



Primitive decomposition of Bott–Chern and Dolbeault harmonic (k, k) -forms on compact almost Kähler manifolds

Tom Holt¹ · Riccardo Piovani¹

Received: 12 January 2023 / Revised: 2 May 2023 / Accepted: 7 July 2023 /
Published online: 8 August 2023
© The Author(s) 2023

Abstract

We consider the primitive decomposition of $\bar{\partial}, \partial$, Bott–Chern and Aeppli-harmonic (k, k) -forms on compact almost Kähler manifolds (M, J, ω) . For any $D \in \{\bar{\partial}, \partial, \text{BC}, \text{A}\}$, it is known that the $L^k P^{0,0}$ component of $\psi \in \mathcal{H}_D^{k,k}$ is a constant multiple of ω^k up to real dimension 6. In this paper we generalise this result to every dimension. We also deduce information on the components $L^{k-1} P^{1,1}$ and $L^{k-2} P^{2,2}$ of the primitive decomposition. Focusing on dimension 8, we give a full description of the spaces $\mathcal{H}_{\text{BC}}^{2,2}$ and $\mathcal{H}_{\text{A}}^{2,2}$, from which follows $\mathcal{H}_{\text{BC}}^{2,2} \subseteq \mathcal{H}_{\bar{\partial}}^{2,2}$ and $\mathcal{H}_{\text{A}}^{2,2} \subseteq \mathcal{H}_{\partial}^{2,2}$. We also provide an almost Kähler 8-dimensional example where the previous inclusions are strict and the primitive components of a harmonic form $\psi \in \mathcal{H}_D^{k,k}$ are not D -harmonic, showing that the primitive decomposition of (k, k) -forms in general does not descend to harmonic forms.

Keywords Bott–Chern Laplacian · Aeppli Laplacian · Dolbeault Laplacian · Primitive decomposition · Almost complex manifold · Harmonic forms

Mathematics Subject Classification 32Q60 · 53C15

The second author is partially supported by GNSAGA of INdAM.

✉ Tom Holt
thomascharles.holt@unipr.it
Riccardo Piovani
riccardo.piovani@unipr.it

¹ Unità di Matematica e Informatica, Dipartimento di Scienze Matematiche, Fisiche e Informatiche, Università degli Studi di Parma, Parco Area delle Scienze 53/A, 43124 Parma, Italy

1 Introduction

A recent answer to a question of Kodaira and Spencer [3, Problem 20] shows that the dimension of the space of Dolbeault harmonic forms depends on the choice of the metric on a given compact almost complex manifold, see [5, 6].

The primitive decomposition of harmonic forms has proven to be useful in describing the spaces of harmonic $(1, 1)$ -forms in dimension 4. In the case of Dolbeault harmonic forms it has been used to show that $h_{\bar{\partial}}^{1,1} := \dim_{\mathbb{C}} \mathcal{H}_{\bar{\partial}}^{1,1}$ is either equal to b^- or $b^- + 1$, depending on the choice of metric, see [4, 5, 12]. Similarly, for Bott–Chern harmonic forms, it yields $h_{BC}^{1,1} := \dim_{\mathbb{C}} \mathcal{H}_{BC}^{1,1} = b^- + 1$ for all metrics, see [4, 10]. See [8, 11, 13] for other related results and [7, 15] for two surveys on the subject.

In this paper, we explore what the primitive decomposition can tell us about harmonic (k, k) -forms in higher dimensions. We start by considering a compact $2n$ -dimensional almost Hermitian manifold (M, J, ω) . The almost complex structure J induces the bidegree decomposition on the space of complex valued k -forms

$$A_{\mathbb{C}}^k = \bigoplus_{p+q=k} A^{p,q}.$$

Additionally, the almost Hermitian structure induces the primitive decomposition on the space of k -forms given by

$$A^k = \bigoplus_{r \geq \max(k-n, 0)} L^r(P^{k-2r}),$$

where $L := \omega \wedge, \Lambda := *^{-1}L*$ and $P^s := \ker \Lambda \cap A^s$ is the space of primitive s -forms, for $s \leq n$ (see e.g., [14, p.26, Théorème 3]). These two decompositions are compatible with each other.

In fact, for Kähler manifolds, i.e., when J is integrable and $d\omega = 0$, the primitive decomposition passes to the space of d -harmonic (p, q) -forms, denoted by $\mathcal{H}_d^{p,q}(M, J) := \ker \Delta_d \cap A^{p,q}$, namely

$$\mathcal{H}_d^{p,q} = \bigoplus_{r \geq \max(p+q-n, 0)} L^r(\mathcal{H}_d^{p-r, q-r} \cap P^{p-r, q-r}), \tag{1}$$

where $P^{p,q} := P_{\mathbb{C}}^m \cap A^{p,q}$.

On Kähler manifolds, we also know that $\mathcal{H}_d^{p,q} = \mathcal{H}_D^{p,q}$ for all $D \in \{\bar{\partial}, \partial, BC, A\}$ (see Sect. 2 for the definitions of these spaces), therefore we have

$$\mathcal{H}_D^{p,q} = \bigoplus_{r \geq \max(p+q-n, 0)} L^r(\mathcal{H}_D^{p-r, q-r} \cap P^{p-r, q-r}). \tag{2}$$

We remark that (1) and (2) have a cohomological meaning in the Kähler setting.

In [2, Corollary 5.4], Cirici and Wilson prove that (1) continues to hold true for almost Kähler manifolds, however it does not directly follow that (2) must also be true. Cattaneo, Tardini and Tomassini prove in [1, Theorem 3.4 and Corollary 3.5] that:

Theorem 1.1 *Let (M, J, ω) be a compact $2n$ -dimensional almost Kähler manifold, then the following decompositions hold:*

$$\begin{aligned} \mathcal{H}_{\bar{\partial}}^{1,1} &= \mathbb{C} \omega \oplus (\mathcal{H}_{\bar{\partial}}^{1,1} \cap P^{1,1}), \\ \mathcal{H}_{\partial}^{1,1} &= \mathbb{C} \omega \oplus (\mathcal{H}_{\partial}^{1,1} \cap P^{1,1}), \\ \mathcal{H}_{\bar{\partial}}^{n-1,n-1} &= \mathbb{C} \omega^{n-1} \oplus L^{n-2}(\mathcal{H}_{\bar{\partial}}^{1,1} \cap P^{1,1}), \\ \mathcal{H}_{\partial}^{n-1,n-1} &= \mathbb{C} \omega^{n-1} \oplus L^{n-2}(\mathcal{H}_{\partial}^{1,1} \cap P^{1,1}). \end{aligned}$$

This means that, on almost Kähler manifolds, $\mathcal{H}_{\bar{\partial}}^{p,q}$ and $\mathcal{H}_{\partial}^{p,q}$ both have primitive decompositions when $(p, q) = (1, 1)$ and, applying the Hodge $*$ operator to the $(1, 1)$ -decompositions, when $(p, q) = (n - 1, n - 1)$.

In [9, Theorems 3.2 and 3.3], Tardini and the second author prove the following results:

Theorem 1.2 *Let (M, J, ω) be a compact $2n$ -dimensional almost Kähler manifold, then the following decompositions hold:*

$$\begin{aligned} \mathcal{H}_{BC}^{1,1} &= \mathbb{C} \omega \oplus (\mathcal{H}_{BC}^{1,1} \cap P^{1,1}), \\ \mathcal{H}_A^{1,1} &= \mathbb{C} \omega \oplus (\mathcal{H}_A^{1,1} \cap P^{1,1}), \\ \mathcal{H}_{BC}^{n-1,n-1} &= \mathbb{C} \omega^{n-1} \oplus L^{n-2}(\mathcal{H}_A^{1,1} \cap P^{1,1}), \\ \mathcal{H}_A^{n-1,n-1} &= \mathbb{C} \omega^{n-1} \oplus L^{n-2}(\mathcal{H}_{BC}^{1,1} \cap P^{1,1}). \end{aligned}$$

We therefore see that, in the almost Kähler setting, $\mathcal{H}_{BC}^{p,q}$ and $\mathcal{H}_A^{p,q}$ both have primitive decompositions when $(p, q) = (1, 1)$ or $(n - 1, n - 1)$.

These two results are sufficient to prove that either (2) or its dual through the Hodge $*$ operator hold for any space of D -harmonic (k, k) -forms on any compact almost Kähler manifold with dimension up to 6. This raises the following question, which we shall answer in this paper: does (2) (or its $*$ dual) hold for (k, k) -forms in general for compact almost Kähler manifolds with dimension 8 or greater? We note that (2) has been shown to fail for dimension 6 in bidegree $(2, 1)$ in [1, Proposition 5.1] for $D \in \{\bar{\partial}, \partial\}$ and in [9, Proposition 5.1] for $D \in \{BC, A\}$.

We also remark that the almost Kähler assumption is necessary for this kind of primitive harmonic decomposition. To see that this is the case in dimension 4 we refer the reader to [10, 12].

