DE GRUYTER Advanced Nonlinear Studies 2023; 23: 20230086

Research Article

Yahui Jiang, Taiyong Chen, Jianjun Zhang, Marco Squassina*, and Nouf Almousa

Ground states of Schrödinger systems with the Chern-Simons gauge fields

https://doi.org/10.1515/ans-2023-0086
received February 3, 2023; accepted July 2, 2023

Abstract: We are concerned with the following coupled nonlinear Schrödinger system:

$$\begin{cases} -\Delta u + u + \left(\int\limits_{|x|}^{\infty} \frac{h(s)}{s} u^2(s)\mathrm{d}s + \frac{h^2(|x|)}{|x|^2}\right) u = |u|^{2p-2}u + b|v|^p|u|^{p-2}u, & x \in \mathbb{R}^2, \\ -\Delta v + \omega v + \left(\int\limits_{|x|}^{\infty} \frac{g(s)}{s} v^2(s)\mathrm{d}s + \frac{g^2(|x|)}{|x|^2}\right) v = |v|^{2p-2}v + b|u|^p|v|^{p-2}v, & x \in \mathbb{R}^2, \end{cases}$$

where $\omega, b > 0$, $p > 1$. By virtue of the variational approach, we show the existence of nontrivial ground-state solutions depending on the parameters involved. Precisely, the aforementioned system admits a positive ground-state solution if $p > 3$ and $b > 0$ large enough or if $p \in (2, 3]$ and $b > 0$ small.

Keywords: Schrödinger systems, ground-states, Chern-Simons gauge fields, variational methods

MSC 2020: 35B09, 35J50, 81T10

1 Introduction

In this article, we consider the following coupled Schrödinger equations with the Chern-Simons gauge fields:

$$\begin{cases} -\Delta u + u + \left(\int\limits_{|x|}^{\infty} \frac{h(s)}{s} u^2(s)\mathrm{d}s + \frac{h^2(|x|)}{|x|^2}\right) u = |u|^{2p-2}u + b|v|^p|u|^{p-2}u, & x \in \mathbb{R}^2, \\ -\Delta v + \omega v + \left(\int\limits_{|x|}^{\infty} \frac{g(s)}{s} v^2(s)\mathrm{d}s + \frac{g^2(|x|)}{|x|^2}\right) v = |v|^{2p-2}v + b|u|^p|v|^{p-2}v, & x \in \mathbb{R}^2, \end{cases} \tag{1.1}$$

where $\omega > 0$, $b > 0$, $p > 1$, and

*** Corresponding author: Marco Squassina,** College of Science, Princess Nourah Bint Abdul Rahman University, PO Box 84428, Riyadh, Saudi Arabia; Dipartimento di Matematica e Fisica, Università Cattolica del Sacro Cuore, Brescia, Via della Garzetta 48, 25133, Italy, e-mail: marsquassina@pnu.edu.sa, marco.squassina@unicatt.it
Yahui Jiang: School of Mathematics, China University of Mining and Technology, Xuzhou, 221116, China, e-mail: 18843111149@163.com
Taiyong Chen: School of Mathematics, China University of Mining and Technology, Xuzhou, 221116, China, e-mail: taiyongchencumt@163.com
Jianjun Zhang: College of Mathematica and Statistics, Chongqing Jiaotong University, Chongqing 400074, China, e-mail: zhangjianjun09@tsinghua.org.cn
Nouf Almousa: College of Science, Princess Nourah Bint Abdul Rahman University, Riyadh, PO Box 84428, Saudi Arabia, e-mail: nmalmousa@pnu.edu.sa

$$h(s) = \int_0^s \frac{r}{2} u^2(r)\mathrm{d}r, \quad g(s) = \int_0^s \frac{r}{2} v^2(r)\mathrm{d}r.$$

When $b = 0$, then System (1.1) is uncoupled and it reduces to two equations of the same type. In recent years, a single nonlinear Schrödinger equation with the Chern-Simons gauge field as follows has received much attention

$$\begin{cases} iD_0\phi + (D_1D_1 + D_2D_2)\phi = -f(\phi), \\ \partial_0 A_1 - \partial_1 A_0 = -\mathrm{Im}(\bar{\phi} D_2 \phi), \\ \partial_0 A_2 - \partial_2 A_0 = \mathrm{Im}(\bar{\phi} D_1 \phi), \\ \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}|\phi|^2, \end{cases} \tag{1.2}$$

where i denotes the imaginary unit, $\partial_0 = \frac{\partial}{\partial t}$, $\partial_1 = \frac{\partial}{\partial x_1}$, $\partial_2 = \frac{\partial}{\partial x_2}$, $(t, x_1, x_2) \in \mathbb{R}^{1+2}$, $\phi : \mathbb{R}^{1+2} \to \mathbb{C}$ is the complex scalar field, $A_\mu : \mathbb{R}^{1+2} \to \mathbb{R}$ is the gauge field, and $D_\mu = -\partial_\mu + iA_\mu$ is the covariant derivative for $\mu = 0, 1, 2$. The Chern-Simons-Schrödinger system consists of Schrödinger equations augmented by the gauge field, which was first proposed and studied in [15,16]. The model was proposed to study vortex solutions, which carry both electric and magnetic charges. This feature of the model is important for the study of the high-temperature superconductor, fractional quantum Hall effect, and Aharovnov-Bohm scattering. For more details about System (1.2), we refer the readers to [10,12,13]. System (1.2) is invariant under gauge transformation

$$\phi \to \phi e^{i\chi}, \quad A_\mu = A_\mu - \partial_\mu \chi,$$

for any arbitrary C^∞ function χ.

Byeon et al. [6] investigated the existence of standing wave solutions for System (1.2) with power-type nonlinearity, that is, $f(u) = \lambda |u|^{p-2}u$ with $p > 2$ and $\lambda > 0$. By using the ansatz

$$\begin{cases} \phi(t, x) = u(|x|)e^{i\omega t}, \quad A_0(t, x) = k(|x|), \\ A_1(t, x) = \frac{x_2}{|x|^2} h(|x|), \quad A_2(t, x) = -\frac{x_1}{|x|^2} h(|x|). \end{cases}$$

Byeon et al. obtained the following nonlocal semilinear elliptic equation:

$$-\Delta u + (\omega + \xi)u + \left(\int_{|x|}^{\infty} \frac{h(s)}{s} u^2(s)\mathrm{d}s + \frac{h^2(|x|)}{|x|^2} \right) u = f(u), \quad x \in \mathbb{R}^2, \tag{1.3}$$

where ξ is a constant and $h(s)$ is defined earlier. Byeon et al. showed that the existence and nonexistence of positive solutions for (1.3) were established depending on the range of $p > 2$ and $\lambda > 0$. For the special case $p = 4$, there exist solutions if $\lambda > 1$. It seems hard to obtain the boundedness of Palais-Smale sequence when $p \in (4, 6)$. They constructed a Nehari-Pohozaev manifold to obtain the boundedness of Palais-Smale sequence. For $p \in (2, 4)$, Pomponio and Ruiz [22] proved the existence and nonexistence of positive solutions for (1.3) under the different range of ω. A series of existence and nonexistence results of solutions for (1.3) have been researched in [7,14,17,19,20,23,30].

Problem (1.1) is a nonlocal problem due to the appearance of the term $\int_{|x|}^{\infty} \frac{h(s)}{s} u^2(s)\mathrm{d}s$, which indicates that (1.1) is not a pointwise identity. This causes some mathematical difficulties that make the study of such a problem particularly interesting. System (1.1) is quite different from the following local scalar field system:

$$\begin{cases} -\Delta u + u = |u|^{2q-2} + b\,|v|^q |u|^{q-2}u, \quad \text{in } \mathbb{R}^N, \\ -\Delta v + \omega^2 v = |v|^{2q-2} + b\,|u|^q |v|^{q-2}v, \quad \text{in } \mathbb{R}^N, \end{cases} \tag{1.4}$$

for $\omega > 0, b \in \mathbb{R}, q \in (2, 2^*)$, which does not depend on the nonlocal term any more. For more results about the existence, multiplicity, and concentration behavior of solutions of the single nonlinear Schrödinger equation, one can refer to [9,29] and references therein. The coupled nonlinear Schrödinger System (1.4) has attracted considerable attention in the past 15 years. Maia et al. [21] by using the variational methods and the ideas of

Rabinowitz [24] investigated the existence of positive ground-state solutions for System (1.4) depending on the parameters b and ω. For more progress in this aspect, we refer to [1–3,5, 18,26,27] and references therein.

The energy functional for System (1.1) $I \in C(E)$ is defined by

$$I(u,v) = \frac{1}{2}\|(u,v)\|_E^2 + \frac{1}{2}(B(u) + B(v)) - \frac{1}{2p}F(u,v),$$

where

$$B(u) = \int_{\mathbb{R}^2} \frac{u^2}{|x|^2}\left(\int_0^{|x|} \frac{s}{2}u^2(s)\mathrm{d}s\right)^2 \mathrm{d}x,$$

$$F(u,v) = \int_{\mathbb{R}^2} (u^{2p} + v^{2p} + 2b|uv|^p)\mathrm{d}x.$$

Here E denotes the subspace of radially symmetric functions in $H_r^1(\mathbb{R}^2) \times H_r^1(\mathbb{R}^2)$ with the norm

$$\|(u,v)\|_E^2 := \int_{\mathbb{R}^2} (|\nabla u|^2 + u^2 + |\nabla v|^2 + \omega v^2)\mathrm{d}x.$$

The critical points of the functional I are weak solutions of (1.1), and elliptic regularity estimates imply that these are classical solutions.

