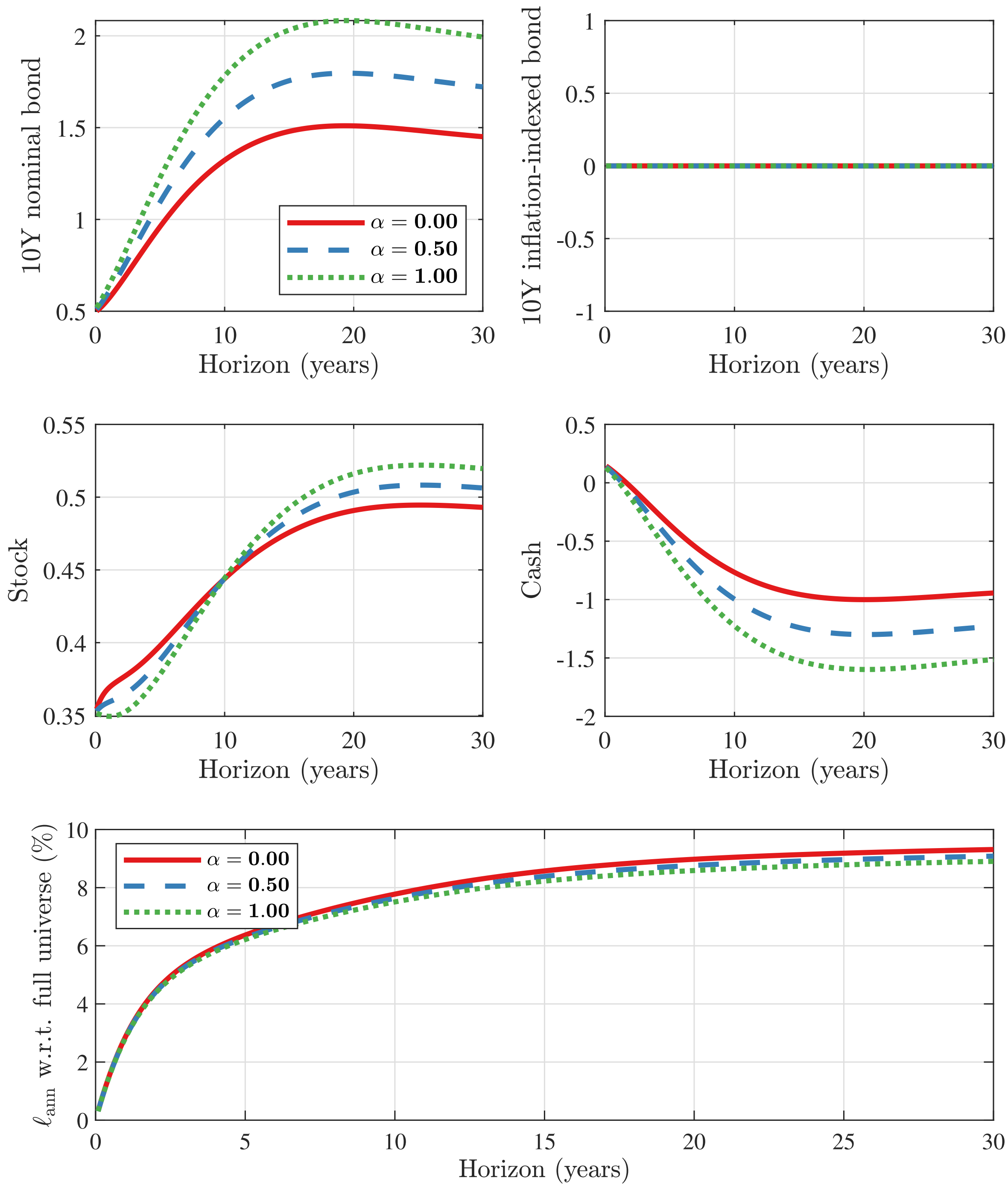


**Figure A.10:** Time series of in-sample optimal portfolio weights and perceived utility losses when the inflation-indexed bond is not available for investors with different degrees of money illusion.

The top graphs show the optimal in-sample portfolio weights, calculated for each date considering the time series of the state variables $\mathbf{X}_t$ and the estimated model parameters. The solid lines refer to an investor that is not money-illusioned ($\alpha = 0$), while the dashed lines refer to a totally money-illusioned investor ($\alpha = 1$). The bottom graphs show the ex-ante annualized certainty equivalent losses, for a strategy starting at the current date, as perceived by an investor with a given degree of money illusion $\alpha$ that is prevented from investing in the inflation-indexed bond. The investment horizon is 30-year long. Panel (a) refers to the model specification with constant risk premia, using the estimates in Table 1a. Panel (b) refers to the model specification with time-varying risk premia constrained in volatility, using the estimates in Table 1b.

**(a)** Constant risk premia | **(b)** Constrained time-varying risk premia

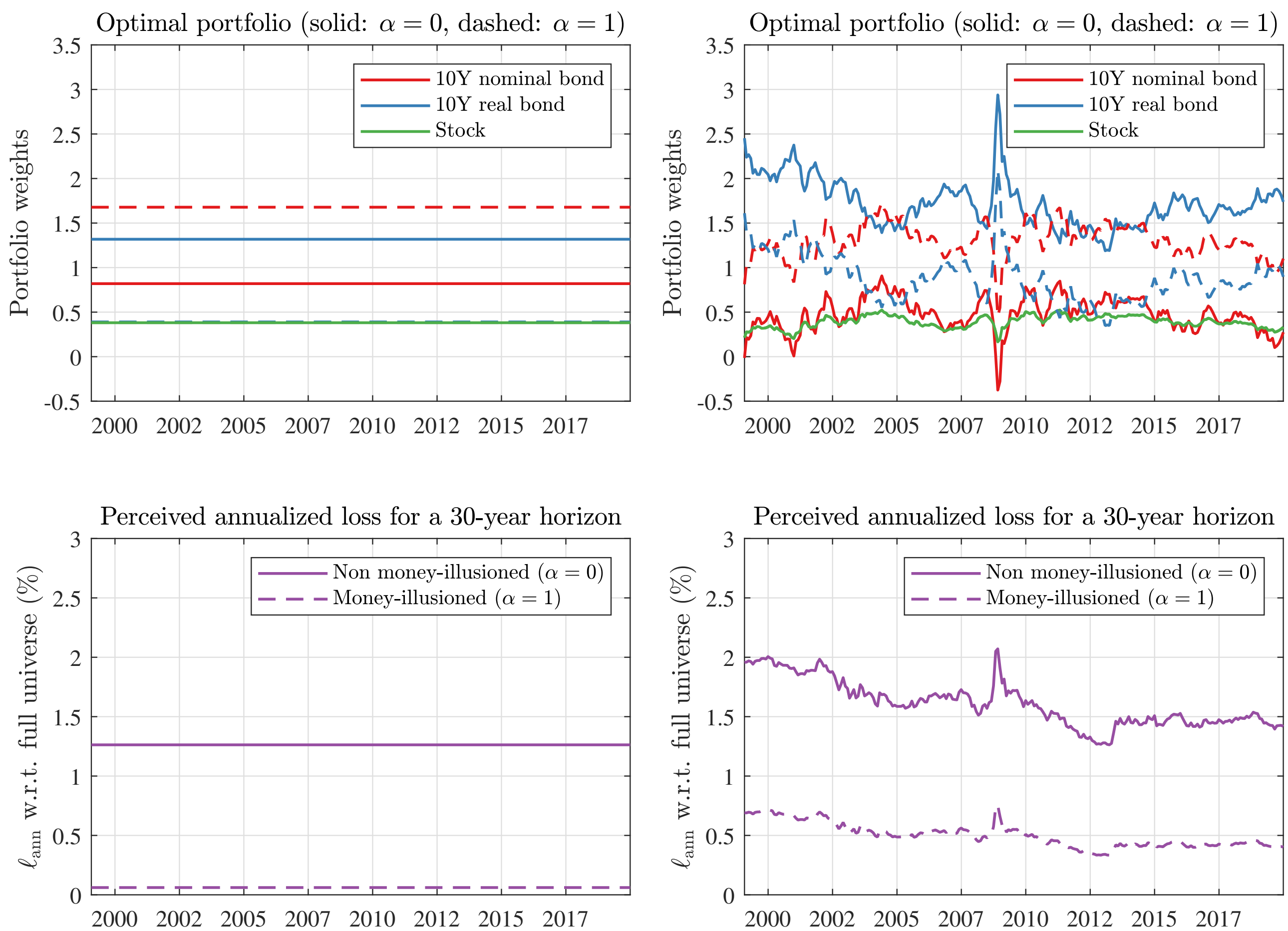


**Figure A.11:** Bond-only analysis. Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$. Utility loss with respect to a non-illusioned investor. Constant risk premia.

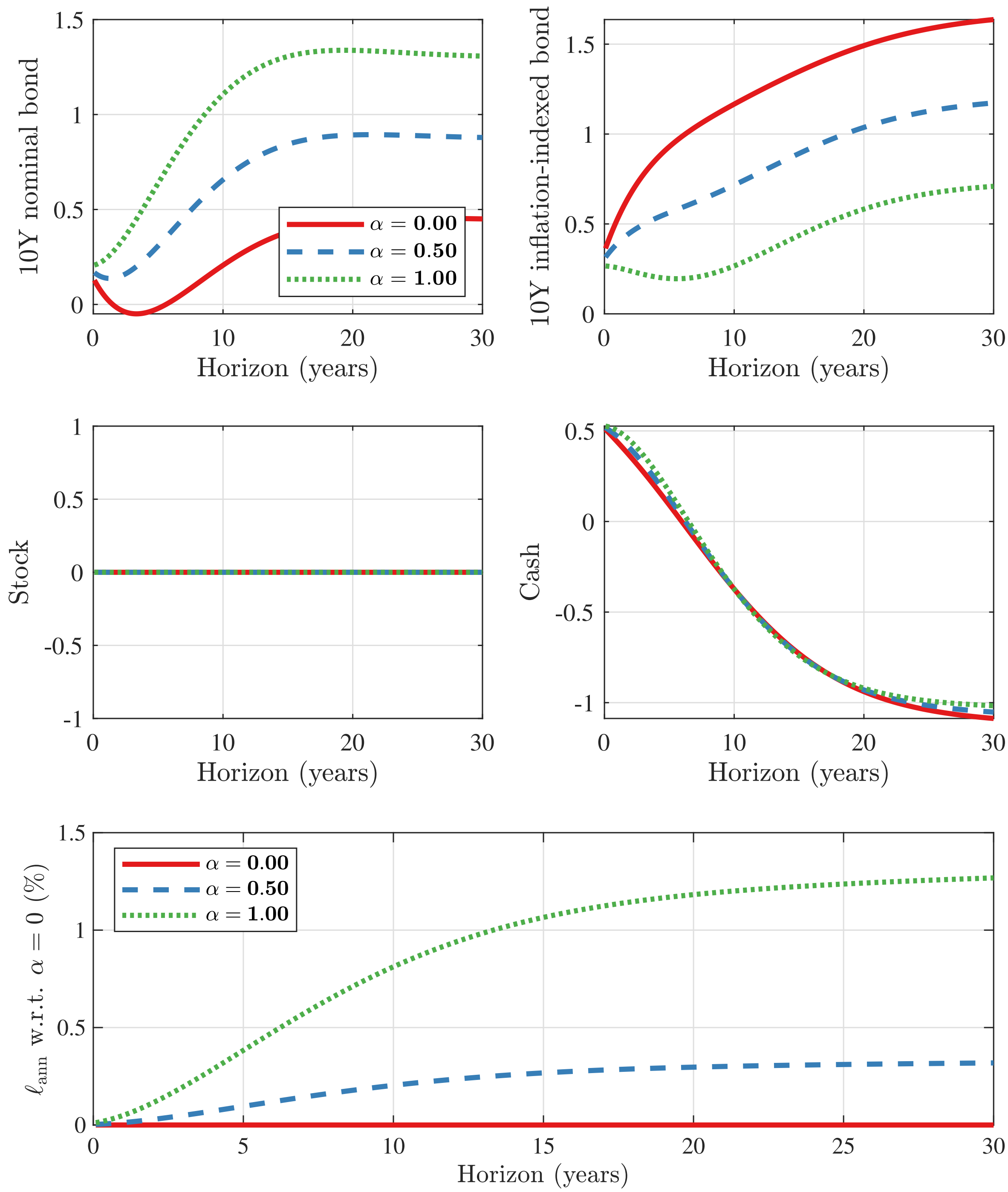


**Figure A.12:** Bond-only analysis. Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$. Utility loss with respect to a non-illusioned investor. Time-varying risk premia.