The structure of this paper is as follows. In Sect. 2 we give a brief overview of some of the basic results which will be used throughout the paper. In Sect. 3 we show that Theorem 1.2 may be partially extended to (k, k) -forms, proving the following concise decomposition.

Theorem 3.8 *Let (M, J, ω) be a compact almost Kähler manifold of real dimension $2n$. For any $k \in \mathbb{N}$ we have*

$$\mathcal{H}_{\text{BC}}^{k,k} = \mathbb{C} \omega^k \oplus (\mathcal{H}_{\text{BC}}^{k,k} \cap \ker L^{n-k})$$

and

$$\mathcal{H}_{\text{A}}^{k,k} = \mathbb{C} \omega^k \oplus (\mathcal{H}_{\text{A}}^{k,k} \cap \ker L^{n-k}).$$

See Theorem 3.7 for a more detailed description. As an application of the previous results, we consider the 8-dimensional case, yielding the following decompositions.

Theorem 3.10 *Let (M, J, ω) be a compact almost Kähler manifold of real dimension 8. We have*

$$\mathcal{H}_{\text{BC}}^{2,2} = \mathbb{C} \omega^2 \oplus \{\omega \wedge \alpha + \beta \mid \alpha \in P^{1,1}, \beta \in P^{2,2}, \omega \wedge \partial\alpha + \partial\beta = \bar{\partial}\alpha = \bar{\partial}\beta = 0\},$$

and

$$\mathcal{H}_{\text{A}}^{2,2} = \mathbb{C} \omega^2 \oplus \{\omega \wedge \alpha + \beta \mid \alpha \in P^{1,1}, \beta \in P^{2,2}, \omega \wedge \partial\alpha - \partial\beta = \bar{\partial}\alpha = \bar{\partial}\beta = 0\}.$$

This follows from almost Kähler identities of [2] and by L^2 integration by parts.

In Sect. 4 we show that Theorem 1.1 may also be partially extended to (k, k) -forms.

Theorem 4.3 *Let (M, J, ω) be a compact almost Kähler manifold of real dimension $2n$. For any $k \in \mathbb{N}$ we have*

$$\mathcal{H}_{\partial}^{k,k} = \mathbb{C} \omega^k \oplus (\mathcal{H}_{\partial}^{k,k} \cap \ker L^{n-k})$$

and

$$\mathcal{H}_{\bar{\partial}}^{k,k} = \mathbb{C} \omega^k \oplus (\mathcal{H}_{\bar{\partial}}^{k,k} \cap \ker L^{n-k}).$$

See Theorem 4.1 for a more detailed description. Looking at the special case of this result in dimension 8, namely Corollary 4.5, along with Theorem 3.10, we directly deduce the following

Corollary 4.6 *Let (M, J, ω) be a compact almost Kähler manifold of real dimension 8. We have*

$$\mathcal{H}_{\text{BC}}^{2,2} \subseteq \mathcal{H}_{\partial}^{2,2}, \quad \mathcal{H}_{\text{A}}^{2,2} \subseteq \mathcal{H}_{\bar{\partial}}^{2,2}.$$

Finally in Sect. 5, we consider a non left invariant almost Kähler structure on the 8-dimensional torus $\mathbb{T}^8 = \mathbb{Z}^8 \backslash \mathbb{R}^8$. We use this example to show that there exists a $(2, 2)$ -form contained in $\mathcal{H}_{\partial}^{2,2}$ but not in $\mathcal{H}_{\text{BC}}^{2,2}$, and likewise there exists a $(2, 2)$ -form in $\mathcal{H}_{\bar{\partial}}^{2,2}$ but not in $\mathcal{H}_{\text{A}}^{2,2}$. We also show that there exists a form $\psi = \omega \wedge \alpha + \beta \in \mathcal{H}_{\text{BC}}^{2,2}$

whose components with respect to the primitive decomposition, $\alpha \in P^{1,1}, \beta \in P^{2,2}$, are not themselves Bott–Chern harmonic. From this we can conclude that the primitive decomposition does not in general apply to D -harmonic $(2, 2)$ -forms in dimension 8, for $D \in \{BC, A, \bar{\partial}, \partial\}$.

Corollary 5.2 *There exists a compact almost Kähler manifold (M, J, ω) of real dimension 8 such that*

$$\mathcal{H}_{BC}^{2,2} \not\subseteq \mathcal{H}_{\bar{\partial}}^{2,2}, \quad \mathcal{H}_A^{2,2} \not\subseteq \mathcal{H}_{\partial}^{2,2}$$

and

$$\mathcal{H}_D^{2,2} \not\subseteq \mathbb{C}\omega^2 \oplus L(P^{1,1} \cap \mathcal{H}_D^{1,1}) \oplus (P^{2,2} \cap \mathcal{H}_D^{2,2}),$$

where $D \in \{BC, A, \bar{\partial}, \partial\}$.

We also consider another 8-dimensional compact nilmanifold, focusing on the subspace of left invariant harmonic forms in $\mathcal{H}_D^{2,2}$. We show that these spaces satisfy (2) for all $D \in \{\bar{\partial}, \partial, BC, A\}$. Furthermore, we show that in this example these spaces are in fact all equal and have dimension 16.

2 Preliminaries

Throughout this paper, we will only consider connected manifolds without boundary. Let (M, J) be an almost complex manifold of dimension $2n$, i.e., a $2n$ -differentiable manifold endowed with an almost complex structure J , that is $J \in \text{End}(TM)$ and $J^2 = -\text{id}$. The complexified tangent bundle $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$ decomposes into the two eigenspaces of J associated to the eigenvalues $i, -i$, which we denote respectively by $T^{1,0}M$ and $T^{0,1}M$, giving us

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M.$$

Denoting by $\Lambda^{1,0}M$ and $\Lambda^{0,1}M$ the dual vector bundles of $T^{1,0}M$ and $T^{0,1}M$, respectively, we set

$$\Lambda^{p,q}M = \wedge^p \Lambda^{1,0}M \wedge \wedge^q \Lambda^{0,1}M$$

to be the vector bundle of (p, q) -forms, and let $A^{p,q} = \Gamma(M, \Lambda^{p,q}M)$ be the space of smooth sections of $\Lambda^{p,q}M$. We denote by $A^k = \Gamma(M, \Lambda^k M)$ the space of k -forms. Note that $\Lambda^k M \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q}M$.

Let $f \in C^\infty(M, \mathbb{C})$ be a smooth function on M with complex values. Its differential df is contained in $A^1 \otimes \mathbb{C} = A^{1,0} \oplus A^{0,1}$. On complex 1-forms, the exterior derivative acts as

$$d: A^1 \otimes \mathbb{C} \rightarrow A^2 \otimes \mathbb{C} = A^{2,0} \oplus A^{1,1} \oplus A^{0,2}.$$

Therefore, it turns out that the derivative operates on (p, q) -forms as

$$d : A^{p,q} \rightarrow A^{p+2,q-1} \oplus A^{p+1,q} \oplus A^{p,q+1} \oplus A^{p-1,q+2},$$

where we denote the four components of d by

$$d = \mu + \partial + \bar{\partial} + \bar{\mu}.$$

From the relation $d^2 = 0$, we derive

$$\begin{cases} \mu^2 = 0, \\ \mu\partial + \partial\mu = 0, \\ \partial^2 + \mu\bar{\partial} + \bar{\partial}\mu = 0, \\ \partial\bar{\partial} + \bar{\partial}\partial + \mu\bar{\mu} + \bar{\mu}\mu = 0, \\ \bar{\partial}^2 + \bar{\mu}\partial + \partial\bar{\mu} = 0, \\ \bar{\mu}\bar{\partial} + \bar{\partial}\bar{\mu} = 0, \\ \bar{\mu}^2 = 0. \end{cases}$$

We also define the operator $d^c := J^{-1}dJ$. It is a straightforward computation to show that

$$d^c = i(\mu - \partial + \bar{\partial} - \bar{\mu}).$$

If the almost complex structure J is induced from a complex manifold structure on M , then J is called integrable. Recall that J is integrable if and only if the exterior derivative decomposes into $d = \partial + \bar{\partial}$.

A Riemannian metric g on M which is preserved by J , i.e. $g(J \cdot, J \cdot) = g(\cdot, \cdot)$, is called almost Hermitian. Let g be an almost Hermitian metric, the 2-form ω such that

$$\omega(u, v) = g(Ju, v) \quad \text{for all } u, v \in \Gamma(TM)$$

is called the fundamental form of g . We will call (M, J, ω) an almost Hermitian manifold. We denote by h the Hermitian extension of g on the complexified tangent bundle $T_{\mathbb{C}}M$, and by the same symbol g the \mathbb{C} -bilinear symmetric extension of g on $T_{\mathbb{C}}M$. Also denote by the same symbol ω the \mathbb{C} -bilinear extension of the fundamental form ω of g on $T_{\mathbb{C}}M$. Thanks to the elementary properties of the two extensions h and g , we may want to consider h as a Hermitian operator $T^{1,0}M \times T^{1,0}M \rightarrow \mathbb{C}$ and g as a \mathbb{C} -bilinear operator $T^{1,0}M \times T^{0,1}M \rightarrow \mathbb{C}$. Note that $h(u, v) = g(u, \bar{v})$ for all $u, v \in \Gamma(T^{1,0}M)$.