Motivated by [21], we try to study the existence of positive ground-state solutions for coupled Schrödinger equations with the Chern-Simons gauge fields (1.1) with suitable conditions on ω and b.

One of the main difficulties is the boundedness of Palais-Smale sequences if we try to use directly the mountain pass theorem to obtain the critical points of I in E. For $p \geq 3$, it is standard to show that Palais-Smale condition holds for I. For $p \in (2,3)$, the functional I has the mountain-pass geometry. However, it seems hard to prove the Palais-Smale condition holds for the functional I. Motivated by [6,25], by using a constrained minimizer on Nehari-Pohozaev manifold, we circumvent this obstacle.

Another problem is the existence of positive ground-state for System (1.1), i.e., a minimal action solution (u,v) with both $u > 0, v > 0$ nontrivial. We point out that System (1.1) also possesses a trivial solution $(0,0)$ and semi-trivial solutions of type $(u,0)$ and $(0,v)$. A solution (u,v) of (1.1) is nontrivial if $u \not\equiv 0$ and $v \not\equiv 0$. Here, we overcome this obstacle by energy estimation.

We now state the main results of this article. The constants b_1, b_2, b_3, b_4, and b_δ involved in the statement depend on the ground-state of the single equation. We will give the corresponding expressions in Section 3.

Theorem 1.1. *Assume that one of the following conditions holds*

(*i*) $p \in (2,3]$ *and* $b \in (0, b_\delta)$ *sufficiently small,*

(*ii*) $p \in (3, 3+\sqrt{6})$ *and* $b > \max\{b_1, b_2\}$,

(*iii*) $p \in [3+\sqrt{6}, \infty)$ *and* $b > \max\{b_3, b_4\}$, *then system* (1.1) *admits a positive vector ground-state.*

Additionally, we prove also the following nonexistence result.

Theorem 1.2. *There exists* $\tilde{b} > 0$ *sufficiently small and* $\tilde{\omega} > 0$ *sufficiently large such that* $b \in (0, \tilde{b})$ *and* $\omega > \tilde{\omega}$, *then the system* (1.1) *has only trivial solution if* $p \in (1,2]$.

Compared with the case of $p > 2$, it seems that the case of $p \in (1,2]$ becomes more complicated and we will consider it in a forthcoming article. The rest of this article is organized as follows. In Section 2, we present some notations and preliminary results and prove the nonexistence result Theorem 1.2. Then, we give the proof of the existence of a positive ground-state in Theorem 1.1. The items (*ii*) and (*iii*) are proved in Section 3, and the item (*i*) is proved in Section 4.

2 Preliminaries

To prove the main results, we use the following notations:

- $E \coloneqq H_r^1(\mathbb{R}^2) \times H_r^1(\mathbb{R}^2)$ with norm

$$\|(u,v)\|_E^2 = \|u\|_{H_r^1(\mathbb{R}^2)}^2 + \|v\|_{H_r^1(\mathbb{R}^2)}^2.$$

- $L^{2p}(\mathbb{R}^2) \times L^{2p}(\mathbb{R}^2)$ for $p > 1$ with the norm

$$\|(u,v)\|_{2p}^{2p} = \|u\|_{2p}^{2p} + \|v\|_{2p}^{2p}.$$

- $\mathbb{L}_{\text{loc}}^{2p}(\mathbb{R}^2) \coloneqq L_{\text{loc}}^{2p}(\mathbb{R}^2) \times L_{\text{loc}}^{2p}(\mathbb{R}^2)$;

Lemma 2.1. [6] *Suppose that a sequence $\{u_n\}$ converges weakly to a function u in $H_r^1(\mathbb{R}^2)$ as $n \to \infty$. Then, for each $\varphi \in H_r^1(\mathbb{R}^2)$, $B(u_n)$, $B'(u_n)\varphi$, and $B'(u_n)u_n$ converge up to a subsequence to $B(u)$, $B'(u)\varphi$, and $B'(u)u$, respectively, as $n \to \infty$.*

Lemma 2.2. [6] *For $u \in H_r^1(\mathbb{R}^2)$, the following inequality holds*

$$\int_{\mathbb{R}^2} |u|^4 \mathrm{d}x \le 4\left(\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} \frac{u^2}{|x|^2}\left(\int_0^{|x|} \frac{s}{2} u^2(s)\mathrm{d}s\right)^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

Furthermore, the equality is attained by a continuum of functions

$$\left\{u_l = \frac{\sqrt{8}\,l}{1+|lx|^2} \in H_r^1(\mathbb{R}^2) | l \in (0,\infty)\right\},$$

$$\frac{1}{4}\int_{\mathbb{R}^2} |u_l|^4 \mathrm{d}x = \int_{\mathbb{R}^2} |\nabla u_l|^2 \mathrm{d}x = \int_{\mathbb{R}^2} \frac{u_l^2}{|x|^2}\left(\int_0^{|x|} \frac{s}{2} u_l^2(s)\mathrm{d}s\right)^2 \mathrm{d}x = \frac{16\pi l^2}{3}.$$

Lemma 2.3. *If (u,v) is a solution of* (1.1) *then it satisfies the Pohozaev identity*:

$$\|u\|_2^2 + \omega\|v\|_2^2 + 2\int_{\mathbb{R}^2}\left(\frac{h^2(|x|)}{|x|^2}u^2 + \frac{g^2(|x|)}{|x|^2}v^2\right)\mathrm{d}x = \frac{1}{p}(\|(u,v)\|_{2p}^{2p} + 2b\|uv\|_p^p). \tag{2.1}$$

Proof. We adopt some ideas in [6]. Assume that $(u,v) \in E$ is a weak solution for Problem (1.1). Similar to [4] and [6], we know that $\int_{|x|}^\infty \frac{h(s)}{s}u^2(s)\mathrm{d}s$, $\int_{|x|}^\infty \frac{g(s)}{s}v^2(s)\mathrm{d}s$, $\frac{h^2(|x|)}{|x|^2}$, $\frac{g^2(|x|)}{|x|^2} \in L^\infty(\mathbb{R}^2)$. Thus, the standard elliptic estimates [11] imply that $u, v \in C_{\text{loc}}^{1,\gamma}(\mathbb{R}^2)$ for some $\gamma > 0$. Then, we obtain $\int_{|x|}^\infty \frac{h(s)}{s}u^2(s)\mathrm{d}s$, $\int_{|x|}^\infty \frac{g(s)}{s}v^2(s)\mathrm{d}s$, $\frac{h^2(|x|)}{|x|^2}$, $\frac{g^2(|x|)}{|x|^2} \in C(\mathbb{R}^2)$. Since $u, v \in H_r^1(\mathbb{R}^2)$, we deduce that $u, v \in C^2(\mathbb{R}^2)$. Then, multiplying the first equation in (1.1) by $x \cdot \nabla u$ and integrating by parts on a ball $B_R = \{x \in \mathbb{R}^2 : |x| < R\}$, then

$$\int_{B_R} \Delta u(\nabla u \cdot x)\mathrm{d}x = \frac{R}{2}\int_{\partial B_R} |\nabla u|^2 \mathrm{d}S_x,$$

$$\int_{B_R} u(\nabla u \cdot x)\mathrm{d}x = -\int_{B_R} u^2 \mathrm{d}x + o_R(1),$$

$$\int_{B_R} |u|^{2p-2}u(\nabla u \cdot x)\mathrm{d}x = -\frac{1}{p}\int_{B_R} u^{2p}\mathrm{d}x + o_R(1),$$

$$\int_{B_R}\left(\int_{|x|}^\infty \frac{h(s)}{s}u^2(s)\mathrm{d}s\right)u(\nabla u \cdot x)\mathrm{d}x + \int_{B_R} \frac{h^2(|x|)}{|x|^2}u(\nabla u \cdot x)\mathrm{d}x$$
$$= \pi h^2(R)u^2(R) + \pi\left(\int_R^{+\infty} \frac{h(s)}{s}u^2(s)\mathrm{d}s\right)u^2(R)R^2 - 2\int_{\mathbb{R}^2} \frac{h^2(|x|)}{|x|^2}u^2\mathrm{d}x + o_R(1).$$

Thus, we deduce that

$$\begin{aligned}&\frac{R}{2}\int_{\partial B_R}|\nabla u|^2\mathrm{d}S_x+\int_{B_R}u^2\mathrm{d}x-\pi h^2(R)u^2(R)-\pi\left(\int_R^{+\infty}\frac{h(s)}{s}u^2(s)\mathrm{d}s\right)u^2(R)R^2+2\int_{\mathbb{R}^2}\frac{h^2(|x|)}{|x|^2}u^2\mathrm{d}x\\&=\frac{1}{p}\int_{B_R}u^{2p}\mathrm{d}x-b\int_{B_R}|v|^p|u|^{p-2}ux\cdot\nabla u\mathrm{d}x.\end{aligned}\tag{2.2}$$

