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A Clebsch portrait for Schrödinger's theory

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Abstract In this note we pursue the investigation initiated in Spera M (in: Nielsen, Barbaresco, (eds) Geometric Science of Information. GSI 2023. Lecture Notes in Computer Science, Springer, Cham, 2023) by addressing geometric and topological issues concerning the zero set of the wave function, provided it is a knot in 3-space. Since, the standard Madelung velocity breaks down thereat, it is necessary to resort to the Clebsch geometry of the probability current shown in the above paper. This leads to considering several tightly interknit symplectic manifolds.

1 Introduction

Vortex structures, while ubiquitous in various physical contexts, emerge, in particular, as *nodal lines* (zero sets) of the wave function of a massive spinless particle ruled by the Schrödinger equation, where the probability density vanishes and the phase is totally undetermined (see e.g. [2, 3]), thus giving rise to both physical and mathematical subtleties. The paper [1] investigated some hydrodynamical aspects of the probability current in Schrödinger's theory, starting from the observation that, outside the nodal line, the latter shares the same (Bohm) trajectories with the Madelung velocity, while exhibiting a regular behaviour; the nodal line motion was found to be closely related to the time derivative of the probability current and the nodal line itself – advected via the hydrodynamical Schrödinger-Madelung equation – arose as a fibre of a Clebsch-type fibration. In the present contribution we further analyse this Clebsch geometry, which turns out to be naturally related to several symplectic manifolds manufactured from wave functions. In particular, we resort to the projective Hilbert space approach of [4, 5] and sketch a portrait akin to the one developed in [6–9] for the Euler equation, ultimately conveyed in diagrammatic form (Theorem 2).

The paper is organised as follows: first, we briefly discuss the standard Madelung-Schrödinger picture, closely following, in particular, the exposition given in [1]; then, we present a series of constructions based on [7–9] and [4, 5] which eventually merge together in the Clebsch portrait of Theorem 2.

2 Preliminaries

Basic references for this section are [10] for quantum hydrodynamics and, specifically concerning geometric aspects, [1, 4, 5, 11–14]. See [15] and [16] as well.

2.1 Quantum hydrodynamics

Let us discuss, for simplicity, the motion of a spinless particle of mass $m > 0$ in 3-space. The quantum wave function depends on x and t: $\psi = \psi(x, t)$, with $x \in \mathbb{R}^3$, $t \in \mathbb{R}$ and obeys the *Schrödinger equation* (set $\hbar = m = 1$)

$$\partial_t \psi = -i\hat{H}\psi := -i(-\frac{1}{2}\Delta + V)\psi \tag{1}$$

Gabriele Barbieri and Mauro Spera have contributed equally to this work.

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with Δ denoting the Laplace operator and $V = V(x)$ a ("classical") potential. Its polar decomposition (as soon as $\rho > 0$) reads

$$\psi = \sqrt{\rho}\, e^{iS}, \qquad \rho = |\psi|^2. \tag{2}$$

However, we shall, whenever expedient, stick to Schrödinger's ψ-formalism, in order to bypass the limitations imposed by the latter.

The Schrödinger equation can be cast into the Madelung-Bohm hydrodynamical form [17–19] (setting $\mathbf{u} = \nabla S$)

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla(V + V_q) \\ \frac{\partial \rho}{\partial t} + \mathrm{div}(\rho\, \mathbf{u}) = 0 \end{cases}, \tag{3}$$

where $V_q = -\frac{1}{2}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}$ is the so-called quantum potential, the first equation is an Euler equation for a *compressible* irrotational fluid ($\mathrm{div}\, \mathbf{u} = \Delta S \neq 0$ in general) and the second one is the continuity equation, involving the probability current

$$\mathbf{j} = \rho\, \mathbf{u} = \mathrm{Im}(\psi^\dagger \nabla \psi) = \frac{1}{2i}(\psi^\dagger \nabla\psi - \psi\nabla\psi^\dagger). \tag{4}$$

We shall freely switch vector fields and differential 1-forms via the musical isomorphisms induced by the Euclidean metric in $\mathbb{R}^3$, so we write for instance $j = \rho\, dS$ and so on. After setting

$$\mathcal{H} := \langle\psi|\hat{H}\psi\rangle = \int_{\mathbf{R}^3} \psi^\dagger \left(-\frac{1}{2}\Delta + V(x)\right)\psi\, d^3x = \int_{\mathbf{R}^3} \left\{\frac{1}{2}\, |\nabla\psi|^2 + V(x)|\psi|^2\right\} d^3x, \tag{5}$$

one can rephrase the above equations in a Hamiltonian fashion, following Bohm [17, 18]:

$$\frac{\partial\rho}{\partial t} = \frac{\delta\mathcal{H}}{\delta S}, \qquad \frac{\partial S}{\partial t} = -\frac{\delta\mathcal{H}}{\delta\rho} \tag{6}$$

or, in complex terms, see [14]:

$$\frac{\partial\psi}{\partial t} = -i\frac{\delta\mathcal{H}}{\delta\psi^\dagger}, \qquad \frac{\partial\psi^\dagger}{\partial t} = i\frac{\delta\mathcal{H}}{\delta\psi}. \tag{7}$$

2.2 Symplectic geometric interpretation

The above discussion can be geometrically reformulated as follows, glossing over analytical subtleties (see [1, 14] for extra bibliography and [20] for a general theory).