Let (M, J, ω) be an almost Hermitian manifold of real dimension $2n$. Denote the extension of h to (p, q) -forms by the Hermitian inner product $\langle \cdot, \cdot \rangle$. Let $*$: $A^{p,q} \rightarrow A^{n-q,n-p}$ be the \mathbb{C} -linear extension of the standard Hodge $*$ operator on Riemannian manifolds with respect to the volume form $\text{Vol} = \frac{\omega^n}{n!}$, i.e., $*$ is defined by the relation

$$\alpha \wedge * \bar{\beta} = \langle \alpha, \beta \rangle \text{Vol} \quad \text{for all } \alpha, \beta \in A^{p,q}.$$

Integrating the pointwise Hermitian inner product on the manifold, we get the standard L^2 product here denoted by

$$\langle\langle \alpha, \beta \rangle\rangle = \int_M \langle \alpha, \beta \rangle \text{Vol} \quad \text{for all } \alpha, \beta \in A^{p,q},$$

which is surely well defined if M is compact. Then the operators

$$d^* = - * d *, \quad \mu^* = - * \bar{\mu} *, \quad \partial^* = - * \bar{\partial} *, \quad \bar{\partial}^* = - * \partial *, \quad \bar{\mu}^* = - * \mu *,$$

are the L^2 formal adjoint operators respectively of $d, \mu, \partial, \bar{\partial}, \bar{\mu}$. Recall that

$$\Delta_d = dd^* + d^*d$$

is the Hodge Laplacian, and, as in the integrable case, set

$$\Delta_\partial = \partial\partial^* + \partial^*\partial, \quad \Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial},$$

respectively as the ∂ and $\bar{\partial}$ Laplacians. Again, as in the integrable case, set

$$\Delta_{BC} = \partial\bar{\partial}\bar{\partial}^*\partial^* + \bar{\partial}^*\partial^*\partial\bar{\partial} + \partial^*\bar{\partial}\bar{\partial}^*\partial + \bar{\partial}^*\partial\partial^*\bar{\partial} + \partial^*\partial + \bar{\partial}^*\bar{\partial},$$

and

$$\Delta_A = \partial\bar{\partial}\bar{\partial}^*\partial^* + \bar{\partial}^*\partial^*\partial\bar{\partial} + \partial\bar{\partial}^*\bar{\partial}\partial^* + \bar{\partial}\partial^*\partial\bar{\partial}^* + \partial\partial^* + \bar{\partial}\bar{\partial}^*,$$

respectively as the Bott–Chern and the Aeppli Laplacians. Note that

$$* \Delta_{BC} = \Delta_A *, \quad \Delta_{BC} * = * \Delta_A. \tag{3}$$

If M is compact, then we easily deduce the following relations:

$$\begin{cases} \Delta_d = 0 & \iff d = 0, d^* = 0, \\ \Delta_\partial = 0 & \iff \partial = 0, \bar{\partial}^* = 0, \\ \Delta_{\bar{\partial}} = 0 & \iff \bar{\partial} = 0, \partial^* = 0, \\ \Delta_{BC} = 0 & \iff \partial = 0, \bar{\partial} = 0, \partial\bar{\partial}^* = 0, \\ \Delta_A = 0 & \iff \partial^* = 0, \bar{\partial}^* = 0, \partial\bar{\partial} = 0, \end{cases} \tag{4}$$

which characterize the spaces of harmonic forms

$$\mathcal{H}_d^k, \mathcal{H}_\partial^{p,q}, \mathcal{H}_{\bar{\partial}}^{p,q}, \mathcal{H}_{BC}^{p,q}, \mathcal{H}_A^{p,q},$$

defined as the spaces of forms which are in the kernel of the associated Laplacians. All these Laplacians are elliptic operators on the almost Hermitian manifold (M, J, ω) (cf. [3, 10]), implying that all the spaces of harmonic forms are finite dimensional when the manifold is compact.

Now we introduce some notation and recall some well-known facts about primitive forms. We denote by

$$L : \Lambda^k M \rightarrow \Lambda^{k+2} M, \quad \alpha \mapsto \omega \wedge \alpha$$

the Lefschetz operator and by

$$\Lambda : \Lambda^k M \rightarrow \Lambda^{k-2} M, \quad \Lambda = *^{-1} L *$$

its adjoint. A differential k -form α on M , for $k \leq n$, is said to be *primitive* if $\Lambda \alpha = 0$, or equivalently if

$$L^{n-k+1} \alpha = 0.$$

Then we have the following vector bundle decomposition (see e.g., [14, p.26, Théorème 3]):

$$\Lambda^k M = \bigoplus_{r \geq \max(k-n, 0)} L^r (P^{k-2r} M),$$

where we use

$$P^s M := \ker(\Lambda : \Lambda^s M \rightarrow \Lambda^{s-2} M)$$

to denote the bundle of primitive s -forms. For any given $\beta \in P^k M$, we have the following formula (cf. [14, p. 23, Théorème 2]) involving the Hodge $*$ operator and the Lefschetz operator:

$$* L^r \beta = (-1)^{\frac{k(k+1)}{2}} \frac{r!}{(n-k-r)!} L^{n-k-r} J \beta. \tag{5}$$

We recall that the map $L^h : \Lambda^k M \rightarrow \Lambda^{k+2h} M$ is injective for $h+k \leq n$ and is surjective for $h+k \geq n$.

Furthermore, the decomposition above is compatible with the bidegree decomposition on the bundle of complex k -forms $\Lambda_{\mathbb{C}}^k M$ induced by J , that is

$$P_{\mathbb{C}}^k M = \bigoplus_{p+q=k} P^{p,q} M,$$

where

$$P^{p,q} M = P_{\mathbb{C}}^k M \cap \Lambda^{p,q} M.$$

In fact, we have

$$\Lambda^{p,q} M = \bigoplus_{r \geq \max(p+q-n, 0)} L^r (P^{p-r, q-r} M).$$

Finally, let us set $P^s := \Gamma(M, P^s M)$ and $P^{p,q} := \Gamma(M, P^{p,q} M)$.

3 Primitive decomposition of Bott–Chern harmonic (k, k) -forms

In order to prove our main result, we will need the following lemmas. The next one is well known, see for instance [9, Theorem 3.2] or [10, Theorem 4.3]. We include an outline of the proof here for the convenience of the reader.

Lemma 3.1 *Let (M, J, ω) be a compact almost Kähler manifold of real dimension $2n$. Let $f \in C^\infty(M, \mathbb{C})$ be a smooth complex valued function. If*

$$\omega^{n-1} \wedge \partial \bar{\partial} f = 0,$$

then $f \in \mathbb{C}$ is a complex constant.

Proof Let V_1, \dots, V_n be a local frame of $T^{1,0}M$, with ϕ^1, \dots, ϕ^n the dual coframe of $\Lambda^{1,0}M$, chosen such that $\omega = i \sum_j \phi^{j\bar{j}}$. We can then write

$$\partial \bar{\partial} f = \partial \left(\sum_{j=1}^n \bar{V}_j f \phi^{\bar{j}} \right) = \sum_{i,j=1}^n V_i \bar{V}_j f \phi^{i\bar{j}} + \sum_{j=1}^n \bar{V}_j f \partial \phi^{\bar{j}}.$$

The wedge product of ω^{n-1} with $\phi^{i\bar{j}}$ is zero unless $i = j$, therefore

$$\omega^{n-1} \wedge \partial \bar{\partial} f = -i(n-1)! \sum_{j=1}^n V_j \bar{V}_j f \text{Vol} + R(f) \text{Vol}$$

where R is a differential operator involving at most first order derivatives. Setting this to zero tells us that f is in the kernel of a strongly elliptic differential operator. In conjunction with the compactness of M , this implies that f must be constant by the maximum principle. □

We will divide the proof of our main result Theorem 3.7 into the following lemmas. In the first one we study the second order differential conditions in the characterisation (4) of Bott–Chern and Aeppli harmonic forms.

