ARTICLE IN PRESS

Mathematical Social Sciences xxx (xxxx) xxx

Contents lists available at ScienceDirect

Mathematical Social Sciences

journal homepage: www.elsevier.com/locate/mss

Piecewise linear feedback rules and weird quasiperiodic dynamics: Some general observations and two economic applications ☆

Laura Gardini [a,b], Davide Radi [b,c], Noemi Schmitt [d], Iryna Sushko [c,e], Frank Westerhoff [d,*]

[a] Department of Economics, Society and Politics, University of Urbino Carlo Bo, Urbino, Italy
[b] Department of Finance, VSB-Technical University of Ostrava, Ostrava, Czech Republic
[c] DiMSEFA, Catholic University of the Sacred Heart, Milan, Italy
[d] Department of Economics, University of Bamberg, Bamberg, Germany
[e] Institute of Mathematics, NAS of Ukraine, Kyiv, Ukraine

ARTICLE INFO

JEL classification:
C63
E32
G12

Keywords:
Economic dynamics
Regulatory policies
Piecewise linear feedback rules
Discrete dynamical systems
Discontinuities
Weird quasiperiodic dynamics

ABSTRACT

We show that regulators who use piecewise linear feedback rules to stabilize the dynamics of otherwise linear discrete dynamical systems of dimension two or higher may (unintentionally) trigger weird quasiperiodic dynamics. This recently discovered type of behavior, which blends features of both quasiperiodic and chaotic motion, may be desirable when it replaces divergent dynamics or undesirable when it suppresses fixed-point dynamics. After demonstrating that weird quasiperiodic dynamics emerge as a surprisingly natural phenomenon in such systems, an observation relevant to many applied disciplines, we discuss how they may arise in economic contexts, using two economic models as examples. In doing so, we uncover novel sources of endogenous fluctuations in dynamic (economic) systems.

1. Introduction

Consider a linear dynamical system in discrete time of dimension two or higher. In general, such a system possesses a unique fixed point, which may be either stable or unstable. In the former case, a regulator may apply a linear feedback rule, targeting, for instance, the change in a state variable or its deviation from the fixed-point value, to accelerate convergence to the fixed point. In the latter case, a regulator may use such a feedback rule to ensure that the system converges to its fixed point. In what follows, we are interested in feedback rules that preserve the system's fixed point.

A notable study in this area is Baumol (1961), who discusses the effects of two linear countercyclical fiscal policy rules within the framework of Samuelson's (1939) multiplier-accelerator model: one that offsets trends in income, and another that adjusts income towards its fixed-point level. Under the first rule, government spending increases above its normal level when income has declined, and decreases below its normal level when income has risen. Under the second rule, government spending is adjusted to offset deviations between current income and its fixed-point level. Neither policy alters the underlying fixed point—government spending is equal to its normal level when income has reached its fixed-point level.

The application of linear feedback rules requires the regulator to intervene continuously in the system's out-of-equilibrium behavior, a level of fine-tuning that may be infeasible or prohibitively costly. As an alternative, a regulator may apply a linear feedback rule only when a certain condition is met. This condition may, for example, refer to a change in a state variable or its deviation from the fixed-point value. The key objective of this paper is to demonstrate that the use of such piecewise linear feedback rules in otherwise linear discrete dynamical systems of dimension two or higher can create weird quasiperiodic dynamics. This newly identified type of behavior, first encountered in Gardini et al. (2025a), blends characteristics of quasiperiodic and chaotic dynamics. Weird quasiperiodic dynamics may be regarded as

☆ This paper was presented at the TI Complexity in Economics Seminars, Amsterdam, Netherlands, October 30, 2025, and at NED 2025, Naples, Italy, June 3–5, 2025. We thank the participants, especially Cees Diks, Jan Tuinstra, and Florian Wagener, for their valuable comments, as well as two anonymous referees and the editors for their constructive feedback.

* Corresponding author at: University of Bamberg, Department of Economics, Feldkirchenstrasse 21, 96045, Bamberg, Germany.
E-mail address: frank.westerhoff@uni-bamberg.de (F. Westerhoff).

https://doi.org/10.1016/j.mathsocsci.2026.102528
Received 12 August 2025; Received in revised form 20 January 2026; Accepted 31 March 2026
Available online 1 April 2026

beneficial when they replace divergent trajectories, but they can also be detrimental if they hinder convergence to the fixed point. As a byproduct, we identify novel mechanisms capable of generating endogenous fluctuations in dynamic (economic) systems.

While our findings apply broadly across disciplines, we illustrate them using two widely studied economic frameworks. First, we examine how piecewise linear feedback rules influence the dynamics of a financial market model with chartists and fundamentalists. This powerful class of models is surveyed by Dieci and He (2018). Second, we show that these rules can generate endogenous business cycle dynamics in Samuelson's (1939) multiplier-accelerator model. See Bischi (2025) for a recent appraisal of Samuelson's contribution. The emergence of weird quasiperiodic attractors in these settings also offers new insights into the nature of financial market turbulence and fluctuations in economic activity.

The dynamics of the regulated system are governed by two different linear maps, each applying to a specific partition of the state space, yet both sharing the same fixed point. Gardini et al. (2025b, 2025c) show that weird quasiperiodic attractors are the only generic type of attractor capable of producing endogenous dynamics in such systems. Technically, a weird quasiperiodic attractor is a closed, invariant, and attracting set on which all the trajectories are dense, possibly exhibiting a complex—indeed, weird—structure. Since it contains no periodic points, a weird quasiperiodic attractor is neither an attracting cycle nor a chaotic attractor; rather, it represents the closure of quasiperiodic trajectories. Although weird quasiperiodic attractors display sensitivity to initial conditions, their largest Lyapunov exponent may be zero. This occurs because nearby trajectories within the attractor can remain close together for extended periods, only separating once they lie on opposite sides of a discontinuity. At that point, different functions apply and they are mapped far apart. Until this separation takes place, the trajectories move in tandem, keeping the largest Lyapunov exponent at zero.

Fig. 1 displays four weird quasiperiodic attractors in the (x, y)-state space. Details regarding the underlying two-dimensional maps and their parameter settings will be provided in Section 2. At first glance, the attractors depicted in red may appear chaotic; however, they are, in fact, weird quasiperiodic. In two of the examples, the systems' fixed points, shown in blue, are also attracting. Matching colors indicate the respective basins of attraction, while initial conditions that lead to divergent dynamics are depicted in light gray. The four panels of Fig. 2 present the time evolution of state variable x associated with the weird quasiperiodic attractors in red, revealing seemingly irregular yet nearly quasiperiodic behaviors. The fixed-point values of state variable x are indicated in blue.

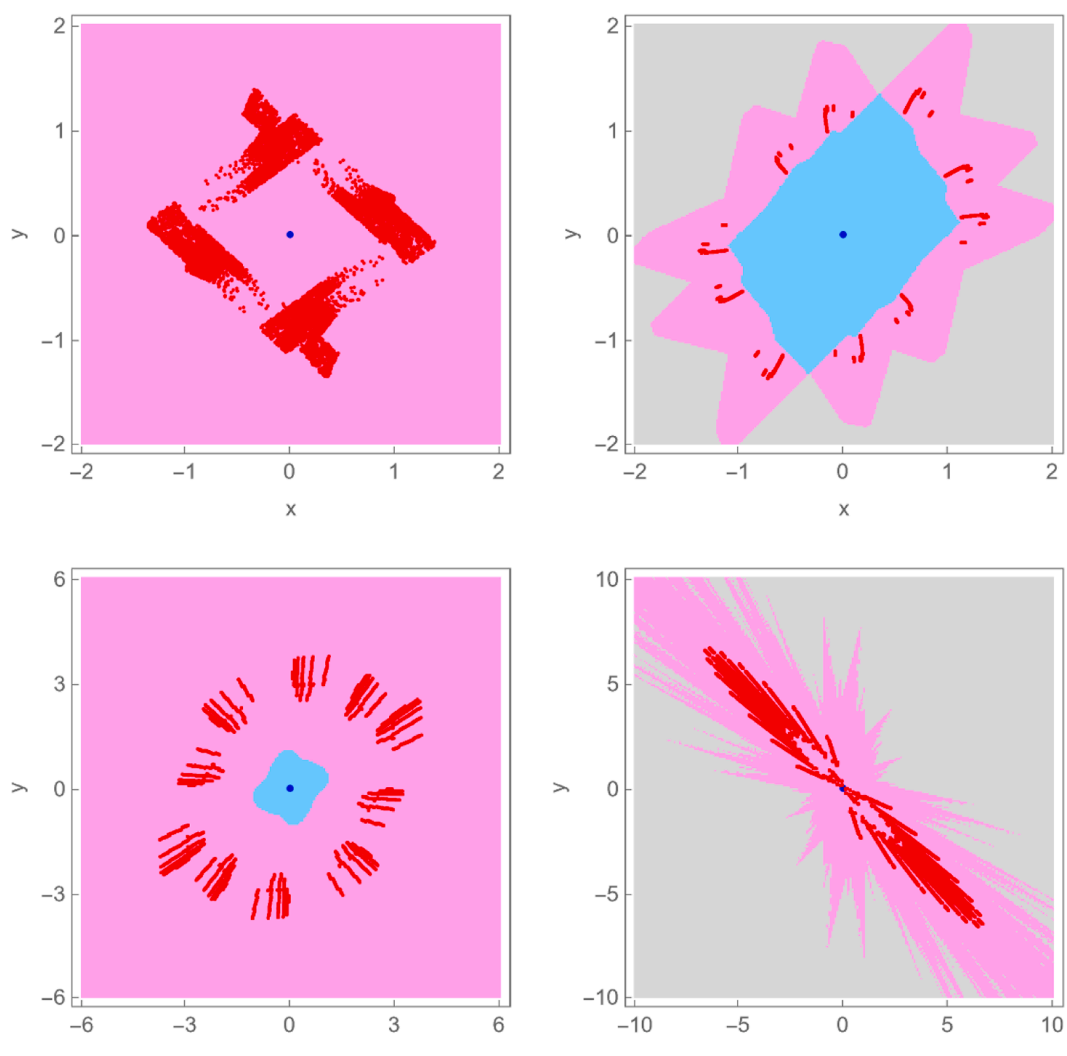

Fig. 1. Weird quasiperiodic attractors in the (x, y)-state space. The blue and red dots represent fixed points and weird quasiperiodic attractors, respectively. Their basins of attraction are shown in matching colors. Initial conditions marked in gray result in divergent dynamics. Top left: map MG_1 with $a = 0.2, b = -1.3,\ c = 0.5$ and $h = 1$. Top right: map MG_1 with $a = 0.15,\ b = -0.55,\ c = -0.5$ and $h = 1$. Bottom left: map MG_4 with $a = 0.15,\ b = -0.9,\ c = -0.5$ and $h = 1$. Bottom right: map MG_2 with $a = -2.5,\ b = -0.55,\ c = -0.5$ and $h = 1$.