Let $\psi : \mathbb{R}^3 \ni x \mapsto (\rho(x), S(x)) \equiv \psi(x) \in \mathbb{R}^2 \cong \mathbb{C}$ be a smooth map (we may use polar coordinates whenever $\rho > 0$, so $S(x) \in \mathbb{R}/2\pi\mathbb{Z} = S^1$). The set $\widetilde{\mathcal{M}}$ of such maps becomes a symplectic manifold as soon as the target space is equipped with the symplectic structure $d\rho \wedge dS$ (and we tacitly compactify $\mathbb{R}^3$ to S^3). The symplectic form Ω on $\widetilde{\mathcal{M}}$ reads

$$\Omega = \int_{\mathbb{R}^3} \delta\rho(x) \wedge \delta S(x) d^3x \tag{8}$$

or, in complex coordinates (*Kähler* structure),

$$\Omega = -i \int_{\mathbb{R}^3} \delta\psi^\dagger(x) \wedge \delta\psi(x) d^3x \tag{9}$$

Let G denote the (connected component of the identity of the) group of *volume preserving* diffeomorphisms of $\mathbb{R}^3$ which rapidly approach the identity at infinity, with "Lie algebra" $\mathfrak{g}$ consisting of divergence-free vector fields ($\mathrm{div}\, \mathbf{b} = 0$) rapidly vanishing at infinity. The space $\mathbb{R}^3$ is equipped with the standard Euclidean metric (allowing the customary identification of vector fields and 1-forms). The symplectic form Ω is *G-invariant* under the natural action of G on $\psi \in \mathcal{M}$ via

$$g(\psi)(x) = \psi(g^{-1}(x)) \tag{10}$$

(since, the Jacobian $J(g) = 1$).

Notice that the probability current can be viewed itself as the velocity field of a new fluid (again compressible), with vorticity

$$\mathbf{w} = \mathrm{curl}\, \mathbf{j} = \nabla\rho \times \nabla S, \quad w = dj = d\rho \wedge dS = -i\, d\psi^\dagger \wedge d\psi, \tag{11}$$

see [1]. The Hamiltonian algebra (Rasetti-Regge (RR) current algebra, [6–9, 21, 22]) associated to the G-coadjoint orbit pertaining to the divergence-free vector field $\mathbf{w}$ consists of functions $\lambda_{\mathbf{b}}$ - for any $\mathbf{b}$ divergence-free - defined as

$$\lambda_{\mathbf{b}} = \int_{\mathbb{R}^3} \mathbf{j} \cdot \mathbf{b} = \int_{\mathbb{R}^3} \mathbf{w} \cdot \mathbf{B}, \tag{12}$$

where $\mathrm{curl}\, \mathbf{B} = \mathbf{b}$. One checks the Lie algebra structure of the RR-current algebra, namely

$$\{\lambda_{\mathbf{b}}, \lambda_{\mathbf{c}}\} = \lambda_{[\mathbf{b},\mathbf{c}]}, \tag{13}$$

the vector field bracket being *minus* the usual one and reading, for divergence-free vector fields

$$[\mathbf{b}, \mathbf{c}] = \mathrm{curl}\,(\mathbf{b} \times \mathbf{c}) \tag{14}$$

and, where the Poisson bracket is the one induced by the symplectic form. Explicitly:

$$\{\lambda_{\mathbf{b}}, \lambda_{\mathbf{c}}\} = \int_{\mathbb{R}^3} \mathbf{j} \cdot [\mathbf{b}, \mathbf{c}] = \int_{\mathbb{R}^3} \mathbf{j} \cdot \mathrm{curl}(\mathbf{b} \times \mathbf{c}) = \int_{\mathbb{R}^3} \mathbf{w} \cdot (\mathbf{b} \times \mathbf{c}). \tag{15}$$

One has the following result in [14], with slight and obvious notational changes.