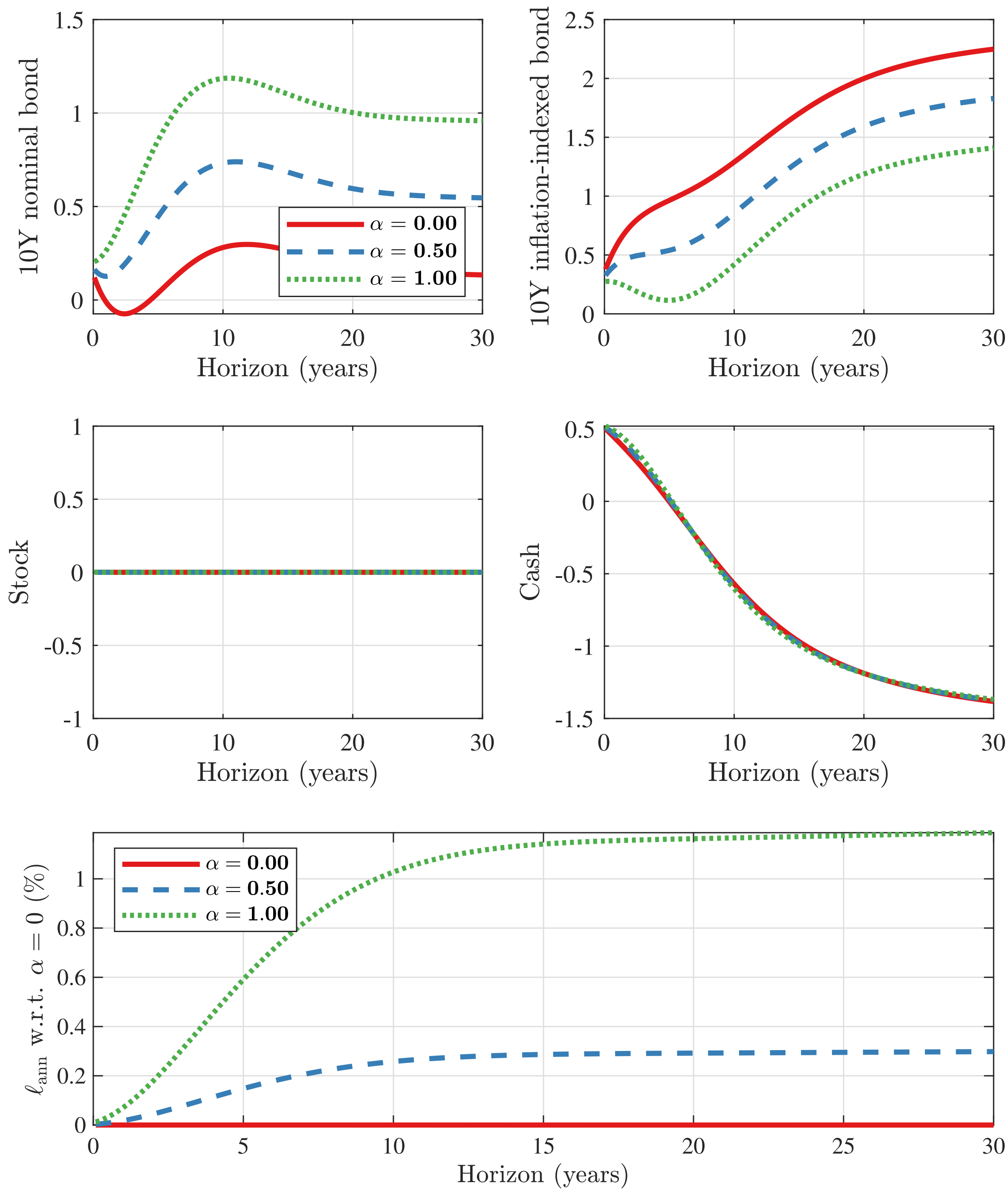


**Figure A.13:** Bond-only analysis. Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$, when the 10-year inflation-indexed bond is excluded from the investable universe. Utility loss with respect to the case where the inflation-indexed bond is available. Constant risk premia.

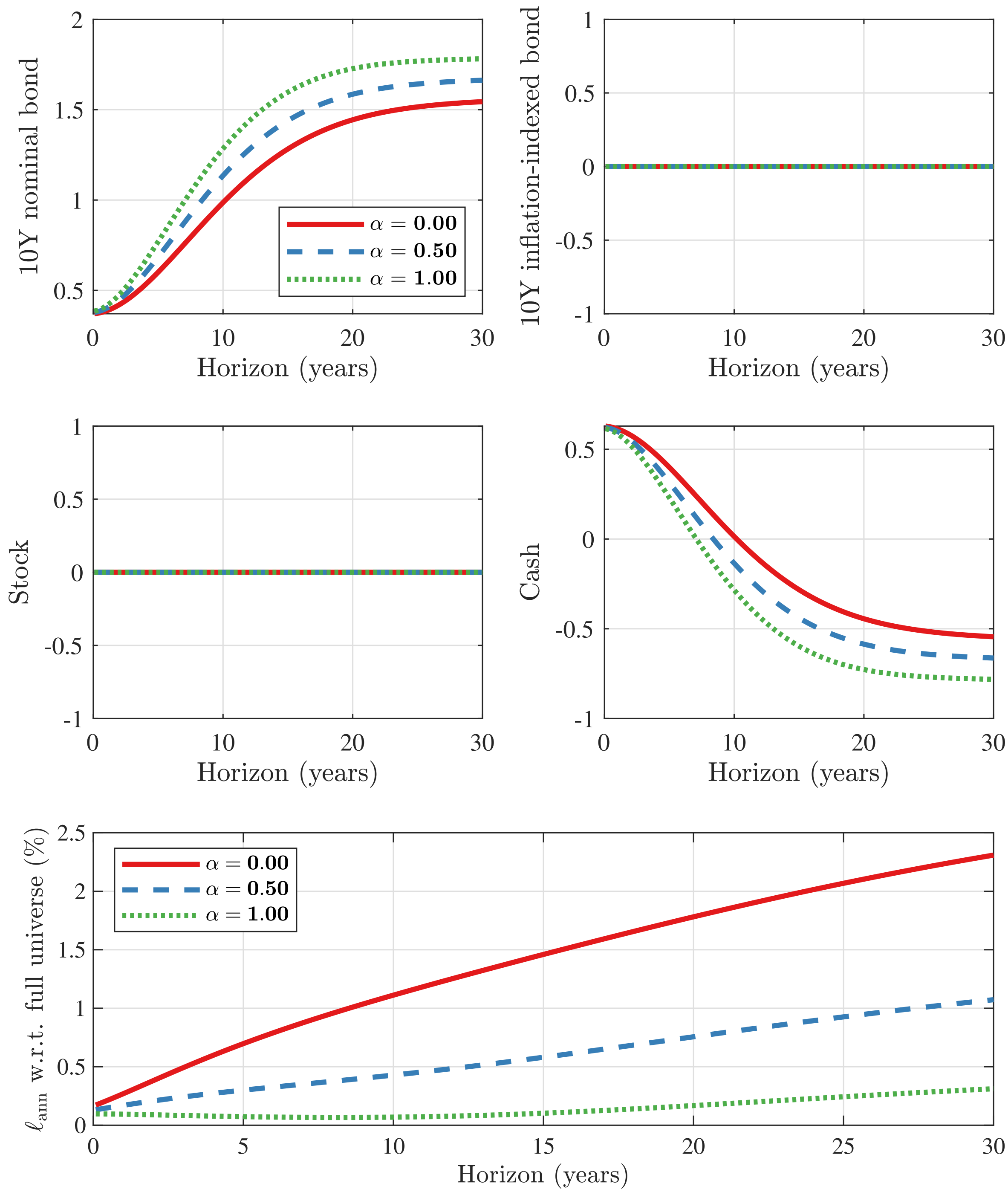


**Figure A.14:** Bond-only analysis. Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$, when the 10-year inflation-indexed bond is excluded from the investable universe. Utility loss with respect to the case where the inflation-indexed bond is available. Time-varying risk premia.

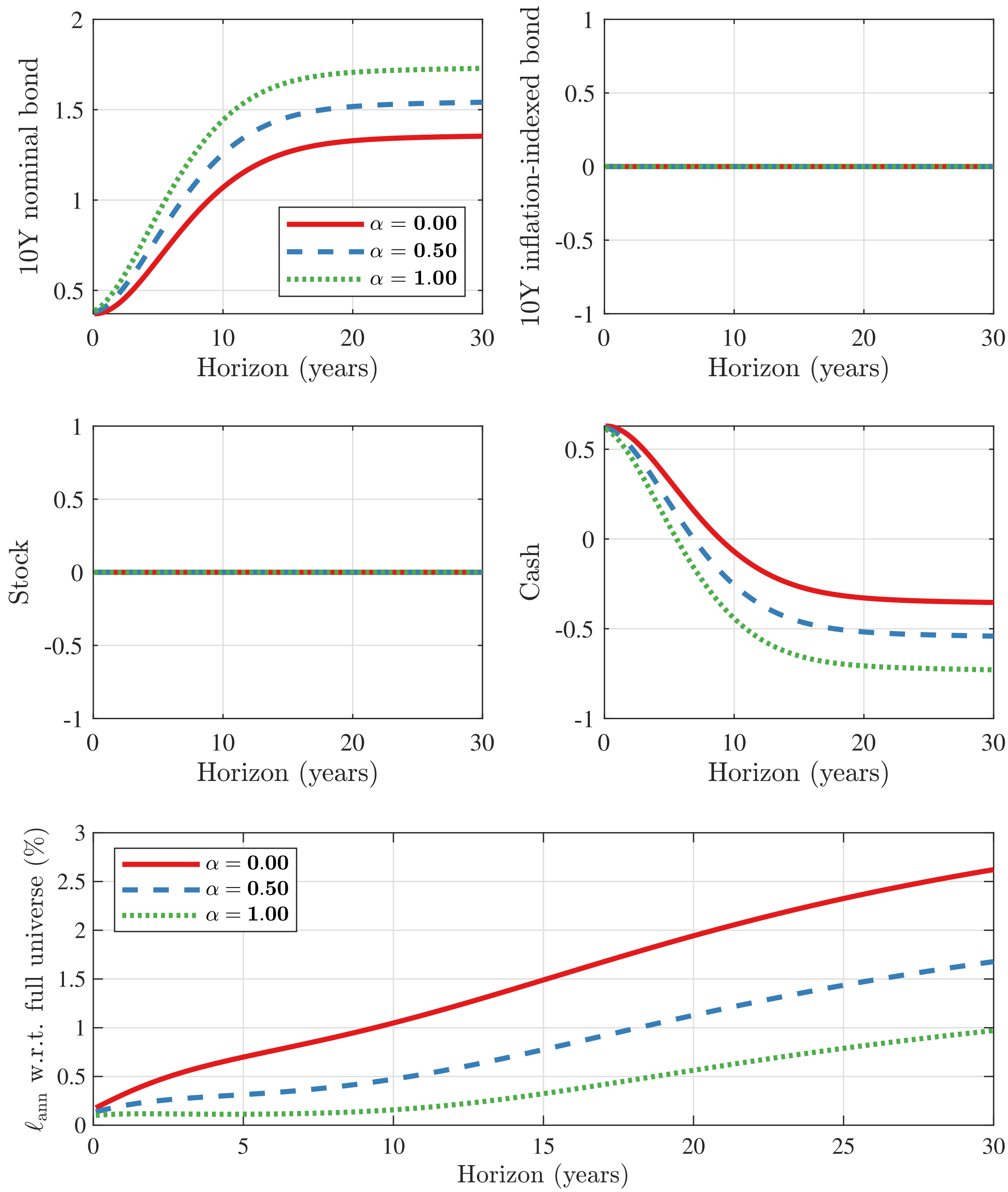


**Figure A.15:** Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$, when the investor can trade 5-year bonds. Utility loss with respect to a non-illusioned investor. Constant risk premia.

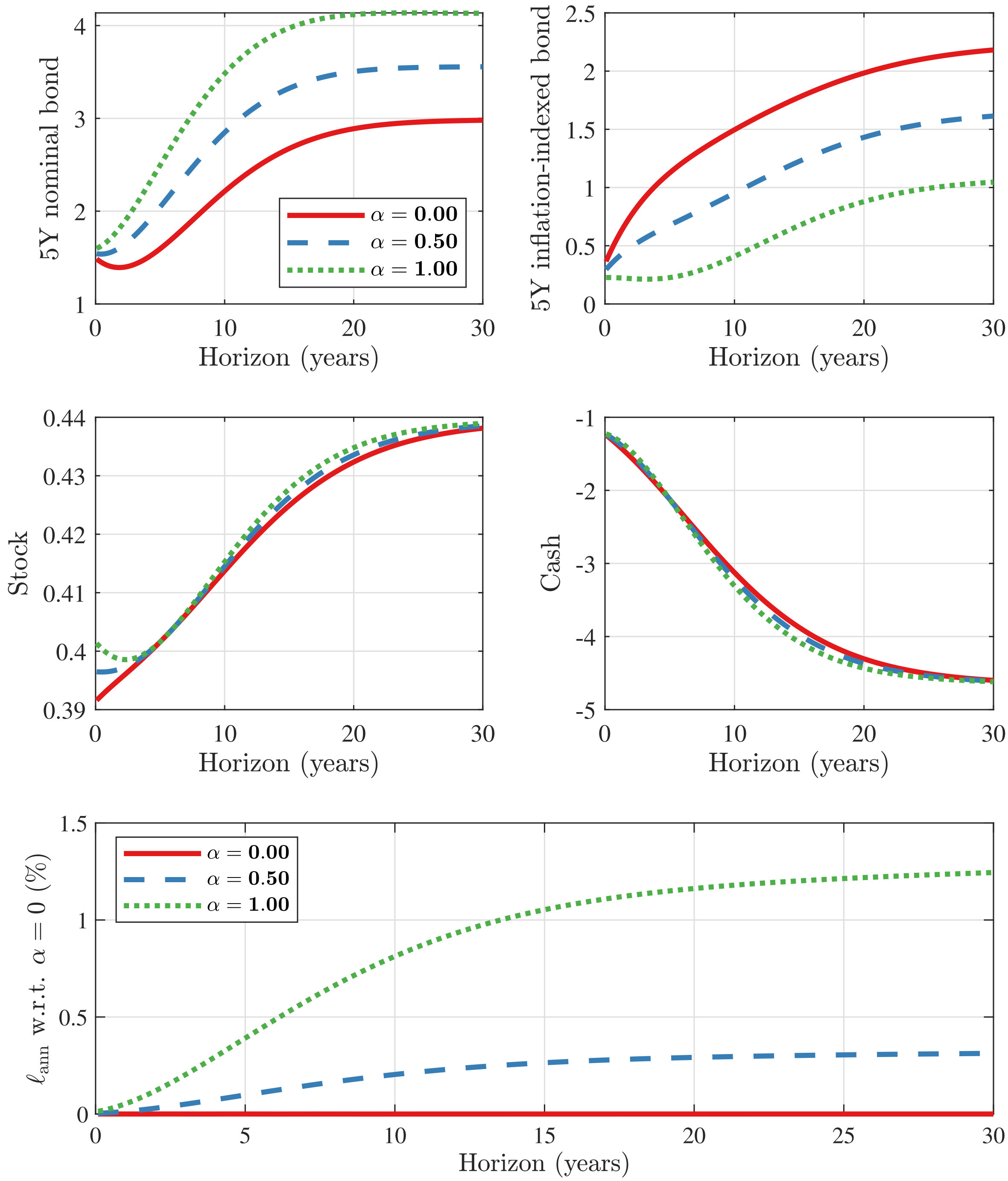


**Figure A.16:** Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$, when the 5-year inflation-indexed bond is excluded from the investable universe. Utility loss with respect to the case where the inflation-indexed bond is available. Constant risk premia.