In a similar way,

$$\begin{aligned}&\frac{R}{2}\int_{\partial B_R}|\nabla v|^2\mathrm{d}S_x+\int_{B_R}\omega v^2\mathrm{d}x-\pi g^2(R)v^2(R)-\pi\left(\int_R^{+\infty}\frac{g(s)}{s}v^2(s)\mathrm{d}s\right)v^2(R)R^2+2\int_{\mathbb{R}^2}\frac{g^2(|x|)}{|x|^2}v^2\mathrm{d}x\\&=\frac{1}{p}\int_{B_R}v^{2p}\mathrm{d}x-b\int_{B_R}|u|^p|v|^{p-2}vx\cdot\nabla v\mathrm{d}x\end{aligned}\tag{2.3}$$

can be obtained. Then, summing up (2.2) and (2.3), we obtain

$$\begin{aligned}&\int_{B_R}(u^2+\omega v^2)\mathrm{d}x+2\int_{\mathbb{R}^2}\left(\frac{h^2(|x|)}{|x|^2}u^2+\frac{g^2(|x|)}{|x|^2}v^2\right)\mathrm{d}x-\frac{1}{p}\int_{B_R}(u^{2p}+v^{2p})\mathrm{d}x-\frac{2b}{p}\int_{B_R}|uv|^p\mathrm{d}x\\&=\pi\left(\int_R^{+\infty}\frac{h(s)}{s}u^2(s)\mathrm{d}s\right)u^2(R)R^2+\pi\left(\int_R^{+\infty}\frac{g(s)}{s}v^2(s)\mathrm{d}s\right)v^2(R)R^2+\pi h^2(R)u^2(R)\\&\quad+\pi g^2(R)v^2(R)-\frac{R}{2}\int_{\partial B_R}(|\nabla u|^2+|\nabla v|^2)\mathrm{d}S_x.\end{aligned}$$

Arguing as in [6, Proposition 2.3.], there exists a suitable sequence $R_n\to\infty$ on which the right-hand side of the aforementioned equation tends to zero. Passing to the limit, we obtain the identity. This completes the proof. □

Proof of Theorem 1.2. Let (u,v) be a solution of (1.1). By Lemma 2.2, one can obtain

$$\begin{aligned}0&=\int_{\mathbb{R}^2}(|\nabla u|^2+u^2+|\nabla v|^2+\omega v^2)\mathrm{d}x+3(B(u)+B(v))-\int_{\mathbb{R}^2}(u^{2p}+v^{2p}+2b\,|uv|^p)\mathrm{d}x\\&\geq\int_{\mathbb{R}^2}\left(u^2+\frac{1}{2}u^4-(1+b)u^{2p}\right)\mathrm{d}x+\int_{\mathbb{R}^2}\left(\omega v^2+\frac{1}{2}v^4-(1+b)v^{2p}\right)\mathrm{d}x.\end{aligned}$$

Denote

$$\begin{aligned}&f_1(t)=t^2+\frac{1}{2}t^4-(1+b)t^{2p},\\&f_1'(t)=2t+2t^3-2p(1+b)t^{2p-1}=2t(1+t^2-2p(1+b)t^{2p-2}).\end{aligned}$$

There exists $\tilde{b}>0$ small enough such that

$$1+(p(1+\tilde{b}))^{\frac{1}{2-p}}(p-1)^{\frac{p-1}{2-p}}(p-2)=0.$$

Then, we have

$$f_1(t)\geq 0,\quad t\in\mathbb{R},\quad b\in(0,\tilde{b}).$$

There exists a $\tilde{\omega}>0$ such that the function $t\mapsto\omega t^2+\frac{1}{2}t^4-(1+b)t^{2p}$ is nonnegative and strictly increases as $\omega>\tilde{\omega}$, $b\in(0,\tilde{b})$. Hence, (u,v) must be identically $(0,0)$. This completes the proof. □

3 Proof of (*ii*) and (*iii*) of Theorem 1.1

Consider the following problem:

$$-\Delta u + \omega u + \left(\int_{|x|}^{\infty} \frac{h(s)}{s} u^2(s)\mathrm{d}s + \frac{h^2(|x|)}{|x|^2}\right) u = |u|^{2p-2}u, \quad x \in \mathbb{R}^2. \tag{3.1}$$

By [6], when $p \in (3, \infty)$, Problem (3.1) admits a positive ground-state solution u_ω. To be more precise, define the associated energy functional by:

$$J_\omega = \frac{1}{2}\int_{\mathbb{R}^2} (|\nabla u|^2 + \omega u^2)\mathrm{d}x + \frac{1}{2}B(u) - \frac{1}{2p}\int_{\mathbb{R}^2} u^{2p}\mathrm{d}x.$$

Denote the ground-state level by:

$$E_\omega \coloneqq \min\{J_\omega(u) : u \in H_r^1(\mathbb{R}^2)\backslash\{0\}, J_\omega'(u)u = 0\}.$$

Moreover,

$$E_\omega \coloneqq \inf_{u \in \mathcal{N}_\omega} J_\omega(u),$$

where

$$\mathcal{N}_\omega \coloneqq \left\{u \in H_r^1(\mathbb{R}^2)\backslash\{0\} : \int_{\mathbb{R}^2} (|\nabla u|^2 + \omega u^2)\mathrm{d}x + 3B(u) = \int_{\mathbb{R}^2} u^{2p}\mathrm{d}x\right\}.$$

Define the Nehari manifold of Problem (1.1) by:

$$\mathcal{N} \coloneqq \{(u, v) \in E\backslash\{(0, 0)\} | \langle I'(u, v), (u, v)\rangle = 0\}.$$

The corresponding ground-state energy is described as:

$$c_{\mathcal{N}} \coloneqq \inf_{(u,v)\in\mathcal{N}} I(u, v).$$

Lemma 3.1. (Mountain-Pass geometry) *Assume* $b > 0$, *then the functional* I *satisfies the following conditions*:
(*i*) *There exists a positive constant* $r > 0$ *such that* $I(u, v) > 0$ *for* $||(u, v)||_E = r$;
(*ii*) *There exists* $(e_1, e_2) \in E$ *with* $||(e_1, e_2)||_E > r$ *such that* $I(e_1, e_2) < 0$.

Proof. Since

$$2b\int_{\mathbb{R}^2} |uv|^p \mathrm{d}x \le b(||u||_{2p}^{2p} + ||v||_{2p}^{2p}),$$

and by the Sobolev embedding theorem, there exists a positive constant C such that

$$I(u, v) = \frac{1}{2}||(u, v)||_E^2 + \frac{1}{2}(B(u) + B(v)) - F(u, v); \quad \ge \frac{1}{2}||(u, v)||_E^2 - C||(u, v)||_E^{2p}.$$

Hence, there exists $r > 0$ such that

$$\inf_{||(u,v)||_E = r} I(u, v) > 0.$$

For any $(u, v) \in E\backslash\{(0, 0)\}$ and $t > 0$,

$$I(tu, tv) = \frac{t^2}{2}||(u, v)||_E^2 + \frac{t^6}{2}(B(u) + B(v)) - \frac{t^{2p}}{2p}F(u, v),$$

which implies that $I(tu, tv) \to -\infty$ as $t \to +\infty$. This completes the proof. □

That is, I satisfies the geometric conditions of the Mountain-Pass theorem. Define

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], E) : I(\gamma(0)) = 0 \text{ and } I(\gamma(1)) < 0\}$.

Lemma 3.2. *For every $(u, v) \in E \backslash \{(0,0)\}$, there exists a unique $\bar{t}_{uv} > 0$ such that $\bar{t}_{uv}(u, v) \in \mathcal{N}$. The maximum of $I(tu, tv)$ for $t \geq 0$ is achieved at $t = \bar{t}_{uv}$.*

Proof. $\forall (u, v) \in E \backslash \{(0,0)\}$ and $t > 0$, let

$$g(t) := I(tu, tv) = \frac{t^2}{2}\|(u, v)\|_E^2 + \frac{t^6}{2}(B(u) + B(v)) - \frac{t^{2p}}{2p}F(u, v).$$

By Lemma 3.1, we deduce that there exists $\bar{t} = \bar{t}_{uv} > 0$ such that

$$g(\bar{t}) = \max_{t>0} g(t).$$

Moreover,

$$\begin{aligned} g'(t) &= t\|(u, v)\|_E^2 + 3t^5(B(u) + B(v)) - t^{2p-1}F(u, v) \\ &= t^{2p-1}\left(\frac{1}{t^{2p-2}}\|(u, v)\|_E^2 + \frac{3}{t^{2p-6}}(B(u) + B(v)) - F(u, v)\right). \end{aligned}$$

As $p > 3$, thus $g'(t)$ is strictly decreasing, the point $t = \bar{t}_{uv}$ is the unique value of $t > 0$ at which $\bar{t}_{uv}(u, v) \in \mathcal{N}$. This completes the proof. □

Lemma 3.3. $c_{\mathcal{N}} = c$.

Proof. Define

$$c_1 := \inf_{(u,v) \in E \backslash \{(0,0)\}} \max_{t \geq 0} I(tu, tv).$$

By Lemma 3.2, we can obtain $c_{\mathcal{N}} = c_1$. Since $I(tu, tv) < 0$ for any t large, it follows that $c \leq c_1$. Since every $\gamma \in \Gamma$ intersects $\mathcal{N}$, $c \geq c_{\mathcal{N}}$. This completes the proof. □

Set

$$b_1 = \left(\frac{(p-1)3^{\frac{p}{p-1}}\|u_1\|_{2p}^{2p}}{pE_\omega}\right)^{p-1}, \quad b_2 = \left(\frac{(p-1)3^{\frac{p}{p-1}}\|u_\omega\|_{2p}^{2p}}{pE_1}\right)^{p-1},$$

$$b_3 = \left(\frac{(p-1)3^{\frac{3}{p-3}}\|u_1\|_{2p}^{2p}}{pE_\omega}\right)^{p-1}, \quad b_4 = \left(\frac{(p-1)3^{\frac{3}{p-3}}\|u_\omega\|_{2p}^{2p}}{pE_1}\right)^{p-1}.$$

Lemma 3.4. *Assume that one of the following conditions holds*

(*i*) $p \in (3, 3 + \sqrt{6})$ *and* $b > \max\{b_1, b_2\}$,

(*ii*) $p \in [3 + \sqrt{6}, \infty)$ *and* $b > \max\{b_3, b_4\}$, *then* $c_{\mathcal{N}} < \min\{E_1, E_\omega\}$, *where* E_1 *is a ground-state energy of problem* (3.1) *with* $\omega = 1$.

