On the relationship between comparisons of risk aversion of different orders

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Abstract

We show conditions which ensure that the comparisons between risk aversion of different orders of two decision makers are related. In particular, we derive a condition ensuring that greater downside risk aversion implies greater risk aversion and a different condition ensuring that the opposite implication holds. We then generalize these results to higher order greater risk aversion, obtaining conditions which make it possible to infer the direction of the comparison for risk aversion of a given order from the knowledge of the direction for a different order.

Keywords: greater risk aversion, greater downside risk aversion, strongly greater downside risk aversion, greater nth degree risk aversion, strongly greater nth degree risk aversion, risk changes, comparison of risk aversion.

1 Introduction

In his seminal paper, Pratt (1964) formalizes the relationship between aversion to risk and the concavity of the utility function. He also introduces a comparison between preferences in terms of local risk aversion and risk premium and shows that one decision maker has *greater risk aversion* than another when the utility function of the first is a concave transformation of the utility function of the second (i.e. the ratio of the marginal utilities of the two decision makers is a decreasing function). Rothschild and Stiglitz (1970) deepen the analysis of risk aversion and identify the concavity of the utility function with the aversion toward mean-preserving spreads in the distribution of random variables. Since that paper, this kind of change in the distribution of wealth has usually been called an *increase in risk*. Diamond and Stiglitz (1974) adapt this notion to the comparison between preferences, reinterpreting greater risk aversion of one decision maker compared to that of another as greater aversion to increases in risk.

Menezes et al. (1980) extend Rothschild and Stiglitz's approach to define *downside risk aversion* as the aversion to changes in risks that shift the distribution of wealth towards the lower tail, preserving its mean and variance. This kind of change in the distribution is called an *increase in downside risk*. Menezes et al. (1980) show that an agent exhibits downside risk aversion if and only if his/her marginal utility is convex. Similarly to Diamond and Stiglitz, Keenan and Snow (2002) extend this approach to the comparison between preferences, showing that one decision maker has *greater downside risk aversion* than another when the utility function of the first decision maker is related to the utility function of the second by a transformation function whose derivative is convex (i.e. the ratio between their marginal utilities is a convex function).

The notion of greater downside risk aversion cannot however be used to rank preferences, because it is neither transitive nor antisymmetric. In order to introduce a (partial) ranking of preferences, Keenan and Snow (2016) introduce the concept of strongly greater downside risk aversion reflecting the case where one decision maker has greater aversion than another to both increases in risk and increases in downside risk. This analysis has also been generalized by Keenan and Snow (2018) and Liu and Wong (2019) with the introduction of the notions of higher order greater risk aversion and higher order strongly greater risk aversion (mixed risk aversion in Liu and Wong (2019)).

An issue which has not as yet been thoroughly investigated is however the relationship between different degrees of greater risk aversion. While it is clear that, by its definition, strongly greater risk aversion of some order implies (both strongly and not) greater risk aversions of the lower orders, to our knowledge little has been written about the converse implication, as well as any kind of relationship between greater risk aversion of different orders. The main goal of this paper is to identify conditions under which a degree of greater risk aversion can be inferred from another one. In this regard, we first discuss a condition on preferences ensuring that greater downside risk aversion implies greater risk aversion. Secondly, we derive a different condition ensuring that greater risk aversion implies greater downside risk aversion for sufficiently large levels of wealth. It is worth noting that, under the derived conditions, greater downside risk aversion and strongly greater downside risk aversion are equivalent. We then extend our conclusions to higher orders: specifically we derive generalized conditions, ensuring both that greater risk aversions of higher orders implies greater risk aversions of lower orders and vice versa.

Following the approach of Diamond and Stiglitz (1974) and Keenan and Snow (2002, 2009) we interpret our results on greater risk aversion in terms of compensated increases in risk and differential risk premium. Moreover, as in Keenan and Snow (2016, 2018) and Liu and Wong (2019), strongly greater risk aversion of a certain order has implications on the response of a decision maker to stochastic dominance shifts of the same order. Our results allow us to show that, under the appropriate conditions, decision makers response to some kinds of shift (compensated or stochastically dominated) in risk is related to decision makers response to shifts of different kinds and orders.

The analysis in the present paper is also related to a different branch of literature, which studies the relationship between different aspects of the attitude toward risk of an agent. In this field, Menegatti (2001) shows conditions ensuring that, for a single agent, risk aversion implies prudence and temperance.[1] Menegatti (2014) examines the opposite direction of this linkage, deriving conditions ensuring that temperance implies prudence and prudence implies risk aversion. Menegatti (2015) generalizes this reasoning to higher orders of risk aversion.[2] Lastly, De Donno and Menegatti (2020) study conditions for the equivalence of risk aversion of different orders. As noted above, all these results show the existence of linkages between the aspects of risk attitude of *a single decision maker*. The analysis in the present paper is thus also complementary to this approach, since it studies the linkages between the aspects of the *comparison of* risk attitude of *two different decision makers*.

The paper proceeds as follows. Section 2 introduces all preliminary concepts and results. Section 3 derives the results involving greater risk aversion, greater downside risk aversion and

[1]The concept of prudence, introduced by Kimball (1990) is associated to a positive third derivative of the utility function, in the same way as the concept of downside risk aversion by Menezes et al. (1980). The two concept are thus substantially equivalent. Temperance, introduced by Kimball (1992), is instead related to a negative fourth derivative of the utility function and is relevant when the decision maker faces a fourth order risk change, as well as in some specific economic problems.

[2]The concept of nth order risk aversion, introduced by Ekern (1980), is related to assumption that the nth derivative of the utility function is positive when n is odd and negative when n is even and is relevant in the case of high order risk changes.

strongly greater downside risk aversion. Section 4 generalizes to nth-order greater and strongly greater risk aversion. Section 5 studies compensated increase in risk, differential risk premia and stochastic dominance shifts. Lastly Section 6 concludes.

2 Assumptions and preliminaries

Let u and v be smooth increasing utility functions defined on an unbounded interval $[a, +\infty)$. The conditions $u'(x) > 0$ and $v'(x) > 0$ for all $x \in [a, +\infty)$ represent the usual non-satiation assumption. We also assume that $\lim_{x \to +\infty} u'(x) \neq +\infty$ and $\lim_{x \to +\infty} v'(x) \neq +\infty$[3]. Note that this condition is automatically satisfied when marginal utilities are bounded.

These assumptions, which have been introduced in Menegatti (2014, 2015)[4] and later used also in De Donno and Menegatti (2020), are compatible with both risk aversion and risk loving, and also with the case where the decision maker is risk averse for certain levels of wealth and a risk loving for other levels. We now briefly discuss their meaning.