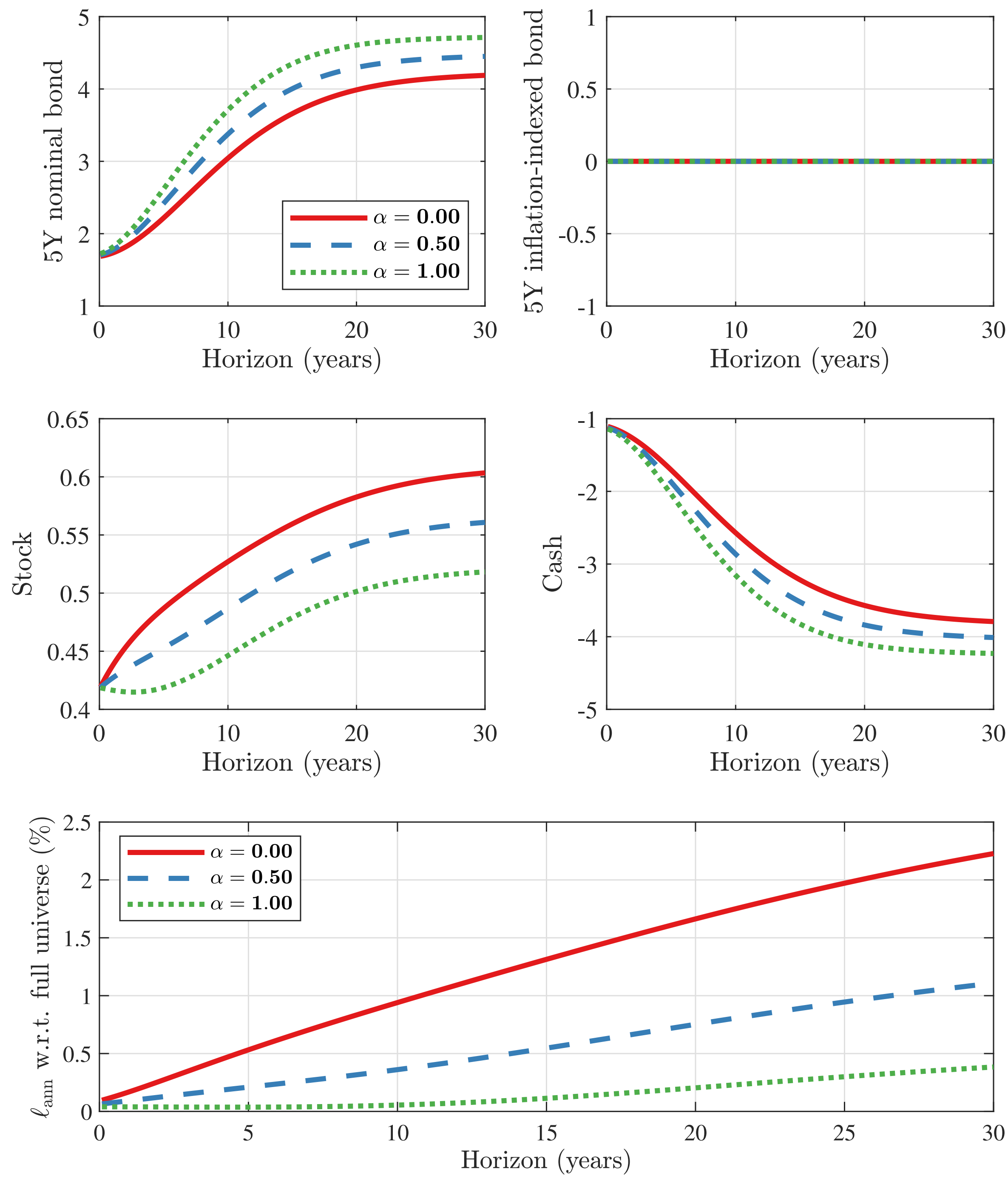


**Figure A.17:** Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$, when the investor can trade 5-year bonds. Utility loss with respect to a non-illusioned investor. Time-varying risk premia (with volatility constraints).

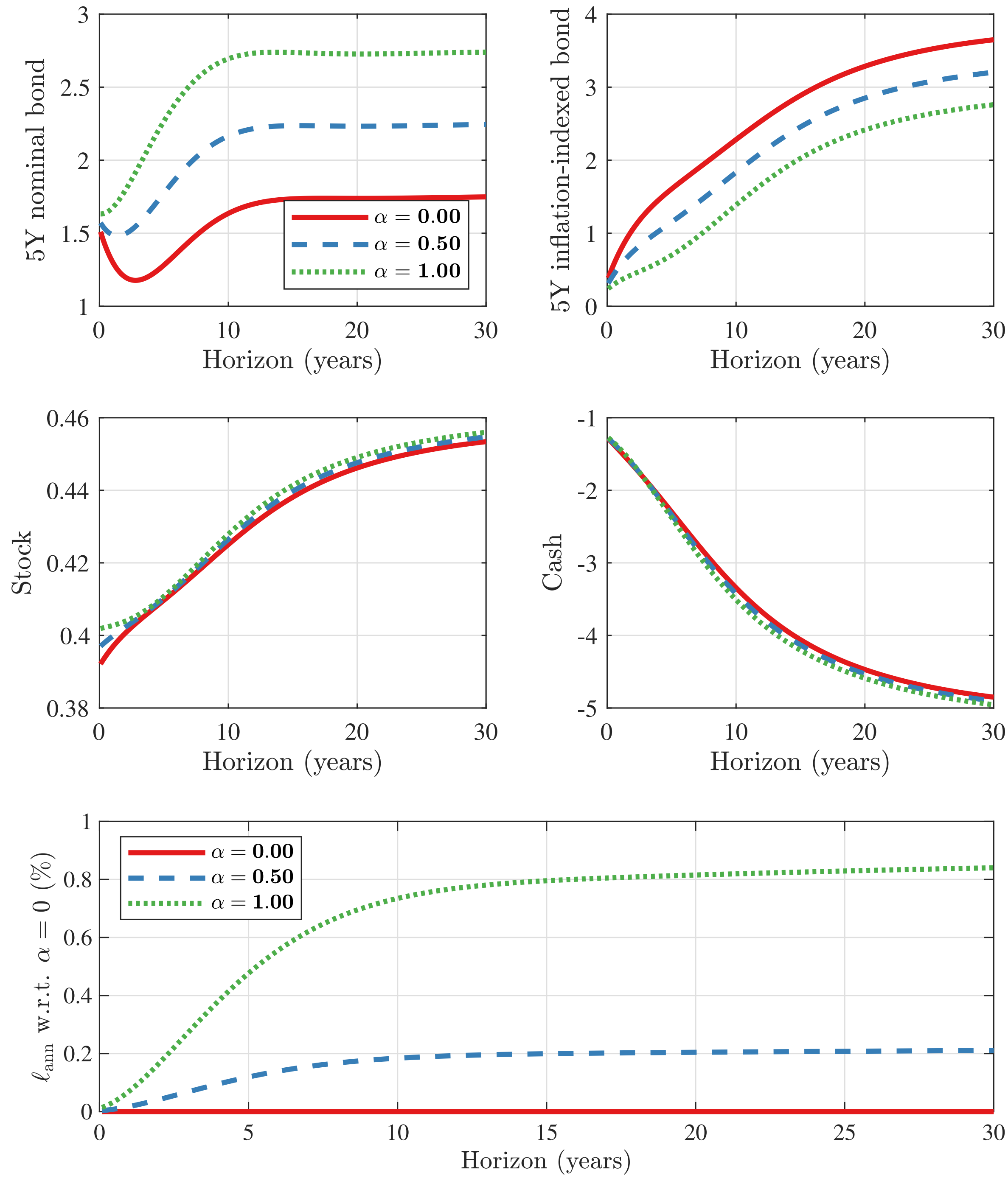


**Figure A.18:** Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$, when the 5-year inflation-indexed bond is excluded from the investable universe. Utility loss with respect to the case where the inflation-indexed bond is available. Time-varying risk premia (with volatility constraints).

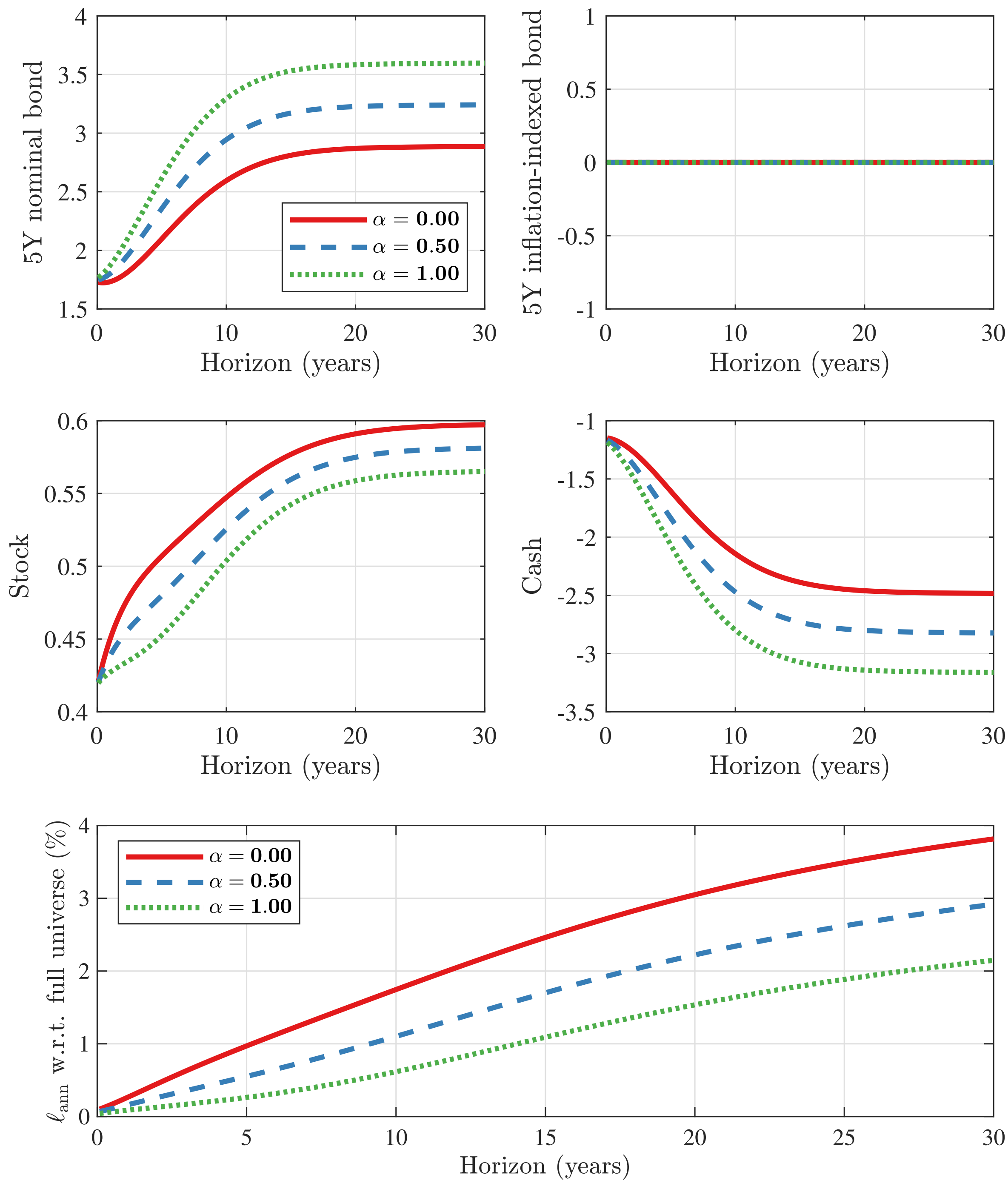


**Figure A.19:** Optimal portfolio strategy for $\gamma = 5$ and different degrees of money illusion $\alpha$. Utility loss with respect to a non-illusioned investor. Constant risk premia.