Theorem 1 *(i) The action of G on $(\widetilde{\mathcal{M}}, \Omega)$ gives rise to an equivariant moment map*

$$\begin{aligned} \mathrm{j} : \widetilde{\mathcal{M}} &\to \mathfrak{g}^* \\ \psi &\mapsto \mathrm{j}(\psi) = \boldsymbol{w} = \nabla\rho \times \nabla S = -i\,(\nabla\psi^\dagger \times \nabla\psi) \end{aligned}$$

(ii) In terms of the ensuing Hamiltonian algebra (RR-current algebra), the Schrödinger equation can be written compactly in the Hamiltonian form

$$\dot{\lambda}_{\boldsymbol{b}} = \{\lambda_{\boldsymbol{b}}, \mathcal{H}\} \tag{16}$$

Remark 1 The assignment $\mathfrak{g} \ni \mathbf{b} \mapsto \lambda_{\mathbf{b}} \in C^\infty(\widetilde{\mathcal{M}})$ actually yields what is called a co-momentum map a standpoint to be adopted in the sequel. Also, $\mathfrak{g} \hookrightarrow \mathfrak{g}^*$, the dual of the "Lie algebra" $\mathfrak{g}$ (naturally accommodating singular objects, as we shall see).

The above result underlines the special role played by the probability current and generalises the standard geometrical description of Euler flows [6–9, 15, 16, 23, and 24]. There are other moment map interpretations of j, in conjunction with ρ, see [4, 5, 11, and 13]. As it was observed in [1], although the equations can be written in a geometric form, the ensuing evolution is *not* a coadjoint motion since the Hamiltonian *does not collectivize* via j (in the sense of Guillemin and Sternberg, see e.g. [25]).

2.3 Circulation quantization

The polar decomposition breaks down at the zeros of ψ: if $\rho(\tilde{x}) = 0$, S is undetermined. Set $K := \psi^{-1}(0)$ (at $t = 0$) and assume it is a knot in 3-space (or, better, in its compactification S^3), and call it *nodal line*. Definiteness of the wave function then requires *quantization* of circulation, see [3, 14]:

$$\int_{\mathcal{C}} dS = 2\pi n, \qquad n \in \mathbb{Z} \tag{17}$$

with $\mathcal{C}$ a closed loop encircling K. This can be interpreted geometrically as follows, via [14]: the *flat* (Maurer-Cartan) $\mathbb{C}^\times$-connection (gauge field)

$$\frac{1}{2\pi i}\, d\log\psi = \frac{1}{2\pi i}\left(\frac{1}{2} d\log\rho + i\,dS\right) \tag{18}$$

defines an element

$$\left[\frac{1}{2\pi i}\, d\log\psi\right] = \left[\frac{dS}{2\pi}\right] \in H^1(S^3 \setminus K; \mathbb{Z}) \cong \mathbb{Z} \tag{19}$$

and, as such, has trivial holonomy. See [14] for gauge theoretic aspects of Schrödinger's theory.

We also wish to point out the recent article [26], wherein circulation quantization together with a zero-helicity condition is recovered within a general Gross-Pitaevskii setting.

2.4 Vortex line motion via probability current

The Madelung velocity ∇S is then *singular* on K, whereby a naive manipulation of the hydrodynamical equations can be misleading and calls for recourse to "multivalued" fields, [2, 27], or, mathematically, to appropriate de Rham currents, [28]. However, in [1], a different route has been pursued. First of all, a formula for the velocity field of the nodal line given in [29] (see also [1]) was related to the time derivative of the probability current: indeed one finds

$$\partial_t j|_K = -\frac{1}{4}[(\Delta\psi^\dagger)d\psi + (\Delta\psi)d\psi^\dagger] \tag{20}$$

which, up to a $\pi/2$-rotation and scaling, coincides with the formula given in [29].

Moreover, the time derivative of the vorticity $w = dj = -i[d\psi^\dagger \wedge d\psi]$ reads, on K:

$$\partial_t w|_K = -\frac{1}{2}[d\psi^\dagger \wedge d(\Delta\psi) - d(\Delta\psi^\dagger) \wedge d\psi]\,. \tag{21}$$

2.5 On Berry's problem and incompressibility

A general solution of the so-called "Berry's problem" [30]: *To construct a wave function ψ pertaining to the hydrogen atom (or to a generic quantum system) having a prescribed nodal line K*, was sketched in [31]. Then, the question was asked concerning the possible *time evolution* of K. Let us briefly review the approach to the above problem outlined in [1] (see also [26] for an independent related approach). View the wave function (at $t = 0$) as a map $\psi : S^3 \to D \subset \mathbb{C} \subset S^2$ (via stereographic projection, D a disc): $\psi = \sqrt{\rho_0} \cdot e^{iS}$. If ψ has a nodal line K, its phase can be adjusted so as to give the new wave function

$$\psi_K(x) = \sqrt{\rho_0} \cdot e^{iS_K} \tag{22}$$

with S_K the *solid angle* function attached to K (defined up to integral multiples of 4π), which is *harmonic* outside K. Then, in terms of *de Rham currents* (distribution-valued forms)