The assumption of unbounded domain for the utility function does not mean that agents have to deal with situations where wealth tends to infinity. It means instead that utility is defined (and thus preferences are described) for every possible (finite) level of wealth. Given this premise, we emphasize that an unbounded domain for the utility function is motivated, in some cases, by the domain of the distribution of the random variable describing wealth (as, for instance, when wealth is normally distributed).[5] Furthermore, notice that even when we are considering problems where random wealth x takes values only in the interval $[x_0, x_1]$, we usually desire that our results hold for every possible values of x_0 and x_1, which suggest that the utility function is defined on an unbounded domain.

The second assumption we introduce in our analysis is non-satiation. This assumption is standard both in microeconomic theory (see, for instance, Mas-Colell et al. 1995) and in growth and intertemporal macroeconomic models (see Romer 2012; Barro and Sala-i-Martin 2004).

Lastly, we exclude that the marginal utility tends to $+\infty$ when the wealth becomes arbitrarily large. This assumption is plausible since we usually presume that, when wealth becomes extremely high, an additional unit of it causes a not too large (or at least a bounded) increase

[3]Here and in what follows, the notation $\lim_{x \to +\infty} f(x) \neq +\infty$ means that either the limit does not exist, or, if it exists, it is finite, namely $\liminf_{x \to +\infty} f(x) < +\infty$.

[4]A similar set of assumptions can also be found in Menegatti (2001) where the risk aversion assumption automatically implies the boundedness of the marginal utility.

[5]Clearly, in this case, the domain is unbounded in both directions and is thus $\mathbb{R}$.

in utility. In fact, economic models often suppose that marginal utility is decreasing, which is an assumption much stronger than ours.[6]

Let f be a transformation function such that $v(x) = f(u(x))$; then, $f' = \frac{v'}{u'} > 0$. In order to compare different risk attitudes, the following definitions are introduced:

Definition 2.1. (i) Agent v has *greater risk aversion than agent u* if $f'' < 0$ (Pratt, 1964).

(ii) Agent v has *greater downside risk aversion than agent u* if $f''' > 0$ (Keenan and Snow, 2002).

(iii) Agent v has *strongly greater downside risk aversion than agent u* if $f'' < 0$ and $f''' > 0$ (Keenan and Snow, 2016).

Pratt (1964) introduced the first definition to show that a risk-averse transformation maps a risk averse utility to a more risk averse one. Keenan and Snow (2009) extended this notion to downside risk aversion. The notion of strongly greater downside risk aversion was introduced by Keenan and Snow (2016), after both they (Keenan and Snow, 2009, 2016) and Liu and Meyer (2012) pointed out that the condition $f''' > 0$ alone does not give a satisfactory notion of downside risk aversion, as it is neither transitive or antisymmetric.

In the following, we show that for a class of utilities, greater downside risk aversion is equivalent to strongly greater downside risk aversion. To do this, we introduce a further hypothesis on the functions $u(x)$, by assuming that it is unbounded (i.e. that $\lim_{x \to +\infty} u(x) = +\infty$). This assumption is needed for the results derived below, because they exploit asymptotic properties of the function f.

The assumption of unbounded utility is more questionable than our other assumptions, because of the well-known St. Petersburg paradox. That is why in decision theory it is often assumed that the utility function is bounded. However, as observed by Toulet (1986) "such a property is in contradiction to current use in many applications", since for instance some of the most used utility functions (such as CRRA and logarithmic utilities), which are realistic from a practical point of view, are asymptotically unbounded. For this reason, several authors have shown conditions under which maximizing expected utility theory is compatible with an unbounded utility, starting from Ryan (1974), Arrow (1974) and Fishburn (1976). This issue has been (and is still) widely discussed in the literature. We recall, among others, the papers by

[6]Growth models often introduce the even stronger assumption that marginal utility tends to 0 when wealth tends to infinity (see Barro and Sala-i-Martin 2004).

Toulet (1986) and Wakker (1993), where axiomatic models of decisions and results on expected utilities are extended to the unbounded case.

3 Greater risk aversion and greater downside risk aversion

3.1 Greater downside risk aversion implies greater risk aversion

We first analyze the existence of a relationship in the direction of linking greater downside risk aversion to greater risk aversion. Indeed, one could expect that greater downside risk aversion always implies greater risk aversion. However, as it was shown by Keenan and Snow (2002), this is not the case in general. Therefore, it seems natural to look for conditions on the utility functions such that this implication holds.

Theorem 3.1. *Let v have greater downside risk aversion than u. If $\lim\limits_{x\to+\infty} \dfrac{v(x)}{u(x)} \neq +\infty$, then v has greater risk aversion than u.*

Proof. As a first step, we prove that if $v(x) = f(u(x))$ is such that $\lim_{x\to+\infty} v(x)/u(x) \neq +\infty$, then $\lim\limits_{y\to+\infty} f'(y) \neq +\infty$.
We prove this step by contradiction. Assume that $\lim\limits_{y\to+\infty} f'(y) = +\infty$. By the chain's rule $\lim_{x\to+\infty} \frac{v'(x)}{u'(x)} = \lim_{x\to+\infty} f'(u(x)) = +\infty$. Moreover, $\lim_{y\to+\infty} f'(y) = +\infty$ also implies that $\lim_{x\to+\infty} v(x) = \lim_{y\to+\infty} f(y) = +\infty$. Since $\frac{v(x)}{u(x)}$ is an indeterminate form ∞/∞, we can apply L'Hôpital's rule, which yields

$$\lim_{x\to+\infty} \frac{v(x)}{u(x)} = \lim_{x\to+\infty} \frac{v'(x)}{u'(x)} = +\infty$$

which contradicts the assumption $\lim_{x\to+\infty} v(x)u(x) \neq +\infty$. Hence it must be $\lim\limits_{y\to+\infty} f'(y) \neq +\infty$.

Greater downside risk aversion ($f''' > 0$) implies that f'' is strictly increasing. If there existed y_0 such that $f''(y) > 0$ for $y > y_0$ on, then f' would be strictly increasing and convex and this is in contradiction with $\lim\limits_{y\to+\infty} f'(y) \neq +\infty$. Therefore $f''(y) < 0$ on $[u(a), +\infty)$.[7] □

An immediate consequence of this theorem is the following:

[7]Proposition 1 (b) in Menegatti (2014) exploits the same argument to show that under non-satiation a prudent agent is always risk averse.