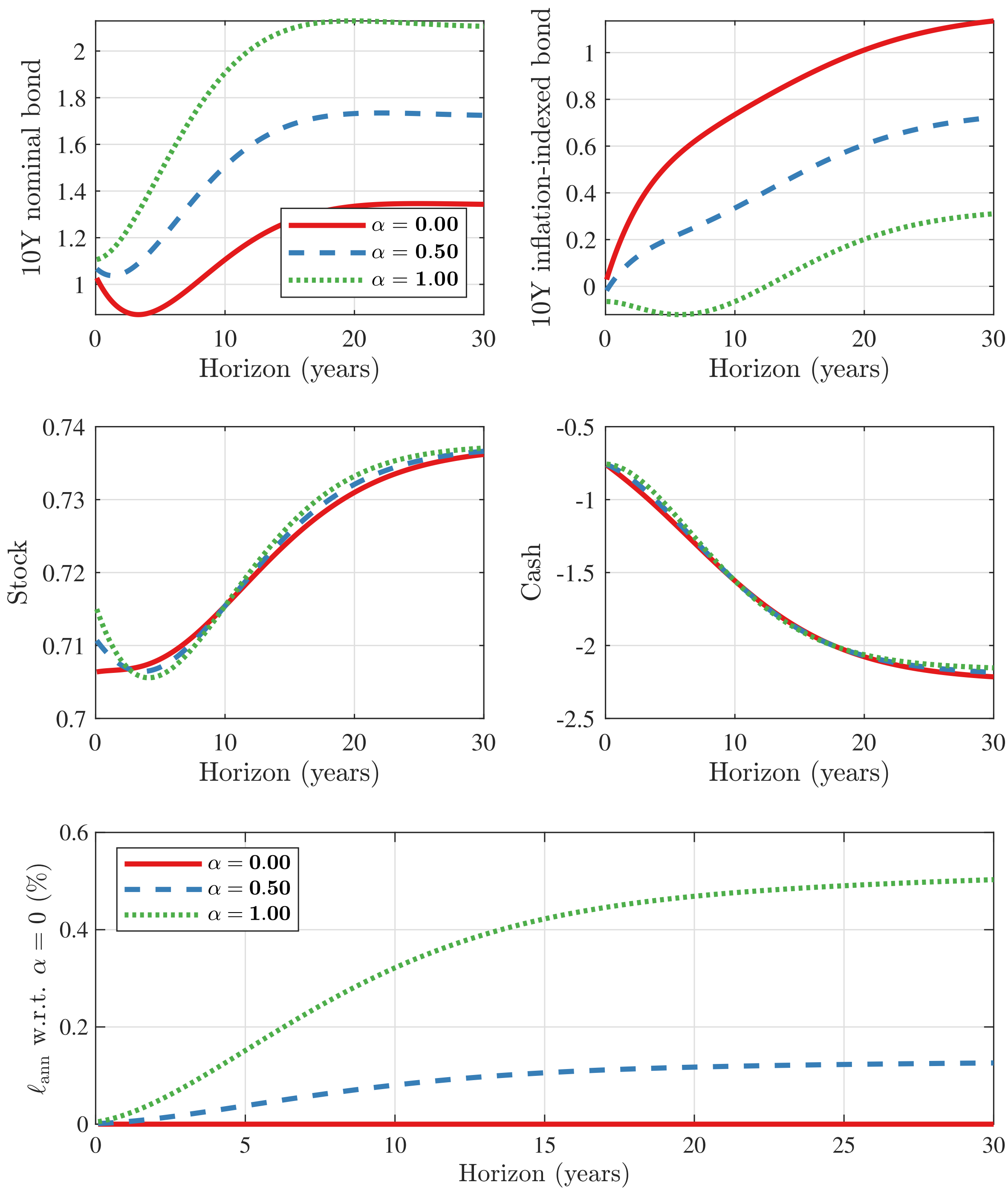


**Figure A.20:** Optimal portfolio strategy for $\gamma = 5$ and different degrees of money illusion $\alpha$, when the 10-year inflation-indexed bond is excluded from the investable universe. Utility loss with respect to the case where the inflation-indexed bond is available. Constant risk premia.

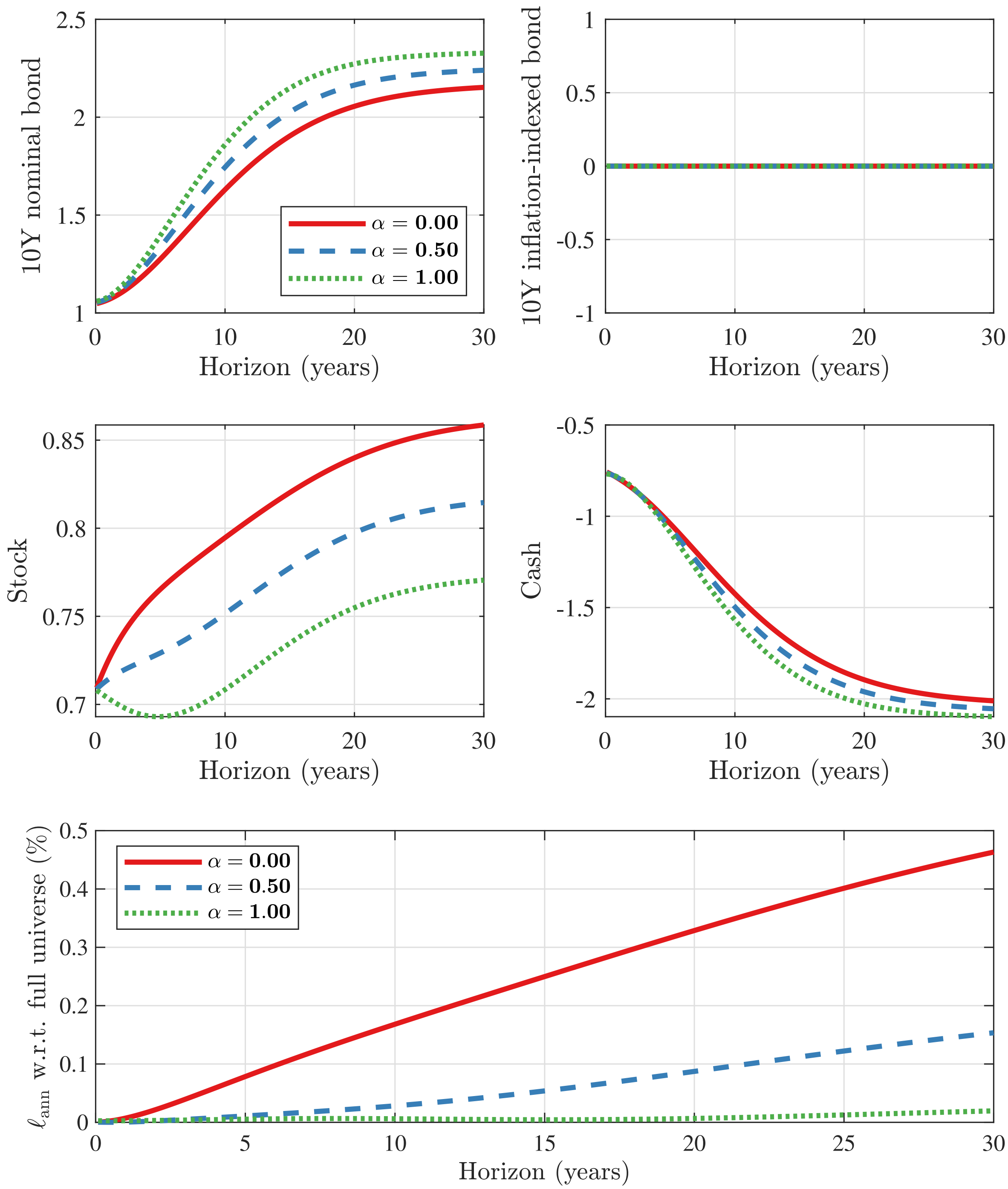


**Figure A.21:** Optimal portfolio strategy for $\gamma = 20$ and different degrees of money illusion $\alpha$. Utility loss with respect to a non-illusioned investor. Constant risk premia.

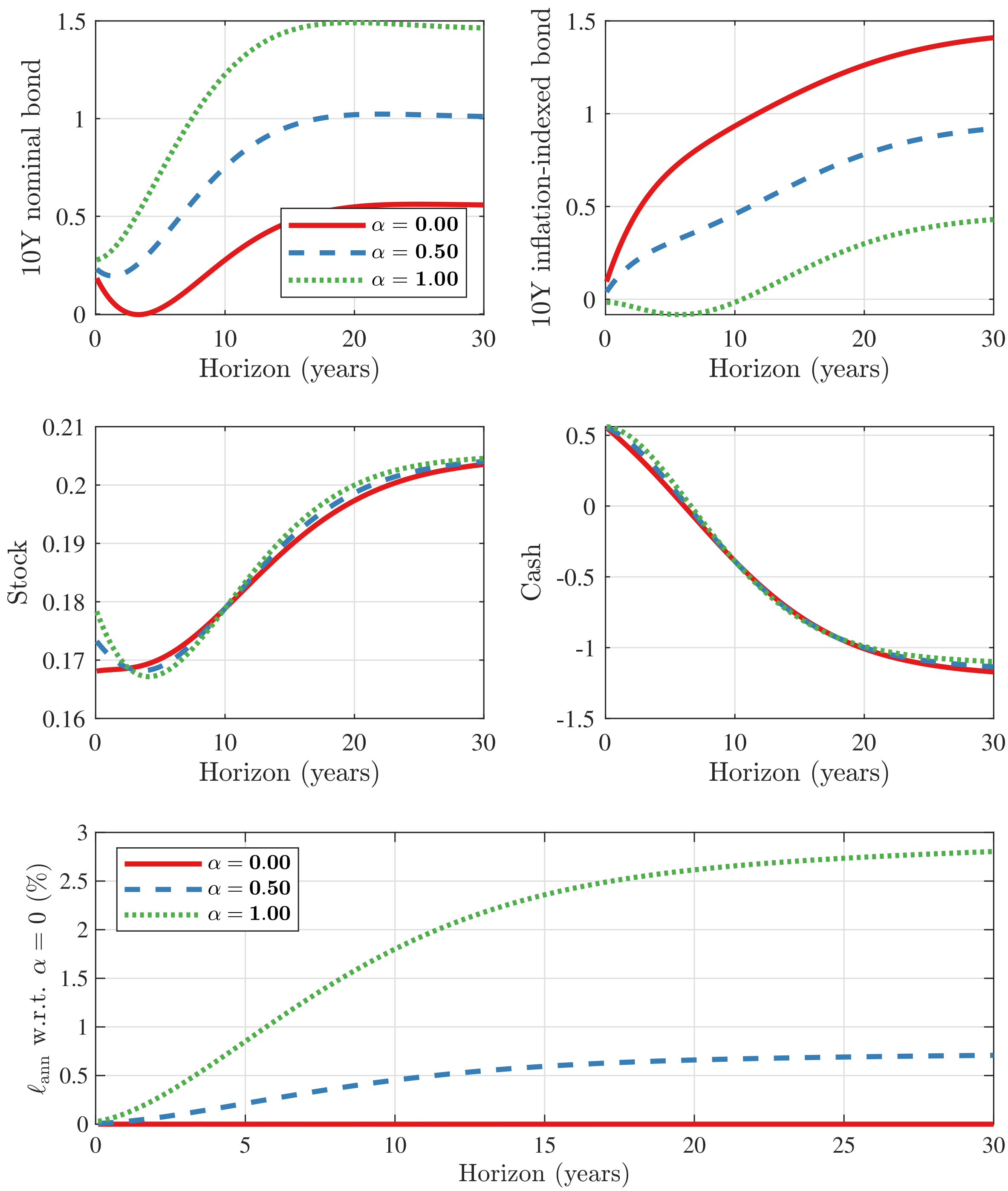


**Figure A.22:** Optimal portfolio strategy for $\gamma = 20$ and different degrees of money illusion $\alpha$, when the 10-year inflation-indexed bond is excluded from the investable universe. Utility loss with respect to the case where the inflation-indexed bond is available. Constant risk premia.

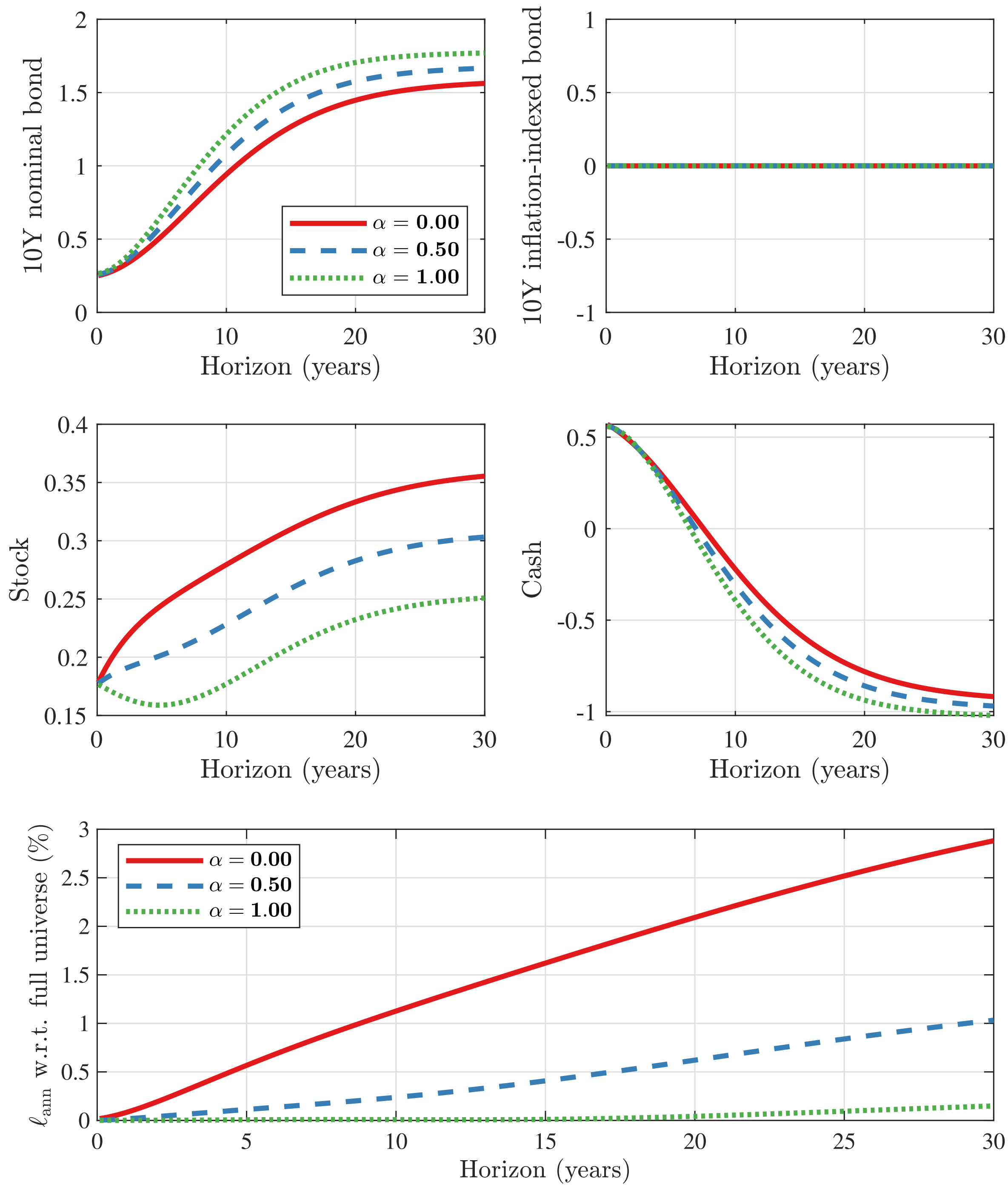


**Figure A.23:** Optimal portfolio strategy for $\gamma = 5$ and different degrees of money illusion $\alpha$. Utility loss with respect to a non-illusioned investor. Time-varying risk premia (with volatility constraints).