$$dS_K = B_K, \qquad dB_K = 2\pi\delta_K \tag{23}$$

with B_K the magnetic field (or fluid velocity) generated by the "wire" (or vortex line) K (see e.g. [32–35] for a historical account and Subsection 3.2 below for amplification). The level surfaces $S_K = c$ provide *Seifert surfaces* for K, i.e. they are oriented surfaces with boundary K (a smooth extension thereof to K being possible). In particular, for $c = 0$ we get a canonical framing, the so-called *solid angle framing*, see [32]. If ψ_K is taken as the initial wave function, its initial density $\rho_0\, d^3x$ (which vanishes exactly on K) evolves via the Schrödinger-Madelung equation (*advection*) according to [11], and so does the nodal line. Explicitly, closely following [11] up to inessential notational changes:

$$\rho(x,t) = \eta_*\rho_0 := \int \rho_0(y)\delta(x - \eta(y,t))d^3y \tag{24}$$

(Eulerian density), where $\eta \in \mathrm{Diff}(\mathbb{R}^3)$ – with velocity vector field $\mathbf{u} = \dot{\eta} \circ \eta^{-1}$ – and η_* denotes push-forward (advection) via η. The curves $t \mapsto \eta(y, t)$ are the Bohmian trajectories of the quantum fluid.

Therefore, the Schrödinger-Madelung evolution via diffeomorphisms η_t (where $\eta_t(y) = \eta(y, t)$ – a Lagrangian path, see [11]) produces knots K_t together with their Seifert surfaces Σ_t.

If, as observed in [1], we require that, at each instant t, via a *U(1)-gauge* (connection), the phase of the wave function equals the solid angle S_{K_t}, then the ensuing velocity $\mathbf{u}_K := \nabla S_{K_t}$ (which is singular on K_t) is divergence-free as S_{K_t} is harmonic. In detail, if

$$\tilde{\psi} := g(x,t)\psi(x,t) \equiv \sqrt{\rho(x,t)}\, e^{iS_{K_t}(x)}, \tag{25}$$

a short calculation then shows that $\tilde{\psi}$ obeys a new "gauged" Schrödinger equation

$$i\partial_t\tilde{\psi} = \tilde{H}\tilde{\psi} \tag{26}$$

with time dependent Hamiltonian

$$\tilde{H} = g^{-1}\partial_t g + g^{-1}H\, g. \tag{27}$$

As it was observed in [1], this is in accordance with Brylinski's theorem ([23], Theorem 3.7.4) stating that, if Y denotes the space of oriented knots in a 3-fold, then its connected components coincide with the orbits of the connected component G^0 of the group G of volume preserving diffeomorphisms of Y and, moreover, any component $\mathcal{C}$ of Y is a homogeneous space G^0/G^0_C for any $C \in \mathcal{C}$ (G^0_C being the isotropy group of C). The above discussion serves as a concrete illustration of the above theorem. One may thus conclude as in [1] that, so long as the nodal line is concerned, its (compressible) Madelung evolution can be traded for an incompressible one. This will be further elaborated on in what follows.

3 The Clebsch picture

The variables ρ and S are of *Clebsch* type, [1, 14]: the level surfaces $\rho = c_1$ and $S = c_2$ (c_1 and $c_2 \in \mathbb{R}$) – i.e. the level surfaces of ψ – give rise, generically, to one-dimensional fibres $S^1 \subset S^3$ everywhere tangent to the vorticity field

$$\mathbf{w} = \nabla\rho \times \nabla S = -i\,(\nabla\psi^\dagger \times \nabla\psi). \tag{28}$$

In particular, if $\rho = 0$, S is undetermined but we still obtain $\psi^{-1}(0) \approx S^1$, namely, the nodal line becomes a fibre of the Clebsch fibration, see [1, 8, 24, 36, and 37].

The probability current $\mathbf{j}$ may be rendered divergence-free by adding a suitable gradient $\nabla\theta$ upon solving an appropriate Poisson equation: indeed, from

$$\tilde{\mathbf{j}} := \mathbf{j} + \nabla\theta, \qquad \mathrm{div}\,\tilde{\mathbf{j}} = 0 \tag{29}$$

we get,

$$\Delta\theta = -\mathrm{div}\,\mathbf{j}, \qquad \theta = -\Delta^{-1}\mathrm{div}\,\mathbf{j}. \tag{30}$$

This does not affect (globally) the vortex lines, since $\tilde{\mathbf{w}} := \mathrm{curl}\ \tilde{\mathbf{j}} = \mathbf{w}$, albeit changing the single Bohmian trajectories. Also, the RR currents do not change.

In order to bypass the polar decomposition, we rephrase the above in terms of ψ (and freely using differential forms), in particular, the vorticity also reads

$$\omega = -i d\psi^{\dagger} \wedge d\psi = dj = d\tilde{j}. \tag{31}$$

We shall establish our Clebsch portrait in a series of steps.