Corollary 3.2. *If* $\lim_{x\to+\infty} \frac{v(x)}{u(x)} \neq +\infty$*, greater downside risk aversion and strongly greater downside risk aversion are equivalent.*

Results in Theorem 3.1 and Corollary 3.2 show when greater downside risk aversion implies greater risk aversion, and, as a consequence, strongly greater downside risk aversion. We emphasize that the theorem establishes a relationship between the risk attitudes of agent u and v, without making any assumption on the risk attitudes of the single agent. More precisely, we do not assume that the two agent are risk averse and our results are also compatible with risk loving. In this case clearly "greater risk aversion" must be interpreted as "lower risk loving".

The condition for greater downside risk aversion to imply risk aversion introduced in Theorem 3.1 and Corollary 3.2 is that $\lim_{x\to+\infty} v(x)/u(x) \neq +\infty$. Note that, as the function v is increasing, it either converges to a finite limit or it diverges to $+\infty$. In the first case $\lim_{x\to+\infty} v(x)/u(x) = 0$, so our assumption is trivially satisfied. We then want to prevent the cases where v is unbounded and it grows faster than u. A possible interpretation of this condition may refer to the fact that, when an agent is risk averse, his utility grows slowly when wealth becomes very large. Similarly, when comparing two agents, the condition $\lim\limits_{x\to+\infty} v(x)/u(x) \neq +\infty$ requires that the utility of agent v does not grow faster than the utility of agent u when wealth becomes very large. This condition may seem restrictive. The following result provides an argument in contrast to this conclusion:

Proposition 3.3. *If* v *has greater risk aversion than* u*, then* $\lim_{x\to+\infty} \frac{v(x)}{u(x)} \neq +\infty$*.*

Proof. If $f''(u) < 0$ for all u , then f' is strictly decreasing and, as a consequence, has a limit. Being it positive, $\lim_{y\to+\infty} f'(y) = L \in [0, +\infty)$. Assume by contradiction that $\lim_{x\to+\infty} v(x)/u(x) = +\infty$. Then $\lim_{x\to+\infty} v(x) = +\infty$ and by L'Hôpital's rule

$$\lim_{x\to+\infty} \frac{v(x)}{u(x)} = \lim_{x\to+\infty} \frac{v'(x)}{u'(x)} = \lim_{x\to+\infty} f'(u(x)) = L$$

which yields a contradiction. □

While the condition $\lim_{x\to+\infty} \frac{v(x)}{u(x)} \neq +\infty$ is not sufficient to assure v has greater risk aversion than u, Proposition 3.3 shows that it is necessary. This means that each time v shows greater risk aversion than u, this condition holds but there may be situations where it holds and nonetheless v does not show greater risk aversion than u. With reference to this second case, consider, for instance, the transformation function $f(u) = 2u + \sin u$: the agent $v = f(u)$ does not exhibit either greater or smaller risk aversion than u since $f''(u) = -\sin u$, although

$\lim_{x\to+\infty} v(x)/u(x) = \lim_{u\to+\infty} f(u)/u = 2 < +\infty$. All this reasoning suggests that, in general, imposing the condition $\lim_{x\to+\infty} v(x)/u(x) \neq +\infty$ is, in a sense, less restrictive than directly imposing greater risk aversion.

On the other hand, we will show that under the assumption of greater downside risk aversion, a dichotomy occurs above certain levels of wealth: either v is more risk averse than u or it is more risk loving, and this depends on the asymptotic behaviour of the ratio between the two utility functions. To see this, we start by underlining an easy consequence of Proposition 3.3: when $\lim_{x\to+\infty} v(x)/u(x) = +\infty$, the utility v cannot have greater risk aversion than u. However, it can show greater downside risk aversion. Consider for example the utility functions $u(x) = \ln x$, $v(x) = (\ln x)^3$ on the interval $[2, +\infty)$: agent v has greater downside risk aversion than u since $v = f(u)$ where $f(y) = y^3$. Nevertheless, $\lim_{x\to+\infty} v(x)/u(x) = +\infty$ and $f''(y) > 0$, so v does not have greater risk aversion than u. This example shows that when v grows faster than u at $+\infty$, the notions of greater downside risk aversion and greater risk aversion can seem inconsistent. Since the notion of risk aversion is reversible (Keenan and Snow, 2021), we can say that in this case v is less risk averse, although it is more downside risk averse, than u. In fact, we will prove that when $\lim_{x\to+\infty} v(x)/u(x) = +\infty$, greater downside risk aversion implies smaller risk aversion, at least for large wealth. To do this, we first introduce the following definition to formalize the notion of risk aversion for large wealth.

Definition 3.1. We say that v has *eventually greater risk aversion* than u if there exists some y_0 such that $f''(y) < 0$ for $y \geq y_0$.

This property has a clear interpretation in terms of risk aversion and risk premium as in Pratt (1964): v has greater risk aversion than u and, as a consequence, the risk premium corresponding to the utility function v is greater than that corresponding to u above a certain level of wealth.[8] In other words, as shown by Diamond and Stiglitz (1974), this means that v dislikes mean-preserving spreads more than u above a certain level of wealth. Given this definition, we can then prove the following:

Proposition 3.4. *Let v have greater downside risk aversion than u. If* $\displaystyle\lim_{x\to+\infty} \frac{v(x)}{u(x)} = +\infty$*, then u has eventually greater risk aversion than v.*

Proof. Since $f'''(y) > 0$, the second derivative f'' is an increasing function. The condition $\displaystyle\lim_{x\to+\infty} v(x)/u(x) = +\infty$ implies $\lim_{x\to+\infty} f'(u(x)) = +\infty$, which in turn entails the existence

[8] If we denote $x_0 = u^{-1}(y_0)$, we can say that $r_v(x) \geq r_u(x)$ for $x \geq x_0$ where r_u is the risk aversion coefficient. In terms of risk premium, we can write $\pi_v(x, \tilde{z}) \geq \pi_u(x, \tilde{z})$ for $x \geq x_0, \tilde{z} \geq 0$.

of some $x_0 \geq a$ such that $f''(y) > 0$ for $y \geq y_0 = u(x_0)$. Thus, if we let $\varphi = f^{-1}$ (so that $u(x) = \varphi(v(x))$), we obtain that $\varphi'(z) = \frac{1}{f'(\varphi(z))} > 0$ on $[f(a), +\infty)$ and

$$\varphi''(z) \;=\; -\frac{1}{[f'(\varphi(z))]^2}\, f''(\varphi(z))\, \varphi'(z) \;=\; -\frac{1}{[f'(\varphi(z))]^3}\, f''(\varphi(z)) \;<\; 0 \quad \text{on } [f(y_0), +\infty).$$

□

Summarizing, Theorem 3.1 and Proposition 3.4 fully describe the comparison between the degree of risk aversion of two agents v and u when v is more downside risk averse than u. More specifically, our results suggest that uniformly greater downside risk aversion imposes a similar degree of uniformity at the lower order of risk aversion: if an agent is more downside risk averse than another, then he will also be either uniformly more risk averse or uniformly more risk loving, at least above certain level of wealth (i.e. in the "eventually" sense introduced in Definition 3.1). The direction of the comparative risk aversion depends on whether the limit $\lim_{x\to+\infty} v(x)/u(x)$ is finite or infinite. This provides a further reason to analyze $\lim_{x\to+\infty} v(x)/u(x)$, besides the fact that this condition is usually easier to check in the applications than directly checking if the utility function satisfies greater risk aversion for all levels of wealth.