Proof. Denote

$$a(u,v) := \|\nabla u\|_2^2 + \|\nabla v\|_2^2,$$
$$b(u,v) := \|u\|_2^2 + \omega\|v\|_2^2,$$
$$c(u,v) := B(u) + B(v).$$

For fixed $(u,v) \in E\backslash\{(0,0)\}$, we have

$$\begin{aligned}
c_{\mathcal{N}} &\le \max_{t\ge0} I(tu,tv)\\
&\le \max_{t\ge0}\left\{\frac{t^2}{2}(a(u,v)+b(u,v)) + \frac{t^6}{2}c(u,v) - \frac{t^{2p}}{2p}F(u,v)\right\}\\
&\le \max_{t\ge0}\left\{\frac{t^2}{2}(a(u,v)+b(u,v)) - \frac{t^{2p}}{4p}F(u,v)\right\} + \max_{t\ge0}\left\{\frac{t^6}{2}c(u,v) - \frac{t^{2p}}{4p}F(u,v)\right\}\\
&\le \frac{p-1}{2p}\left(\frac{2(a(u,v)+b(u,v))^p}{F}\right)^{\frac{1}{p-1}} + \frac{p-3}{2p}\left(\frac{6c(u,v)^{\frac{p}{3}}}{F}\right)^{\frac{3}{p-3}}.
\end{aligned}$$

Assume $\omega \le 1$, $E_1 \ge E_\omega$. Choosing (u_1,u_1) such that

$$c_{\mathcal{N}} \le \max_{t\ge0} I(tu_1,tu_1) < E_\omega,$$

where u_1 is a positive ground-state solution of (3.1) with $\omega = 1$. Then,

$$\begin{aligned}
c_{\mathcal{N}} &\le \max_{t\ge0} I(tu_1,tu_1)\\
&\le \frac{p-1}{2p}\left(\frac{2(2\|u_1\|^2)^p}{2(1+b)\|u_1\|_{2p}^{2p}}\right)^{\frac{1}{p-1}} + \frac{p-3}{2p}\left(\frac{6(2B(u_1))^{\frac{p}{3}}}{2(1+b)\|u_1\|_{2p}^{2p}}\right)^{\frac{3}{p-3}}\\
&\le \max\{2^{\frac{p}{p-1}}, 3^{\frac{3}{p-3}}\}\frac{p-1}{p}\|u_1\|_{2p}^{2p}\left(\frac{1}{1+b}\right)^{\frac{1}{p-1}}\\
&< \max\{2^{\frac{p}{p-1}}, 3^{\frac{3}{p-3}}\}\frac{p-1}{p}\|u_1\|_{2p}^{2p}\left(\frac{1}{b}\right)^{\frac{1}{p-1}}.
\end{aligned}$$

Due to

$$\max\{2^{\frac{p}{p-1}}, 3^{\frac{3}{p-3}}\} = \begin{cases} 3^{\frac{p}{p-1}}, & p \in [3+\sqrt{6},\infty),\\ 3^{\frac{3}{p-3}}, & p \in (3,3+\sqrt{6}),\end{cases}$$

we deduce that

$$b > \begin{cases} \left(\dfrac{(p-1)3^{\frac{p}{p-1}}\|u_1\|_{2p}^{2p}}{pE_\omega}\right)^{p-1}, & p \in [3+\sqrt{6},\infty),\\[2ex] \left(\dfrac{(p-1)3^{\frac{3}{p-3}}\|u_1\|_{2p}^{2p}}{pE_\omega}\right)^{p-1}, & p \in (3,3+\sqrt{6}).\end{cases}$$

In a similar fashion, if $\omega \ge 1$, it follows that

$$b > \begin{cases} \left(\dfrac{(p-1)3^{\frac{p}{p-1}}\|u_\omega\|_{2p}^{2p}}{pE_1}\right)^{p-1}, & p \in [3+\sqrt{6},\infty),\\[2ex] \left(\dfrac{(p-1)3^{\frac{3}{p-3}}\|u_\omega\|_{2p}^{2p}}{pE_1}\right)^{p-1}, & p \in (3,3+\sqrt{6}).\end{cases}$$

In conclusion, $c_{\mathcal{N}} < \min\{E_1, E_\omega\}$ provided

$$b > \begin{cases} \max\{b_1, b_2\} & p \in [3+\sqrt{6}, \infty), \\ \max\{b_3, b_4\} & p \in (3, 3+\sqrt{6}). \end{cases}$$

This completes the proof. □

Proof of (ii) and (iii) of Theorem 1.1. We divide the proof in several steps.

Step 1. We show that the existence of ground-state solutions of Problem (1.1). By the Ekeland variational principle [28], there exists a sequence $\{(u_n, v_n)\} \subset E$ such that

$$I(u_n, v_n) \to c, \quad I'(u_n, v_n) \to 0 \quad \text{in } E'.$$

By computing $I(u_n, v_n) - \frac{1}{6}\langle I'(u_n, v_n), (u_n, v_n)\rangle$, it is easy to obtain that $\{(u_n, v_n)\}$ is bounded in E. Then, there exists $(u, v) \in E$ such that, up to a subsequence

$$\begin{aligned} &(u_n, v_n) \to (u, v) && \text{weakly in } E, \\ &(u_n, v_n) \to (u, v) && \text{strongly in } L^{2p}(\mathbb{R}^2) \times L^{2p}(\mathbb{R}^2) \quad \text{for } p \in (3, +\infty), \\ &(u_n, v_n) \to (u, v) && \text{a.e. in } \mathbb{R}^2. \end{aligned}$$

Then, $I'(u, v) = 0$ and $I(u, v) \le c$. If $(u, v) \ne (0, 0)$, it follows that $(u, v) \in \mathcal{N}$, $I(u, v) \ge c$. Then, $I(u, v) = c$. Now, it remains to prove $(u, v) \ne (0, 0)$. Assume by the contrary that $(u, v) = (0, 0)$, then

$$\begin{aligned} c + o_n(1)\|(u_n, v_n)\|_E &= I((u_n, v_n)) - \frac{1}{2}\langle I'(u_n, v_n), (u_n, v_n)\rangle \\ &= -(B(u + B(v))) + \left(\frac{1}{2} - \frac{1}{2p}\right)F(u, v) + o_n(1) \\ &= o_n(1), \end{aligned}$$

which is a contradiction. Thus, (u, v) is a ground-state solution of System (1.1).

Step 2. We show that $u \ne 0$ and $v \ne 0$. Without loss of generality, we assume that $u = 0$ and $v \not\equiv 0$. Multiplying the first equation in (1.1) by $(u_n, 0)$, we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} (|\nabla u_n|^2 + u_n^2)\mathrm{d}x = 0.$$

Multiplying the second equation in (1.1) by $(0, v_n)$,

$$\int_{\mathbb{R}^2} (|\nabla v_n|^2 + \omega v_n^2)\mathrm{d}x + 3B(v_n) = \int_{\mathbb{R}^2} v_n^{2p}\mathrm{d}x + o_n(1). \tag{3.2}$$

Therefore, there exists t_n such that

$$\frac{1}{t_n^2}\int_{\mathbb{R}^2} (|\nabla v_n|^2 + \omega v_n^2)\mathrm{d}x + 3B(v_n) = t_n^{2p-6}\int_{\mathbb{R}^2} v_n^{2p}\mathrm{d}x. \tag{3.3}$$

Combining (3.2) and (3.3), it follows that $t_n \to 1$, as $n \to \infty$. Hence,

$$\lim_{n\to\infty} I(u_n, v_n) \to I(0, v) \ge E_\omega,$$

which contradicts the fact that $c_{\mathcal{N}} < E_\omega$. Similarly, if $v = 0$ and $u \not\equiv 0$, we can obtain $I(u_n, v_n) \to I(u, 0)$, as $n \to \infty$, which contradicts the fact that $c_{\mathcal{N}} < E_1$.