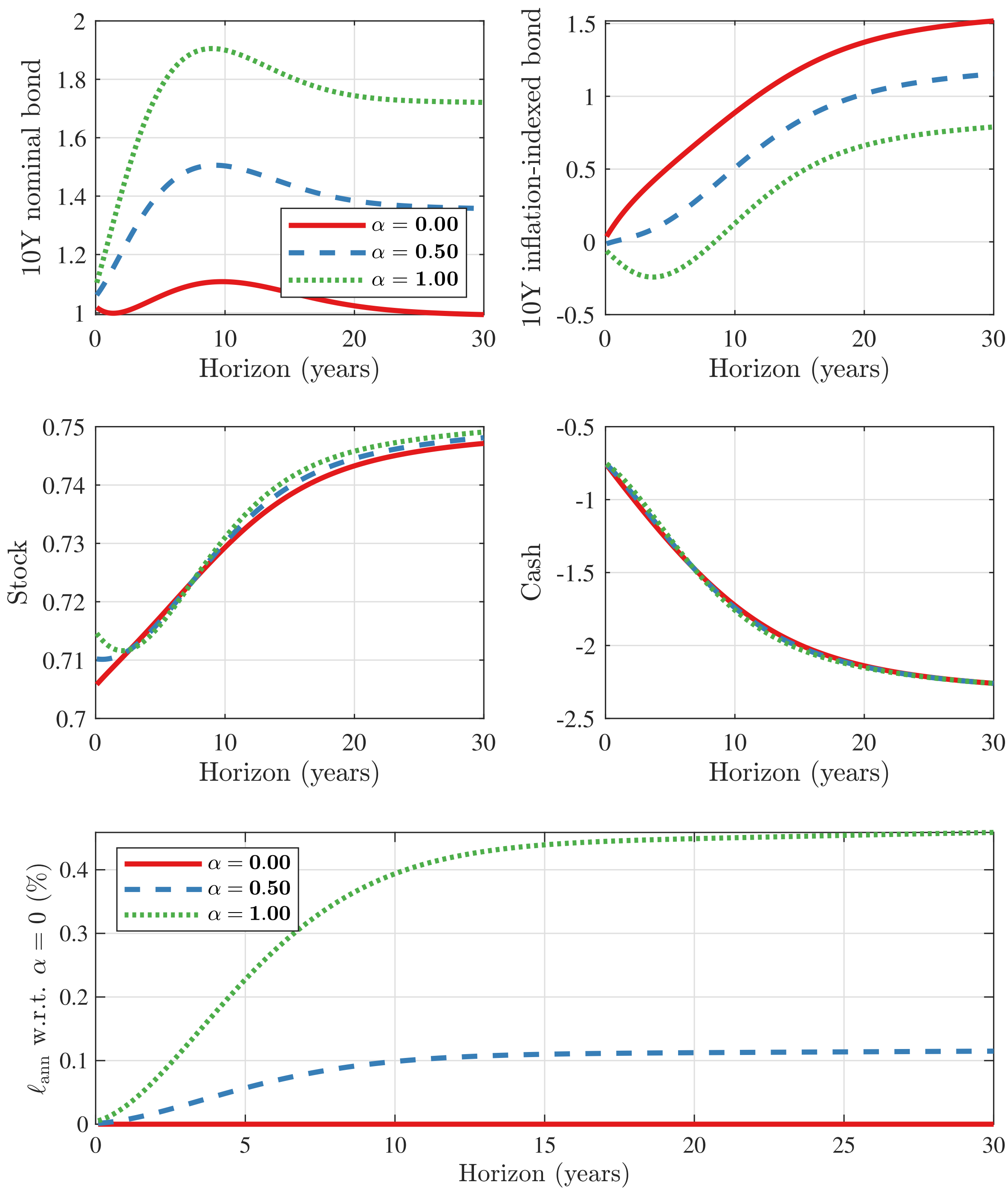


**Figure A.24:** Optimal portfolio strategy for $\gamma = 5$ and different degrees of money illusion $\alpha$, when the 10-year inflation-indexed bond is excluded from the investable universe. Utility loss with respect to the case where the inflation-indexed bond is available. Time-varying risk premia (with volatility constraints).

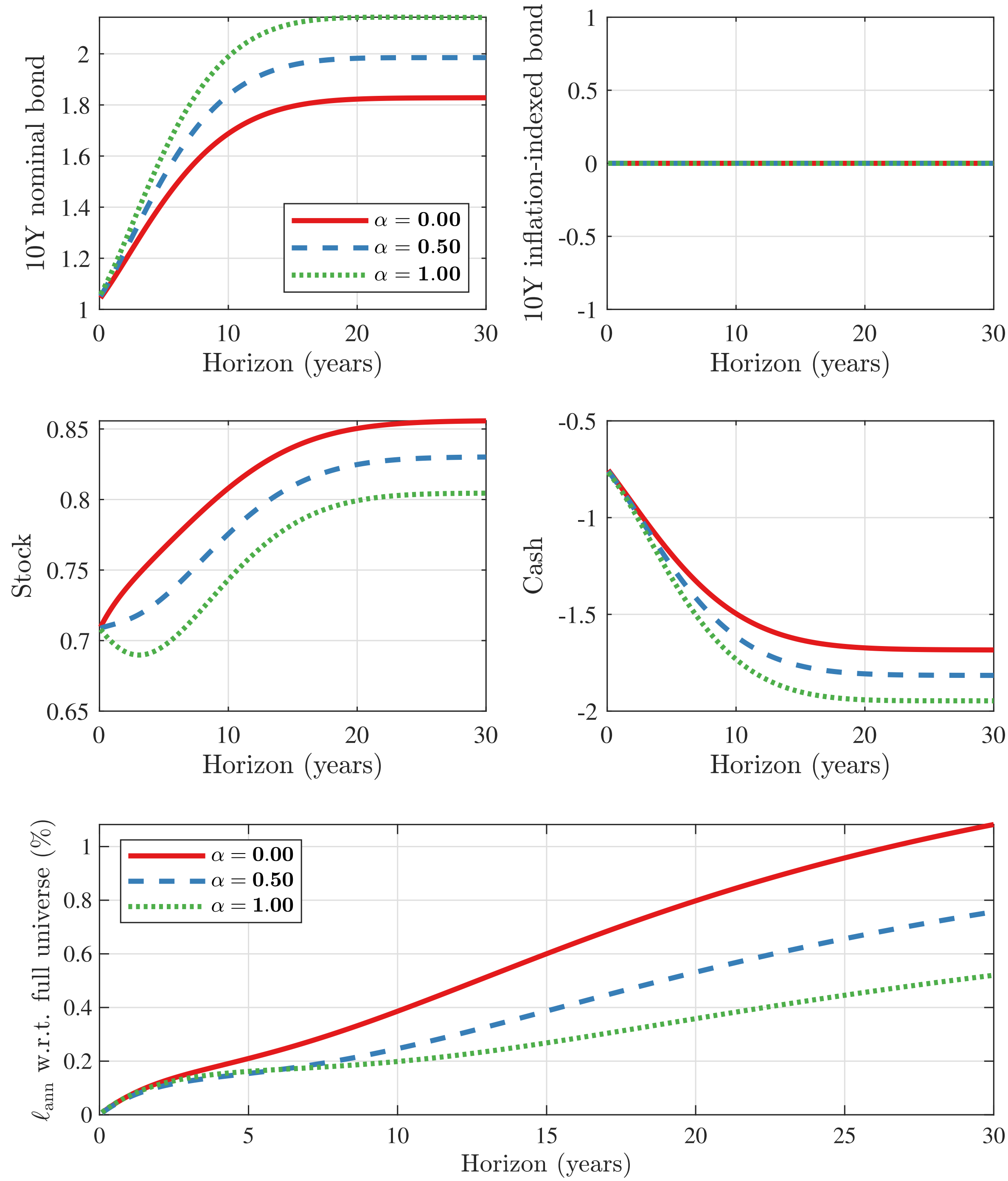


**Figure A.25:** Optimal portfolio strategy for $\gamma = 20$ and different degrees of money illusion $\alpha$. Utility loss with respect to a non-illusioned investor. Time-varying risk premia (with volatility constraints).

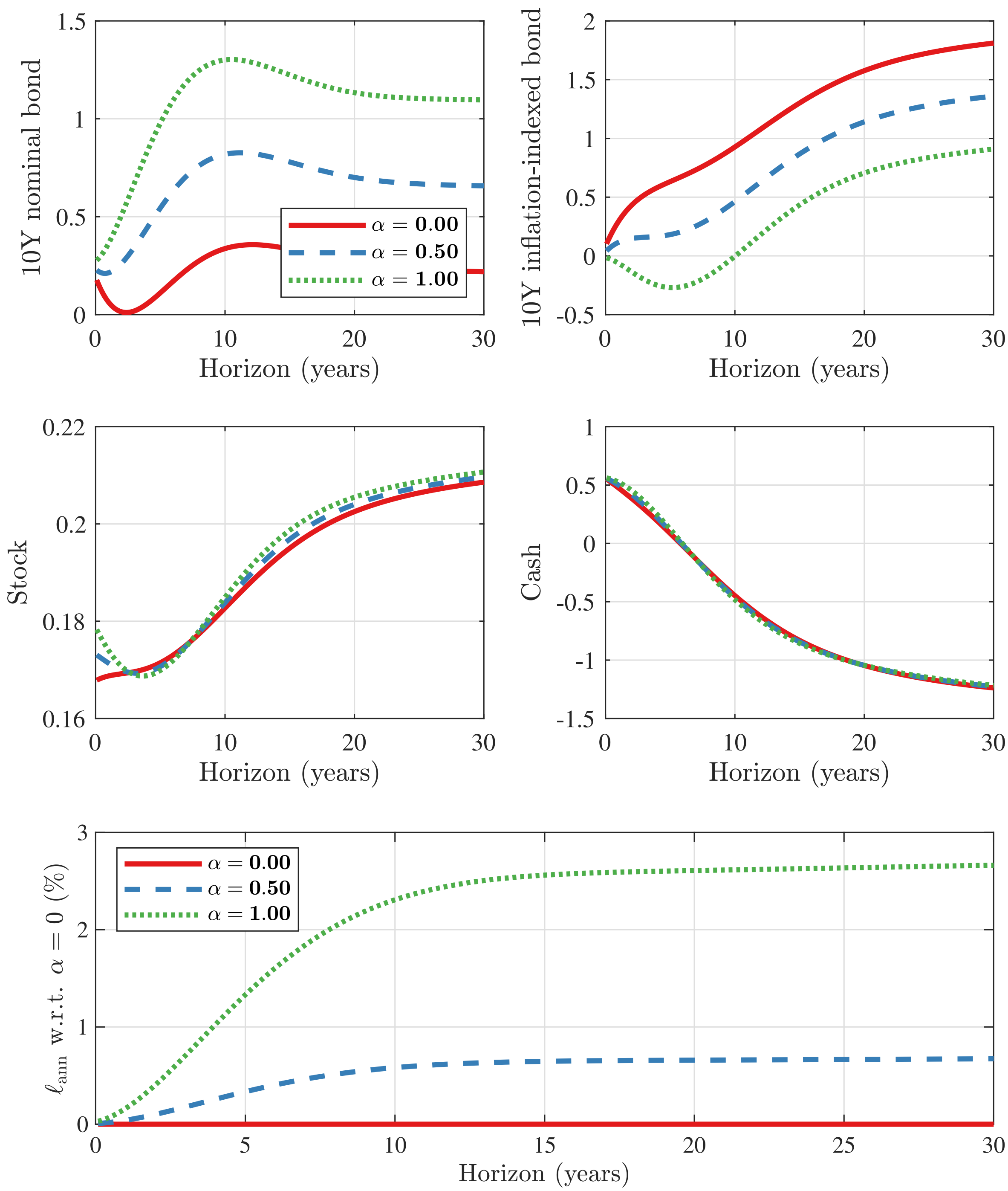


**Figure A.26:** Optimal portfolio strategy for $\gamma = 20$ and different degrees of money illusion $\alpha$, when the 10-year inflation-indexed bond is excluded from the investable universe. Utility loss with respect to the case where the inflation-indexed bond is available. Time-varying risk premia (with volatility constraints).