Lastly we emphasize the parallelism between the results in this subsection and those by Menegatti (2014), related to the attitude toward risk of a single agent. Menegatti (2014) shows conditions ensuring that, for a single agent, prudence/downside risk aversion implies risk aversion. The results in this subsection provide similar findings for the comparison of attitude toward risk of two agents, deriving conditions ensuring that greater downside risk aversion implies greater risk aversion.

3.2 Greater risk aversion implies greater downside risk aversion

In Subsection 3.1 we derived a condition which makes it possible to infer conclusion on the comparison between the degree of risk aversion of two agents from their degree of downside risk aversion. We now provide a similar reasoning in the opposite direction, looking for possible inference on the comparison of downside risk aversion starting from information on the comparison of risk aversion.

Along the lines of Definition 3.1, we introduce the following:

Definition 3.2. (i) We say that v has *eventually greater downside risk aversion* than u if there exists some y_0 such that $f'''(y) > 0$ for $y \geq y_0$.

(ii) We say that v has *eventually strongly greater downside risk aversion* than u if there exists some y_0 such that $f''(y) < 0$ and $f'''(y) > 0$ for $y \geq y_0$.

We also introduce the following definition in analogy with Definition 1 in De Donno and Menegatti (2020).

Definition 3.3. A function f has a (well-defined) *second order asymptotic elasticity* if f is 3-times differentiable and

$$AE_2(f) = \lim_{y \to +\infty} -\frac{yf'''(y)}{f''(y)}$$

exists (it can be possibly be infinite).

In De Donno and Menegatti (2020), a similar notion was introduced for the utility of a single decision maker function, for which the quantity AE_2 was called asymptotic relative prudence. Since in the present framework f is not a utility function, but a transformation function, we identify this quantity with the elasticity of its second derivative, in line with the notion introduced by Kramkov and Schachermayer (1999). Under this regularity assumption of the transformation function, greater downside risk aversion can be inferred from greater risk aversion, at least above some level of wealth:

Theorem 3.5. *Assume that the function f has a (well-defined) second order asymptotic elasticity. If v has greater risk aversion than u, then v has eventually greater downside risk aversion (hence eventually strongly greater downside risk aversion) than u.*

Proof. As was proved in Theorem 1 (a) in De Donno and Menegatti (2020), under the assumptions of the proposition, if $f''(y) < 0$, then there exists y_0 such that $f'''(y) > 0$ for $y \geq y_0$. For sake of the reader, we sketch the proof below. As a first step, one can show that if $f''(y) < 0$, then $\lim_{y \to +\infty} -yf'''(y)/f''(y) = AE_2(f) \geq 0$: if this is not the case, then f''' is necessarily strictly negative from some point on and this in contradiction with Proposition 1(a) in Menegatti (2014). If $AE_2(f) = +\infty$, then the claim is immediately proved. If $0 \leq AE_2(f) < +\infty$, then there exists some y^* such that the function is bounded by some constant C for $y \geq y^*$. Then we can write the following chain of inequalities

$$\begin{aligned} |xf''(x) - yf''(y)| &= \left| \int_y^x \left(f''(t) + tf'''(t)\right) dt \right| \leq \int_y^x \left| f''(t) \left(1 + \frac{tf'''(t)}{f''(t)}\right) \right| dt \\ &\leq (1 + C) \int_y^x \left|f''(t)\right| dt = (1 + C)|f'(x) - f'(y)|. \end{aligned} \quad (3.1)$$

Since $f''(y) < 0$, $f'(y)$ is strictly decreasing and strictly positive, hence it decreases towards a finite limit $l \geq 0$. The above inequalities imply that that $yf''(y)$ converges as well to some limit $L \leq 0$. We claim that $L = 0$. Consider indeed the function $g(y) = y(f'(y) - l)$. Its derivative $g'(y) = f'(y) - l + yf''(y)$ tends to L as $y \to +\infty$. If $L < 0$, there exists some $\tilde{y}$ such that $g'(y) < 0$, hence the function g is strictly decreasing, for $y \geq \tilde{y}$. Since $g(y) \geq 0$ for all y, it admits a finite (non-negative) limit as $y \to +\infty$. But, if this is the case, for the asymptote criterion, $g'(y)$ must tend to 0 contradicting the fact that $L < 0$. Therefore $L = 0$. Since both $yf''(y)$ and $f'(y) - l$ tend to 0 and $-yf'''(y)/f''(y)$ has a finite limit, we can apply L'Hôpital's rule to obtain

$$0 \leq \lim_{y \to +\infty} -\frac{yf''(y)}{f'(y) - l} = \lim_{y \to +\infty} \left(-1 - \frac{yf'''(y)}{f''(y)} \right),$$

namely $AE_2(f) \geq 1$. This implies that there exists some positive constant c and some y_0 such that $-f'''(y)/f''(y)$ is greater than c for $y \geq y_0$ and, as a consequence $f'''(y) > 0$ for $y \geq y_0$. □

Differently from the condition exploited in Theorem 3.1, which being necessary and sufficient, allowed us to categorize all possible cases, the condition $\lim_{y \to +\infty} -y\frac{f'''(y)}{f''(y)}$ introduced in Theorem 3.5 is sufficient but not necessary in order to obtain eventual downside greater risk aversion as a consequence of greater risk aversion. Indeed, consider for instance the following transformation function defined on $[1, +\infty)$:

$$f(y) = 2y - \int_1^y (y - t)\, e^{-\int_1^t \frac{3 - \sin u}{u}\, du}\, dt.$$

The derivatives of f are respectively

$$f'(y) = 2 - \int_1^y e^{-\int_1^t \frac{3 - \sin u}{u}\, du}\, dt > 0;$$

$$f''(y) = -e^{-\int_1^y \frac{3 - \sin u}{u}\, du} < 0;$$

$$f'''(y) = \frac{3 - \sin y}{y}\, e^{-\int_1^y \frac{3 - \sin u}{u}\, du} > 0.$$

Therefore, an agent $v = f(u)$ displays both greater risk aversion and greater downside risk aversion than u. Nonetheless, the function $-\dfrac{yf'''(y)}{f''(y)} = 3 - \sin y$ does not admit a limit as $y \to +\infty$.