Lemma 3.2 *Let (M, J, ω) be a compact almost Kähler manifold of real dimension $2n$. For any $k \in \mathbb{N}$ such that $2k \leq n$, write any (k, k) -form ψ as*

$$\psi = \sum_{m=0}^k \omega^{k-m} \wedge \alpha^{m,m},$$

where every (m, m) -form $\alpha^{m,m}$ is primitive. If ψ satisfies both

$$\partial\bar{\partial}\psi = 0 \quad \text{and} \quad \partial\bar{\partial}*\psi = 0,$$

then $\alpha^{0,0} \in \mathbb{C}$ is a complex constant. Moreover,

$$\omega^{n-3} \wedge \partial\bar{\partial}\alpha^{1,1} = 0,$$

i.e., $\partial\bar{\partial}\alpha^{1,1}$ is primitive, and

$$\omega^{n-4} \wedge \partial\bar{\partial}\alpha^{2,2} = 0.$$

Proof Fix $2k \leq n$. Every form $\alpha^{m,m}$ is primitive, i.e.,

$$\omega^{n-2m+1} \wedge \alpha^{m,m} = 0. \quad (6)$$

By (5), we have the formula

$$*\psi = \sum_{m=0}^k (-1)^m \frac{(k-m)!}{(n-k-m)!} \omega^{n-k-m} \wedge \alpha^{m,m}.$$

Now, assume that $\partial\bar{\partial}\psi = \partial\bar{\partial}*\psi = 0$. Since $d\omega = 0$, it follows

$$0 = \partial\bar{\partial}\psi = \sum_{m=0}^k \omega^{k-m} \wedge \partial\bar{\partial}\alpha^{m,m}, \quad (7)$$

$$0 = \partial\bar{\partial}*\psi = \sum_{m=0}^k (-1)^m \frac{(k-m)!}{(n-k-m)!} \omega^{n-k-m} \wedge \partial\bar{\partial}\alpha^{m,m}. \quad (8)$$

We want to compare (7) and (8). Note that

$$k-m \leq n-k-m \iff 2k \leq n,$$

therefore we can compute the wedge product between (7) and ω to the power $n-k-m-(k-m) = n-2k$ and obtain

$$0 = \omega^{n-2k} \wedge \partial\bar{\partial}\psi = \sum_{m=0}^k \omega^{n-k-m} \wedge \partial\bar{\partial}\alpha^{m,m}. \quad (9)$$

If we take the wedge product of ω^{k-1} with both equations (8) and (9), we find

$$\begin{aligned} 0 &= \omega^{k-1} \wedge \partial \bar{\partial} * \psi = \sum_{m=0}^k (-1)^m \frac{(k-m)!}{(n-k-m)!} \omega^{n-m-1} \wedge \partial \bar{\partial} \alpha^{m,m} \\ &= \frac{k!}{(n-k)!} \omega^{n-1} \wedge \partial \bar{\partial} \alpha^{0,0} - \frac{(k-1)!}{(n-k-1)!} \omega^{n-2} \wedge \partial \bar{\partial} \alpha^{1,1} \end{aligned}$$

by (6), since $n - 2m + 1 \leq n - m - 1$ iff $m \geq 2$. In the same way,

$$\begin{aligned} 0 &= \omega^{k-1} \wedge \omega^{n-2k} \wedge \partial \bar{\partial} \psi = \sum_{m=0}^k \omega^{n-m-1} \wedge \partial \bar{\partial} \alpha^{m,m} \\ &= \omega^{n-1} \wedge \partial \bar{\partial} \alpha^{0,0} + \omega^{n-2} \wedge \partial \bar{\partial} \alpha^{1,1} \end{aligned}$$

by (6). Now, thanks to the last two equations, we easily deduce

$$\omega^{n-1} \wedge \partial \bar{\partial} \alpha^{0,0} = 0$$

and

$$\omega^{n-2} \wedge \partial \bar{\partial} \alpha^{1,1} = 0.$$

In particular, this yields that $\alpha^{0,0} \in \mathbb{C}$ is a complex constant by Lemma 3.1.

If we take the wedge product of ω^{k-2} with both equations (8) and (9), we find

$$\begin{aligned} 0 &= \omega^{k-2} \wedge \partial \bar{\partial} * \psi = \sum_{m=1}^k (-1)^m \frac{(k-m)!}{(n-k-m)!} \omega^{n-m-2} \wedge \partial \bar{\partial} \alpha^{m,m} \\ &= - \frac{(k-1)!}{(n-k-1)!} \omega^{n-3} \wedge \partial \bar{\partial} \alpha^{1,1} + \frac{(k-2)!}{(n-k-2)!} \omega^{n-4} \wedge \partial \bar{\partial} \alpha^{2,2} \end{aligned}$$

by (6). In the same way,

$$\begin{aligned} 0 &= \omega^{k-2} \wedge \omega^{n-2k} \wedge \partial \bar{\partial} \psi = \sum_{m=1}^k \omega^{n-m-2} \wedge \partial \bar{\partial} \alpha^{m,m} \\ &= \omega^{n-3} \wedge \partial \bar{\partial} \alpha^{1,1} + \omega^{n-4} \wedge \partial \bar{\partial} \alpha^{2,2} \end{aligned}$$

by (6). Now, thanks to the last two equations, we easily deduce

$$\omega^{n-3} \wedge \partial \bar{\partial} \alpha^{1,1} = 0$$

and

$$\omega^{n-4} \wedge \partial \bar{\partial} \alpha^{2,2} = 0,$$

which is the claim. \square

Remark 3.3 If we take the wedge product of ω^{k-l} , for $3 \leq l \leq k-1$, with both equations (8) and (9), we find similar sums, but this time we have three or more addends. This does not imply, in general, that every addend is equal to 0.

In the next lemma we study the first order differential conditions in the characterisation (4) of Bott–Chern harmonic forms.

Lemma 3.4 *Let (M, J, ω) be a compact almost Kähler manifold of real dimension $2n$. For any $k \in \mathbb{N}$ such that $2k \leq n$, write any (k, k) -form ψ as*

$$\psi = \sum_{m=0}^k \omega^{k-m} \wedge \alpha^{m,m},$$

where every (m, m) -form $\alpha^{m,m}$ is primitive. Assume that $\alpha^{0,0} \in \mathbb{C}$ is a complex constant. If $\partial\psi = 0$, then $\partial\alpha^{1,1}$ is primitive, i.e.,

$$\omega^{n-2} \wedge \partial\alpha^{1,1} = 0.$$

If $\bar{\partial}\psi = 0$, then $\bar{\partial}\alpha^{1,1}$ is primitive, i.e.,

$$\omega^{n-2} \wedge \bar{\partial}\alpha^{1,1} = 0.$$

Proof Fix $2k \leq n$. Assume that $\partial\psi = 0$ and $\alpha^{0,0} \in \mathbb{C}$. Since $d\omega = 0$, it follows

$$0 = \partial\psi = \sum_{m=1}^k \omega^{k-m} \wedge \partial\alpha^{m,m}. \quad (10)$$

If we take the wedge product of ω^{n-k-1} and (10), we find

$$0 = \omega^{n-k-1} \wedge \partial\psi = \sum_{m=1}^k \omega^{n-m-1} \wedge \partial\alpha^{m,m} = \omega^{n-2} \wedge \partial\alpha^{1,1},$$

by (6).

Assume now that $\bar{\partial}\psi = 0$ and $\alpha^{0,0} \in \mathbb{C}$. Since $d\omega = 0$, it follows

$$0 = \bar{\partial}\psi = \sum_{m=1}^k \omega^{k-m} \wedge \bar{\partial}\alpha^{m,m}. \quad (11)$$

If we take the wedge product of ω^{n-k-1} and (11), we find

$$0 = \omega^{n-k-1} \wedge \bar{\partial}\psi = \sum_{m=1}^k \omega^{n-m-1} \wedge \bar{\partial}\alpha^{m,m} = \omega^{n-2} \wedge \bar{\partial}\alpha^{1,1},$$

by (6), and this ends the proof. □

Remark 3.5 If we take the wedge product of ω^{n-k-l} , for $2 \leq l \leq n - k - 1$, with both equations (10) and (11), we find similar sums, but this time we have two or more addends. This does not imply, in general, that every addend is equal to 0.

Finally, in the following lemma we study the first order differential conditions in the characterisation (4) of Aeppli harmonic forms.

Lemma 3.6 *Let (M, J, ω) be a compact almost Kähler manifold of real dimension $2n$. For any $k \in \mathbb{N}$ such that $2k \leq n$, write any (k, k) -form ψ as*

$$\psi = \sum_{m=0}^k \omega^{k-m} \wedge \alpha^{m,m},$$

where every (m, m) -form $\alpha^{m,m}$ is primitive. Assume that $\alpha^{0,0} \in \mathbb{C}$ is a complex constant. If $\partial * \psi = 0$, then $\partial\alpha^{1,1}$ is primitive, i.e.,

$$\omega^{n-2} \wedge \partial\alpha^{1,1} = 0.$$

If $\bar{\partial} * \psi = 0$, then $\bar{\partial}\alpha^{1,1}$ is primitive, i.e.,

$$\omega^{n-2} \wedge \bar{\partial}\alpha^{1,1} = 0.$$

Proof Fix $2k \leq n$. By (5), we have the formula

$$* \psi = \sum_{m=0}^k (-1)^m \frac{(k-m)!}{(n-k-m)!} \omega^{n-k-m} \wedge \alpha^{m,m}.$$

Assume that $\partial * \psi = 0$ and $\alpha^{0,0} \in \mathbb{C}$. Since $d\omega = 0$, it follows

$$0 = \partial * \psi = \sum_{m=1}^k (-1)^m \frac{(k-m)!}{(n-k-m)!} \omega^{n-k-m} \wedge \partial\alpha^{m,m}. \tag{12}$$

If we take the wedge product of ω^{k-1} and (12), we find

$$\begin{aligned} 0 &= \omega^{k-1} \wedge \partial * \psi = \sum_{m=1}^k (-1)^m \frac{(k-m)!}{(n-k-m)!} \omega^{n-m-1} \wedge \partial \alpha^{m,m} \\ &= -\frac{(k-1)!}{(n-k-1)!} \omega^{n-2} \wedge \partial \alpha^{1,1}, \end{aligned}$$

by (6), and this is equivalent to the first claim.