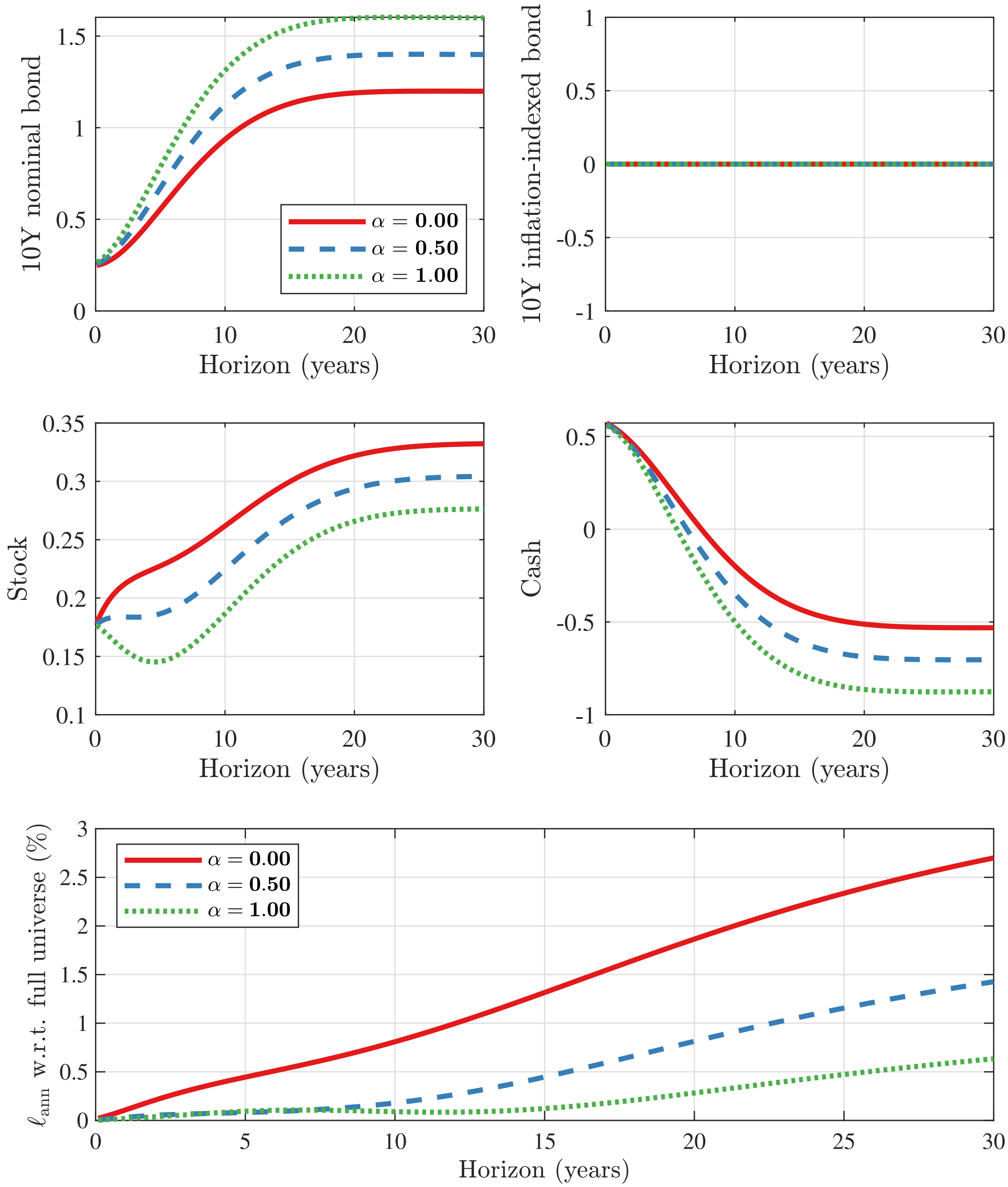


**Figure A.27:** Baseline model estimated observing also survey-based inflation expectations. Constant risk premia. Time series of relevant model-implied and observed variables.

The top graphs show the model-implied instantaneous nominal rate, real rate and break-even inflation. The second graphs from the top show the model-implied 10-year nominal yield, real yield, break-even inflation, expected inflation and the 10-year CPI inflation rate forecast from the Survey of Professional Forecasters. The third graphs from the top show the model-implied risk premia of the risky assets and of the unexpected inflation. The bottom graphs show the portfolio allocation of a nominal mean-variance investor ($\gamma = 10$) and the corresponding maximum ex-ante Sharpe ratio (considering a 10-year nominal bond, a 10-year inflation-indexed bond and the stock index).

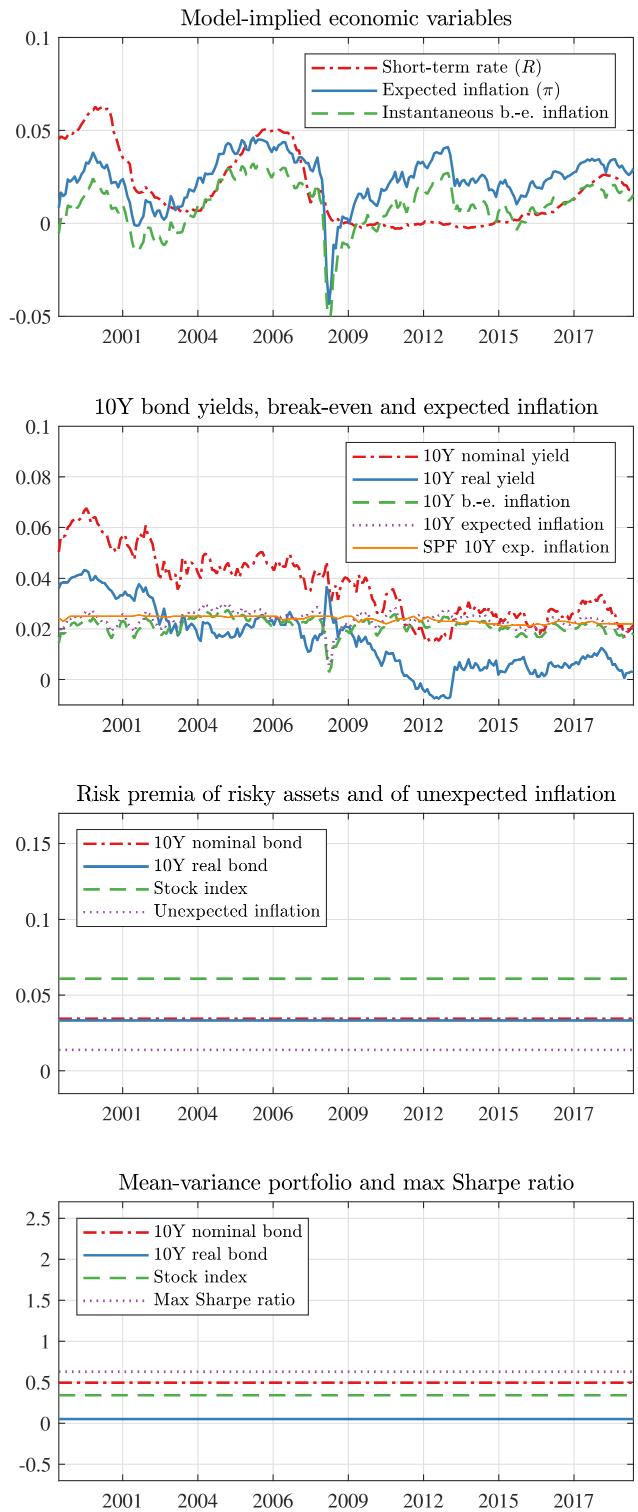


**Figure A.28:** Baseline model estimated observing also survey-based inflation expectations. Constant risk premia. Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$. Utility loss with respect to a non-illusioned investor.

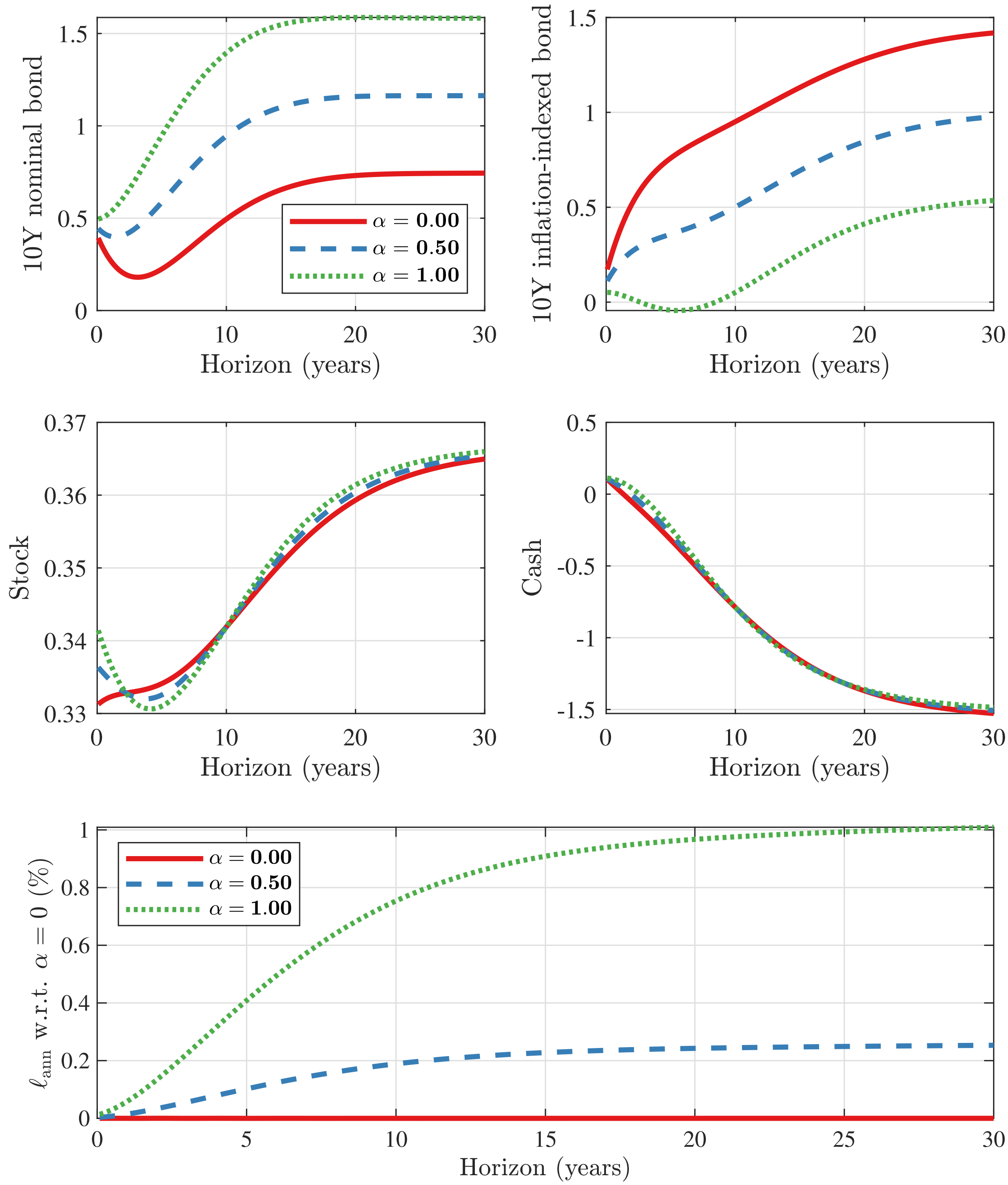


**Figure A.29:** Baseline model estimated observing also survey-based inflation expectations. Constant risk premia. Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$, when the 10-year inflation-indexed bond is excluded from the investable universe. Utility loss with respect to the case where the inflation-indexed bond is available.

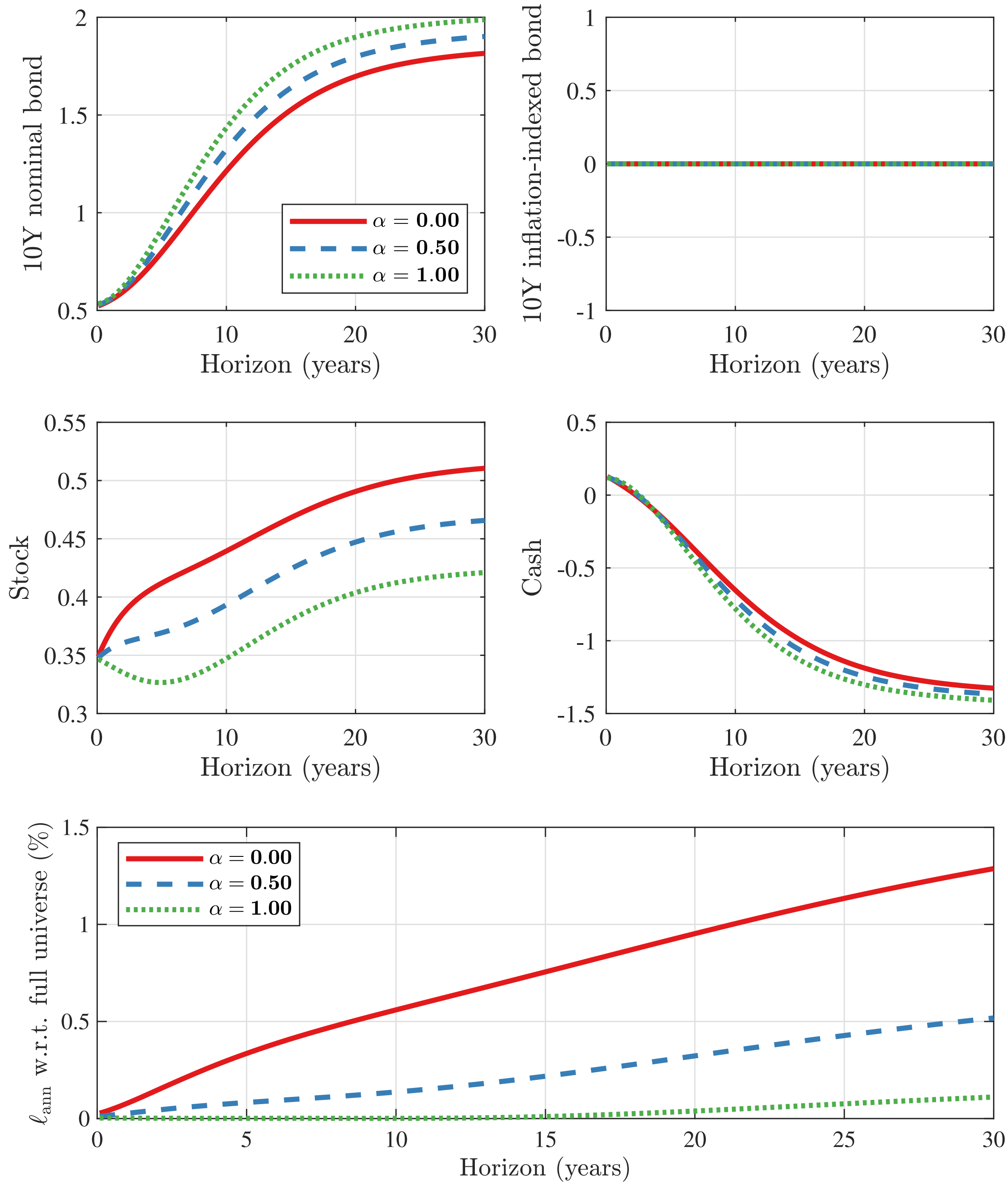


**Figure A.30:** 4-factor affine no-arbitrage Nelson-Siegel term structure specification with constant risk premia. Time series of relevant model-implied and observed variables.