Note that the result shows that our limit condition implies a kind of uniformity in greater downside risk aversion resulting from consistently greater risk aversion. In particular, it excludes that the function f''' eventually oscillates implying that it cannot continue to have alternating signs. This eventual invariant sign for f''' means that the attitude towards downside risk of an agent with respect to another does not continue to change as wealth increases, whatever the direction of this attitude is. The significant point is that our result shows that this direction too is automatically determined given greater risk aversion. Furthermore, the existence of the asymptotic elasticity, in most cases, does not even require the computation of the derivatives. Indeed, all rational functions composed of exponential, logarithmic, polynomial functions or inverse tangent have a, possibly infinite, limit at $+\infty$. Therefore, if f has any of these forms, no computation is needed to prove that the condition is fulfilled (see also De Donno, Menegatti (2020)).

As in Subsection 3.1, one can notice the parallelism between these results and those obtained for the case of a single agent. In this framework, Menegatti (2001) identifies conditions ensuring that risk aversion implies prudence/downside risk aversion, whereas in this subsection, conditions are derived for the comparison of the risk attitude of two agents, ensuring that greater risk aversion implies greater downside risk aversion. Similarly, the results obtained above for sufficiently high levels of wealth, related to the asymptotic elasticity of function f, exhibit some analogies with those obtained by De Donno and Menegatti (2020) for the case of a single agent.

We conclude this section with a remark on the Schwarzian measure. Since greater downside risk aversion does not induce a partial order, Keenan and Snow introduce the Schwarzian measure $S_u = \frac{u'''}{u'} - \frac{3}{2}\left(\frac{u''}{u'}\right)^2$ in order to rank preferences. They show indeed that if $S_v(x) > S_u(x)$, then $f''' > 0$, although the converse implication does not hold, unless small changes in risk are considered. In other words, in general, the Schwarzian measure is a refinement of the notion of greater downside risk aversion. Below we characterize a class of transformation functions for which greater risk aversion implies the Schwarzian ranking for large risks. For this class of functions, the ranking coincides with the ranking induced by strongly greater downside risk aversion.

Proposition 3.6. *Assume that the function $-yf'''(y)/f''(y)$ has a (possibly infinite) limit as $y \to +\infty$ and that v has greater risk aversion than u. There thus exists x_s such that $S_v(x) > S_u(x)$ for all $x \geq x_s$ if one of the following conditions hold:*

(a) $\lim_{y\to+\infty} f'(y) > 0$

(b) $\lim_{y\to+\infty} -\frac{yf'''(y)}{f''(y)} < 3.$

Proof. As shown by Keenan and Snow (2009, Lemma 3), the condition $S_v(x) > S_u(x)$ for $x \geq x_s$ is equivalent to $S_f(y) > 0$ for $y \geq u(x_s)$. With some manipulations, we can write

$$S_f(y) = \frac{1}{y^2}\left(-\frac{yf''(y)}{f'(y)}\right)\left[\left(-\frac{yf'''(y)}{f''(y)}\right) - \frac{3}{2}\left(-\frac{yf''(y)}{f'(y)}\right)\right]$$

where $\frac{1}{y^2}\left(-\frac{yf''(y)}{f'(y)}\right) > 0$. With the same arguments as in the proof of Theorem 3.5, one can show that either $k = \lim_{y\to+\infty} -yf'''(y)/f''(y) = +\infty$ or

$$1 \leq k = \lim_{y\to+\infty} -\frac{yf'''(y)}{f''(y)} = 1 + \lim_{y\to+\infty} -\frac{yf''(y)}{f'(y) - l}.$$

If condition (a) is satisfied, i.e if $l > 0$, then, $\lim_{y\to+\infty} yf''(y)/f'(y) = 0$ and

$$\left(-\frac{yf'''(y)}{f''(y)}\right) - \frac{3}{2}\left(-\frac{yf''(y)}{f'(y)}\right) \to k \geq 1$$

as $y \to +\infty$. On the other hand, if $l = 0$ but $k < 3$

$$\left(-\frac{yf'''(y)}{f''(y)}\right) - \frac{3}{2}\left(-\frac{yf''(y)}{f'(y)}\right) \to k - \frac{3}{2}(k-1) = \frac{3}{2} - \frac{k}{2} > 0.$$

In both cases, there exists $y_s = u(x_s)$ such that this quantity is strictly positive for all $y \geq y_s$. □

Consider as an example the two functions $f_1(y) = y - e^{-y}$ and $f_2(y) = 1 - e^{-y}$ on $[1, +\infty)$. For both of them, the limit in condition (b) is infinite, but f_1 satisfies condition (a) while f_2 does not. Since $f_1'' = f_2'' < 0$ and $f_1''' = f_2''' > 0$, $v_1 = f_1(u)$ and $v_2 = f_2(u)$ have the same behaviour in comparison to u in terms of risk aversion, that is they have both strongly greater downside risk aversion than u, this ranking is coherent with the ranking yielded by the Schwarzian measure only for v_1. As a further example, we can compare $f_3(y) = \sqrt{y}$ and $f_4(y) = 1 - y^{-2}$. The derivatives of these two functions go to 0 as $y \to +\infty$ hence condition (a) is not satisfied. Moreover $\lim_{y\to+\infty} -yf_3'''(y)/f_3''(y) = 3/2$ while $\lim_{y\to+\infty} -yf_4'''(y)/f_4''(y) = 4$, meaning that f_3 satisfies condition (b) while f_4 does not. As in the previous example, both functions exhibit strongly greater downside risk aversion but only f_3 determines an increase in the Schwarzian measure.

4 Higher order risk attitudes

The results presented in the previous sections can be extended to higher order risk attitudes. Following Keenan and Snow (2018), we introduce the notion of strongly greater aversion of nth-order. As usual, we denote with $f^{(n)}$ the nth-derivative of f and assume that all functions are sufficiently smooth for the mentioned derivatives to be well-defined.