Now, assume that $\bar{\partial} * \psi = 0$ and $\alpha^{0,0} \in \mathbb{C}$. Since $d\omega = 0$, it follows

$$0 = \bar{\partial} * \psi = \sum_{m=1}^k (-1)^m \frac{(k-m)!}{(n-k-m)!} \omega^{n-k-m} \wedge \bar{\partial} \alpha^{m,m}. \tag{13}$$

If we take the wedge product of ω^{k-1} and (13), we find

$$\begin{aligned} 0 &= \omega^{k-1} \wedge \bar{\partial} * \psi = \sum_{m=1}^k (-1)^m \frac{(k-m)!}{(n-k-m)!} \omega^{n-m-1} \wedge \bar{\partial} \alpha^{m,m} \\ &= -\frac{(k-1)!}{(n-k-1)!} \omega^{n-2} \wedge \bar{\partial} \alpha^{1,1}, \end{aligned}$$

by (6), and this is equivalent to the second claim. □

We can now prove the following properties of Bott–Chern and Aeppli harmonic (k, k) -forms on a compact almost Kähler manifold.

Theorem 3.7 *Let (M, J, ω) be a compact almost Kähler manifold of real dimension $2n$. For any $k \in \mathbb{N}$, write any (k, k) -form ψ as*

$$\psi = \sum_{m=0}^{\min(k,n-k)} \omega^{k-m} \wedge \alpha^{m,m},$$

where every (m, m) -form $\alpha^{m,m}$ is primitive. If ψ is Bott–Chern or Aeppli harmonic, then $\alpha^{0,0} \in \mathbb{C}$ is a complex constant, $\partial \alpha^{1,1}, \bar{\partial} \alpha^{1,1}, \partial \bar{\partial} \alpha^{1,1}$ are primitive, i.e.,

$$\omega^{n-2} \wedge \partial \alpha^{1,1} = \omega^{n-2} \wedge \bar{\partial} \alpha^{1,1} = \omega^{n-3} \wedge \partial \bar{\partial} \alpha^{1,1} = 0,$$

and

$$\omega^{n-4} \wedge \partial \bar{\partial} \alpha^{2,2} = 0.$$

Proof Let us begin with the case $2k \leq n$. Note that if ψ is Bott–Chern or Aeppli harmonic, then ψ satisfies both

$$\partial \bar{\partial} \psi = 0 \quad \text{and} \quad \partial \bar{\partial} * \psi = 0,$$

and thus we can apply Lemma 3.2. Finally, if ψ is Bott–Chern harmonic, we can apply Lemma 3.4. Conversely, if ψ is Aeppli harmonic, we can apply Lemma 3.6. This concludes the proof of the case $2k \leq n$.

Conversely, assume now that $2k \geq n$. By (5), we have the formula

$$*\psi = \sum_{m=0}^{n-k} (-1)^m \frac{(k-m)!}{(n-k-m)!} \omega^{n-k-m} \wedge \alpha^{m,m}.$$

Set $l := n - k$ and

$$\beta^{m,m} := (-1)^m \frac{(k-m)!}{(n-k-m)!} \alpha^{m,m}.$$

We note that $2l \leq n$ and the primitive decomposition of $*\psi \in A^{l,l}$ is

$$*\psi = \sum_{m=0}^l \omega^{l-m} \wedge \beta^{m,m}.$$

By (3), we know that ψ is Bott–Chern harmonic iff $*\psi$ is Aeppli harmonic, and ψ is Aeppli harmonic iff $*\psi$ is Bott–Chern harmonic. Therefore by the first part of the theorem applied to $*\psi$ we conclude that $\beta^{0,0} \in \mathbb{C}$, $\partial\beta^{1,1}$, $\bar{\partial}\beta^{1,1}$, $\partial\bar{\partial}\beta^{1,1}$ are primitive and

$$\omega^{n-4} \wedge \partial\bar{\partial}\beta^{2,2} = 0.$$

It is an observation that the same holds respectively for $\alpha^{0,0}$, $\alpha^{1,1}$, $\alpha^{2,2}$, ending the proof. □

Since the coefficient of ω^k is constant in the primitive decomposition of any Bott–Chern or Aeppli harmonic (k, k) -form, we can state the following characterisations of the spaces of Bott–Chern and Aeppli harmonic (k, k) -forms.

Theorem 3.8 *Let (M, J, ω) be a compact almost Kähler manifold of real dimension $2n$. For any $k \in \mathbb{N}$ we have*

$$\mathcal{H}_{\text{BC}}^{k,k} = \mathbb{C} \omega^k \oplus (\mathcal{H}_{\text{BC}}^{k,k} \cap \ker L^{n-k})$$

and

$$\mathcal{H}_{\text{A}}^{k,k} = \mathbb{C} \omega^k \oplus (\mathcal{H}_{\text{A}}^{k,k} \cap \ker L^{n-k}).$$

Proof Let us consider the primitive decomposition of a Bott–Chern or Aeppli harmonic (k, k) -form ψ , i.e.,

$$\psi = \sum_{m=0}^{\min(k,n-k)} \omega^{k-m} \wedge \alpha^{m,m}.$$

Thanks to Theorem 3.7, we know that the coefficient of ω^k , denoted by $\alpha^{0,0}$, is a complex constant. Since $\omega^{n-2m+1} \wedge \alpha^{m,m} = 0$, therefore

$$\omega^{n-k} \wedge \omega^{k-m} \wedge \alpha^{m,m} = \omega^{n-m} \wedge \alpha = 0$$

for any $m \geq 1$. This proves the two inclusions \subseteq . The other two inclusions \supseteq are trivial. □

Remark 3.9 When we consider the case of Theorem 3.8 when $k = 1$, we recover the results of Tardini and the second author, [9, Theorems 3.2 and 3.3]. Namely, we have a complete decomposition into primitive harmonic forms

$$\begin{aligned} \mathcal{H}_{BC}^{1,1} &= \mathbb{C} \omega \oplus (\mathcal{H}_{BC}^{1,1} \cap P^{1,1}), \\ \mathcal{H}_A^{1,1} &= \mathbb{C} \omega \oplus (\mathcal{H}_A^{1,1} \cap P^{1,1}). \end{aligned}$$

Corollary 5.2 will show that a complete decomposition into primitive harmonic forms does not hold in higher bidegrees (k, k) with $k \geq 2$.

The previous theorems can be further specialized in real dimension 8 for bidegree $(2, 2)$. Using almost Kähler identities of [2] and considering the L^2 inner product of forms, we are able to prove the following decompositions which are somewhat surprising in their lack of symmetry between ∂ and $\bar{\partial}$. This is mainly due to the fact that ∂ and $\bar{\partial}$ do not anticommute in the non integrable case.

Theorem 3.10 *Let (M, J, ω) be a compact almost Kähler manifold of real dimension 8. We have*

$$\mathcal{H}_{BC}^{2,2} = \mathbb{C} \omega^2 \oplus \{ \omega \wedge \alpha + \beta \mid \alpha \in P^{1,1}, \beta \in P^{2,2}, \omega \wedge \partial \alpha + \partial \beta = \bar{\partial} \alpha = \bar{\partial} \beta = 0 \},$$

and

$$\mathcal{H}_A^{2,2} = \mathbb{C} \omega^2 \oplus \{ \omega \wedge \alpha + \beta \mid \alpha \in P^{1,1}, \beta \in P^{2,2}, \omega \wedge \partial \alpha - \partial \beta = \bar{\partial} \alpha = \bar{\partial} \beta = 0 \}.$$

Proof Let us prove the Bott–Chern case. The Aepli case is proved by a similar argument. By Theorem 3.8, we know

$$\mathcal{H}_{BC}^{2,2} = \mathbb{C} \omega^2 \oplus (\mathcal{H}_{BC}^{2,2} \cap \ker L^2),$$

therefore we want to prove that

$$\mathcal{H}_{BC}^{2,2} \cap \ker L^2 = \{ \omega \wedge \alpha + \beta \mid \alpha \in P^{1,1}, \beta \in P^{2,2}, \omega \wedge \partial \alpha + \partial \beta = \omega \wedge \bar{\partial} \alpha = \bar{\partial} \beta = 0 \}.$$

The inclusion \supseteq is straightforward. Let us then take a form $\psi \in \mathcal{H}_{BC}^{2,2} \cap \ker L^2$. Its primitive decomposition is

$$\psi = \omega \wedge \alpha + \beta,$$

with $\alpha \in P^{1,1}$ and $\beta \in P^{2,2}$. Since ψ is Bott–Chern harmonic, we know $\partial\psi = \bar{\partial}\psi = 0$, that is

$$\omega \wedge \partial\alpha + \partial\beta = \omega \wedge \bar{\partial}\alpha + \bar{\partial}\beta = 0.$$