Therefore, $u \not\equiv 0$ and $v \not\equiv 0$, (u, v) is a nontrivial ground-state solution of (1.1). In fact, since $(|u|, |v|) \in \mathcal{N}$ and $c_{\mathcal{N}} = I(|u|, |v|)$, we conclude that $(|u|, |v|)$ is a nonnegative solution of (1.1). Using the strong maximum principle, we infer that $|u|, |v| > 0$. Thus, $(|u|, |v|)$ is a positive least energy solution of (1.1). This completes the proof. □

4 Proof of (i) of Theorem 1.1

Given $(u, v) \in E\backslash\{(0, 0)\}$, consider the path

$$\gamma_{u,v}(t) := (t^{\alpha}u(t\,\cdot), t^{\alpha}v(t\,\cdot)), \quad t \geq 0,$$

where $\alpha > 1$ such that $\frac{1}{p-1} < \alpha < \frac{1}{3-p}$ for $p \in (2, 3)$ and $\alpha > 1$ for $p = 3$. Then,

$$I(\gamma_{u,v}(t)) = \frac{t^{2\alpha}}{2}(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) + \frac{t^{2(\alpha-1)}}{2}(\|u\|_2^2 + \omega\|v\|_2^2) + \frac{t^{6\alpha-4}}{2}(B(u) + B(v)) - \frac{t^{2p\alpha-2}}{2p}F(u, v).$$

By differentiating both sides with respect to t at 1, we obtain the following constraint:

$$J(u, v) = \alpha(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) + (\alpha - 1)(\|u\|_2^2 + \omega\|v\|_2^2) + (3\alpha - 2)(B(u) + B(v)) - \frac{p\alpha - 1}{p}F(u, v).$$

Define a constraint manifold of Pohozaev-Nehari type

$$\mathcal{M}_b := \{(u, v) \in E\backslash\{(0, 0)\} | J(u, v) = 0\}.$$

The corresponding ground-state energy is described as:

$$c_b := \inf_{(u,v)\in\mathcal{M}_b} I(u, v).$$

By [6], when $p \in (2, 3]$, Problem (3.1) admits a positive ground-state solution $\tilde{u}_{\omega}$. To be more precise, denote the ground-state level by:

$$\widetilde{E}_{\omega} := \inf_{u\in\mathcal{M}_{\omega}} J_{\omega}(u),$$

where

$$\mathcal{M}_{\omega} := \left\{u \in H_r^1(\mathbb{R}^2)\backslash\{0\} : \int_{\mathbb{R}^2}(\alpha\,|\nabla u|^2 + (\alpha - 1)\omega u^2)\mathrm{d}x + (3\alpha - 2)B(u) = \frac{p\alpha - 1}{p}\int_{\mathbb{R}^2} u^{2p}\mathrm{d}x\right\}.$$

Define the set of ground-state solutions of (3.1) by:

$$S_{\omega} := \{u \in H_r^1(\mathbb{R}^2)\backslash\{0\} : J_{\omega}'(u) = 0, J_{\omega}(u) = \widetilde{E}_{\omega}\}.$$

S_{ω} is nonempty.

Lemma 4.1. *The set S_{ω} is compact in $H_r^1(\mathbb{R}^2)$. More precisely, any sequence $\{u_n\} \subset S_{\omega}$ has subsequences $\{u_{jn}\} \subset S_{\omega}$ and $u \in S_{\omega}$ such that $u_{jn} \to u$ strongly in $H_r^1(\mathbb{R}^2)$ as $n \to \infty$.*

Proof. For any $\{u_n\} \subset S_{\omega}$,

$$\begin{aligned}
E_{\omega} &= \frac{1}{2}\int_{\mathbb{R}^2}(|\nabla u_n|^2 + \omega u_n^2)\mathrm{d}x + \frac{1}{2}B(u_n) - \frac{1}{2p}\int_{\mathbb{R}^2} u_n^{2p}\mathrm{d}x \\
&= \left(\frac{1}{2} - \frac{\alpha}{2(p\alpha - 1)}\right)\int_{\mathbb{R}^2}|\nabla u_n|^2\mathrm{d}x + \left(\frac{1}{2} - \frac{\alpha - 1}{2(p\alpha - 1)}\right)\omega\int_{\mathbb{R}^2} u_n^2\mathrm{d}x + \left(\frac{1}{2} - \frac{3\alpha - 2}{2(p\alpha - 1)}\right)B(u_n) \\
&\geq \left(\frac{1}{2} - \frac{\alpha}{2(p\alpha - 1)}\right)\int_{\mathbb{R}^2}(|\nabla u_n|^2 + \omega u_n^2)\mathrm{d}x.
\end{aligned}$$

Therefore, $\{u_n\}$ is bounded in S_{ω}. There exists $u \in H_r^1(\mathbb{R}^2)$, such that, up to a subsequence, $u_n \to u$ weakly in $H_r^1(\mathbb{R}^2)$, strongly in $L^p(\mathbb{R}^2)$ for $p \in (2, 3]$ as $n \to \infty$. Moreover, due to $u_n \in \mathcal{M}_{\omega}$ for any n, we can obtain

$$\liminf_{n\to\infty} \|u_n\|_{H_r^1(\mathbb{R}^2)} > 0. \tag{4.1}$$

First, we prove that $u \not\equiv 0$. Indeed, if $u = 0$, by Lemma 2.1, we obtain $u_n \to 0$ strongly in $H_r^1(\mathbb{R}^2)$, as $n \to \infty$, which contradicts (4.1). Thus, $u \not\equiv 0$. Next, we show that $u \in S_\omega$. Since u_n satisfies Problem (3.1), we obtain that u is a solution of Problem (3.1). By the semicontinuity of the norms,

$$\widetilde{E}_\omega \leq J_\omega(u) \leq \liminf_{n\to\infty} J_\omega(u_n) = \widetilde{E}_\omega.$$

Therefore, $u \in S_\omega$ and $u_n \to u$ strongly in $H_r^1(\mathbb{R}^2)$, as $n \to \infty$. That is, S_ω is compact in $H_r^1(\mathbb{R}^2)$. □

Lemma 4.2. *For given positive constants a, b, c, and d, a function $f(t) = at^{2\alpha} + bt^{2(\alpha-1)} + ct^{6\alpha-4} - \mathrm{d}t^{2p\alpha-2}$ has exactly one critical point on $(0, +\infty)$, the maximum.*

Proof. The proof is similar to [6]; we omit it here. □

Lemma 4.3. *For any $(u, v) \in E\backslash\{(0, 0)\}$, there exists a unique $t_{u,v} > 0$ such that $\gamma_{u,v}(t_{u,v}) \in \mathcal{M}_b$ and*

$$c_b = \inf_{(u,v)\in E\backslash\{(0,0)\}} \max_{t>0} I(\gamma_{u,v}(t)).$$

Proof. The proof is standard, we omit it here. □

Lemma 4.4. *If $b > 0$, then $c_b < \widetilde{E}_1 + \widetilde{E}_\omega$, where $\widetilde{E}_1$ is a ground-state energy of Problem (3.1) with, $\omega = 1$.*

Proof. Let $\widetilde{u}_1$ and $\widetilde{u}_\omega$ be a positive ground-state solution associated with the level $\widetilde{E}_1$ and $\widetilde{E}_\omega$, respectively. Denote $f(t) = I(t^\alpha\widetilde{u}_1(t\,\cdot), t^\alpha\widetilde{u}_\omega(t\,\cdot))$, that is,

$$\begin{aligned} f(t) = {}& \frac{t^{2\alpha}}{2}(\|\nabla\widetilde{u}_1\|_2^2 + \|\nabla\widetilde{u}_\omega\|_2^2) + \frac{t^{2(\alpha-1)}}{2}(\|\widetilde{u}_1\|_2^2 + \omega\|\widetilde{u}_\omega\|_2^2) \\ & + \frac{t^{6\alpha-4}}{2}(B(\widetilde{u}_1) + B(\widetilde{u}_\omega)) - \frac{t^{2p\alpha-2}}{2p}F(\widetilde{u}_1, \widetilde{u}_\omega). \end{aligned}$$

According to Lemma 4.2, there exists unique $t_0 > 0$ such that $(t_0^\alpha\widetilde{u}_1(t_0\,\cdot), t_0^\alpha\widetilde{u}_\omega(t_0\,\cdot)) \in \mathcal{M}_b$. Hence,

$$\begin{aligned} c_b &\leq I(t_0^\alpha\widetilde{u}_1(t_0\,\cdot), t_0^\alpha\widetilde{u}_\omega(t_0\,\cdot)) = I(t_0\widetilde{u}_1(t_0\,\cdot), 0) + I(0, t_0\widetilde{u}_\omega(t_0\,\cdot)) - 2bt_0^{2p}\int_{\mathbb{R}^2}|\widetilde{u}_1(t_0x)\widetilde{u}_\omega(t_0x)|^p\,\mathrm{d}x \\ &< I(t_0\widetilde{u}_1(t_0\,\cdot), 0) + I(0, t_0\widetilde{u}_\omega(t_0\,\cdot)) \\ &= \widetilde{E}_1 + \widetilde{E}_\omega. \end{aligned}$$

This completes the proof. □

Lemma 4.5. $\liminf_{b\to 0} c_b > 0$.

Proof. Suppose by contradiction the lemma does not hold. Then, there exists $\{b_k\}$ such that $b_k \to 0$ and $c_{b_k} \to 0$, as $k \to \infty$. Moreover, there exists $\{(u_k, v_k)\} \subset \mathcal{M}_{b_k}$ such that $I(u_k, v_k) \to 0$, as $k \to \infty$, that is,

$$\begin{aligned} o_k(1) &= \frac{1}{2}\int_{\mathbb{R}^2}(|\nabla u_k|^2 + u_k^2 + |\nabla v_k|^2 + \omega v_k^2)\mathrm{d}x + \frac{1}{2}(B(u_k) + B(v_k)) - \frac{1}{2p}F(u_k, v_k) \\ &\geq \left(\frac{1}{2} - \frac{\alpha}{2(p\alpha - 1)}\right)\|(u_k, v_k)\|_E^2. \end{aligned}$$

We deduce that $\|(u_k, v_k)\|_E \to 0$, as $k \to \infty$. Since $\{(u_k, v_k)\} \subset \mathcal{M}_{b_k}$,

$$(\alpha - 1)\|(u_k, v_k)\|_E^2 \leq C\|(u_k, v_k)\|_E^{2p}.$$

Thus, we have $\|(u_k, v_k)\|_E \geq (\frac{\alpha-1}{C})^{\frac{1}{2p-2}}$, which is a contradiction. This completes the proof. □

Given $\delta > 0$, let

$$(S_\omega)^\delta := \{u \in H_r^1(\mathbb{R}^2) | u = \tilde{u} + \bar{u}, \tilde{u} \in S_\omega, \|\bar{u}\|_{H_r^1(\mathbb{R}^2)} \leq \delta\}$$

be the neighborhood of S_ω of radius δ.