Definition 4.1. (i) Agent v is *more nth order risk averse than agent u* if $(-1)^{n-1}f^{(n)} > 0$.

(ii) Agent v has a *strongly greater nth order risk aversion than agent u* if $(-1)^{k-1}f^{(k)} > 0$ for $k = 1, \ldots, n$.

Notice that the notion of strongly greater nth order risk aversion coincides with Liu and Wong's (2019) notion of (m, n) mixed risk aversion with $m = 1$. We can immediately extend Theorem 3.1 to this case:

Theorem 4.1. *Let v be more nth order risk averse than u. If $\lim_{x \to +\infty} v(x)/u(x) \neq +\infty$ then v is more kth order risk averse than u for $k = 2, \ldots, n$. As a further consequence, it has a strongly greater kth order risk aversion than u for $k = 2, \ldots, n$.*

Proof. The proof follows that of Theorem 3.1 to show that $\lim_{y \to +\infty} f'(y) \neq +\infty$. Then we apply Proposition 3 in Menegatti (2015) to function f. □

Theorem 4.1 shows conditions under which the direction of the comparison of the degree of risk aversion of two agents of a given order determines the direction of the same comparisons of lower orders. In particular, this result allows us to understand whether a transformation yields strongly greater nth order aversion to risk by looking only at the nth derivative of the transformation.

Again the theorem establishes a relationship between the risk attitudes of agent u and v, without making any assumption on the risk attitude of the single agent. If however agent u has some degree of risk aversion, and v is strongly greater nth order risk averse, then v inherits the same degree of risk aversion as u.

We recall that an agent u exhibits nth-order mixed risk aversion if $(-1)^{k+1}u^{(k)}(x) \geq 0$ for all $x \in [a, +\infty)$ and for all $k = 1, \ldots, n$ (Caballé and Pomarski, 1996). We can thus state the following:

Corollary 4.2. *Let v be more nth-degree risk averse than u and $\lim_{x \to +\infty} v(x)/u(x) \neq +\infty$. If, in addition, $(-1)^{n-1}u^{(n)}(x) > 0$ for all $x \in [a, +\infty)$, then both u and v exhibit nth order mixed risk aversion.*

Proof. Proposition 3 in Menegatti (2015) implies that $(-1)^{k-1}u^{(k)}(x) > 0$ for all $x \in [a, +\infty)$, for all $k = 2, \ldots, n-1$, i.e. u is nth order mixed risk averse. The claim then follows applying Lemma 1 in Keenan and Snow (2018) (see also Proposition 1 (i) in Liu and Wong (2019)). □

Theorem 4.1 makes possible inference from the comparisons of higher order risk aversion to the comparisons of lower order risk aversion. Now, as we did in Subsection 3.2, we try to infer a relationship in the opposite direction (i.e. from lower to higher orders of risk aversion), under some additional information on the transformation function. First, we need to extend some definitions.

Definition 4.2. (i) We say that v has *eventually greater nth order risk aversion* than u if there exists some y_0 such that $(-1)^{n-1}f^{(n)}(y) > 0$ for $y \geq y_0$.

(ii) We say that v has *eventually strongly greater nth order risk aversion than u* if there exists some y_0 such that $(-1)^{k-1}f^{(k)}(y) > 0$ for $k = 1, \ldots, n$ for $y \geq y_0$.

Moreover, in concert with what was done in Section 3, we introduce the following definition in analogy with Definition 1 in De Donno and Menegatti (2020), where a similar property of the utility function of a single decision maker was called asymptotic nth degree risk aversion.

Definition 4.3. A function f has a (well-defined) *nth order asymptotic elasticity* if f is $n+1$-times differentiable and

$$AE_n(f) = \lim_{y \to +\infty} -\frac{y\, f^{(n+1)}(y)}{f^{(n)}(y)}$$

exists (it can be possibly be infinite).

Under this assumption on the transformation function, a strongly higher order comparison of risk attitude function can be inferred from a lower one:

Theorem 4.3. *Assume that* $\lim_{x \to +\infty} v(x)/u(x) \neq +\infty$ *and that the transformation function f has a well-defined hth order asymptotic elasticity for $h = 2, \ldots, n-1$. Then if v has eventually greater kth order risk aversion than u for some $k \geq 2$, it has eventually strongly greater hth-order risk aversion than u for $h = 3, \ldots, n$.*

Proof. Assume that $(-1)^{k-1}f^{(k)}(y) > 0$ for some $y \geq y_0$, namely v is more kth order risk averse than u on $[u^{-1}(y_0), +\infty)$. Then by Theorem 4.1, v has a strongly greater kth order (and as a consequence hth order for $h \leq k$) risk aversion than u.

To show that greater kth order risk aversion can be extended to higher orders above certain levels of wealth, we prove now that if $k = 3$, then $f^{(4)}(y) < 0$ for some $y \geq y_1$, namely has eventually strongly greater 4th order risk aversion than u. The argument can then be extended to the higher order cases. First of all, we observe that $f'''(y) > 0$ for $y \geq y_0$ implies that

$f''(y) < 0$ for $y \geq y_0$ and $\lim_{y \to +\infty} f''(y) = 0$. Then we can proceed as in the proof of Theorem 3.5. Indeed, Proposition 2 (b) in Menegatti (2014) implies[9] that $\lim_{y \to +\infty} -yf^{(4)}(y)/f'''(y) \geq 0$. If this limit is $+\infty$, then we can immediately conclude. Otherwise, we can write a chain of inequalities analogous to (3.1) to show that $yf'''(y)$ admits a limit $L \geq 0$ as $y \to +\infty$. This implies that the derivative of the function $g(y) = yf''(y)$ tends to L. If $L > 0$, the function g is strictly increasing and strictly negative for y greater than some $z > 0$, therefore it admits a finite limit, and for the asymptote criterion, its derivative must tend to 0, contradicting the fact that $L > 0$. It follows that $yf'''(y)$ converges to 0. Then, mimicking the argument in Theorem 3.5 we can apply L'Hôpital's rule to show that $AE_3(f) \geq 1$, and we can conclude as in Theorem 3.5. □

In the previous theorem we exploit the property of having nth order asymptotic elasticity for f, to prove that its derivatives show alternating signs above some level of wealth. Caballé and Pomanski (1996) showed that any function exhibiting all derivatives of alternating sign can be expressed as a mixture of exponential functions. As far as these functions are concerned, the nth order asymptotic elasticity is always well-defined, and we can think of our assumption as a reasonable condition to obtain our result.