Moreover, by Theorem 3.7, we have

$$\partial\bar{\partial}\beta = 0.$$

Let us compute the pointwise inner product between $\omega \wedge \bar{\partial}\alpha$ and $\bar{\partial}\beta$. By the almost Kähler identities of [2, Proposition 3.1], in particular $[\Lambda, \bar{\partial}] = -i\partial^*$, we have

$$\langle L\bar{\partial}\alpha, \bar{\partial}\beta \rangle = \langle \bar{\partial}\alpha, \Lambda\bar{\partial}\beta \rangle = -i\langle \bar{\partial}\alpha, \partial^*\beta \rangle = i\langle \bar{\partial}\alpha, *\bar{\partial}\beta \rangle.$$

Now we integrate this pointwise inner product on the manifold to get the usual L^2 product between forms, obtaining

$$\langle\langle L\bar{\partial}\alpha, \bar{\partial}\beta \rangle\rangle = i\langle\langle \bar{\partial}\alpha, *\bar{\partial}\beta \rangle\rangle = i\langle\langle \alpha, \bar{\partial}^* *\bar{\partial}\beta \rangle\rangle = i\langle\langle \alpha, *\partial\bar{\partial}\beta \rangle\rangle = 0.$$

Since $L\bar{\partial}\alpha = -\bar{\partial}\beta$ and they are L^2 orthogonal, they must both be equal to zero. Now, by the Lefschetz isomorphism, $L\bar{\partial}\alpha = 0$ if and only if $\bar{\partial}\alpha = 0$. This ends the proof. \square

Example 5.1 will show that these two characterisations of the spaces $\mathcal{H}_{BC}^{2,2}$ and $\mathcal{H}_A^{2,2}$ in general cannot be further improved requiring that $\partial\alpha = \bar{\partial}\beta = \bar{\partial}\alpha = \partial\beta = 0$.

4 Primitive decomposition of Dolbeault harmonic (k, k) -forms

The next theorem yields similar conclusions for the Dolbeault case to the ones in the Bott–Chern and Aepli case.

Theorem 4.1 *Let (M, J, ω) be a compact almost Kähler manifold of real dimension $2n$. Let ψ denote a (k, k) -form, for some $k \in \mathbb{N}$. We can write*

$$\psi = \sum_{m=0}^{\min(k,n-k)} \omega^{k-m} \wedge \alpha^{m,m},$$

with $\alpha^{m,m} \in P^{m,m}$. If ψ is Dolbeault harmonic then $\alpha^{0,0} \in \mathbb{C}$ is a complex constant and $\partial\alpha^{1,1}, \bar{\partial}\alpha^{1,1}$ are primitive.

Proof We start by considering the case when $2k \leq n$, using a similar argument to the one used in [1, Theorem 3.4].

Note that ψ is Dolbeault harmonic if and only it satisfies $\bar{\partial}\psi = 0$ and $\partial*\psi = 0$. Since ω is almost Kähler, when we write these conditions out using the primitive

decomposition of ψ we get

$$0 = \bar{\partial}\psi = \sum_{m=0}^k \omega^{k-m} \wedge \bar{\partial}\alpha^{m,m}, \tag{14}$$

$$0 = \partial*\psi = \sum_{m=0}^k (-1)^m \frac{(k-m)!}{(n-k-m)!} \omega^{n-k-m} \wedge \partial\alpha^{m,m}. \tag{15}$$

Then, by taking the wedge product of ω^{n-k-1} with equation (14) we find that

$$\sum_{m=0}^k \omega^{n-m-1} \wedge \bar{\partial}\alpha^{m,m} = \omega^{n-1} \wedge \bar{\partial}\alpha^{0,0} + \omega^{n-2} \wedge \bar{\partial}\alpha^{1,1} = 0, \tag{16}$$

since $\omega^{n-2m+1} \wedge \alpha^{m,m} = 0$ for all $m \in \mathbb{N}$. Similarly, by taking the wedge product of ω^{k-1} with equation (15) we find

$$\begin{aligned} \sum_{m=0}^k (-1)^m \frac{(k-m)!}{(n-k-m)!} \omega^{n-m-1} \wedge \partial\alpha^{m,m} \\ = \frac{k!}{(n-k)!} \omega^{n-1} \wedge \partial\alpha^{0,0} - \frac{(k-1)!}{(n-k-1)!} \omega^{n-2} \wedge \partial\alpha^{1,1} = 0. \end{aligned}$$

Multiplying this by $\frac{(n-k-1)!}{(k-1)!}$, we have

$$\frac{k}{n-k} \omega^{n-1} \wedge \partial\alpha^{0,0} - \omega^{n-2} \wedge \partial\alpha^{1,1} = 0. \tag{17}$$

We can then sum the equations (16) and (17), to get

$$\omega^{n-1} \wedge \left(\bar{\partial} + \frac{k}{n-k} \partial \right) \alpha^{0,0} + \omega^{n-2} \wedge (\bar{\partial} - \partial) \alpha^{1,1} = 0. \tag{18}$$

Now, by making use of the operator

$$d^c = i(\mu - \partial + \bar{\partial} - \bar{\mu})$$

and the fact that $\mu = \bar{\mu} = 0$ when acting on an $(n-1, n-1)$ -form, we can rewrite equation (18) as

$$\omega^{n-1} \wedge \left(\bar{\partial} + \frac{k}{n-k} \partial \right) \alpha^{0,0} = \text{id}^c(\omega^{n-2} \wedge \alpha^{1,1}).$$

Applying $-id^c$ the right-hand side vanishes and we are left with

$$\omega^{n-1} \wedge (\mu - \partial + \bar{\partial} - \bar{\mu}) \left(\bar{\partial} + \frac{k}{n-k} \partial \right) \alpha^{0,0} = \left(\frac{k}{n-k} + 1 \right) \omega^{n-1} \wedge \partial \bar{\partial} \alpha^{0,0} = 0.$$

In particular, we have $\omega^{n-1} \wedge \partial \bar{\partial} \alpha^{0,0} = 0$, which implies that $\alpha^{0,0}$ is constant by Lemma 3.1. Now, looking at (18), we deduce that $\partial \alpha^{1,1}$ and $\bar{\partial} \alpha^{1,1}$ are primitive.

The result in the case when $2k \geq n$ follows simply from the first case by Serre duality. Namely, we have

$$\mathcal{H}_{\bar{\partial}}^{k,k} = \overline{* \mathcal{H}_{\bar{\partial}}^{n-k,n-k}}.$$

Therefore, any element of $\mathcal{H}_{\bar{\partial}}^{k,k}$ with $2k \geq n$ can be written as

$$* \bar{\psi} = \sum_{m=0}^{n-k} (-1)^m \frac{(n-k-m)!}{(k-m)!} \omega^{k-m} \wedge \overline{\alpha^{m,m}}$$

for some $\psi \in \mathcal{H}_{\bar{\partial}}^{n-k,n-k}$, and since $2(n-k) \leq n$ we conclude that $\alpha^{0,0}$ is constant and $\partial \alpha^{1,1}$ and $\bar{\partial} \alpha^{1,1}$ are primitive. □

Remark 4.2 Note that the same statement of Theorem 4.1 also holds for Δ_{∂} -harmonic (k, k) -forms. The proof of this is equivalent to the one above up to conjugation.

The above result allows the primitive decomposition of (k, k) -forms to descend partially to Dolbeault harmonic (k, k) -forms in the following way, using the same proof as for Theorem 3.8.

Theorem 4.3 *Let (M, J, ω) be a compact almost Kähler manifold of real dimension $2n$. For any $k \in \mathbb{N}$ we have*

$$\mathcal{H}_{\bar{\partial}}^{k,k} = \mathbb{C} \omega^k \oplus (\mathcal{H}_{\bar{\partial}}^{k,k} \cap \ker L^{n-k})$$

and

$$\mathcal{H}_{\partial}^{k,k} = \mathbb{C} \omega^k \oplus (\mathcal{H}_{\partial}^{k,k} \cap \ker L^{n-k}).$$

Remark 4.4 When we consider the case of Theorem 4.3 with $k = 1$, we recover the results of Cattaneo, Tardini and Tomassini, [1, Theorem 3.4 and Corollary 3.5]. Namely, we have a complete decomposition into primitive harmonic forms

$$\begin{aligned} \mathcal{H}_{\bar{\partial}}^{1,1} &= \mathbb{C} \omega \oplus (\mathcal{H}_{\bar{\partial}}^{1,1} \cap P^{1,1}), \\ \mathcal{H}_{\partial}^{1,1} &= \mathbb{C} \omega \oplus (\mathcal{H}_{\partial}^{1,1} \cap P^{1,1}). \end{aligned}$$

Corollary 5.2 will show that a complete decomposition into primitive harmonic forms does not hold in higher bidegrees (k, k) with $k \geq 2$.

If we now restrict to real dimension 8, we obtain the following.