3.1 The Fubini-Study Kähler form

Let us now turn to the projective version (call it $\mathcal{M}$) of our initial manifold $\widetilde{\mathcal{M}}$ by taking norm one ψ's up to a phase – using the notation of [4, 5]; specifically, $\mathcal{M} = \mathbb{P}(\mathcal{H})$, with $\mathcal{H}$ be a Hilbert-Sobolev space of wave functions (however using an L^2-inner product thereon, as in the quoted references). Then let $\dot{\psi}$ be a tangent vector at $[\psi]$: as such, it is determined up to a summand $ic\,\psi$, $c \in \mathbb{R}$, which vanishes upon restriction to $\psi^{-1}(0) = K \subset \mathbb{R}^3$; this can be quickly ascertained as follows: from the requirement $\langle \psi + \epsilon\dot{\psi}, \psi + \epsilon\dot{\psi}\rangle = 1$ at first order in ϵ, we get

$$\Re\langle \psi, \dot{\psi}\rangle = 0 \tag{32}$$

and this condition persists via alteration of vectors by a phase factor and upon replacement of $\dot{\psi}$ by $\dot{\psi} + ic\,\psi$. The manifold $\mathcal{M}$ comes equipped with the Fubini-Study (FS) symplectic Kähler form

$$\Omega^{FS}_{\mathcal{M}\,[\psi]}(\dot{\psi}_1, \dot{\psi}_2) := \Im\int_{\mathbb{R}^3} \overline{\dot{\psi}_1}\,\dot{\psi}_2\ d^3x. \tag{33}$$

The G-equivariant moment map j induces an analogous map on $\mathcal{M}$, in view of (global) phase invariance of the vorticity.

3.2 The map $\mathscr{K}$

Let us now introduce the map

$$\mathscr{K} : \mathcal{M} \ni [\psi] \mapsto K = \psi^{-1}(0) \equiv \omega_K \in \mathcal{O}_{\omega_K}, \tag{34}$$

where ω_K is the (singular) Poincaré dual of K, see [38], i.e. a de Rham current -corresponding to a δ-like vorticity concentrated on K, previously denoted by δ_K in (23): in vector terms $\mathbf{w}_K(x) = \int_K \delta^3(x - y(s))\,dy(s) \in \mathfrak{g}^*$, and $\mathcal{O}_{\omega_K}$ is the coadjoint orbit pertaining to it.

Let us then "localize" on K by pulling back wave functions to K and by replacing $\int_{\mathbb{R}^3}$ by $\int_K$ throughout, in particular in evaluating the Fubini-Study form.

The crucial observation is the following: let K be parametrized by its arc-length s and $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ be the standard Frénet trihedron (assuming absence of inflection points in K; in any case, the latter yield measure zero sets); then posit

$$\dot{\psi} = \Re\dot{\psi} + i\,\Im\dot{\psi}\ \equiv\ \dot{\psi}_n + i\,\dot{\psi}_b\,, \tag{35}$$

abutting at the vector field on K

$$\mathbf{a}_{\dot{\psi}}(s) := \dot{\psi}_n(s)\,\mathbf{n}(s) + \dot{\psi}_b(s)\,\mathbf{b}(s)\,. \tag{36}$$

Therefore, we may set up the identification

$$\dot{\psi}|_K \equiv \mathbf{a}_{\dot{\psi}} \tag{37}$$

and, given $\dot{\psi}_1$, $\dot{\psi}_2$, we get (obvious notation)

$$\int_K \mathbf{t}\cdot(\mathbf{a}_{\dot{\psi}_1} \times \mathbf{a}_{\dot{\psi}_2})\,ds = \Omega_{\omega_K}(\mathbf{a}_{\dot{\psi}_1}, \mathbf{a}_{\dot{\psi}_2}) = \widetilde{\Omega^{FS}}_{\mathcal{M}}(\dot{\psi}_1|_K, \dot{\psi}_2|_K) \equiv \Im\int_K \overline{\dot{\psi}_1|_K}\,\dot{\psi}_2|_K\,. \tag{38}$$

(after an appropriate integral replacement in the FS-form, signalled by$^{\sim}$). The formally integrable complex structure of $\mathcal{O}_{\omega_K}$, see [23], and is then compatible with the standard complex structure of $\mathcal{M}$. The map $\mathscr{K}$ becomes, with the above proviso, a G-equivariant moment map (see below, Subsection 3.5).

Remark 2 The "Marsden-Weinstein-Meyer quotient"

$$\mathscr{K}^{-1}(K)/G_{\omega_K} \cong \{K\}. \tag{39}$$

Indeed, one considers wave functions having the same zero set K; then, the G_{ω_K}-action just reparametrizes K and it is quite arbitrary elsewhere (see also below). These functions give rise to the same integral cohomology class $[\frac{1}{2\pi i} d \log \psi] \in H^1(S^3 \backslash K; \mathbb{Z})$.