Lemma 4.6. *For any $\delta > 0$, there exists $b_\delta > 0$ such that for any $b \in (0, b_\delta)$, up to a subsequence, there exists $(u_n^b, v_n^b) \subset \mathcal{M}_b$ satisfying*

$$I(u_n^b, v_n^b) \to c_b, \quad I'(u_n^b, v_n^b) \to 0, \text{as } n \to \infty, \tag{4.2}$$

and $\{u_n^b\} \subset (S_1)^\delta$ and $\{v_n^b\} \subset (S_\omega)^\delta$.

Proof. We adopt some ideas in [8]. Suppose by contradiction the lemma does not hold. Then, for $\delta_0 > 0$, there exists $\{b_k\} \subset \mathbb{R}^+$ such that $b_k \to 0$, as $k \to \infty$, and for any $\{(u_n^{b_k}, v_n^{b_k})\} \subset \mathcal{M}_{b_k}$ satisfying (4.2), there holds $\{u_n^{b_k}\} \subset H_r^1(\mathbb{R}^2) \setminus (S_1)^{\delta_0}$ or $\{v_n^{b_k}\} \subset H_r^1(\mathbb{R}^2) \setminus (S_\omega)^{\delta_0}$. For any k, there exists n_k such that

$$|I(u_{n_k}^{b_k}, v_{n_k}^{b_k}) - c_{b_k}| \leq 1/k.$$

Let $\tilde{u}_k = u_{n_k}^{b_k}$ and $\tilde{v}_k = v_{n_k}^{b_k}$. By Lemma 4.4, we have

$$\limsup_{k\to\infty} I(\tilde{u}_k, \tilde{v}_k) \leq \limsup_{k\to\infty} c_{b_k} \leq \widetilde{E}_1 + \widetilde{E}_\omega. \tag{4.3}$$

Since $\{(\tilde{u}_k, \tilde{v}_k)\} \subset \mathcal{M}_{b_k}$,

$$\begin{aligned} c_{b_k} &= \frac{1}{2}\|(\tilde{u}_k, \tilde{v}_k)\|_E^2 + \frac{1}{2}(B(\tilde{u}_k) + B(\tilde{v}_k)) - \frac{1}{2p}F(\tilde{u}_k, \tilde{v}_k) \\ &\geq \left(\frac{1}{2} - \frac{\alpha}{2(p\alpha - 1)}\right)\|(\tilde{u}_k, \tilde{v}_k)\|_E^2, \end{aligned}$$

we deduce that $\{(\tilde{u}_k, \tilde{v}_k)\}$ is bounded in E. Up to a subsequence, $\tilde{u}_k \to u$ and $\tilde{v}_k \to v$ weakly in $H_r^1(\mathbb{R}^2)$, strongly in $L^{2p}(\mathbb{R}^2)$ for $p \in (2, 3]$, as $k \to \infty$. By Lemma 4.5, we have

$$\liminf_{k\to\infty} \min\{\|\tilde{u}_k\|_{H_r^1(\mathbb{R}^2)}, \|\tilde{v}_k\|_{H_r^1(\mathbb{R}^2)}\} > 0.$$

Note that $b_k > 0$ and

$$o_k(1) = \int_{\mathbb{R}^2} |\nabla\tilde{u}_k|^2 + \tilde{u}_k^2 + 3B(\tilde{u}_k) - \int_{\mathbb{R}^2} \tilde{u}_k^{2p},$$

there exists t_k such that

$$\frac{1}{t_k^{4\alpha-4}}\int_{\mathbb{R}^2} |\nabla\tilde{u}_k|^2 + \frac{1}{t_k^{4\alpha-2}}\int_{\mathbb{R}^2} \tilde{u}_k^2 + 3B(\tilde{u}_k) = t_k^{2+2p\alpha-6\alpha}\int_{\mathbb{R}^2} \tilde{u}_k^{2p},$$

that is, $t_k^\alpha\tilde{u}_k(t_k\,\cdot) \in \mathcal{N}_1$. Similarly, there exists s_k such that $s_k^\alpha\tilde{v}_k(t_k\,\cdot) \in \mathcal{N}_\omega$.

Step 1. We claim that $t_k \to 1$ and $s_k \to 1$ as $k \to \infty$. We only give the proof of $t_k \to 1$, as the second convergence being similar. We consider two cases:

Case I. $u \neq 0$. If $\limsup_{k\to\infty} t_k > 1$, then we can assume that $t_k > 1$ for all k, we have

$$\begin{aligned} o_k(1) &= (t_k^{2+2p\alpha-6\alpha} - 1)\int_{\mathbb{R}^2} \tilde{u}_k^{2p} - \left(\frac{1}{t_k^{4\alpha-4}} - 1\right)\int_{\mathbb{R}^2} |\nabla\tilde{u}_k|^2 - \left(\frac{1}{t_k^{4\alpha-2}} - 1\right)\int_{\mathbb{R}^2} \tilde{u}_k^2 \\ &\geq (t_k^{2+2p\alpha-6\alpha} - 1)\int_{\mathbb{R}^2} \tilde{u}_k^{2p}, \end{aligned}$$

which yields $t_k \to 1$ as $k \to \infty$. This is a contradiction. So $\limsup_{k\to\infty} t_k \leq 1$. Similarly, $\liminf_{k\to\infty} t_k \geq 1$. Then, $\lim_{k\to\infty} t_k = 1$.

Case II. $u = 0$. If $\limsup_{k\to\infty} t_k > 1$, then we can assume that $t_k > 1$ for all k, we have $\limsup_{k\to\infty} \|\tilde{u}_k\|_{H_r^1(\mathbb{R}^2)} = 0$, which contradicts (4.3). So $\limsup_{k\to\infty} t_k \le 1$. Similarly, $\liminf_{k\to\infty} t_k \ge 1$. Then, $\lim_{k\to\infty} t_k = 1$.

Step 2. Let $\bar{u}_k = t_k^\alpha \tilde{u}_k(t_k\,\cdot)$ and $\bar{v}_k = t_k^\alpha \tilde{v}_k(t_k\,\cdot)$, then $\bar{u}_k \to u$ and $\bar{v}_k \to v$ weakly in $H_r^1(\mathbb{R}^2)$, as $k \to \infty$. Next, we show $u \in S_1$, $v \in S_\omega$ and $\bar{u}_k \to u$, $\bar{v}_k \to v$ in $H_r^1(\mathbb{R}^2)$, as $k \to \infty$. This will be a contradiction.

Since $\tilde{u}_k \in H_r^1(\mathbb{R}^2)$, there exists $U_k \in C_0(\mathbb{R}^2)$ and $V_k \in C_0(\mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} |\nabla\tilde{u}_k - U_k|^2 \mathrm{d}x < \varepsilon, \quad \int_{\mathbb{R}^2} |\tilde{u}_k - V_k|^2 \mathrm{d}x < \varepsilon.$$

Therefore,

$$\begin{aligned}\|\nabla(\bar{u}_k - \tilde{u}_k)\|_2^2 &= \int_{\mathbb{R}^2} |\nabla(t_k^\alpha \tilde{u}_k(t_k x) - \tilde{u}_k(x))|^2 \mathrm{d}x \\ &\le 2\int_{\mathbb{R}^2} |\nabla(t_k^\alpha \tilde{u}_k(t_k x)) - U_k(x)|^2 \mathrm{d}x + 2\int_{\mathbb{R}^2} |\nabla\tilde{u}_k(x) - U_k(x)|^2 \mathrm{d}x \\ &= 2\int_{\mathbb{R}^2} |t_k^{\alpha+1}\nabla(\tilde{u}_k(t_k x)) - U_k(x)|^2 \mathrm{d}x + 2\int_{\mathbb{R}^2} |\nabla\tilde{u}_k(x) - U_k(x)|^2 \mathrm{d}x \\ &\le 4t_k^{2\alpha+2}\int_{\mathbb{R}^2} |U_k(t_k x) - U_k(x)|^2 \mathrm{d}x + 2\,|t_k^{\alpha+1} - 1|^2 \int_{\mathbb{R}^2} U_k^2(x)\mathrm{d}x + (4t_k^{2\alpha} + 2)\varepsilon \\ &= 12\varepsilon\end{aligned}$$

and

$$\begin{aligned}\|\bar{u}_k - \tilde{u}_k\|_2^2 &= \int_{\mathbb{R}^2} |t_k^\alpha \tilde{u}_k(t_k x) - \tilde{u}_k(x)|^2 \mathrm{d}x \\ &\le 2\int_{\mathbb{R}^2} |t_k^\alpha \tilde{u}_k(t_k x) - V_k(x)|^2 \mathrm{d}x + 2\int_{\mathbb{R}^2} |\tilde{u}_k(x) - V_k(x)|^2 \mathrm{d}x \\ &\le 4t_k^{2\alpha}\int_{\mathbb{R}^2} |V_k(t_k x) - V_k(x)|^2 \mathrm{d}x + 2\,|t_k^\alpha - 1|^2 \int_{\mathbb{R}^2} V_k^2(x)\mathrm{d}x + (4t_k^{2\alpha-2} + 2)\varepsilon \\ &= 12\varepsilon.\end{aligned}$$

It follows that $\|\bar{u}_k - \tilde{u}_k\|_{H_r^1(\mathbb{R}^2)} \to 0$ and $\|\bar{v}_k - \tilde{v}_k\|_{H_r^1(\mathbb{R}^2)} \to 0$ as $k \to \infty$. So

$$I(\tilde{u}_k, \tilde{v}_k) = I(\bar{u}_k, \bar{v}_k) + o_k(1) \ge \widetilde{E}_1 + \widetilde{E}_\omega + o_k(1).$$