Corollary 4.5 *Let (M, J, ω) be a compact almost Kähler manifold of real dimension 8. We have*

$$\mathcal{H}_{\bar{\partial}}^{2,2} = \mathbb{C} \omega^2 \oplus \{ \omega \wedge \alpha + \beta \mid \alpha \in P^{1,1}, \beta \in P^{2,2}, \omega \wedge \bar{\partial} \alpha + \bar{\partial} \beta = \omega \wedge \partial \alpha - \partial \beta = 0 \},$$

and

$$\mathcal{H}_{\partial}^{2,2} = \mathbb{C} \omega^2 \oplus \{ \omega \wedge \alpha + \beta \mid \alpha \in P^{1,1}, \beta \in P^{2,2}, \omega \wedge \partial \alpha + \partial \beta = \omega \wedge \bar{\partial} \alpha - \bar{\partial} \beta = 0 \}.$$

Proof The result follows immediately from Theorem 4.3, from the characterisation (4) of Dolbeault and ∂ -harmonic forms and from formula (5). \square

From Theorem 3.10 and Corollary 4.5, we deduce the following inclusions of the spaces of harmonic forms in dimension 8.

Corollary 4.6 *Let (M, J, ω) be a compact almost Kähler manifold of real dimension 8. We have*

$$\mathcal{H}_{BC}^{2,2} \subseteq \mathcal{H}_{\bar{\partial}}^{2,2}, \quad \mathcal{H}_A^{2,2} \subseteq \mathcal{H}_{\partial}^{2,2}.$$

5 Examples

In this section we present two 8-dimensional examples of nilmanifolds and study their harmonic $(2, 2)$ -forms.

Example 5.1 We consider a similar construction to the one in [1, Example 4.9]. Let $\mathbb{T}^8 = \mathbb{Z}^8 \backslash \mathbb{R}^8$ be the 8-dimensional torus with real coordinates $(x^1, y^1, x^2, y^2, x^3, y^3, x^4, y^4)$ on \mathbb{R}^8 . Let $g = g(x^4, y^4)$ be a non-constant function on \mathbb{T}^8 . We define an almost complex structure J by setting

$$\begin{aligned} \phi^1 &= e^g dx^1 + i e^{-g} dy^1, \\ \phi^2 &= dx^2 + i dy^2, \\ \phi^3 &= dx^3 + i dy^3, \\ \phi^4 &= dx^4 + i dy^4 \end{aligned}$$

to be a global coframe of $(1, 0)$ -forms. Denote by V_1, V_2, V_3, V_4 the global frame of vector fields dual to $\phi^1, \phi^2, \phi^3, \phi^4$. Then, the structure equations are

$$\begin{aligned} d\phi^1 &= V_4(g) \phi^{4\bar{1}} - \bar{V}_4(g) \phi^{\bar{1}4}, \\ d\phi^2 &= 0, \\ d\phi^3 &= 0, \\ d\phi^4 &= 0. \end{aligned}$$

We endow (\mathbb{T}^8, J) with the almost Kähler metric given by the fundamental form

$$\omega = i(\phi^{1\bar{1}} + \phi^{2\bar{2}} + \phi^{3\bar{3}} + \phi^{4\bar{4}}).$$

We consider the volume form

$$\frac{\omega^4}{4!} = \phi^{1\bar{1}2\bar{2}3\bar{3}4\bar{4}} = \phi^{1234\bar{1}\bar{2}\bar{3}\bar{4}}.$$

We want to show that the inclusion $\mathcal{H}_{BC}^{2,2} \subseteq \mathcal{H}_\partial^{2,2}$ of Corollary 4.6 is strict. Let us consider the form $\phi^{12\bar{2}\bar{3}}$. We compute

$$\begin{aligned} \partial\phi^{12\bar{2}\bar{3}} &= 0, \\ \bar{\partial}*\phi^{12\bar{2}\bar{3}} &= \bar{\partial}\phi^{14\bar{3}\bar{4}} = 0, \\ \bar{\partial}\phi^{12\bar{2}\bar{3}} &= V_4(g)\phi^{4\bar{1}\bar{2}\bar{2}\bar{3}} \neq 0. \end{aligned}$$

Therefore, $\phi^{12\bar{2}\bar{3}} \in \mathcal{H}_\partial^{2,2} \setminus \mathcal{H}_{BC}^{2,2}$, proving our claim. We note that the same holds for the form $\phi^{13\bar{2}\bar{3}}$. By duality (see (4)), also note that $*\psi \in \mathcal{H}_\partial^{2,2} \setminus \mathcal{H}_A^{2,2}$.

We also want to show that in general the primitive decomposition of $(2, 2)$ -forms does not descend to the spaces of Bott–Chern, Aeppli, Dolbeault and ∂ -harmonic forms. Namely, we want to find a $(2, 2)$ -form

$$\psi = \omega \wedge \alpha + \beta,$$

where $\alpha \in P^{1,1}$ and $\beta \in P^{2,2}$, such that α and β are not Bott–Chern and ∂ -harmonic. Considering $\bar{\psi}$, the same can then be shown for the cases of Aeppli and Dolbeault. Let us consider the form $\psi = 2\phi^{2\bar{1}4\bar{4}}$. Its primitive decomposition is

$$2\phi^{2\bar{1}4\bar{4}} = (\phi^{2\bar{1}4\bar{4}} + \phi^{2\bar{1}3\bar{3}}) + (\phi^{2\bar{1}4\bar{4}} - \phi^{2\bar{1}3\bar{3}}),$$

where

$$\begin{aligned} \phi^{2\bar{1}4\bar{4}} + \phi^{2\bar{1}3\bar{3}} &= -i\omega \wedge \phi^{2\bar{1}} \in L(P^{1,1}), \\ \phi^{2\bar{1}4\bar{4}} - \phi^{2\bar{1}3\bar{3}} &\in P^{2,2}. \end{aligned}$$

Set $\beta = \phi^{2\bar{1}4\bar{4}} - \phi^{2\bar{1}3\bar{3}}$ and $\alpha = -i\phi^{2\bar{1}}$. Then we have

$$\begin{aligned} \bar{\partial}\alpha &= 0, \\ \bar{\partial}\beta &= 0, \\ \omega \wedge \partial\alpha + \partial\beta &= 0, \\ \partial\alpha &= -i\bar{V}_4(g)\phi^{2\bar{1}\bar{4}} \neq 0, \end{aligned}$$

$$\partial\beta = -\overline{V_4}(g)\phi^{21\overline{4}3\overline{3}} \neq 0,$$

therefore $\psi \in \mathcal{H}_{BC}^{2,2} \cap \mathcal{H}_{\partial}^{2,2}$, while $\alpha \notin \mathcal{H}_{BC}^{1,1} \cup \mathcal{H}_{\partial}^{1,1}$ and $\beta \notin \mathcal{H}_{BC}^{2,2} \cup \mathcal{H}_{\partial}^{2,2}$. Note that this also shows that the two results of Theorem 3.10 cannot be strengthened by asking that $\omega \wedge \partial\alpha = \partial\beta = 0$, instead of $\omega \wedge \partial\alpha + \partial\beta = 0$ or $\omega \wedge \partial\alpha - \partial\beta = 0$.

Summing up the results from the above example, we state the following corollary.

Corollary 5.2 *There exists a compact almost Kähler manifold (M, J, ω) of real dimension 8 such that*

$$\mathcal{H}_{BC}^{2,2} \not\subseteq \mathcal{H}_{\partial}^{2,2}, \quad \mathcal{H}_A^{2,2} \not\subseteq \mathcal{H}_{\partial}^{2,2}$$

and

$$\mathcal{H}_D^{2,2} \not\subseteq \mathbb{C}\omega^2 \oplus L(P^{1,1} \cap \mathcal{H}_D^{1,1}) \oplus (P^{2,2} \cap \mathcal{H}_D^{2,2}),$$

where $D \in \{BC, A, \overline{\partial}, \partial\}$.

We remark that the almost Kähler structure of Example 5.1 is not left invariant with respect to the usual Lie group structure of the torus. In fact, we do not have any example of an 8-dimensional manifold with a left invariant almost Kähler structure which satisfies the conditions of Corollary 5.2. Below we present one such example of a manifold (described in [1, Example 4.3]), with a left invariant almost Kähler structure. We show that in this example all left invariant harmonic forms $\psi \in \mathcal{H}_D^{2,2}$ have a primitive decomposition such that each component is also contained in $\mathcal{H}_D^{2,2}$, where $D \in \{BC, A, \partial, \overline{\partial}\}$.