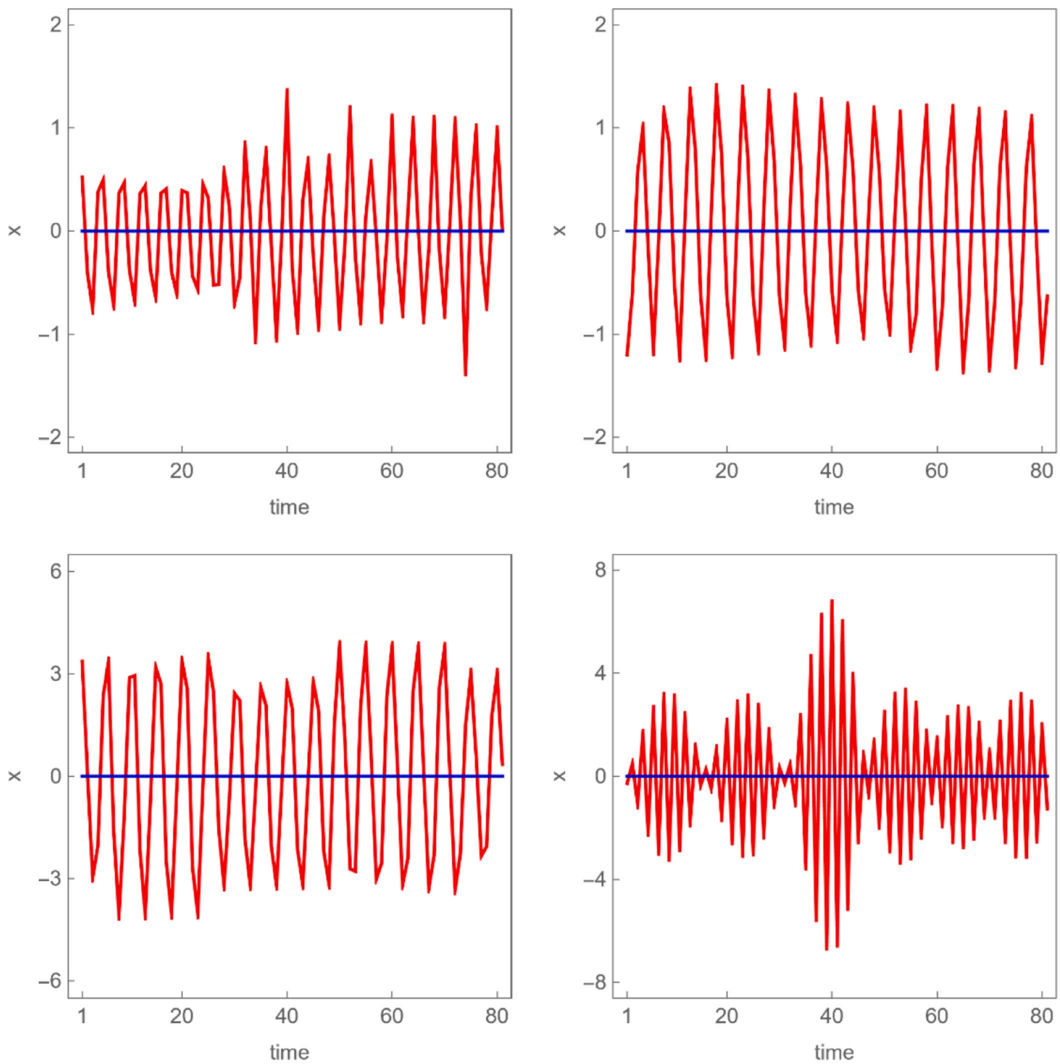

Fig. 2. Weird quasiperiodic dynamics in the time domain. The red lines show the evolution of the state variable x of the weird quasiperiodic attractors presented in Fig. 1. The blue lines mark their fixed-point levels.

Remarkably, weird quasiperiodic attractors may emerge in different dynamic constellations. To see this, recall that the dynamical systems under consideration involve two linear sub-maps: one that applies when the regulator is inactive, and another that applies when the regulator is active. The fixed points of both sub-maps are equal. The regulator is inactive when the system's dynamics are in the vicinity of these fixed points. We therefore refer to this map as the internal sub-map, and the other as the external sub-map. Due to the partitioning of the state space, the fixed point of the internal sub-map is a real fixed point of the dynamical system, while that of the external sub-map is considered virtual. Clearly, a virtual fixed point is, by definition, a fixed point that lies outside the domain to which its map applies. Depending on whether the fixed points of the internal and external sub-maps are stable or unstable, four distinct constellations may give rise to weird quasiperiodic attractors.

Let us revisit Fig. 1. In all four examples, the fixed points are located at the origin. In the top-left panel, the fixed point of the internal sub-map is unstable, while the fixed point of the external sub-map is stable. Trajectories neither converge to the origin nor diverge. Instead, nearly all initial conditions trigger dynamics that converge to the weird quasiperiodic attractor. In the top-right panel, the fixed point of the internal sub-map is stable, rendering the real fixed point of the dynamical system at least locally stable. Despite the fact that the fixed point of the external sub-map is unstable, a weird quasiperiodic attractor nevertheless exists. Moreover, divergent dynamics are observable for more distant initial conditions. In the bottom-left panel, both the fixed points of the internal and external sub-maps are stable. In this case, a fixed-point attractor coexists with a weird quasiperiodic attractor, and no divergent behavior occurs. In the bottom-right panel, the fixed points of the internal and external sub-maps are unstable. The origin is not attracting, yet a weird quasiperiodic attractor with a nontrivial basin of attraction still exists. Apparently, weird quasiperiodic attractors can arise under a wide range of dynamical conditions.

We proceed as follows. In Section 2, we show that piecewise linear feedback rules can generate weird quasiperiodic dynamics in otherwise linear discrete dynamical systems. In Sections 3 and 4, we explore how these rules manifest in financial markets and the real economy, respectively. Section 5 concludes the paper.

2. General observations

Due to our economic applications, we focus on second-order discontinuous piecewise linear difference equations that can be rewritten as two-dimensional discontinuous piecewise linear maps. In the following, we consider only the generic behavior of such maps. Nongeneric cases and higher-order systems are examined in Gardini et al. (2025b, 2025c).

2.1. A general setup

Let R_t describe a regulator's manipulation of a discrete dynamical system. The state variable X_t evolves according to

$$X_t = aX_{t-1} + bX_{t-2} + A + R_{t-1}, \tag{2.1}$$

where a, b and A are parameters, with $b \neq 0$. For

$$R_t = R = 0, \tag{2.2}$$

we obtain the second-order linear difference equation

$$X_t = aX_{t-1} + bX_{t-2} + A. \tag{2.3}$$

For ease of exposition, we introduce the auxiliary variable $Y_{t-1} = X_{t-2}$, allowing us to rewrite (2.3) as the two-dimensional linear map

$$LG = \begin{cases} X' = aX + bY + A \\ Y' = X \end{cases}, \tag{2.4}$$

where the prime symbol denotes the unit time advancement operator. Ignoring the nongeneric case $1 - a - b \neq 0$, map LG has the unique fixed point

$$\overline{X} = \overline{Y} = \frac{A}{1-a-b}. \tag{2.5}$$

The Jacobian matrix of map LG is

$$J(LG) = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}, \tag{2.6}$$

with trace $tr(LG) = a$ and determinant $det(LG) = -b$. Based on the well-known stability conditions (i) $1 + tr + det > 0$, (ii) $1 - tr + det > 0$, and (iii) $1 - det > 0$, as reported, e.g., in Medio and Lines (2001), the fixed point of map LG is globally stable when the following inequalities hold jointly: (i) $a > b - 1$, (ii) $a < 1 - b$, (iii) and $b > -1$. In the (b, a)-parameter space, these conditions define a stability triangle, which will be discussed further in the sequel. Roughly speaking, the fixed point of map LG is globally stable when neither parameter a nor parameter b is too large in absolute value.

2.2. Piecewise linear feedback rules

In the following, we consider two general types of piecewise linear feedback rules, each presented in two variations, yielding four rules in total. While Ott et al. (1990) apply such rules to control chaos, Lü et al. (2002) show that they can also generate chaotic behavior. For a review, see Schöll and Schuster (2007). We demonstrate that piecewise linear feedback rules may produce weird quasiperiodic dynamics.

2.2.1. Intervention rule CC

The first type of intervention rule responds to the change of a system's state variable when the magnitude of this change exceeds a threshold $h > 0$. Indexing this rule as CC, indicating that the intervention is conditioned on the change of the state variable and activated when this change becomes sufficiently large, we obtain

$$R_t = R_t^{CC} = \begin{cases} -c(X_t - X_{t-1}) & \text{if } |X_t - X_{t-1}| > h \\ 0 & \text{if } |X_t - X_{t-1}| \leq h \end{cases}. \tag{2.7}$$

Parameter c represents the strength of the regulator's intervention. For $c > 0$, the regulator uses a negative feedback rule, acting against the current change of the state variable X_t. For $c < 0$, the regulator uses a positive feedback rule, reinforcing the current change of the state variable X_t. The regulator is inactive when $|X_t - X_{t-1}| \leq h$.

Inserting (2.7) into (2.1), the dynamics of the regulated system evolve as

$$MG_0 : \begin{cases} MG_{E,0} = \begin{cases} X' = aX + bY + A - c(X - Y) & \text{if } |X - Y| > h \\ Y' = X \end{cases} \\ MG_{I,0} = \begin{cases} X' = aX + bY + A & \text{if } |X - Y| \leq h \\ Y' = X \end{cases} \end{cases}, \tag{2.8}$$

constituting a two-dimensional discontinuous piecewise linear map. The dynamics of map MG_0 follow the external sub-map $MG_{E,0}$ when the magnitude of the change of the state variable X_t exceeds the threshold parameter $h > 0$. Otherwise, they follow the internal sub-map $M_{I,0}$. Both linear sub-maps $MG_{E,0}$ and $MG_{I,0}$ share the same fixed point as the linear map LG.

It is convenient to express the state variables of map MG_0 in deviations from their fixed-point values. Defining $x := X - \overline{X}$ and $y := Y - \overline{Y}$ turns map MG_0 into

$$MG_1 : \begin{cases} MG_{E,1} = \begin{cases} x' = (a-c)x + (b+c)y & \text{if } |x - y| > h \\ y' = x \end{cases} \\ MG_{I,1} = \begin{cases} x' = ax + by & \text{if } |x - y| \leq h \\ y' = x \end{cases} \end{cases}. \tag{2.9}$$

Both linear sub-maps $MG_{E,1}$ and $MG_{I,1}$ have the fixed point $\overline{x} = \overline{y} = 0$. The fixed point of the external sub-map $MG_{E,1}$ is a virtual fixed point for map MG_1, as it is located outside the region for which the external sub-map $MG_{E,1}$ is defined. The fixed point of the internal sub-map $MG_{I,1}$ is a real fixed point for map MG_1. The left panel of Fig. 3 illustrates the partition of the (x,y)-state space of map MG_1.