Recalling that $\limsup_{k\to\infty} I(\tilde{u}_k, \tilde{v}_k) \le \widetilde{E}_1 + \widetilde{E}_\omega$, we obtain

$$\lim_{k\to\infty} J_1(\bar{u}_k) = \widetilde{E}_1, \quad \lim_{k\to\infty} J_\omega(\bar{v}_k) = \widetilde{E}_\omega.$$

Arguing as in the proof of Lemma 4.1, we deduce that $u \not\equiv 0$. Thanks to the lower semicontinuity of norms,

$$J_1(u) \le \liminf_{k\to\infty} J_1(\bar{u}_k) = \widetilde{E}_1.$$

If $J_1(u) = \widetilde{E}_1$, it yields that $\bar{u}_k \to u$ strongly in $H_r^1(\mathbb{R}^2)$ and $u \in S_1$. If not, we have

$$\|u\|_{H_r^1(\mathbb{R}^2)} < \liminf_{k\to\infty} \|\bar{u}_k\|_{H_r^1(\mathbb{R}^2)}.$$

It follows that $u \notin \mathcal{M}_1$. Then, there exists a unique $t_0 \in (0, 1)$ such that $J(t_0^\alpha u(t_0\,\cdot)) = 0$. Thus, we have

$$J_1(t_0^\alpha u(t_0\,\cdot)) < \lim_{n\to\infty}\left(\frac{t_0^{2\alpha}}{2}\|\nabla\bar{u}_k\|_2^2 + \frac{t_0^{2(\alpha-1)}}{2}\|\bar{u}_k\|_2^2 + \frac{t_0^{6\alpha-4}}{2}B(\bar{u}_k) - \frac{t_0^{2p\alpha-2}}{2p}\|\bar{u}_k\|_{2p}^{2p}\right).$$

Since $J_1(t^\alpha \bar{u}_k(t\,\cdot))$ has the maximum value at $t = 1$ for all k, it follows that

$$J_1(t_0^\alpha u(t_0\,\cdot)) < \lim_{n\to\infty}\left(\frac{1}{2}\|\nabla\bar{u}_k\|_2^2 + \frac{1}{2}\|\bar{u}_k\|_2^2 + \frac{1}{2}B(\bar{u}_k) - \frac{1}{2p}\|\bar{u}_k\|_{2p}^{2p}\right) = \widetilde{E}_1,$$

which is a contradiction. That is, $u \in \mathcal{M}_1$ and $J_1(u) = \widetilde{E}_1$, which yields $u \in S_1$, and $\bar{u}_k \to u$ strongly in $H_r^1(\mathbb{R}^2)$ as $k \to \infty$. Finally, we can similarly prove $v \in S_\omega$ and $\bar{v}_k \to v$ strongly in $H_r^1(\mathbb{R}^2)$ as $k \to \infty$. By Step 1, we know that $\tilde{u}_k \to u$ and $\tilde{v}_k \to v$ strongly in $H_r^1(\mathbb{R}^2)$, as $k \to \infty$. This is a contradiction with the fact that $\tilde{u}_k \in H_r^1(\mathbb{R}^2)\backslash(S_1)^{\delta_0}$ or $\tilde{v}_k \in H_r^1(\mathbb{R}^2)\backslash(S_\omega)^{\delta_0}$. This completes the proof. □

Proof of (i) of Theorem 1.1. We divide the proof into several steps.

Step 1. $\mathcal{M}_b$ is nonempty. For each $(u, v) \in E\backslash\{(0, 0)\}$, $J(t^\alpha u(t\,\cdot), t^\alpha v(t\,\cdot))$ is of the form $at^{2\alpha} + bt^{2(\alpha-1)} + ct^{6\alpha-4} - dt^{2p\alpha-2}$, which is positive for small t and negative for large t. Thus, there exists $\tilde{t}_{uv} > 0$ such that $J(\tilde{t}_{uv}^\alpha u(\tilde{t}_{uv}\,\cdot), \tilde{t}_{uv}^\alpha v(\tilde{t}_{uv}\,\cdot)) = 0$. Thus, $\mathcal{M}_b$ is not empty.

Step 2. $\mathcal{M}_b$ is bounded away from zero, i.e., $(0, 0) \notin \partial\mathcal{M}_b$. For each $(u, v) \in \mathcal{M}_b$,

$$F(u, v) = \frac{p}{p\alpha - 1}(\alpha a(u, v) + (\alpha - 1)b(u, v) + (3\alpha - 2)c(u, v)) \geq \frac{p}{p\alpha - 1}(\alpha - 1)\|(u, v)\|_E^2. \tag{4.4}$$

By the Sobolev embedding theorem, there exists a constant $C > 0$ such that for any $(u, v) \in \mathcal{M}_b$, $\|(u, v)\|_E^{2p} \geq C\|(u, v)\|_E^2$. Therefore, $\|(u, v)\|_E \geq \rho > 0$ and the conclusion holds.

Step 3. $c_b > 0$. For each $(u, v) \in \mathcal{M}_b$, combining (4.4)

$$\begin{aligned}
I(u, v) &= \frac{1}{2}\|(u, v)\|_E^2 + \frac{1}{2}(B(u) + B(v)) - \frac{1}{2p}F(u, v)\\
&= \frac{1}{2}\|(u, v)\|_E^2 + \frac{1}{2}(B(u) + B(v)) - \frac{1}{2(p\alpha - 1)}(\alpha a(u, v) + (\alpha - 1)b(u, v) + (3\alpha - 2)c(u, v))\\
&= \left(\frac{1}{2} - \frac{\alpha}{2(p\alpha - 1)}\right)a(u, v) + \left(\frac{1}{2} - \frac{\alpha - 1}{2(p\alpha - 1)}\right)b(u, v) + \left(\frac{1}{2} - \frac{3\alpha - 2}{2(p\alpha - 1)}\right)c(u, v)\\
&\geq \left(\frac{1}{2} - \frac{\alpha}{2(p\alpha - 1)}\right)\|(u, v)\|_E^2.
\end{aligned}$$

Then, taking into account Step 2 and $p\alpha - 1 > \alpha$, one can obtain $c_b > 0$.

Step 4: If $\{(u_n, v_n)\}$ is a minimizing sequence for I on $\mathcal{M}_b$, then it is bounded. Let $\{(u_n, v_n)\} \subset \mathcal{M}_b$ such that $I(u_n, v_n) \to c_b$. As in Step 3, we obtain

$$I(u_n, v_n) = \left(\frac{1}{2} - \frac{\alpha}{2(p\alpha - 1)}\right)a(u_n, v_n) + \left(\frac{1}{2} - \frac{\alpha - 1}{2(p\alpha - 1)}\right)b(u_n, v_n) + \left(\frac{1}{2} - \frac{3\alpha - 2}{2(p\alpha - 1)}\right)c(u_n, v_n).$$

Since the coefficients of $a(u_n, v_n)$, $b(u_n, v_n)$ and $c(u_n, v_n)$ are positive, then

$$I(u_n, v_n) \geq \left(\frac{1}{2} - \frac{\alpha}{2(p\alpha - 1)}\right)\|(u_n, v_n)\|_E^2,$$

it follows that $\{(u_n, v_n)\}$ is bounded in E. Thus, there exists $(u, v) \in E$ such that, up to a subsequence

$$\begin{aligned}
&(u_n, v_n) \to (u, v) \quad \text{weakly in } E,\\
&(u_n, v_n) \to (u, v) \quad \text{strongly in } L^{2p}(\mathbb{R}^2) \times L^{2p}(\mathbb{R}^2) \quad \text{for } p \in (2, 3],\\
&(u_n, v_n) \to (u, v) \quad \text{a.e. in } \mathbb{R}^2.
\end{aligned}$$

If $a(u, v) + b(u, v) = \liminf_{n\to\infty} a(u_n, v_n) + b(u_n, v_n)$, then it follows $(u_n, v_n) \to (u, v)$ strongly in E as $n \to \infty$ and $(u, v) \neq (0, 0)$, then c_b is attained by (u, v).

If $a(u, v) + b(u, v) < \liminf_{n\to\infty} a(u_n, v_n) + b(u_n, v_n)$, by Lemma 2.1 and $J(u_n, v_n) = 0$, we deduce that $J(u, v) < 0$, then it follows that $(u, v) \notin \mathcal{M}_b$ and $(u, v) \neq (0, 0)$. Then, there exists a unique $t_0 \in (0, 1)$ such that $J(t_0^\alpha u(t_0\,\cdot), t_0^\alpha v(t_0\,\cdot)) = 0$. Thus, we have

$$I(t_0^\alpha u(t_0\,\cdot), t_0^\alpha v(t_0\,\cdot)) < \lim_{n\to\infty}\left(\frac{t_0^{2\alpha}}{2}a(u_n, v_n) + \frac{t_0^{2(\alpha-1)}}{2}b(u_n, v_n) + \frac{t_0^{6\alpha-4}}{2}c(u_n, v_n) - \frac{t_0^{2p\alpha-2}}{2p}F(u_n, v_n)\right).$$

Since $J(t^\alpha u_n(t\,\cdot), t^\alpha v_n(t\,\cdot))$ has the maximum value at $t = 1$ for all n, it follows that

$$I(t_0^\alpha u(t_0\,\cdot), t_0^\alpha v(t_0\,\cdot)) < \lim_{n\to\infty}\left(\frac{1}{2}a(u_n, v_n) + \frac{1}{2}b(u_n, v_n) + \frac{1}{2}c(u_n, v_n) - \frac{1}{2p}F(u_n, v_n)\right) = c_b,$$

which is a contradiction.