3.3 Localising to K

We have another "localization" map ℓ which is, at least formally, consistently defined in view of the analysis carried out in [9] (and one has $\ell \circ j = \mathscr{K}$); it is given explicitly as follows ("$\cdot$" denoting coadjoint action)

$$\ell : \mathcal{O}_\omega \ni g \cdot \omega \mapsto g \cdot \omega_K \in \mathcal{O}_{\omega_K} \tag{40}$$

which is well-defined, since for the respective vorticity isotropy groups one has

$$G_\omega \subset G_{\omega_K} \tag{41}$$

upon remembering that, at the Lie algebra level:

$$\mathfrak{g}_\omega = \{\mathbf{u} \in \mathfrak{g} \mid [\mathbf{u}, \mathbf{w}] = 0\}, \tag{42}$$

see [8]. Concretely, one has a "migration" from a diffuse to a string-like vorticity ($\omega \to \omega_K$ in $\mathfrak{g}^*$), reflected in the corresponding RR-currents $\lambda_{\mathbf{b}}$, see Subsection 3.5 for details.

The same will hold for the other localization map $\hat{\ell}$ which will be shortly introduced.

3.4 The n-field representation

According to [8] (see also [36, 38]) smooth maps $\mathbf{n} : S^3 = \mathbb{R}^3 \cup \{\infty\} \to S^2$ (in a fixed homotopy class and such that $\mathbf{n}(\infty) = \mathbf{n}_\infty$, fixed once for all) give rise to a symplectic Kähler manifold $\widetilde{\mathcal{N}}$ (Clebsch manifold) and any such map determines a Hopf number (helicity)

$$\mathscr{H}(\mathbf{n}) = \int \tilde{j} \wedge d\tilde{j} \;\; \in \mathbb{Z}. \tag{43}$$

representing the linking number of two generic fibres (diffeomorphic to S^1).

In detail, let

$$\mathbf{n} = (\sin\vartheta \cos\varphi,\ \sin\vartheta \sin\varphi,\ \cos\vartheta) \in S^2 \tag{44}$$

(with ϑ and φ denoting, as usual, colatitude and longitude, the latter counted from the x-axis). Set

$$\psi = \tan\frac{\vartheta}{2}\, e^{i\varphi} = \sqrt{\rho}\, e^{iS}, \tag{45}$$

Notice, that upon a phase change $\psi \mapsto e^{i\alpha}\psi$, the vorticity ω does not change and $\mathbf{n}$ rotates around the polar axis by an angle α, whence we get a map

$$\mathscr{I} : \mathcal{M} \ni [\psi] \mapsto [\mathbf{n}] \in \mathcal{N} \equiv \widetilde{\mathcal{N}}/S^1. \tag{46}$$

Let us define the new vorticity (also phase-invariant)

$$\hat{\omega} = -i\, \frac{d\psi^\dagger \wedge d\psi}{(1 + |\psi|^2)^2}, \tag{47}$$

pointwise proportional to ω and, in particular, coinciding with it on K: $\omega|_K = \hat{\omega}|_K$, and tangent to it. Then it is easily checked that

$$\hat{\omega} = \frac{1}{2\pi}\mathbf{n}^*\sigma \tag{48}$$

with σ the normalised area form on S^2. Correspondingly, we have the G-equivariant moment map

$$\nu : \psi \mapsto \hat{\omega} \tag{49}$$

introduced in [8], yielding a map (same notation)

$$\nu : \mathcal{N} \supset \mathscr{I}(\mathcal{O}_{[\psi]}) \to \mathcal{O}_{\hat{\omega}} \tag{50}$$

($\mathcal{O}_{[\psi]}$ being the G-orbit of $[\psi]$ in $\mathcal{M}$). One has

$$d\hat{j} = \hat{\omega} \tag{51}$$

for some $\hat{j}$, determined up to an exact 1-form, as the cohomology group $H^2(S^3) = 0$; thus one has a G-equivariant moment map $\hat{\mathrm{j}}$ analogous to the above j. The symplectic Kähler structure of $\widetilde{\mathcal{N}}$ and $\mathcal{N}$ reads, for generic smooth vector fields **b** and **c**, [8, 9]:

$$\Omega^{\mathcal{N}}_{\mathbf{n}}(X_{\mathbf{b}}, X_{\mathbf{c}}) \equiv \int_{\mathbb{R}^3} \mathbf{n}(x) \cdot (\mathbf{b}\cdot\nabla\, \mathbf{n}(x) \times \mathbf{c}\cdot\nabla\, \mathbf{n}(x))\, d^3x = \int_{\mathbb{R}^3} \sigma_{\mathbf{n}(x)}(\mathbf{b}\cdot\nabla\, \mathbf{n}(x), \mathbf{c}\cdot\nabla\, \mathbf{n}(x))\, d^3x. \tag{52}$$

The Hopf number $\int \hat{j} \wedge \hat{\omega}$ equals the former one (the Clebsch fibres being the same).

As anticipated, we also have another obvious localization map

$$\hat{\ell} : \mathcal{O}_{\hat{\omega}} \to \mathcal{O}_{\omega_K} \tag{53}$$

enjoying the same properties as the previous one.