The stability conditions of the fixed point of the internal sub-map $MG_{I,1}$, considered in isolation, are identical to those for map LG. Since the Jacobian matrix for the external sub-map $MG_{E,1}$

$$J(MG_{E,1}) = \begin{pmatrix} a-c & b+c \\ 1 & 0 \end{pmatrix}, \tag{2.10}$$

has trace $tr(MG_{E,1}) = a - c$ and determinant $det(MG_{E,1}) = -b - c$, its fixed point, considered in isolation, is globally stable when (i) $a > b + 2c - 1$, (ii) $a < 1 - b$ and (iii) $b > -1 - c$ jointly hold. Compared to the stability conditions of the fixed point of the internal sub-map $MG_{I,1}$, the following differences can be observed. For $c > 0$, the first stability condition becomes more binding, while the third stability condition becomes less binding. The second stability condition is independent of parameter c. In (b, a)-parameter space, we observe an upwards-left movement of the resulting stability triangle as parameter c increases. The opposite occurs for $c < 0$.

To investigate the dynamics of map MG_1, we conduct computer simulations. Fig. 4 presents two-dimensional bifurcation diagrams for map MG_1 with $h = 1$. As shown by Gardini et al. (2025b, 2025c), a general property of the maps considered in this paper is that only three generic outcomes are possible. Throughout the paper, we use the following color coding to distinguish between them: parameter combinations that yield fixed-point dynamics are shown in light blue; those that lead to weird quasiperiodic dynamics are indicated in light red; and those that result in divergent dynamics are depicted in light gray. The blue and red lines mark the stability triangles of the fixed points of the internal sub-map $MG_{I,1}$ and of the external sub-map $MG_{E,1}$, respectively. Accordingly, parameter regions labeled A, B, C, and D correspond to the following stability combinations of the fixed points of sub-maps $MG_{I,1}$ and $MG_{E,1}$: (i) stable and stable, (ii) stable and unstable, (iii) unstable and stable, and (iv) unstable and unstable, respectively.

Fig. 4 provides the following insights. In the top-left panel, we set $c = 0.5$ and choose initial conditions close to the origin. All parameter combinations within regions A and B lead to fixed-point dynamics. Parameter combinations within region C result in weird quasiperiodic dynamics. In the top-right panel, we again set $c = 0.5$, but now use

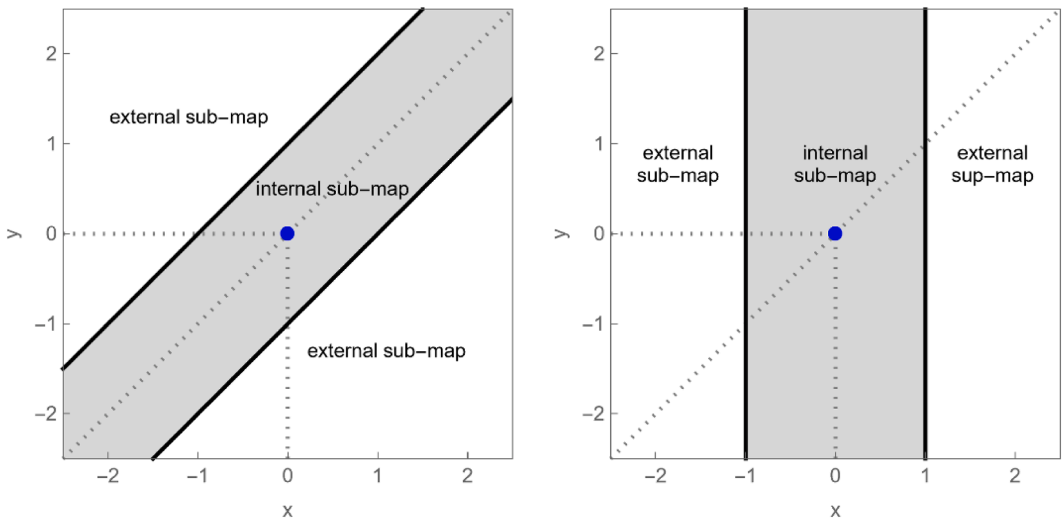

Fig. 3. Partition of the (x,y)-state space for $h = 1$. The left panel shows the regions where the external and internal sub-maps of maps MG_1 and MG_4 apply. The right panel shows the same for maps MG_2 and MG_3. The blue dot marks the real and virtual fixed point of these maps, located in the region of the internal sub-maps (gray region), and always given by the origin.

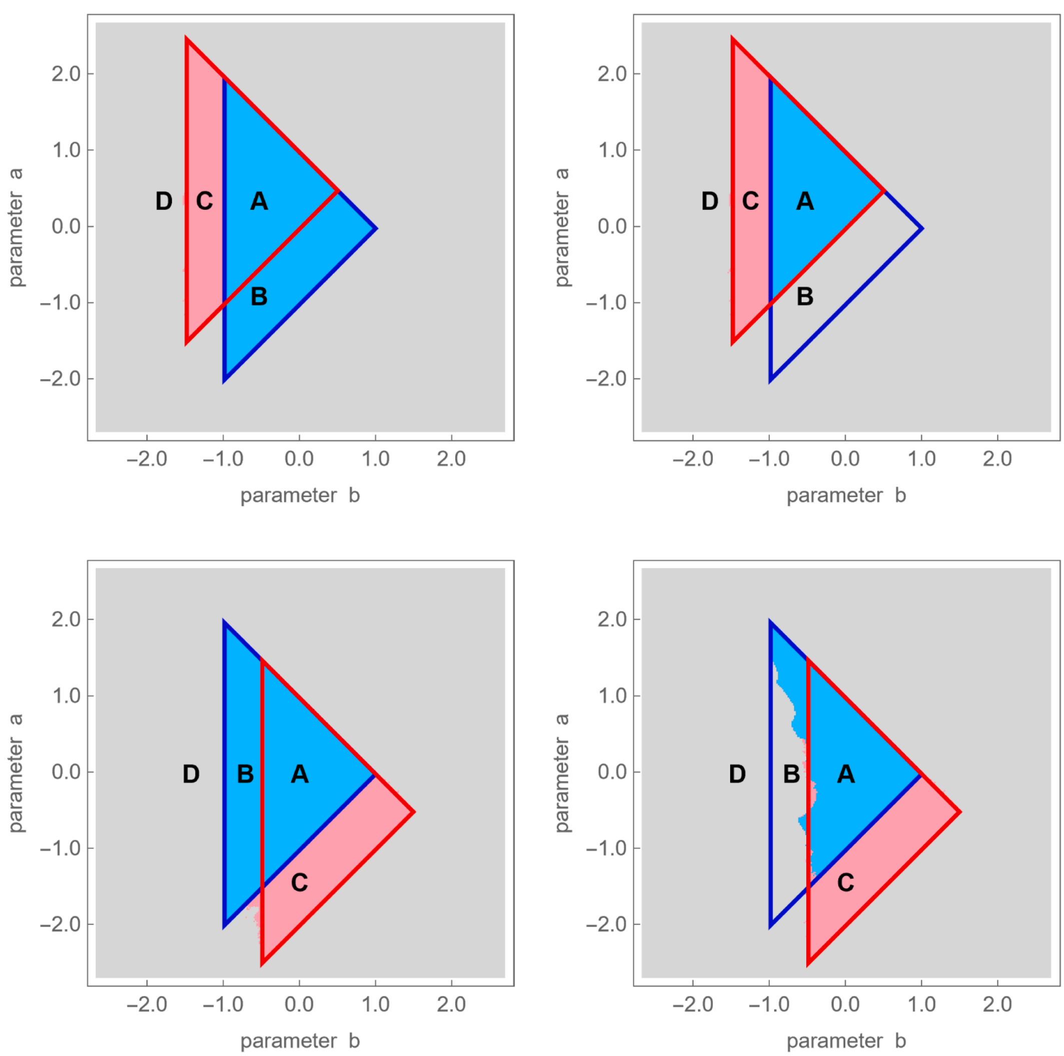

Fig. 4. Two-dimensional bifurcation diagrams for map MG_1 with $h = 1$. Light blue parameter combinations yield fixed-point dynamics, light red indicate weird quasiperiodic dynamics, and light gray correspond to divergent dynamics. Top left: $c = 0.5$ and initial conditions close to the origin. Top right: $c = 0.5$ and initial conditions distant from the origin. Bottom left: $c = -0.5$ and initial conditions close to the origin. Bottom right: $c = -0.5$ and initial conditions distant from the origin. The stability triangles of the internal and external sub-maps $MG_{I,1}$ and $MG_{E,1}$ are depicted in blue and red, respectively.

initial conditions that are distant from the origin. Compared to the previous experiment, parameter combinations within region B may now produce divergent dynamics, indicating that the real fixed point of map MG_1 is only locally stable in this region. In the bottom-left panel, we set $c = -0.5$ and again choose initial conditions close to the origin, whereas in the bottom-right panel, we set $c = -0.5$ and use initial conditions distant from the origin. A closer inspection reveals that weird quasiperiodic dynamics may emerge in regions A, B, C, and D.

Two examples of weird quasiperiodic attractors are depicted in the top-left and top-right panels of Figs. 1 and 2. The example in the top-left panels is based on a parameter setting from region C with $c = 0.5$. For this setting, almost all initial conditions converge to the weird quasiperiodic attractor. The example in the top-right panels corresponds to a parameter setting from region B with $c = -0.5$. In this case, the system's behavior depends on initial conditions: it may exhibit fixed-point dynamics, weird quasiperiodic dynamics, or divergent dynamics.

2.2.2. *Intervention rule CD*

Let us now assume that the regulator reacts to the change of the system's state variable X_t when this state variable becomes too distant from its fixed-point value. Using the index *CD* for this rule, indicating that the intervention is conditioned on the change of the state variable and activated when the state variable is sufficiently distant from its fixed point, the regulator's behavior accumulates to

$$R_t = R_t^{CD} = \begin{cases} -c(X_t - X_{t-1}) \text{ if } |\overline{X} - X_t| > h \\ 0 \qquad\qquad\quad \text{ if } |\overline{X} - X_t| \leq h \end{cases}. \tag{2.11}$$

The threshold parameter h indicates when the regulator becomes active, while parameter c controls the regulator's aggressiveness. For $c > 0$, the regulator uses a negative feedback rule; for $c < 0$, the regulator uses a positive feedback rule.

Inserting (2.11) into (2.1), the dynamics of the regulated system, expressed in deviations from the fixed point, are now governed by the map

$$MG_2 : \begin{cases} MG_{E,2} = \begin{cases} x' = (a-c)x + (b+c)y \text{ if } |x| > h \\ y' = x \end{cases} \\ MG_{I,2} = \begin{cases} x' = ax + by \qquad\quad \text{ if } |x| \leq h \\ y' = x \end{cases} \end{cases}. \tag{2.12}$$

Since $MG_{E,2} = MG_{E,1}$ and $MG_{I,2} = MG_{I,1}$, the fixed points and their stability properties remain as before. Due to the alternative partition of the state space, visualized in the right panel of Fig. 4, the dynamics of map MG_2 differ from those of map MG_1.