As a final remark, note that Theorems 4.1 and 4.3 considered together imply that, under some mild conditions, once we know the direction of the comparison between the degrees of risk aversion of two agents of a given order, we can infer conclusions on the same comparison of all other orders (both higher and lower), at least for sufficiently large levels of wealth. Again, we can observe that these findings mirror those derived for a single agent. In particular, Menegatti (2015) provides conditions implying that risk aversion of different orders are related, also showing that, under these conditions, an agent's behaviour in case of changes in risk of lower orders can be inferred by their behaviour in case of risk changes of higher orders. For the converse direction, results involving sufficiently high levels of wealth and asymptotic elasticity for the case of a single agent, which show similarities with the above results, can be found in De Donno and Menegatti (2020).

5 Compensated increases in risk and stochastic dominance shifts

Keenan and Snow (2002, 2009) characterize downside risk aversion as aversion to compensated increases in downside risk. We now extend this characterization to higher order risk aversion and unbounded domain and interpret our results in this framework.

[9]For the higher orders one can use Proposition 3 in Menegatti (2015).

Consider all cumulative distribution functions with support contained in $[a, +\infty)$ such that u has finite moments up to order n. Given two distribution functions F and G in this class, we denote $T_0(x) = G(x) - F(x)$ and then define recursively

$$T_k(x) = \int_a^x u'(y) T_{k-1}(y) dy \tag{5.1}$$

for $k = 1, \ldots, n-1$. We define an nth order compensated increase in risk for u as follows: [10]

Definition 5.1. Given the two distribution functions F and G, we call T_0 an *nth order compensated increase in risk* for u if

(a) $\lim_{x \to +\infty} u(x) T_0(x) = 0$;

(b) $\lim_{x \to +\infty} T_k(x) = 0$ for $k = 1, \ldots, n-1$;

(c) $T_{n-1}(x) \geq 0$ for all $x \in [a, +\infty)$.

As for the second and third order cases analysed by Keenan and Snow, an increase in nth order risk aversion is equivalent to the dislike of nth order compensated increases in risk.

Proposition 5.1. *Assume that* $\lim_{x \to +\infty} v(x)/u(x) \neq +\infty$. *All nth order compensated increases in risk for u result in lower expected utility for v if and only if v is more nth order risk averse than u.*

Proof. We extend to our framework the argument of Keenan and Snow (2009). To prove sufficiency, we first observe that, under the condition $\lim_{x \to +\infty} v(x)/u(x) \neq +\infty$, the expected value of v is well defined with respect to the distribution functions considered. Indeed, since $\lim_{x \to +\infty} v(x)/u(x) = L \geq 0$, for $\varepsilon > 0$, there exists $M \geq 0$ such that $|v(x)/u(x) - L| < \varepsilon$ for $x \geq M$. Then

$$\begin{aligned} \int_a^{+\infty} |v(x)| dF(x) &= \int_a^M |v(x)| dF(x) + \int_M^{+\infty} |v(x)| dF(x) \\ &\leq F(M) \sup_{a \leq x \leq M} |v(x)| + \int_M^{+\infty} |v(x) - Lu(x)| dF(x) + L \int_M^{+\infty} |u(x)| dF(x) \\ &\leq F(M) \sup_{a \leq x \leq M} |v(x)| + (\varepsilon + M) \int_a^{+\infty} |u(x)| dF(x) < +\infty. \end{aligned}$$

[10] Hanoch and Levy (1969) and Tesfatsion (1976) analyze the extension to risk with unbounded domain of the results on stochastic dominance. In particular, the results by Tesfatsion (1976) imply that all the improper integrals in our paper are well defined.

Now we write
$$\int_a^{+\infty} v(y)d(G-F)(y) = \lim_{x\to+\infty}\int_a^x v(y)d(G-F)(y)$$
and apply repeatedly integration by parts' to obtain
$$\int_a^x v(y)d(G-F)(y) = v(x)T_0(x) - f'(u(x))T_1(x) + \cdots \tag{5.2}$$
$$+(-1)^{n-1}f^{(n-1)}(u(x))\,T_{n-1}(x) + \int_a^x (-1)^n f^{(n)}(u(y))\,u'(y)T_{n-1}(y)dy. \tag{5.3}$$

As $\lim_{x\to+\infty}\frac{v(x)}{u(x)} \neq +\infty$, condition (a) implies that $\lim_{x\to+\infty} v(x)T_0(x) = \lim_{x\to+\infty}\frac{v(x)}{u(x)}u(x)T_0(x) = 0$. Furthermore, $\lim_{x\to+\infty} f^{(k)}(u(x))T_k(x) = 0$ for $k = 1,\ldots,n-1$ because of (b) and the fact that $f^{(k)}(u(x))$ are bounded[11] at $+\infty$. As a result, we have that
$$\int_a^{+\infty} v(y)d(G-F)(y) = \lim_{x\to+\infty}\int_a^x (-1)^n f^{(n)}(u(y))\,u'(y)\,T_{n-1}(y)dy \leq 0$$
if $(-1)^{n+1}f^{(n)}(y) \geq 0$ for all y. To prove necessity one can follow a similar argument as in Lemma 1 in Keenan and Snow (2009). ☐

The equivalence between nth degree risk aversion and stronger risk aversion can then be reformulated as follows:

Theorem 5.2. *Assume that* $\lim_{x\to+\infty} v(x)/u(x) \neq +\infty$. *If all nth order compensated increases in risk for u result in lower expected utility for v, then all kth order compensated increases in risk for u result in lower expected utility for v for $k \leq n$.*

Proof. It follows from Proposition 5.1 and Theorem 4.1 ☐

Further results are obtained by relating compensated changes in risk of lower orders with those of higher orders. To study this issue, we say that a compensated increase in risk T_0 is concentrated on $[a^*,+\infty)$ with $a^* \geq a$, if its support is contained[12] in $[a^*,+\infty)$.

Theorem 5.3. *Assume that* $\lim_{x\to+\infty} v(x)/u(x) \neq +\infty$ *and that the transformation function f has kth-degree asymptotic elasticity for $k = 2,\ldots,n-1$. If all 2nd order*[13] *compensated increases in risk for u result in lower expected utility for v, there exists $x^* \geq a$ such that all kth ($k \leq n$) order compensated increases in risk for u concentrated on $[x^*,+\infty)$ result in lower expected utility for v.*

[11] As we have already observed f' has a limit $l \geq 0$, since it is strictly decreasing; for $k \geq 2$ it can be proved that $f^{(k)}(y)$ go to 0 as $y \to +\infty$ (see for instance Lemma 1 in De Donno, Magnani, Menegatti (2020)).