3.5 RR-picture

As anticipated, we have the following (co-momentum) description in terms of RR-currents.

First

$$\psi \mapsto \{\mathbf{b} \mapsto \lambda^{\mathscr{K}}_{\mathbf{b}} := \int_K \mathbf{B} = \lambda^{\mathcal{O}_{\omega_K}}_{\mathbf{b}}\}, \qquad K = \psi^{-1}(0), \tag{54}$$

(again $\mathbf{b} = \mathrm{curl}\,\mathbf{B}$). Moreover, localization via ℓ yields

$$\lambda^{\mathcal{O}_\omega}_{\mathbf{b}} = \int \mathbf{w} \cdot \mathbf{B} \to \int_K \mathbf{B} = \lambda^{\mathcal{O}_{\omega_K}}_{\mathbf{b}} \tag{55}$$

and, similarly, for $\hat{\ell}$,

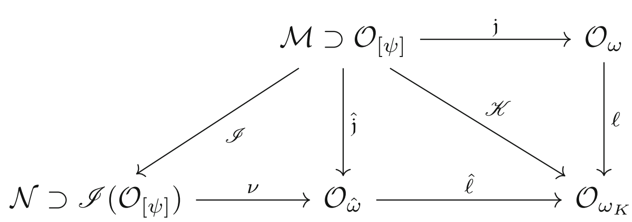
$$\lambda^{\mathcal{O}_{\hat{\omega}}}_{\mathbf{b}} = \int \widehat{\mathbf{w}} \cdot \mathbf{B} \to \lambda^{\mathcal{O}_{\omega_K}}_{\mathbf{b}} \tag{56}$$

3.6 The main result

We collect the previous findings in a theorem, whose proof immediately follows from the above preparations.

Theorem 2 *(Clebsch portrait).*

(i) With the above notation, we have a commutative diagram:



where all maps other than $\mathscr{I}$ are G-equivariant moment maps and $\mathscr{K}$, ℓ, $\hat{\ell}$ all "localize" to K.

(ii) The overall diagram is compatible with the dynamics, provided we replace the Schrödinger Hamiltonian H by the time dependent Hamiltonian $\tilde{H}$, thus switching to a perfect fluid flow induced by the gradient of the solid angle function, thereby keeping track of the vortex line evolution only ("slow variable" motion).

(iii) The complex structures on $\mathcal{M}$, $\mathcal{N}$, $\mathcal{O}_{\omega_K}$ are compatible, at fixed time.

Remark 3 The use of two different vorticities ω and $\hat{\omega}$ was motivated by the fact that here the Clebsch variables (ρ and S) are dictated by the quantum mechanical context and the ensuing **n**-field representation is ancillary. In the papers [6, 8, and 36], the latter was the primary object, whereby vorticities arose.

4 Conclusions and outlook

In this paper we delved into a Clebsch picture for the Schrödinger equation, pursuing the investigation in [1], focussing on evolution of the zero set of a wave function, assuming it is a knot in 3-space (vortex line). Its Schrödinger motion can be traded for its Euler evolution via the (divergence-free) current probability field $\tilde{j}$, which, unlike the Madelung velocity, is regular on K, though sharing the same trajectories with the latter outside K. A somewhat intricate but vivid network of symplectic manifolds arose. The upshot is that the standard quantum mechanical complex formalism, complemented by the projective geometry of the pure state space, naturally bypasses the limitations of the polar decomposition of the wave function. The emerging concept of helicity of the latter may yield further insights in quantum mechanical issues. In particular, establishment of a direct connection with the approach outlined in [26] would be desirable.