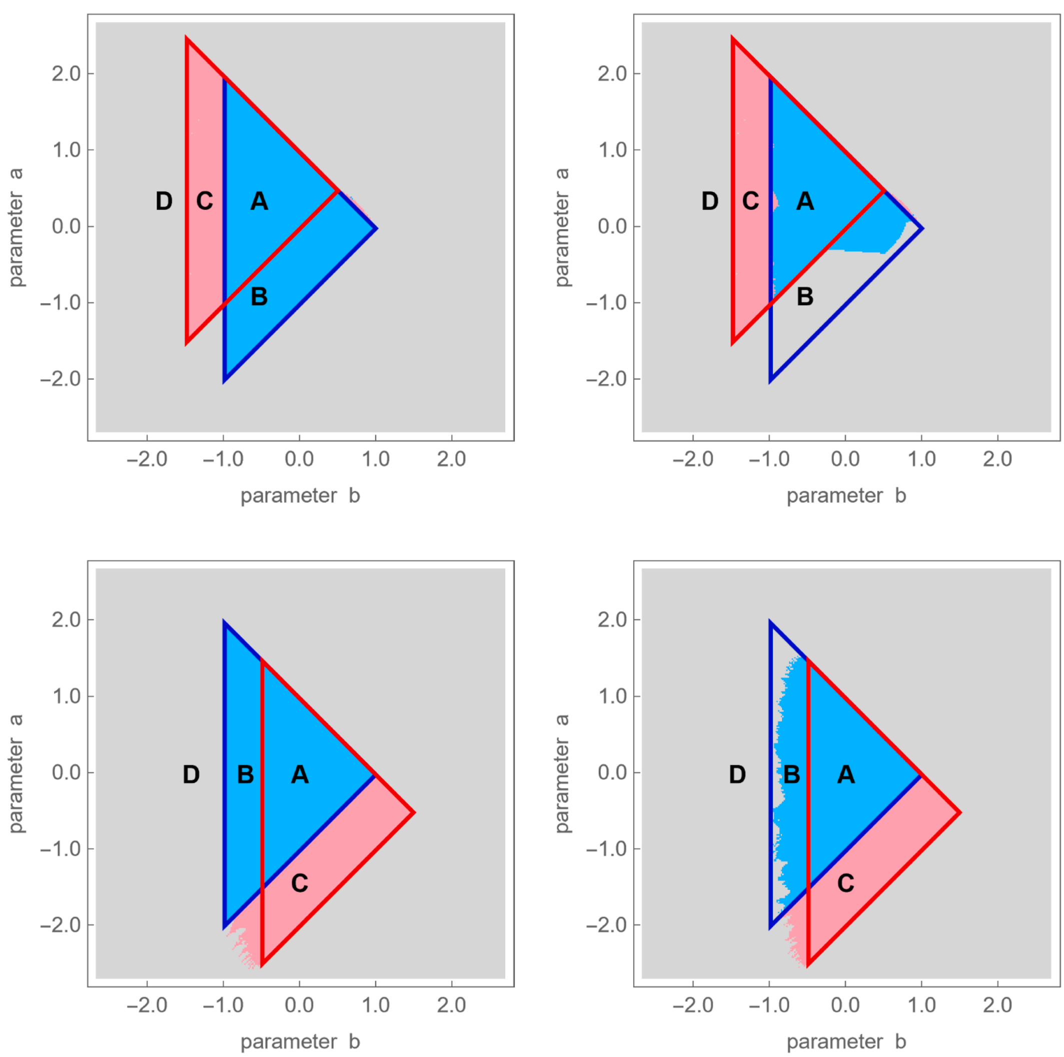

Fig. 5. Two-dimensional bifurcation diagrams for map MG_2 with $h = 1$. Light blue parameter combinations yield fixed-point dynamics, light red indicate weird quasiperiodic dynamics, and light gray correspond to divergent dynamics. Top left: $c = 0.5$ and initial conditions close to the origin. Top right: $c = 0.5$ and initial conditions distant from the origin. Bottom left: $c = -0.5$ and initial conditions close to the origin. Bottom right: $c = -0.5$ and initial conditions distant from the origin. The stability triangles of the internal and external sub-maps $MG_{I,2}$ and $MG_{E,2}$ are depicted in blue and red, respectively.

The two-dimensional bifurcation diagrams presented in Fig. 5 offer several insights. In the top-left panel, we set $c = 0.5$ and use initial conditions close to the origin. In the top-right panel, we again set $c = 0.5$, but now choose initial conditions that are distant from the origin. In the bottom-left and bottom-right panels, we repeat these simulations for $c = -0.5$. Notably, weird quasiperiodic dynamics always seem to emerge in region C, i.e., for parameter combinations that lie within the stability triangle of the fixed point of the external sub-map $MG_{E,1}$, but outside the stability triangle of the fixed point of the internal sub-map $MG_{I,1}$. However, weird quasiperiodic dynamics may also occur in regions A, B, and D.

An example of a weird quasiperiodic attractor based on a parameter setting from region D with $c = -0.5$ is shown in the bottom-right panels of Figs. 1 and 2. Depending on the initial conditions, the system's generic trajectory exhibits either weird quasiperiodic dynamics or divergent behavior.

2.2.3. Intervention rule DD

Instead of acting on the current change of a system's state variable, the regulator may also intervene in the system's dynamics by responding to the deviation of a system's state variable from its fixed-point value. Let us first assume that the regulator follows this principle when the system's state variable X_t is sufficiently distant from its fixed-point value. Indexing this rule as DD, indicating that the intervention is conditioned on the deviation of the state variable from its fixed point and activated when this deviation becomes sufficiently large, we obtain

$$R_t = R_t^{DD} = \begin{cases} c(\overline{X} - X_t) & \text{if } |\overline{X} - X_t| > h \\ 0 & \text{if } |\overline{X} - X_t| \leq h \end{cases}. \tag{2.13}$$

The regulator becomes active when the state variable X_t deviates by more than the threshold parameter h from its fixed-point value $\overline{X}$. For $c > 0$, the regulator seeks to accelerate the convergence of the system's state variable X_t towards its fixed-point value $\overline{X}$, while for $c < 0$, the regulator seeks to delay this movement.

Using (2.13), the regulated system's dynamics now evolve according to the map

$$MG_3 : \begin{cases} MG_{E,3} = \begin{cases} x' = (a-c)x + by & \text{if } |x| > h \\ y' = x \end{cases} \\ MG_{I,3} = \begin{cases} x' = ax + by & \text{if } |x| \leq h \\ y' = x \end{cases} \end{cases}. \tag{2.14}$$

The right panel of Fig. 3 shows the partition of the (x,y)-state space of map MG_3. Considered in isolation, the fixed point of the internal sub-map $MG_{I,3}$, being a real fixed point of map MG_3, has the same stability conditions as the fixed point of map LG. Considered in isolation, the

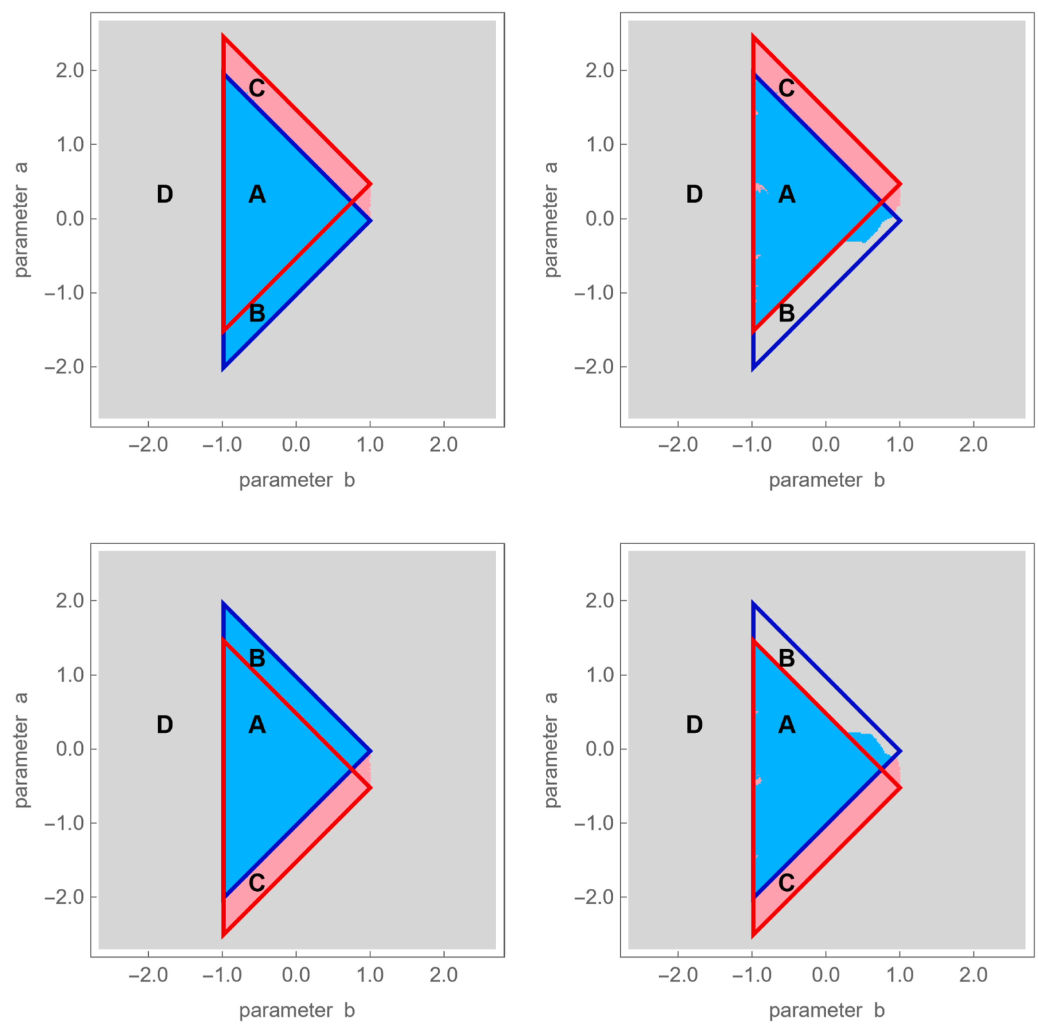

Fig. 6. Two-dimensional bifurcation diagrams for map MG_3 with $h = 1$. Light blue parameter combinations yield fixed-point dynamics, light red indicate weird quasiperiodic dynamics, and light gray correspond to divergent dynamics. Top left: $c = 0.5$ and initial conditions close to the origin. Top right: $c = 0.5$ and initial conditions distant from the origin. Bottom left: $c = -0.5$ and initial conditions close to the origin. Bottom right: $c = -0.5$ and initial conditions distant from the origin. The stability triangles of the internal and external sub-maps $MG_{I,3}$ and $MG_{E,3}$ are depicted in blue and red, respectively.

fixed point of the external sub-map $MG_{E,3}$, being a virtual fixed point of map MG_3, is globally stable when (i) $a > -1 + b + c$, (ii) $a < 1 - b + c$, and (iii) $b > -1$ hold. Relative to the stability conditions of the fixed point of the internal sub-map $MG_{I,3}$, the following differences arise. For $c > 0$, the first stability condition becomes less binding, while the second stability condition becomes more binding. The third stability condition is independent of parameter c. In (b, a)-parameter space, we observe an upward movement of the resulting stability triangle as parameter c increases. For $c < 0$, opposite occurs.

The two-dimensional bifurcation diagrams presented in Fig. 6 provide the following insights. In the top-left panel, we set $c = 0.5$ and use initial conditions close to the origin, whereas we set $c = 0.5$ with initial conditions distant from the origin in the top-right panel. These simulations are repeated for $c = -0.5$ in the bottom-left and bottom-right panels. Once again, it is evident that weird quasiperiodic dynamics consistently emerge in region C, although such dynamics may also appear in regions A, B, and D.