Example 5.3 We start by defining

$$\mathbb{H}(1, 2) := \left\{ \left(\begin{array}{cccc} 1 & 0 & x_1 & z_1 \\ 0 & 1 & x_2 & z_2 \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 0 \end{array} \right) \middle| x_1, x_2, y, z_1, z_2 \in \mathbb{R} \right\}.$$

Then, if we let $\Gamma \subset \mathbb{H}(2, 1)$ be the subgroup of elements with integer valued entries, we can define the compact 8-manifold $X := \Gamma \backslash \mathbb{H}(2, 1) \times \mathbb{T}^3$. A left invariant coframe on X can be given by

$$\begin{aligned} e^1 &= dx_2, & e^2 &= dx_1, & e^3 &= dy, & e^4 &= du, \\ e^5 &= dz_1 - x_1dy, & e^6 &= dz_2 - x_2dy, & e^7 &= dv, & e^8 &= dw, \end{aligned}$$

where u, v, w parametrise \mathbb{T}^3 . From this coframe, we derive the structure equations

$$\begin{aligned} de^1 &= de^2 = de^3 = de^4 = de^7 = de^8 = 0, \\ de^5 &= -e^{23}, \quad de^6 = -e^{12}. \end{aligned}$$

An almost Hermitian structure is then defined so that

$$\begin{aligned} \phi^1 &= e^1 + ie^5, & \phi^2 &= e^2 + ie^6, \\ \phi^3 &= e^3 + ie^7, & \phi^4 &= e^4 + ie^8 \end{aligned}$$

are orthonormal $(1, 0)$ -forms, with structure equations given by

$$\begin{aligned} d\phi^1 &= -\frac{i}{4}(\phi^{23} + \phi^{2\bar{3}} - \phi^{3\bar{2}} + \phi^{\bar{2}\bar{3}}), \\ d\phi^2 &= -\frac{i}{4}(\phi^{13} + \phi^{1\bar{3}} - \phi^{3\bar{1}} + \phi^{\bar{1}\bar{3}}), \\ d\phi^3 &= d\phi^4 = 0. \end{aligned}$$

The fundamental form corresponding to this almost Hermitian structure is

$$\omega = i(\phi^{1\bar{1}} + \phi^{2\bar{2}} + \phi^{3\bar{3}} + \phi^{4\bar{4}}).$$

This fundamental form is d -closed and so the structure we have defined is almost Kähler.

It is then a trivial (although computationally tedious) task to compute the space of left invariant forms contained in $\mathcal{H}_{\partial}^{2,2}$. This space is spanned by the following sixteen $(2, 2)$ -forms:

$$\begin{array}{cccc} \phi^{12\bar{1}\bar{2}}, & \phi^{12\bar{1}\bar{3}} + \phi^{13\bar{1}\bar{2}}, & \phi^{123\bar{4}} + \phi^{132\bar{4}}, & \phi^{122\bar{3}} + \phi^{23\bar{1}\bar{2}}, \\ \phi^{132\bar{3}} + \phi^{23\bar{1}\bar{3}}, & \phi^{132\bar{4}} + \phi^{23\bar{1}\bar{4}}, & \phi^{13\bar{1}\bar{3}} + \phi^{232\bar{3}}, & \phi^{13\bar{1}\bar{4}} + \phi^{232\bar{4}}, \\ \phi^{142\bar{3}} + \phi^{24\bar{1}\bar{3}}, & \phi^{142\bar{4}} + \phi^{24\bar{1}\bar{4}}, & \phi^{14\bar{1}\bar{3}} + \phi^{242\bar{3}}, & \phi^{14\bar{1}\bar{4}} + \phi^{242\bar{4}}, \\ \phi^{24\bar{1}\bar{3}} + \phi^{34\bar{1}\bar{2}}, & \phi^{143\bar{4}} + \phi^{34\bar{1}\bar{4}}, & \phi^{243\bar{4}} + \phi^{342\bar{4}}, & \phi^{343\bar{4}}. \end{array}$$

Furthermore, when we compute the spaces of left invariant forms contained in $\mathcal{H}_{\partial}^{2,2}$, $\mathcal{H}_{BC}^{2,2}$ and $\mathcal{H}_A^{2,2}$, we find that these spaces are all equal to the space of left invariant forms in $\mathcal{H}_{\partial}^{2,2}$. We claim that this implies that the primitive decomposition descends to harmonic $(2, 2)$ -forms in all of these spaces, i.e., if $\psi \in \mathcal{H}_D^{2,2}$ is left invariant then

$$\psi \in \mathbb{C}\omega^2 \oplus L(P^{1,1} \cap \mathcal{H}_D^{1,1}) \oplus (P^{2,2} \cap \mathcal{H}_D^{2,2}),$$

for any $D \in \{\partial, \bar{\partial}, BC, A\}$.

To see why this is the case, consider the $(2, 2)$ -form $\psi \in A^{2,2}(X)$ with primitive decomposition $\psi = c\omega^2 + \alpha \wedge \omega + \beta$, where $c \in \mathbb{C}$, $\alpha \in P^{1,1}$, $\beta \in P^{2,2}$. Let ψ be contained in both $\mathcal{H}_{\partial}^{2,2}$ and $\mathcal{H}_{\bar{\partial}}^{2,2}$. From $\psi \in \mathcal{H}_{\partial}^{2,2}$ we obtain

$$\begin{aligned} \bar{\partial}\alpha \wedge \omega + \bar{\partial}\beta &= 0, \\ \partial\alpha \wedge \omega - \partial\beta &= 0, \end{aligned}$$

and from $\psi \in \mathcal{H}_D^{2,2}$ we obtain

$$\begin{aligned}\bar{\partial}\alpha \wedge \omega - \bar{\partial}\beta &= 0, \\ \partial\alpha \wedge \omega + \partial\beta &= 0.\end{aligned}$$

Combining these results, we see that

$$\partial\alpha \wedge \omega = \bar{\partial}\alpha \wedge \omega = \partial\beta = \bar{\partial}\beta = 0.$$

This is sufficient to imply that $\omega \wedge \alpha, \beta \in \mathcal{H}_D^{2,2}$ or in other words

$$\psi \in \mathbb{C}\omega^2 \oplus L(P^{1,1} \cap \mathcal{H}_D^{1,1}) \oplus (P^{2,2} \cap \mathcal{H}_D^{2,2})$$

for all $D \in \{\partial, \bar{\partial}, BC, A\}$.

Acknowledgements The authors would like to express their sincere gratitude to Adriano Tomassini for several interesting discussions.

Author Contributions Both authors wrote and reviewed the manuscript.

Funding Open access funding provided by Università degli Studi di Parma within the CRUI-CARE Agreement.

Declarations

Conflict of interest The authors declare no competing interests.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Cattaneo, A., Tardini, N., Tomassini, A.: Primitive decompositions of Dolbeault harmonic forms on compact almost-Kähler manifolds (2022). [arXiv:2201.09273](https://arxiv.org/abs/2201.09273) (to appear in Rev. Un. Mat. Argentina)
2. Cirici, J., Wilson, S.O.: Topological and geometric aspects of almost Kähler manifolds via harmonic theory. *Sel. Math. (N.S.)* **26**(3), 35 (2020)
3. Hirzebruch, F.: Some problems on differentiable and complex manifolds. *Ann. Math.* **60**, 213–236 (1954)
4. Holt, T.: Bott–Chern and $\bar{\partial}$ harmonic forms on almost Hermitian 4-manifolds. *Math. Z.* **302**(1), 47–72 (2022)
5. Holt, T., Zhang, W.: Harmonic forms on the Kodaira-Thurston manifold. *Adv. Math.* **400**, 108277 (2022)
6. Holt, T., Zhang, W.: Almost Kähler Kodaira–Spencer problem. *Math. Res. Lett.* **29**(6), 1685–1700 (2022)

7. Piovani, R.: Harmonic $(1, 1)$ -forms on compact almost Hermitian 4-manifolds. *Riv. Math. Univ. Parma (N.S.)* **13**(2), 671–692 (2022)
8. Piovani, R.: Dolbeault harmonic $(1, 1)$ -forms on 4-dimensional compact quotients of Lie groups with a left invariant almost Hermitian structure. *J. Geom. Phys.* **180**, 104639 (2022)
9. Piovani, R., Tardini, N.: Bott–Chern harmonic forms and primitive decompositions on compact almost Kähler manifolds. *Ann. Mat. Pura Appl.* **1923**, 1–23 (2022)
10. Piovani, R., Tomassini, A.: Bott–Chern Laplacian on almost Hermitian manifolds. *Math. Z.* **301**(3), 2685–2707 (2022)
11. Piovani, R., Tomassini, A.: On the dimension of Dolbeault harmonic $(1, 1)$ -forms on almost Hermitian 4-manifolds. *Pure Appl. Math. Q.* **18**(3), 1187–1201 (2022)
12. Tardini, N., Tomassini, A.: $\bar{\partial}$ -Harmonic forms on 4-dimensional almost-Hermitian manifolds (2021). [arXiv:2104.10594](https://arxiv.org/abs/2104.10594) (to appear in *Math. Res. Lett.*)
13. Tardini, N., Tomassini, A.: Almost-complex invariants of families of six-dimensional solvmanifolds. *Complex Manifolds* **9**(1), 238–260 (2022)
14. Weil, A.: *Introduction à L'Étude des Variétés Kählériennes*. Publications de l'Institut de Mathématique de l'Université de Nancago VI. Hermann, Paris (1958)
15. Zhang, W.: Almost complex Hodge theory. *Riv. Math. Univ. Parma (N.S.)* **13**(2), 481–504 (2022)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.