The top graphs show the model-implied instantaneous nominal rate, real rate and break-even inflation. The second graphs from the top show the model-implied 10-year nominal yield, real yield, break-even inflation, expected inflation and the 10-year CPI inflation rate forecast from the Survey of Professional Forecasters. The third graphs from the top show the model-implied risk premia of the risky assets and of the unexpected inflation. The bottom graphs show the portfolio allocation of a nominal mean-variance investor ($\gamma = 10$) and the corresponding maximum ex-ante Sharpe ratio (considering a 10-year nominal bond, a 10-year inflation-indexed bond and the stock index).

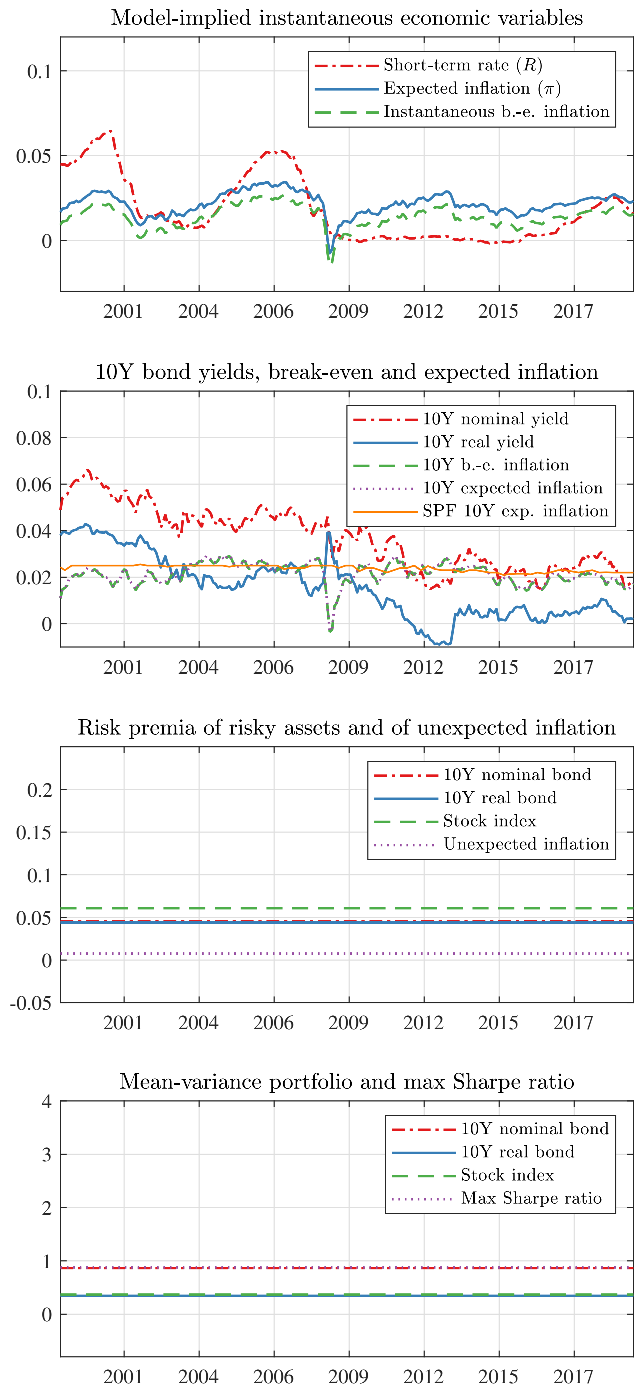


**Figure A.31:** 4-factor affine no-arbitrage Nelson-Siegel term structure specification with constant risk premia. Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$. Utility loss with respect to a non-illusioned investor.

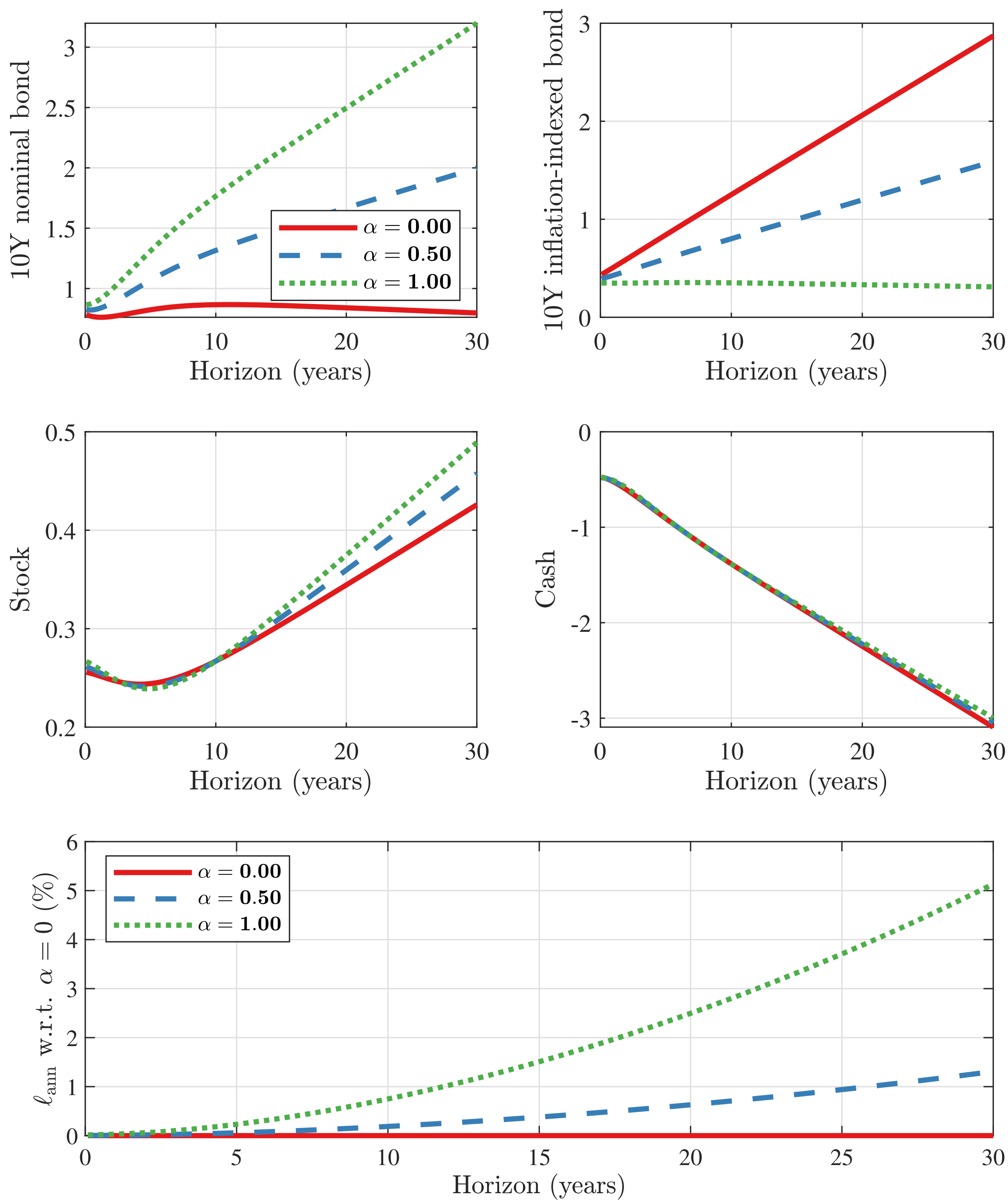


**Figure A.32:** 4-factor affine no-arbitrage Nelson-Siegel term structure specification with constant risk premia. Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$, when the 10-year inflation-indexed bond is excluded from the investable universe. Utility loss with respect to the case where the inflation-indexed bond is available.

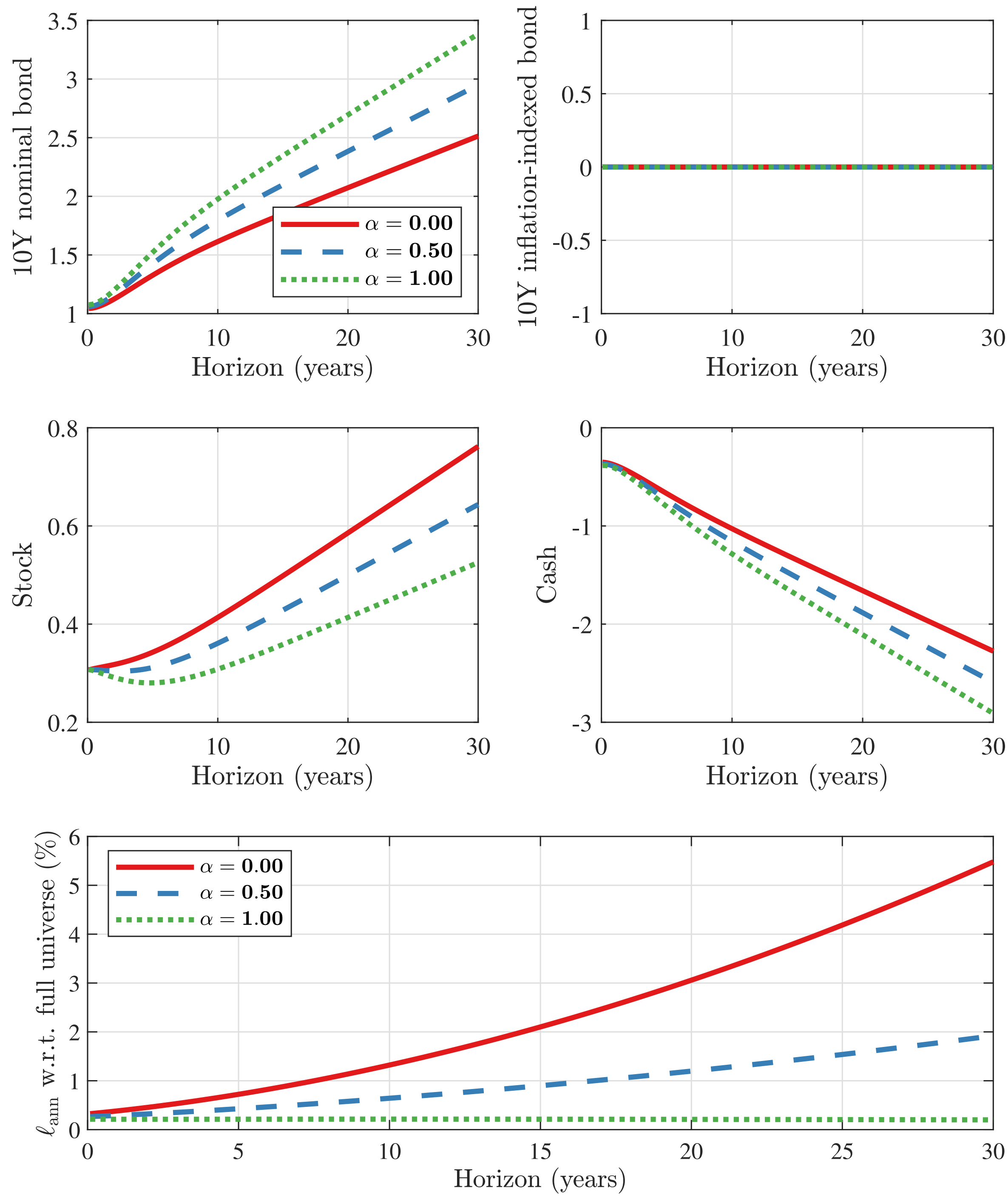


**Figure A.33:** UK-based analysis. Time series of relevant model-implied and observed variables.

The top graphs show the model-implied instantaneous nominal rate, real rate and break-even inflation. The second graphs from the top show the model-implied 10-year nominal yield, real yield, break-even inflation and expected inflation. The third graphs from the top show the model-implied risk premia of the risky assets and of the unexpected inflation. The bottom graphs show the portfolio allocation of a nominal mean-variance investor ($\gamma = 10$) and the corresponding maximum ex-ante Sharpe ratio (considering a 10-year nominal bond, a 10-year inflation-indexed bond and the stock index).