2.2.4. Intervention rule DC

Finally, we consider that the regulator acts on the deviation of a system's state variable from its fixed-point value when the change of this state variable becomes larger than the threshold parameter h. Indexing this rule as DC, indicating that the intervention is conditioned on the deviation of the state variable from its fixed point and activated when the change of the state variable becomes sufficiently large, we obtain

$$R_t = R_t^{DC} = \begin{cases} c(\overline{X} - X_t) & \text{if } |X_t - X_{t-1}| > h \\ 0 & \text{if } |X_t - X_{t-1}| \le h \end{cases}, \tag{2.15}$$

where c is again the regulator's control parameter and h the rule's activation threshold.

Based on (2.15), the dynamics of the new system are governed by

$$MG_4 : \begin{cases} MG_{E,4} = \begin{cases} x' = (a - c)x + by & \text{if } |x - y| > h \\ y' = x \end{cases} \\ MG_{I,4} = \begin{cases} x' = ax + by & \text{if } |x - y| \le h \\ y' = x \end{cases} \end{cases}. \tag{2.16}$$

The fixed points of sub-maps $MG_{E,4}$ and $MG_{I,4}$ and their stability properties are the same as for sub-maps $MG_{E,3}$ and $MG_{I,3}$. The differences in the dynamics between maps MG_3 and MG_4 are caused by their different state-space partitions, as portrayed in Fig. 3.

Fig. 7 visualizes several implications of map MG_4. The two-dimensional bifurcation diagram uses the same simulation as before.

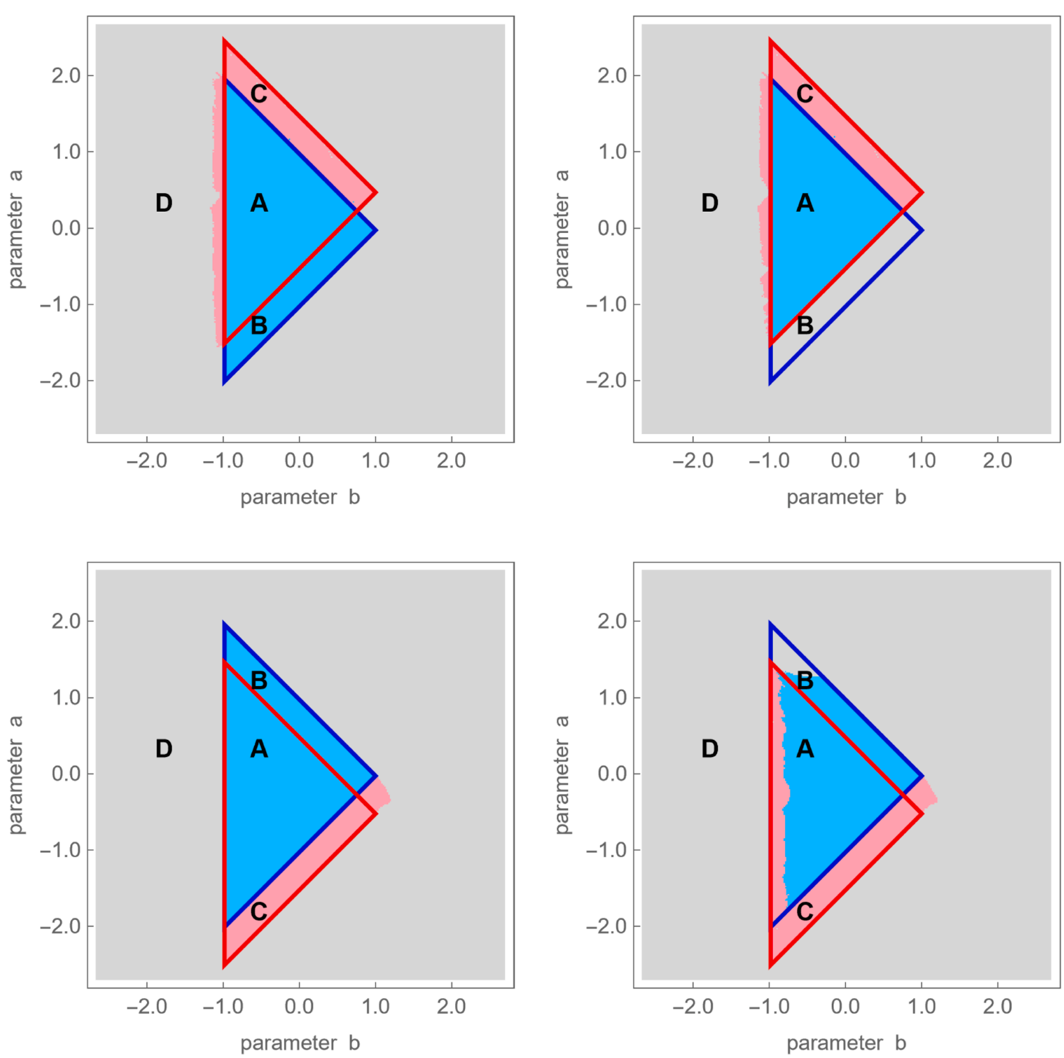

Fig. 7. Two-dimensional bifurcation diagrams for map MG_4 with $h = 1$. Light blue parameter combinations yield fixed-point dynamics, light red indicate weird quasiperiodic dynamics, and light gray correspond to divergent dynamics. Top left: $c = 0.5$ and initial conditions close to the origin. Top right: $c = 0.5$ and initial conditions distant from the origin. Bottom left: $c = -0.5$ and initial conditions close to the origin. Bottom right: $c = -0.5$ and initial conditions distant from the origin. The stability triangles of the internal and external sub-maps $MG_{I,4}$ and $MG_{E,4}$ are depicted in blue and red, respectively.

Parameter combinations from region C give rise to weird quasiperiodic dynamics; however, such dynamics may also occur in regions A, B, and D.

An example of a weird quasiperiodic attractor coexisting with a fixed-point attractor is shown in the bottom-left panels of Figs. 1 and 2. This example is based on a parameter setting from region A with $c = -0.5$. As usual, interesting attractor-switching dynamics may evolve in the presence of exogenous noise, an aspect that deserves more attention in future research.

2.3. Summary

Let us summarize the main observations we have made so far by reviewing the dynamic behaviors of maps MG_1 to MG_4, as depicted in Figs. 4–7. All four maps are capable of producing weird quasiperiodic dynamics. Considering regions A, B, C, and D, we draw the following conclusions.

- Region A: The fixed points of both the internal and external sub-maps are stable. Two generic outcomes are possible. First, for all initial conditions, trajectories converge to the real fixed point. Second, depending on the initial conditions, trajectories converge either to the real fixed point or to a weird quasiperiodic attractor. Divergent dynamics do not occur in this region.
- Region B: The fixed point of the internal sub-map is stable, while the fixed point of the external sub-map is unstable. Two generic outcomes are possible. First, depending on the initial conditions, trajectories either converge to the real fixed point or diverge. Second, trajectories may converge to the real fixed point, to a coexisting weird quasiperiodic attractor, or exhibit divergent behavior.
- Region C: The fixed point of the internal sub-map is unstable, whereas the fixed point of the external sub-map is stable. For almost all initial conditions, trajectories converge to a weird quasiperiodic attractor. Neither fixed-point nor divergent dynamics are observed in this region.
- Region D: The fixed points of both the internal and external sub-maps are unstable. Two generic outcomes are possible. First, depending on the initial conditions, trajectories either converge to a weird quasiperiodic attractor or diverge. Second, for almost all initial conditions, trajectories exhibit divergent behavior. Fixed-point dynamics are not possible in this region.

In summary, our simulations demonstrate that regulators can generate weird quasiperiodic dynamics by applying intervention rules

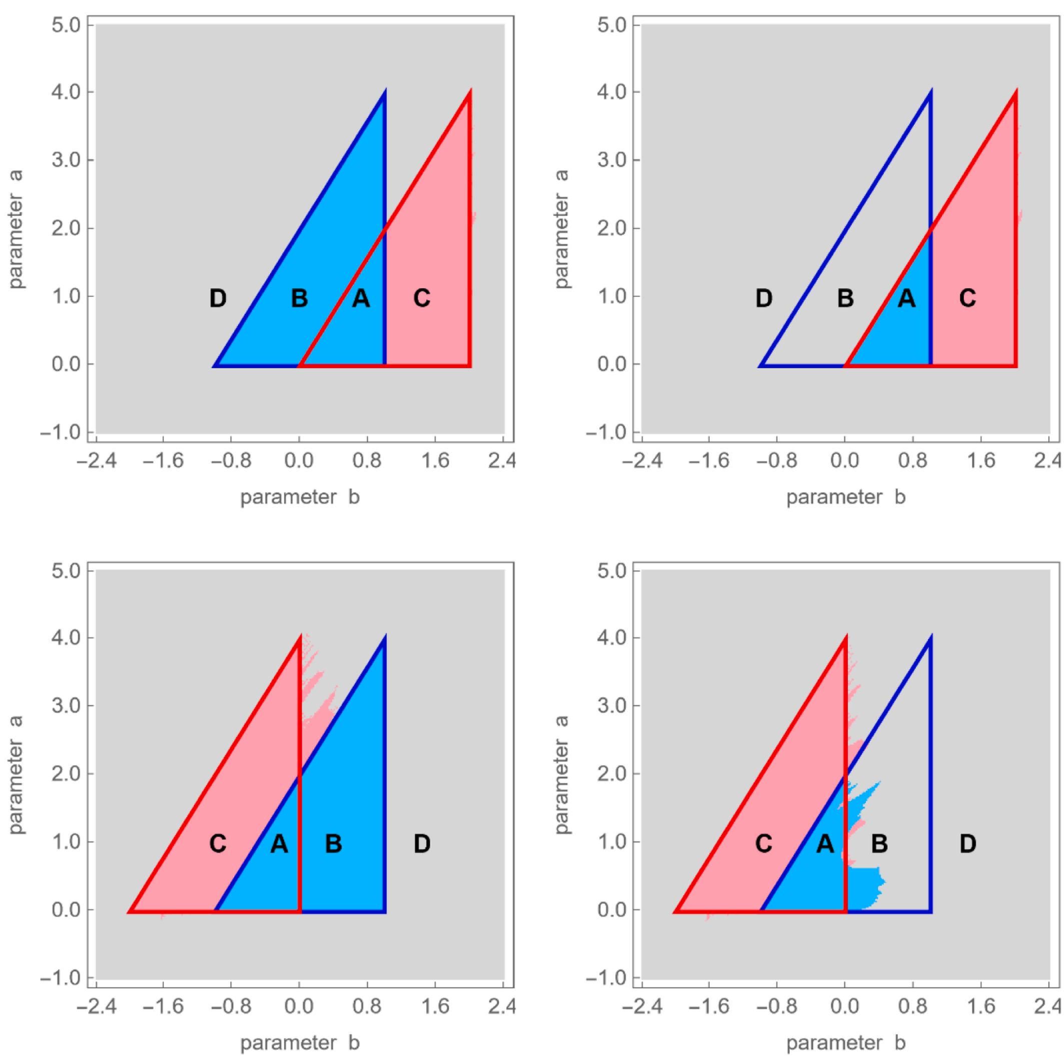

Fig. 8. Two-dimensional bifurcation diagrams for map MF_1 with $h = 1$. Light blue parameter combinations yield fixed-point dynamics, light red indicate weird quasiperiodic dynamics, and light gray correspond to divergent dynamics. Top left: $c = 1$ and initial conditions close to the origin. Top right: $c = 1$ and initial conditions distant from the origin. Bottom left: $c = -1$ and initial conditions close to the origin. Bottom right: $c = -1$ and initial conditions distant from the origin. The stability triangles of the internal and external sub-maps $MF_{I,1}$ and $MF_{E,1}$ are depicted in blue and red, respectively.