Step 5. The minimizer (u, v) is a regular point of $\mathcal{M}_b$, i.e., $J'(u, v) \neq 0$. To the contrary, suppose that $J'(u, v) = 0$. For $(u_t, v_t) = (t^\alpha u(tx), t^\alpha v(tx))$, one has

$$J(u_t, v_t) = t\frac{\mathrm{d}}{\mathrm{d}t}I(u_t, v_t), \quad \frac{\mathrm{d}}{\mathrm{d}t}J(u_t, v_t) = \frac{\mathrm{d}}{\mathrm{d}t}I(u_t, v_t) + t\cdot\frac{\mathrm{d}^2}{dt^2}I(u_t, v_t).$$

Since $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1} J(u_t, v_t) = 0$, it follows that

$$2\alpha^2 a(u, v) + 2(\alpha-1)^2 b(u, v) + 2(3\alpha-2)^2 c(u, v) - \frac{2(p\alpha-1)^2}{p}F(u, v) = 0.$$

Then, combining with $J(u, v) = 0$, we obtain

$$0 = (\alpha^2 - \alpha(p\alpha-1))a(u, v) + ((\alpha-1)^2 - (\alpha-1)(p\alpha-1))b(u, v) + ((3\alpha-2)^2 - (3\alpha-2)(p\alpha-1))c(u, v).$$

The coefficients of $a(u, v)$, $b(u, v)$, and $c(u, v)$ in the aforementioned identity are negative, which is a contradiction.

Step 6. $I'(u, v) = 0$. Thanks to Lagrange multiplier rule, there exists $\mu \in \mathbb{R}$ such that

$$I'(u, v) = \mu J'(u, v). \tag{4.5}$$

We claim $\mu = 0$. There holds

$$\begin{cases} \alpha a + (\alpha-1)b + (3\alpha-2)c - \dfrac{p\alpha-1}{p}d = 0; \\ (1-2\alpha\mu)a + (1-2\mu(\alpha-1))b + 3(1-\mu(6\alpha-4))c - (1-\mu(2p\alpha-2))d = 0; \\ (2\mu(\alpha-1)-1)b + 2(\mu(6\alpha-4)-1)c - \dfrac{\mu(2p\alpha-2)-1}{p}d = 0. \end{cases}$$

The first equation holds since $J(u, v) = 0$. The second one follows by multiplying (4.5) by (u, v) and integrating. The third one comes from Pohozaev equality. It follows that

$$\begin{aligned} 0 = \mu((2\alpha^2 - (2p\alpha-2)\alpha)a(u, v) &+ (2(\alpha-1)^2 - 2(p\alpha-1)(\alpha-1))b(u, v) \\ &+ (2(3\alpha-2)^2 - 2(p\alpha-1)(3\alpha-2))c(u, v)). \end{aligned}$$

All coefficients of $a(u, v)$, $b(u, v)$, $c(u, v)$ in the aforementioned identity are negative. This implies that $\mu = 0$.

Step 7. Thanks to Lemma 4.6, the minimization $\{(u_n, v_n)\}$ can be chosen in $(S_1)^\delta \times (S_\omega)^\delta$, where $\delta > 0$ is small such that $0 \notin (S_1)^\delta$ and $0 \notin (S_\omega)^\delta$. Hence, $u \not\equiv 0$ and $v \not\equiv 0$, (u, v) is a nontrivial ground-state solution of (1.1). In fact, since $(|u|, |v|) \in \mathcal{N}$ and $c_b = I(|u|, |v|)$, we conclude that $(|u|, |v|)$ is a nonnegative solution of (1.1). Using the strong maximum principle, we infer that $|u|, |v| > 0$. Thus, $(|u|, |v|)$ is a positive ground-state solution of (1.1). This completes the proof. □

Acknowledgment: The authors would like to express their sincere gratitude to the anonymous referee for his/her valuable suggestions and comments.

Funding information: J. J. Zhang was partially supported by NSFC (No. 11871123). Almousa and Squassina are supported by Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP-HC2023/3), Princess Nourah bint Abdulrahman University, Saudi Arabia.

Conflict of interest: The authors state no conflict of interest.

References

[1] A. Ambrosetti and E. Colorado, *Bound and ground-states of coupled nonlinear Schrödinger equations*, C.R. Math. **342** (2006), 453–458.

[2] A. Ambrosetti and E. Colorado, *Standing waves of some coupled nonlinear Schrödinger equations*, J. London Math. Soc. **75** (2007), 67–82.

[3] X.-M. An and J. Yang, *Existence and asymptotic behavior of solitary waves for a weakly coupled Schrödinger system*, Adv. Nonlinear Stud. **22** (2022), 159–183.

[4] A. Azzollini and A. Pomponio, *Positive energy static solutions for the Chern-Simons-Schrödinger system under a large-distance fall-off requirement on the gauge potentials*, Calc. Var. Partial Differ. Equ. **60** (2021), 1–30.

[5] T. Bartsch and Z.-Q. Wang, *Note on ground-states of nonlinear Schrödinger systems*, J. Partial Differ. Equ. **19** (2006), 200–207.

[6] J. Byeon, H. Huh, and J. Seok, *Standing waves of nonlinear Schrödinger equations with the gauge field*, J. Funct. Anal. **263** (2012), 1575–1608.

[7] J. Byeon, H. Huh, and J. Seok, *On standing waves with a vortex point of order N for the nonlinear Chern-Simons-Schrödinger equations*, J. Differ. Equ. **261** (2016), 1285–1316.

[8] D. Cassani, H. Tavares, and J.-J. Zhang, *Bose fluids and positive solutions to weakly coupled systems with critical growth in dimension two*, J. Differ. Equ. **269** (2020), 2328–2385.

[9] X.-P. Chen and C.-L. Tang, *Least energy sign-changing solutions for Schrödinger-Poisson systems with potential well*, Adv. Nonlinear Stud. **22** (2022), 390–415.

[10] G.V. Dunne, *Self-Dual Chern-Simons Theories*, Springer, Berlin, 1995.

[11] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd edition, Grundlehren Math. Wiss., vol. 224, Springer, Berlin, 1983.

[12] C.R. Hagen, *A new gauge theory without an elementary photon*, Ann. Phys. **157** (1984), 342–359.

[13] C.R. Hagen, *Rotational anomalies without anyons*, Phys. Rev. **31** (1985), 2135–2136.

[14] H. Huh, *Standing waves of the Schrödinger equation coupled with the Chern-Simons gauge field*, J. Math. Phys. **53** (2012), 063702.

[15] R. Jackiw and S.-Y. Pi, *Classical and quantal nonrelativistic Chern-Simons theory*, Phys. Rev. D. **42** (1990), 3500–3513.

[16] R. Jackiw and S.-Y. Pi, *Soliton solutions to the guaged nonlinear Schrödinger equations on the plane*, Phys. Rev. Lett. **64** (1990), 2969–2972.

[17] C. Ji and F. Fang, *Standing waves for the Chern-Simons-Schrödinger equation with critical exponential growth*, J. Math. Anal. Appl. **450** (2017), 578–591.

[18] T.-C. Lin and J.-C. Wei, *Ground state of N coupled nonlinear Schrödinger equations in $\mathbb{R}^n$, $n \leq 3$*, Comm. Math. Phys. **255** (2005), 629–653.

[19] Z.-F. Liu and C.-L. Tang, *Sign-changing Solutions for the Chern-Simons-Schrödinger Equation with Concave-convex Nonlinearities*, Qual. Theory Dyn. Syst. **3** (2022), 88.

[20] Z.-S. Liu, Z. Ouyang, and J.-J. Zhang, *Existence and multiplicity of sign-changing standing waves for a gauged nonlinear Schrödinger equation in $\mathbb{R}^2$*, Nonlinearity **32** (2019), 3082–3111.

[21] L.A. Maia, E. Montefusco, and B. Pellacci, *Positive solutions for a weakly coupled nonlinear Schrödinger system*, J. Differ. Equ. **229** (2006), 743–767

[22] A. Pomponio and D. Ruiz, *A variational analysis of a gauged nonlinear Schrödinger equation*, J. Eur. Math. Soc. **17** (2015), 1463–1486.

[23] A. Pomponio and D. Ruiz, *Boundary concentration of a gauged nonlinear Schrödinger equation on large balls*, Calc. Var. Partial Differ. Equ. **53** (2015), 289–316.

[24] P.H. Rabinowitz, *On a class of nonlinear Schrödinger equations*, Z. Angew. Math. Phys. **43** (1992), 270–291.

[25] D. Ruiz, *The Schrödinger-Poisson equation under the effect of a nonlinear local term*, J. Funct. Anal. **237** (2006), 655–674.

[26] B. Sirakov, *Least energy solitary waves for a system of nonlinear Schrödinger equations in $\mathbb{R}^n$*, Comm. Math. Phys. **271** (2007), 199–221.

[27] J. Wang, X. Wang, and S. Wei, *Existence of normalized solutions for the coupled elliptic system with quadratic nonlinearity*, Adv. Nonlinear Stud. **22** (2022), 203–227.

[28] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.

[29] Z.-Y. Xu and J.-F. Yang, *Multiple solutions to multi-critical Schrödinger equations*, Adv. Nonlinear Stud. **22** (2022), 273–288.

[30] J. Zhang, W. Zhang, and X. Xie, *Infinitely many solutions for a gauged nonlinear Schrödinger equation*, Appl. Math. Lett. **88** (2019), 21–27.