**(a)** Constant risk premia | **(b)** Constrained time-varying risk premia

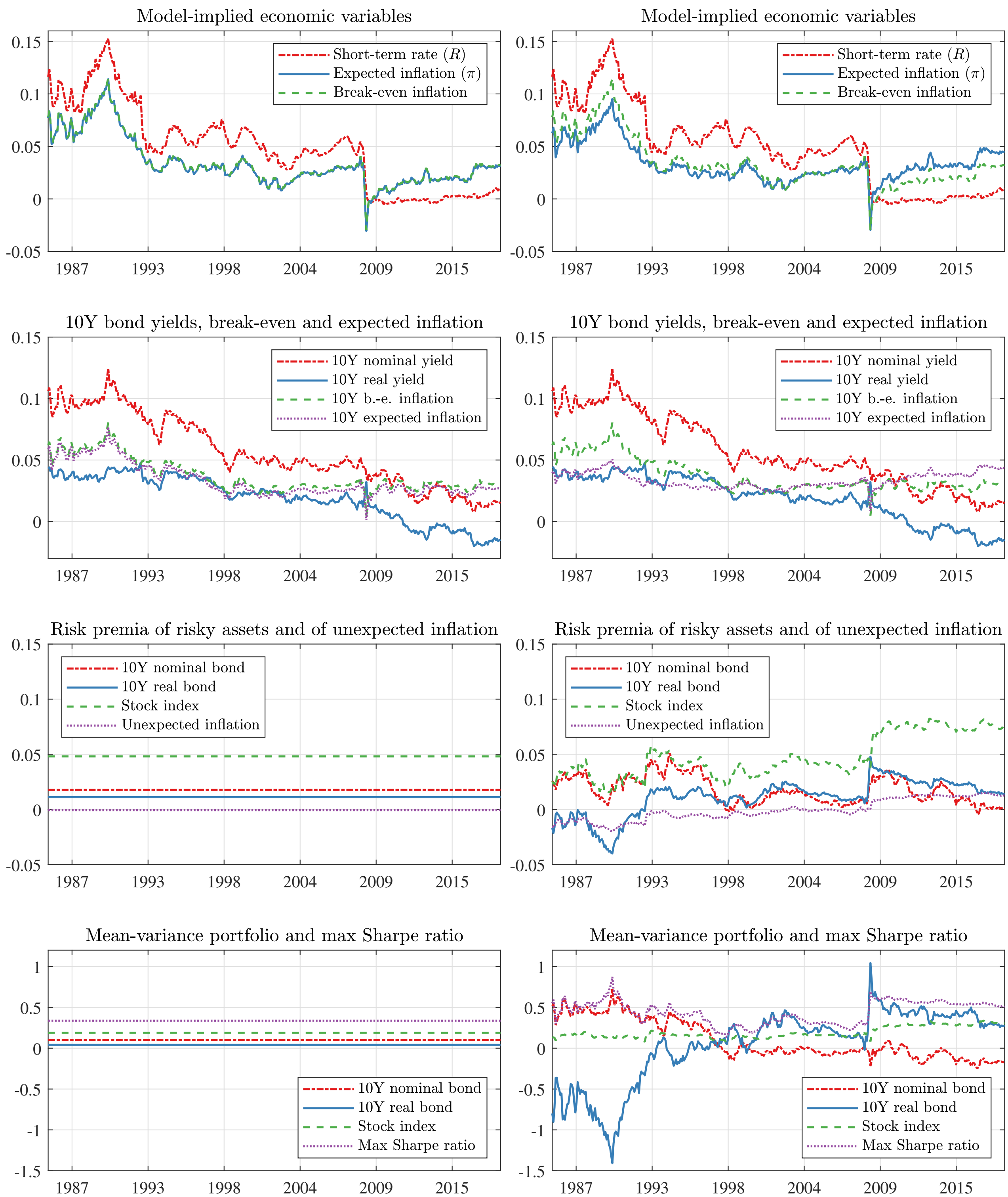


**Figure A.34:** UK-based analysis. Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$. Utility loss with respect to a non-illusioned investor. Constant risk premia.

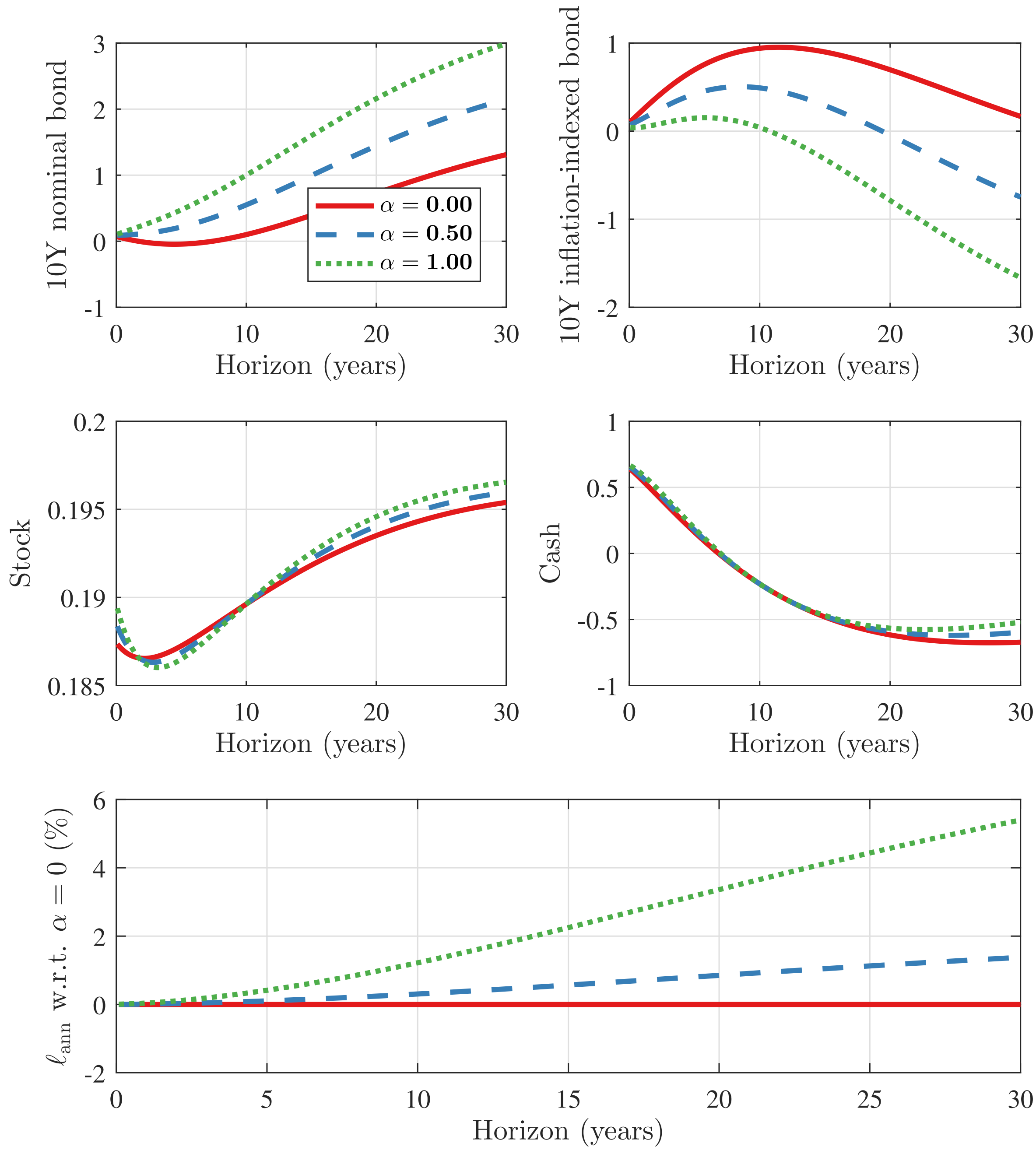


**Figure A.35:** UK-based analysis. Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$. Utility loss with respect to a non-illusioned investor. Time-varying risk premia.

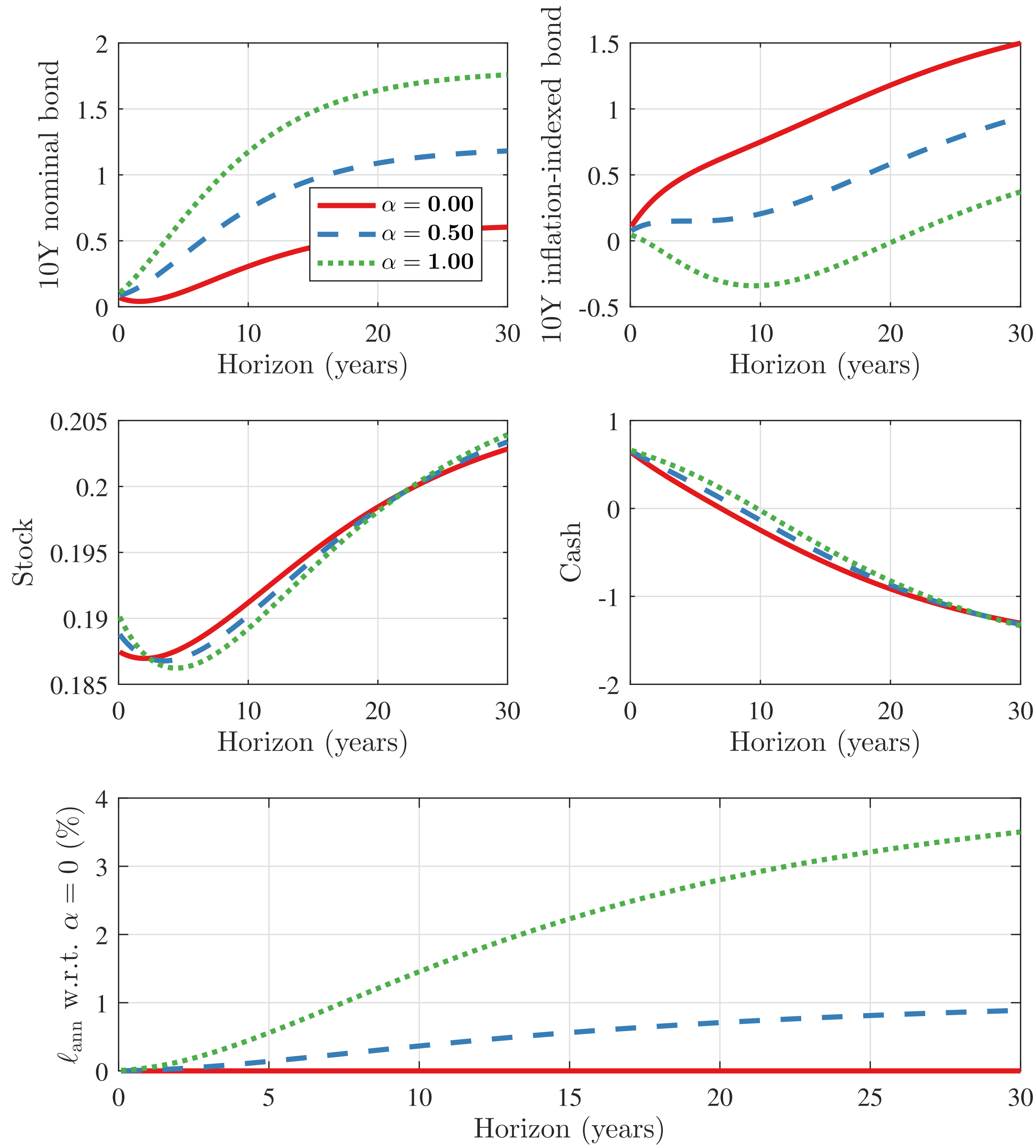


**Figure A.36:** UK-based analysis. Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$, when the 10-year inflation-indexed bond is excluded from the investable universe. Utility loss with respect to the case where the inflation-indexed bond is available. Constant risk premia.

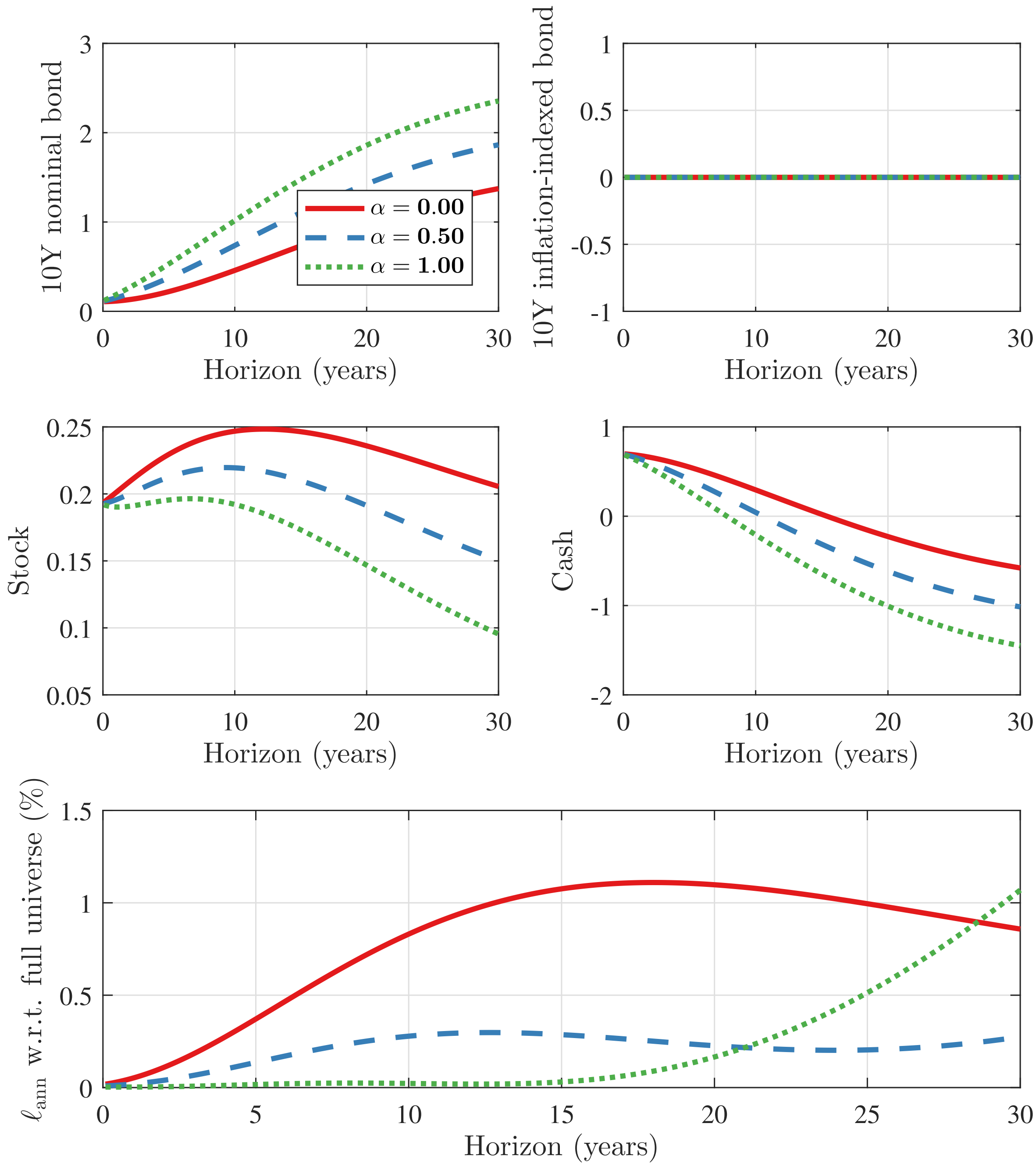


**Figure A.37:** UK-based analysis. Optimal portfolio strategy for $\gamma = 10$ and different degrees of money illusion $\alpha$, when the 10-year inflation-indexed bond is excluded from the investable universe. Utility loss with respect to the case where the inflation-indexed bond is available. Time-varying risk premia.