CC, CD, DD, or DC within regions A, B, C, and D. Similar patterns are observed in the next section, documented in Figs. 8–11, where we also show that multiple weird quasiperiodic attractors may coexist. A deeper mathematical analysis is left for future research.

3. Regulating financial markets

In our first application, we consider an order-driven financial market model in which a market maker adjusts prices in response to current excess demand. This demand comprises the orders placed by chartists D_t^C, fundamentalists D_t^F, positive feedback traders D_t^P, negative feedback traders D_t^N, and the regulator R_t. While speculators' trading rules are linear, the regulator's intervention rule is piecewise linear. In Section 3.1, we first analyze the case in which the regulator is inactive. In Section 3.2, we then examine the effects of four piecewise linear intervention rules. Several examples are presented in Section 3.3. To place our findings in context, we recall that weird quasiperiodic dynamics were first observed in the chartist-fundamentalist model by Gardini et al. (2025a), albeit in a different type of dynamical system and under a different economic mechanism. Other related chartist-fundamentalist models featuring dynamics driven by piecewise linear maps include Anufriev et al. (2020); Dieci et al. (2022); Gardini et al. (2022); Jungeilges et al. (2021); Panchuk et al. (2018); Tramontana et al. (2013); and Sushko and Tramontana (2025). These models do not exhibit weird quasiperiodic dynamics, as their sub-maps do not share the same fixed points.

3.1. A financial market setup

Let us turn to the details of the model. The market maker quotes the price for period $t+1$ as

$$P_{t+1} = P_t + m\left(D_t^C + D_t^F + D_t^P + D_t^N + R_t\right). \tag{3.1}$$

For ease of exposition, we set the market maker's price adjustment parameter to $m = 1$. Accordingly, the market maker increases (decreases) the price when the sum of the market participants' orders is positive (negative).

Chartists believe in the persistence of bull and bear markets. Let F be the fundamental value of the market. Chartists perceive a bull market when the price is above its fundamental value and, in such a situation, optimistically submit buy orders. In a bear market, when the price is below its fundamental value, they pessimistically submit sell orders. We formalize the orders placed by chartists as

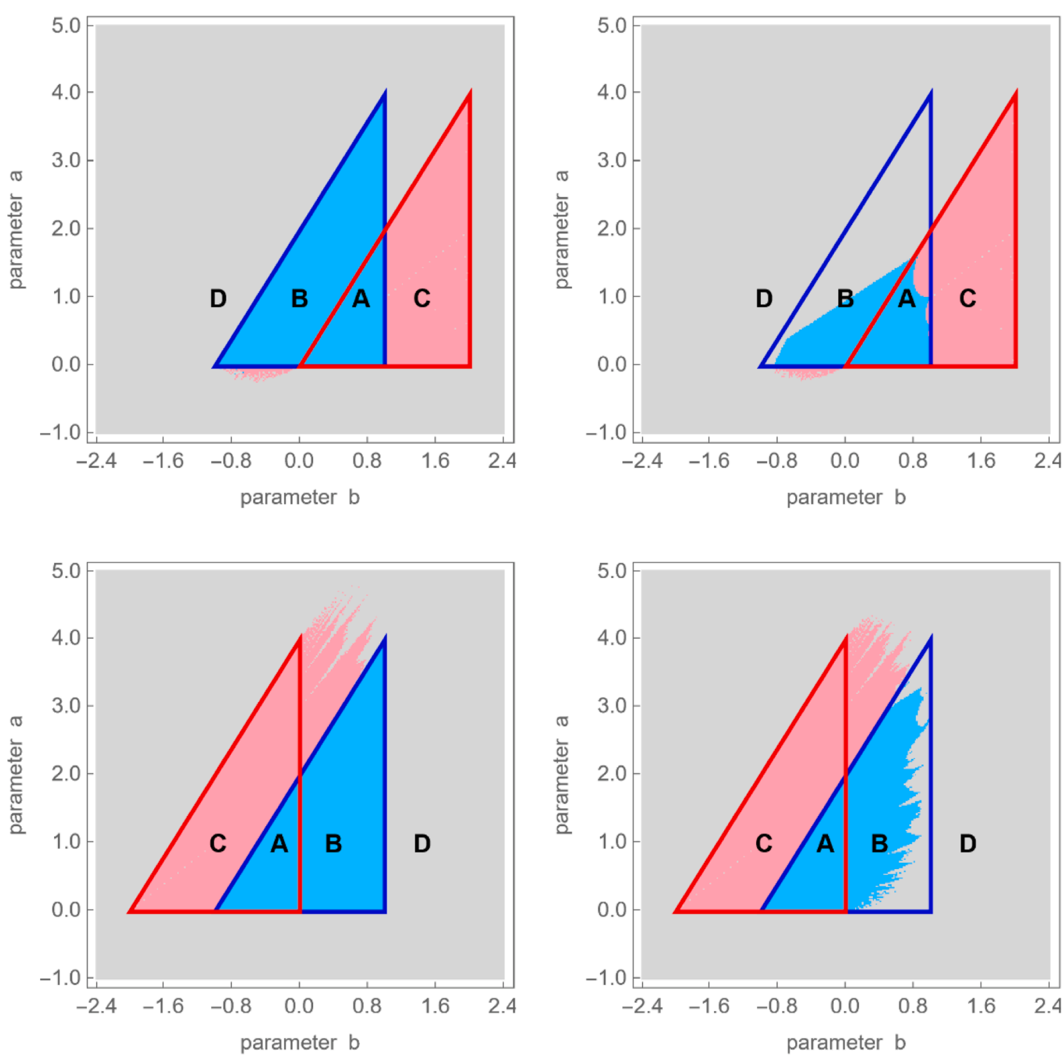

Fig. 9. Two-dimensional bifurcation diagrams for map MF_2 with $h = 1$. Light blue parameter combinations yield fixed-point dynamics, light red indicate weird quasiperiodic dynamics, and light gray correspond to divergent dynamics. Top left: $c = 1$ and initial conditions close to the origin. Top right: $c = 1$ and initial conditions distant to the origin. Bottom left: $c = -1$ and initial conditions close to the origin. Bottom right: $c = -1$ and initial conditions distant to the origin. The stability triangles of the internal and external sub-maps $MF_{I,2}$ and $MF_{E,2}$ are depicted in blue and red, respectively.

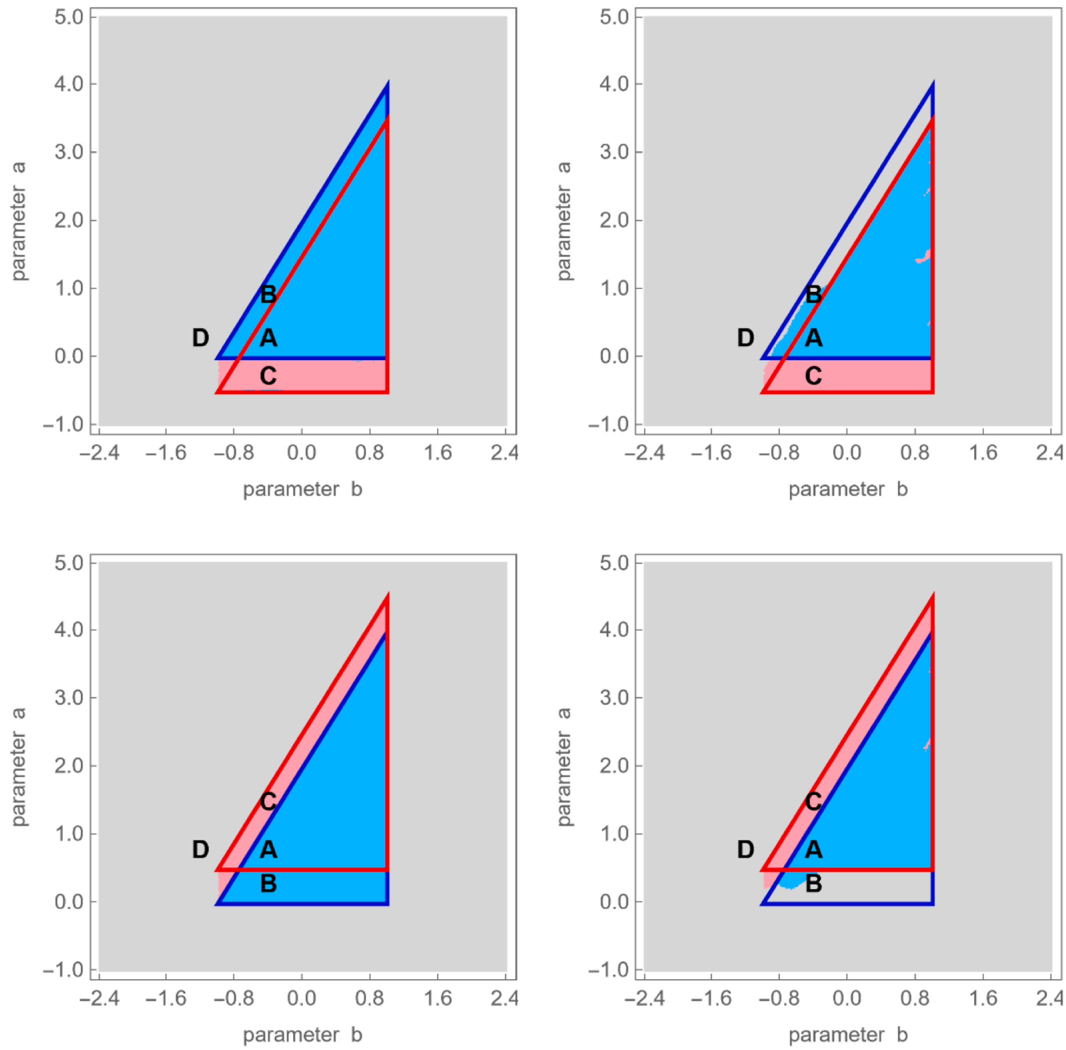

Fig. 10. Two-dimensional bifurcation diagrams for map MF_3 with $h = 1$. Light blue parameter combinations yield fixed-point dynamics, light red indicate weird quasiperiodic dynamics, and light gray correspond to divergent dynamics. Top left: $c = 0.5$ and initial conditions close to the origin. Top right: $c = 0.5$ and initial conditions distant to the origin. Bottom left: $c = -0.5$ and initial conditions close to the origin. Bottom right: $c = -0.5$ and initial conditions distant to the origin. The stability triangles of the internal and external sub-maps $MF_{I,3}$ and $MF_{E,3}$ are depicted in blue and red, respectively.

$$D_t^C = a'(P_t - F), \tag{3.2}$$

where parameter $a' > 0$ captures their trading aggressiveness. Fundamentalists believe that the price will revert to its fundamental value. Their orders are expressed as

$$D_t^F = a''(F - P_t), \tag{3.3}$$

where parameter $a'' > 0$ captures their trading aggressiveness. Fundamentalists trade in the opposite direction as chartists: they submit buy orders when the market is undervalued and sell orders when it is overvalued.