[12] This is equivalent to saying that $F(x) = G(x)$ for $a \leq x \leq a^*$.

[13] D&S in the language of Keenan and Snow.

Proof. Theorem 4.3 implies that there exists y^* such that $(-1)^{k+1}f^{(k)}(y) \geq 0$ for all $2 \leq k \leq n, u \geq u^*$. Taking $x^* = u^{-1}(y^*)$ we prove the claim. □

As emphasized by Keenan and Snow (2002, 2009), compensated increase in risk is related to the concept of differential risk premium. In particular, the differential risk premium is the amount that an agent is willing to pay to avoid a compensated increase in risk (of a given order). A formal definition of differential risk premium for an increase in risk described by Rotschild and Stiglitz (order 2) and an increase in downside risk (order 3) are provided by Keenan and Snow (2002, 2009). A formal definition for the general case of nth order differential risk premium is the following:

Definition 5.2. Given two distribution functions F and G, the *nth order differential risk premium* π_u^n for u bearing risk G over F is defined by

$$\int_a^{+\infty} u(x - \pi_u^n)dF(x) = \int_a^{+\infty} u(x)dG(x)$$

where $T_0(x) = G(x) - F(x + \pi_u^n)$ is a nth order compensated increase in risk.

From this definition we obtain the following conclusions, which generalize the findings by Keenan and Snow (2009) for changes in risk of orders 2 and 3:

Proposition 5.4. *Assume that* $\lim_{x\to+\infty} v(x)/u(x) \neq +\infty$. *All nth order compensated increases in risk for u result in larger differential risk premium for v if and only if v is more nth order risk averse than u.*

Combining this result with the results of Theorem 4.1 we obtain:

Theorem 5.5. *Assume that* $\lim_{x\to+\infty} v(x)/u(x) \neq +\infty$. *If all nth order compensated increases in risk for u result in larger differential risk premium for v, then all kth order compensated increases in risk for u result in larger differential risk premia for v for $k \leq n$.*

Keenan and Snow (2016) relate the notion of strongly greater risk aversion to the dislike of change in probabilities which induces a third order stochastic dominance deterioration in the distribution for u. Keenan and Snow (2018) extend this property to strongly greater aversion of nth order. Liu and Wong (2019) provide a version of this equivalence for (m, n) mixed aversion which coincides with Keenan and Snow's result when $m = 1$. We adapt their definition and result to our framework and analyze the linkages with the dislike for compensated increases in risk defined above.

Given the two distribution functions F and G, let T_k $(k \geq 0)$ be defined as above.

Definition 5.3. We say that T_0 induces a *nth order stochastic dominance shift (NSD)* in the distribution for u if

(a) $\lim_{x \to +\infty} u(x)T_0(x) = 0$;

(b) $\lim_{x \to +\infty} T_k(x) = t_k \in [0, +\infty)$ for $k = 1, \ldots, n-1$;

(c) $T_{n-1}(x) \geq 0$ for all $x \in [a, +\infty)$.

It is clear from this definition that the set of changes in probabilities that induces NSD shifts is larger than the set of distributions which induce nth order compensated increases in risk. A stronger assumption on the relationship between v and u is therefore needed to obtain a lower expected utility (or equivalently a lower risk premium) for v as a consequence of a NSD shift in distribution for u. Paraphrasing Ekern (1980), who made similar observations comparing increase in nth order risk and nth order stochastic dominance for utility functions, the stochastic dominance rule puts stricter restrictions on the transformation function, while increase in risk puts stricter restrictions on the distribution functions of risk. Proposition 1 in Keenan and Snow (2018) (Proposition 3 in Liu and Wong (2019)) can be restated as follows:

Proposition 5.6. *Utility v never likes a change in income risk that induces a NSD shift in the distribution for utility u if and only if v has strongly greater nth order risk aversion than u.*

Proof. The proof is again based on the integration by parts' formula (5.2) and is very similar to the bounded case, so we omit most of it. Note only that from Proposition 3.3, we obtain that $\lim_{x \to +\infty} v(x)/u(x) \neq +\infty$ hence v is integrable. Moreover, $\lim_{x \to +\infty} f'(u(x))T_1(x) \geq 0$ because of condition (b) in Definition 5.3 and the fact that f' is decreasing, while $\lim_{x \to +\infty} f^{(k)}(u(x))T_k(x) = 0$ for $2 \leq k \leq n-1$, since T_k is bounded. □

This result suggests that in general, the dislike for NSD shifts implies the dislike for compensated increases in risk. In our framework, the opposite implication also holds true.

Theorem 5.7. *Assume that $\lim_{x \to +\infty} v(x)/u(x) \neq +\infty$. Utility v never likes an nth order compensated increase in risk for u if and only if it never likes a change in income risk that induces an NDS shift in the distribution for utility u.*

Proof. Follows from Propositions 5.1 and 5.6 together with Theorem 4.1. □

6 Conclusions

Starting from Pratt (1964) a series of papers (Keenan and Snow, 2002, 2009, 2016, 2018 and 2021, and Li and Wong, 2019) studies the comparison of agents' preferences in the case of changes in risk of different order, by introducing the concepts of greater risk aversion, greater downside risk aversion, strongly greater downside risk aversion, greater nth order risk aversion and strongly greater nth order risk aversion. A different series of papers (Menegatti, 2001, 2014, 2015 and De Donno and Menegatti, 2020) studies the linkages between the different aspects of a single agent's preferences. The intuition that choices made in cases of different changes in risk are related underpins the single agent analysis of the second strand of literature and, in this paper, is applied to the comparison of agents analyzed in the first strand. This allows us to show the existence of some linkages between the comparisons of agents' degree of aversion to changes in risk of different order.

In particular, we derive two different conditions ensuring respectively that greater downside risk aversion implies greater risk aversion and that greater risk aversion implies greater downside risk aversion above some levels of wealth. Moreover, we generalize these results to the case of high order greater risk aversion, showing that, under different specific conditions, there exist linkages between the comparison of preferences of order n and those of higher and lower orders. The implications of these results for strongly greater downside risk aversion and nth order strongly greater risk aversion are also derived.

We also apply our results to compensated increases in risk and to the related concepts of differential risk premia and stochastic dominance shifts. Our results in this field show that the comparisons of agents' behaviour when facing choices of different orders on these variables are related.

Lastly, we emphasize that the conditions under which our conclusions are obtained are fairly weak, as they are satisfied by large classes of utility functions and, specifically, by the most frequently used ones. This confirms the significance of the linkages described in the paper, which are fairly robust to the choice of function used to describe preferences.

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