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References

1. M. Spera, Hydrodynamics of the probability current in Schrödinger theory. In Nielsen, F., Barbaresco, F. (eds.) Geometric Science of Information. GSI 2023. Lecture Notes in Computer Science, vol 14072. Springer, Cham, Springer Nature (2023), 272-281. Online 1 (2023)
2. F.E.A. dos Santos, Hydrodynamics of vortices in Bose-Einstein condensates: a defect-gauge field approach. Phys. Rev. A **94**(6), 063633 (2016)
3. T. Takabayasi, Hydrodynamical Formalism of Quantum Mechanics and Aharonov-Bohm Effect. Prog. Theor. Phys. **69**(5), 1323–1344 (1983)
4. B. Khesin, G. Misiołek, K. Modin, Geometric hydrodynamics via Madelung transform. PNAS **115**(24), 6165–6170 (2018)
5. B. Khesin, G. Misiołek, K. Modin, Geometry of the Madelung transform. Arch. Rational Mech. Anal. **234**, 549–573 (2019)
6. V. Penna, M. Rasetti, M. Spera, Quantum Dynamics of 3-D Vortices, in "Secondary Calculus and Cohomological Physics", M. Henneaux, J. Krasil'shchik, A. Vinogradov (Eds.), Contemporary Mathematics **219**, 173–193 (1998)
7. V. Penna, M. Spera, A geometric approach to quantum vortices. J. Math. Phys. **30**(12), 2778–2784 (1989)
8. V. Penna, M. Spera, On coadjoint orbits of rotational perfect fluids. J. Math. Phys. **33**(3), 901–909 (1992)
9. V. Penna, M. Spera, String limit of vortex current algebra. Phys. Rev. B **62**, 14547–14553 (2000)
10. R.E. Wyatt, *Quantum Dynamics with Trajectories* (Springer, New York, 2005)
11. M.S. Foskett, D.D. Holm, C. Tronci, Geometry of nonadiabatic quantum hydrodynamics. Acta Appl. Math. **162**, 63–103 (2019)
12. M.S. Foskett, C. Tronci, Holonomy and vortex structures in quantum hydrodynamics, in *Hamiltonian systems: dynamics, analysis, applications*. ed. by A. Fathi, P.J. Morrison, T.M. Seara, S. Tabachnikov (Cambridge University Press, Cambridge, 2022)
13. D. Fusca, The Madelung transform as a momentum map. J. Geom. Mech. **9**(2), 157–165 (2017)
14. M. Spera, Moment map and gauge geometric aspects of the Schrödinger and Pauli equations. Int. J. Geom. Methods Mod. Phys. **13**(4), 1630004 (2016)
15. D.D. Holm, T. Schmah, C. Stoica, *Geometric mechanics and symmetry: from finite to infinite dimensions* (Oxford University Press, Oxford, 2009)
16. V.I. Arnold, B.A. Khesin, *Topological methods in hydrodynamics*, 2nd edn. (Springer Nature, Switzerland, 2021)
17. D. Bohm, A Suggested Interpetation of the Quantum Theory in Terms of "Hidden" Variables I. Phys. Rev. **85**(2), 166–179 (1952)
18. D. Bohm, A Suggested Interpetation of the Quantum Theory in Terms of "Hidden" Variables II. Phys. Rev. **85**(2), 180–193 (1952)
19. E. Madelung, Quantentheorie in hydrodynamischer form. Zeit. F. Phys. **40**, 322–326 (1927)
20. S.K. Donaldson, Moment maps and diffeomorphisms. Asian J. Math. **3**(1), 1–16 (1999)
21. G.A. Goldin, R. Menikoff, D.H. Sharp, Particle statistics from induced representations of a local current group. J. Math. Phys. **21**(4), 650–664 (1980)
22. M. Rasetti, T. Regge, Vortices in He II, current algebras and quantum knots. Phys. A **80**(3), 217–233 (1975)
23. J.L. Brylinski, *Loop spaces* (Characteristic Classes and Geometric Quantization. Birkhäuser, Boston, Basel, Berlin, 1993)
24. J. Marsden, A. Weinstein, Coadjoint orbits, vortices and Clebsch variables for incompressible fluids. Physica **7D**, 305–323 (1983)
25. V. Guillemin, S. Sternberg, *Symplectic techniques in physics* (Cambridge University Press, Cambridge, 1984)
26. A. Belloni, R.L. Ricca, On the zero helicity condition for quantum vortex defects. J. Fluid Mech. **963**, R-21-R−29 (2023)
27. H. Kleinert, *Multivalued fields in condensed matter, electromagnetism, and gravitation* (World Scientific, Singapore, 2008)
28. G. de Rham, *Differentiable manifolds* (Springer, Berlin, 1984)
29. I. Bialynicki-Birula, Z. Bialynicka-Birula, C. Śliwa, Motion of vortex lines in quantum mechanics. Phys. Rev. A **61**(7), 032110 (2000)
30. M. Berry, Knotted zeros in the quantum states of hydrogen. Found. Phys. **31**(4), 659–667 (2001)
31. L.H. Kauffman, S.J. Lomonaco, Jr. Quantum knots and knotted zeros. Proc. SPIE 10984, Quantum Information Science, Sensing, and Computation XI, 109840A (2019)
32. J. Binysh, G.P. Alexander, Maxwell's theory of solid angle and the construction of knotted fields. J. Phys. A: Math. Theor. **51**(20), 385202 (2018)
33. M. Borodzik, S. Dangskul, A. Ranicki, Solid angles and Seifert hypersurfaces. Ann. Glob. Anal. Geom. **57**, 415–454 (2020)
34. R. Fenn, *Techniques of geometric topology* (Cambridge University Press, Cambridge, 1983)
35. R.L. Ricca, B. Nipoti, Gauss' linking number revisited. J. Knot Theory Ramif. **20**(10), 1325–1343 (2011)
36. E.A. Kuznetsov, A.V. Mikhailov, On the topological meaning of canonical Clebsch variables. Phys. Lett. A **77**(1), 37–38 (1980)
37. H. Lamb, *Hydrodynamics* (Cambridge University Press, Cambridge, 1932)
38. R. Bott, L. Tu, *Differential forms in algebraic topology* (Springer, Berlin, 1982)