Positive feedback traders believe that the current price trend will continue. Their orders are given by

$$D_t^P = b'(P_t - P_{t-1}), \tag{3.4}$$

where parameter $b' > 0$ indicates how aggressively they react to the current price trend. Positive feedback traders are ready to buy when the market increases and ready to sell when it decreases. Negative feedback traders believe in trend reversals. Their orders are given by

$$D_t^P = b''(P_{t-1} - P_t), \tag{3.5}$$

where parameter $b'' > 0$ denotes their aggressiveness. Negative feedback traders submit buy orders when the market falls and sell orders when it rises. Obviously, also positive and negative feedback traders have opposing views on the market's future direction.

Let us first assume that the regulator is inactive, i.e.,

$$R_t = R = 0. \tag{3.6}$$

Then, the financial market setup implies that the price evolves as

$$P_t = (1 - a + b)P_{t-1} - bP_{t-2} + aF, \tag{3.7}$$

where parameter $a = a' - a''$ aggregates the reaction parameters of chartists and fundamentalists and parameter $b = b' - b''$ the reaction parameters of positive and negative feedback traders. Both aggregate parameters may be positive or negative.

When the dynamics is at rest, the price equals its fundamental value reflecting efficient financial markets.

$$\overline{P} = F, \tag{3.8}$$

Expressing the price in deviations from its fixed point, i.e., $x_t = P_t - \overline{P}$, and introducing the auxiliary variable $y_t = x_{t-1}$, we obtain the two-dimensional linear map

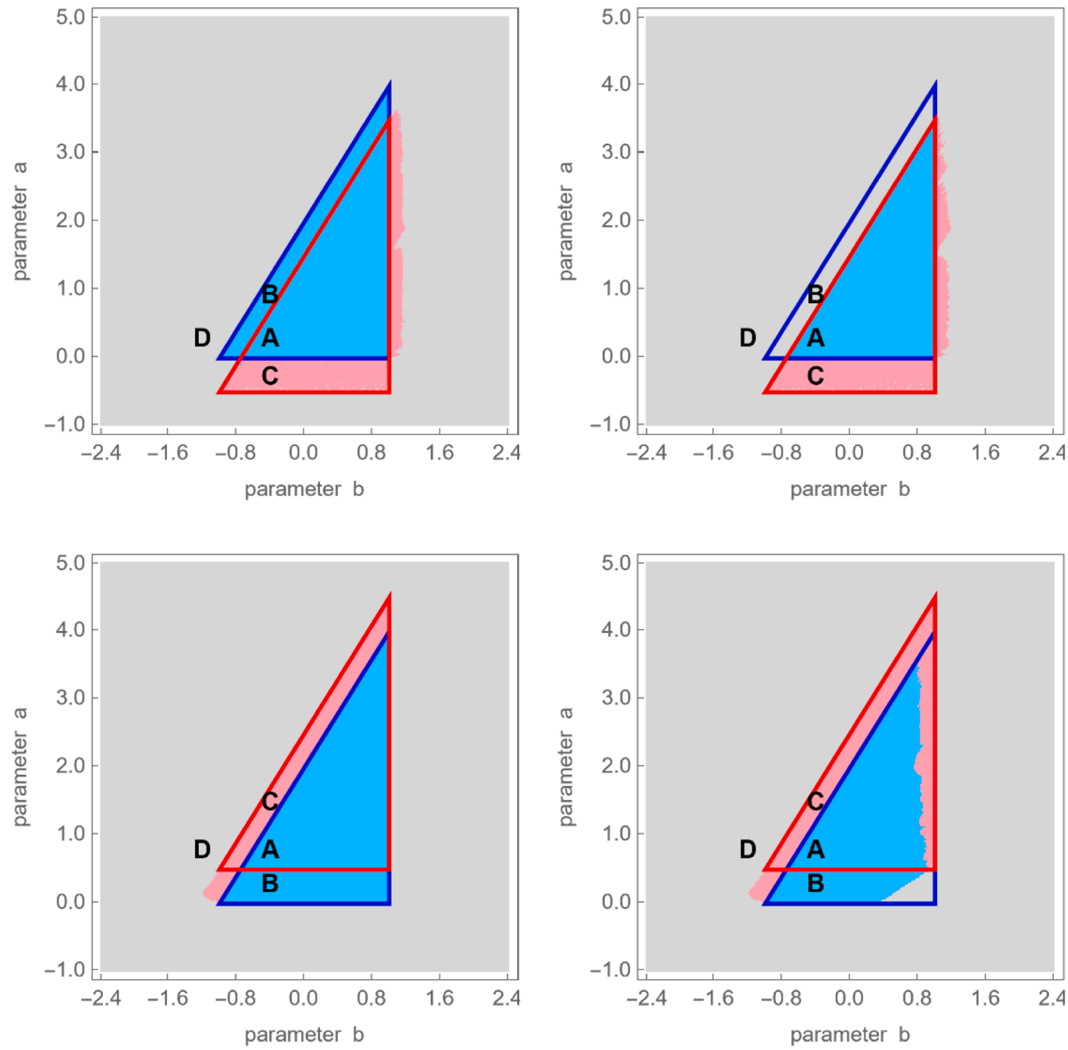

Fig. 11. Two-dimensional bifurcation diagrams for map MF_4 with $h = 1$. Light blue parameter combinations yield fixed-point dynamics, light red indicate weird quasiperiodic dynamics, and light gray correspond to divergent dynamics. Top left: $c = 0.5$ and initial conditions close to the origin. Top right: $c = 0.5$ and initial conditions distant to the origin. Bottom left: $c = -0.5$ and initial conditions close to the origin. Bottom right: $c = -0.5$ and initial conditions distant to the origin. The stability triangles of the internal and external sub-maps $MF_{I,4}$ and $MF_{E,4}$ are depicted in blue and red, respectively.

$$LF = \begin{cases} x' = (1-a+b)x - by \\ y' = x \end{cases}. \tag{3.9}$$

Map LF has a unique fixed point, given by

$$\bar{x} = \bar{y} = 0. \tag{3.10}$$

The Jacobian matrix of map LF,

$$J(LF) = \begin{pmatrix} 1-a+b & -b \\ 1 & 0 \end{pmatrix}, \tag{3.11}$$

has trace $tr(LF) = 1 - a + b$ and determinant $det(LF) = b$. The fixed point of map LF is globally stable when aggregate parameters a and b are inside the stability triangle defined by the inequalities (i) $a < 2 + 2b$, (ii) $a > 0$, and (iii) $b < 1$.

3.2. Piecewise linear feedback rules

We now consider four piecewise linear feedback rules for the regulator. For reviews on the effectiveness of linear feedback rules within chartist-fundamentalist models, see Westerhoff (2008) and Westerhoff and Franke (2018). As previously noted, implementing linear feedback rules requires the regulator to intervene continuously in the system's out-of-equilibrium behavior, a level of fine-tuning that may be infeasible or prohibitively costly in practice. The piecewise linear feedback rules introduced below may therefore be viewed as a more robust and practical alternative.

3.2.1. Intervention rule CC

According to the first rule, the regulator intervenes in the market as follows

$$R_t = R_t^{CC} = \begin{cases} -c(P_t - P_{t-1}) & if\ |P_t - P_{t-1}| > h \\ 0 & if\ |P_t - P_{t-1}| \le h \end{cases}. \tag{3.12}$$

For finite values of the threshold parameter h, the regulator seeks to weaken the market impact of positive feedback traders by setting $c > 0$, or of negative feedback traders by setting $c < 0$.

Replacing (3.6) with (3.12) reveals that the dynamics of the regulated financial market are given by

$$MF_1:\begin{cases} MF_{E,1}=\begin{cases} x'=(1-a+b-c)x-(b-c)y \text{ if } |x-y|>h \\ y'=x \end{cases} \\ MF_{I,1}=\begin{cases} x'=(1-a+b)x-by \quad \text{ if } |x-y|\le h \\ y'=x \end{cases} \end{cases}. \tag{3.13}$$

The dynamics of map MF_1 are governed by the external sub-map $MF_{E,1}$ when the change of the price exceeds the threshold parameter $h > 0$. Otherwise, they are governed by the internal sub-map $MF_{I,1}$. The fixed points of sub-maps $MF_{E,1}$ and $MF_{I,1}$ are both given by $\overline{x} = \overline{y} = 0$, being a virtual or a real fixed point of map MF_1, respectively.

Considered in isolation, the stability conditions of the fixed point of sub-map $MF_{I,1}$ are given by those for map LF. Since the Jacobian matrix for sub-map $MF_{E,1}$

$$J(MF_{E,1})=\begin{pmatrix} 1-a+b-c & -b+c \\ 1 & 0 \end{pmatrix}, \tag{3.14}$$

has trace $tr(MF_{E,1}) = a-c$ and determinant $det(MF_{E,1}) = b-c$, its fixed point is globally stable when (i) $a < 2+2b-2c$, (ii) $a > 0$, and (iii) $b < 1+c$ jointly hold. For $c > 0$, the first condition becomes more binding, while the third condition becomes less binding. The second condition is independent of parameter c. In (b,a)-parameter space, the corresponding stability triangle shifts to the right. The opposite occurs for $c < 0$, i.e., the stability triangle shifts to the left.

The two-dimensional bifurcation diagrams shown in Fig. 8, which follow the same simulation design as those discussed in Section 2, illustrate the behavior of map MF_1 with $h = 1$. In the top-left and top-right panels, we set $c = 1$ and use initial conditions that are close to the origin and distant from the origin, respectively. In the bottom-left and bottom-right panels, we repeat the computations for $c = -1$.

Most importantly, the regulator's interventions may trigger weird quasiperiodic dynamics in region C. One could argue that the resulting price fluctuations are desirable, as they replace divergent dynamics. However, the regulator may also trigger weird quasiperiodic dynamics in region A, an outcome that may be regarded as undesirable, as it replaces fixed-point dynamics. Moreover, in region B, the regulator may induce weird quasiperiodic dynamics or even divergent behavior, again replacing fixed-point dynamics. It is also worth noting that in some instances within region D, divergent dynamics may transition into weird quasiperiodic dynamics.

3.2.2. Intervention rule CD

Let us now assume that the regulator reacts to the change of the price when the mispricing in the market becomes too extreme, i.e.,

$$R_t = R_t^{CD} = \begin{cases} -c(P_t-P_{t-1}) \text{ if } |P_t-F|>h \\ 0 \qquad\qquad\quad \text{ if } |P_t-F|\le h \end{cases}. \tag{3.15}$$

The meaning of parameter c is as before, and parameter h marks the intervention threshold. We now obtain the map

$$MF_2:\begin{cases} MF_{E,2}=\begin{cases} x'=(1-a+b-c)x-(b-c)y \text{ if } |x|>h \\ y'=x \end{cases} \\ MF_{I,2}=\begin{cases} x'=(1-a+b)x-by \quad \text{ if } |x|\le h \\ y'=x \end{cases} \end{cases}. \tag{3.16}$$

Note that $MF_{E,2} = MF_{E,1}$ and $MF_{I,2} = MF_{I,1}$. Hence, the fixed points and their stability triangles remain as before. Differences in the dynamics of maps MG_2 and MG_1 are due to their alternative state-space partitions.

The two-dimensional bifurcation diagrams presented in Fig. 9 illustrate the dynamics of map MF_2. As before, we set $c = 1$ in the top panels, where initial conditions are selected close to the origin in the top-left panel and farther from the origin in the top-right panel. In the bottom-left and bottom-right panels, we repeat these simulations for $c = -1$. Once again, weird quasiperiodic dynamics persistently emerge in region C, although they may also occur in regions A, B, and D.

3.2.3. Intervention rule DD

According to the third intervention rule, the regulator acts on the market's mispricing when its mispricing becomes too strong. The regulator's behavior now reads

$$R_t = R_t^{DD} = \begin{cases} c(F-P_t) \text{ if } |P_t-F|>h \\ 0 \qquad\quad\ \text{ if } |P_t-F|\le h \end{cases}, \tag{3.17}$$

resulting in the map

$$MF_3:\begin{cases} MF_{E,3}=\begin{cases} x'=(1-a+b-c)x-by \text{ if } |x|>h \\ y'=x \end{cases} \\ MF_{I,3}=\begin{cases} x'=(1-a+b)x-by \quad \text{ if } |x|\le h \\ y'=x \end{cases} \end{cases}. \tag{3.18}$$

The real fixed point of map MF_3 is given by the fixed point of map $MF_{I,3}$ and has the same stability conditions as the fixed point of map LF. The fixed point of $MF_{E,3}$ is a virtual fixed point of map MF_3, and is globally stable when (i) $a < 2+2b-c$, (ii) $a > -c$, and (iii) $b < 1$ hold. In (b, a)-parameter space, the corresponding stability triangle shifts downwards for $c > 0$ and upwards for $c < 0$.

The two-dimensional bifurcation diagrams presented in Fig. 10 illustrate the dynamics of map MF_3 with $h = 1$. In the top panels, the regulator's intervention force is set to $c = 0.5$, and in the bottom panels to $c = -0.5$. In line with our previous insights, we observe that region C consistently yields quasiperiodic dynamics, although such behavior may also occur in regions A, B, and D.

3.2.4. Intervention rule DC

Finally, we consider the case where the regulator acts on the market's mispricing when the price of the market changes too strongly. To be precise, we assume that the regulator uses the intervention rule

$$R_t = R_t^{DC} = \begin{cases} c(F-P_t) \text{ if } |P_t-P_{t-1}|>h \\ 0 \qquad\quad\ \text{ if } |P_t-P_{t-1}|\le h \end{cases}. \tag{3.19}$$

Accordingly, the dynamics of the regulated financial market are due to

$$MF_4:\begin{cases} MF_{E,4}=\begin{cases} x'=(1-a+b-c)x-by \text{ if } |x-y|>h \\ y'=x \end{cases} \\ MF_{I,4}=\begin{cases} x'=(1-a+b)x-by \quad \text{ if } |x-y|\le h \\ y'=x \end{cases} \end{cases}. \tag{3.20}$$

Since the fixed points of maps $MF_{E,4}$ and $MF_{I,4}$ and their stability properties are the same as for maps $MF_{E,3}$ and $MF_{I,3}$, the differences in the dynamics between maps MF_3 and MF_4 are caused by their different state-space partitions.

The two-dimensional bifurcation diagrams for map MF_4 with $h = 1$, presented in Fig. 11, lead to the following conclusions. Regardless of whether the regulator's intervention force is positive (top panels with $c = 0.5$) or negative (bottom panels with $c = -0.5$), weird quasiperiodic dynamics are observed in region C. Once again, such behavior may also occur in regions A, B, and D.

3.3. Emergence of financial market turbulences

Using piecewise linear feedback rules, the regulator may set weird quasiperiodic dynamics in motion. Let us examine a few illustrative examples.

The top panels of Fig. 12 are based on map MF_1 with $a = 0.5,\ b = 1.2,\ c = 1$ and $h = 1$. The blue and red dots in the left panel represent the fixed point and the weird quasiperiodic attractor in (x,y)-state space,

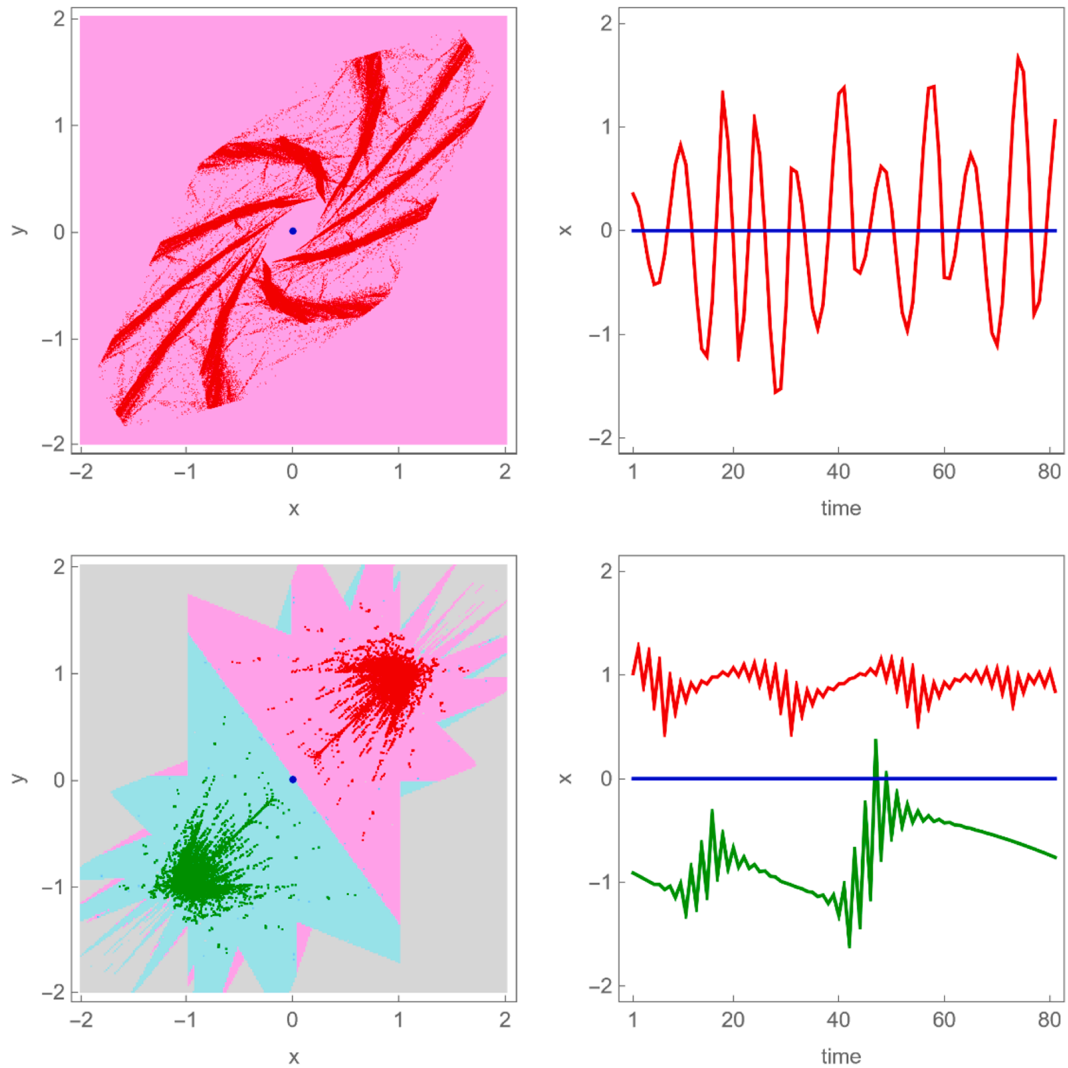

Fig. 12. First set of examples for endogenous financial market dynamics. Left panels: The blue dots represent the fixed points, while the red and green dots indicate weird quasiperiodic attractors, with their basins of attraction shown in matching colors. Right panels: The blue, red, and green lines show the corresponding dynamics in the time domain. Top panels: Map MF_1 with $a = 0.5$, $b = 1.2$, $c = 1$ and $h = 1$. Bottom panels: Map MF_2 with $a = -0.05$, $b = -0.75$, $c = 1$ and $h = 1$.

respectively, with their basins of attraction shown in matching colors. Since this parameter combination falls within region C, the fixed point of the internal sub-map $MF_{I,1}$ is unstable, while the fixed point of the external sub-map $MF_{I,1}$ is stable. As a result, the weird quasiperiodic attractor is the only attractor that persists. The blue and red lines in the right panel illustrate the corresponding dynamics in the time domain. The financial market is characterized by price fluctuations that circle around its fundamental value. The bottom panels are based on map MF_2 with $a = -0.05$, $b = -0.75$, $c = 1$ and $h = 1$. This parameter combination lies in region D. In this case, the fixed points of the internal and external sub-maps $MF_{I,2}$ and $MF_{E,2}$ are both unstable. This scenario presents an example in which two weird quasiperiodic attractors, shown in red and green, coexist. Depending on the initial conditions, the market is, on average, either overvalued or undervalued. However, there are episodes in which the market's overvaluation or undervaluation briefly reverses.

Fig. 13 presents two further examples of endogenous financial market dynamics for lower values of parameters c. The top panels are based on map MF_3 with $a = 1.7$, $b = 0.985$, $c = 0.35$ and $h = 1$. This parameter combination belongs to region A, and a weird quasiperiodic attractor coexists with a fixed-point attractor. Their basins of attraction exhibit truly weird structures. The bottom panels are based on map MF_3 with $a = -0.11$, $b = 0.95$, $c = 0.35$ and $h = 1$. Since this parameter combination lies in region C, only a weird quasiperiodic attractor exists. Notably, the price dynamics of the financial market display intricate bull and bear market phases, a phenomenon first studied by Day and Huang (1990). In contrast to their model, in which such dynamics are chaotic, we find these price fluctuations to be weird quasiperiodic.

4. Regulating the real economy

In our second application, we examine a demand-driven economy in the spirit of Samuelson (1939). As Bischi (2025) notes, this model continues to serve as a valuable workhorse for identifying key drivers of business cycle dynamics. In contrast to Baumol (1961), who assumes that the government influences income through a linear countercyclical fiscal policy rule, we consider a piecewise linear policy rule. Related piecewise linear macroeconomic models include Matsuyama (2007); Gardini et al. (2008); Matsuyama et al. (2016), and Gardini et al. (2023), amongst others. However, we provide the first example of a macroeconomic model that can generate weird quasiperiodic business cycle dynamics. We continue as follows. In Section 4.1, we present a standard linear Keynesian goods market model. In Section 4.2, we extend this model by allowing government expenditures to follow a piecewise linear rule. In Section 4.3, we provide several examples that illustrate weird quasiperiodic business cycle dynamics.

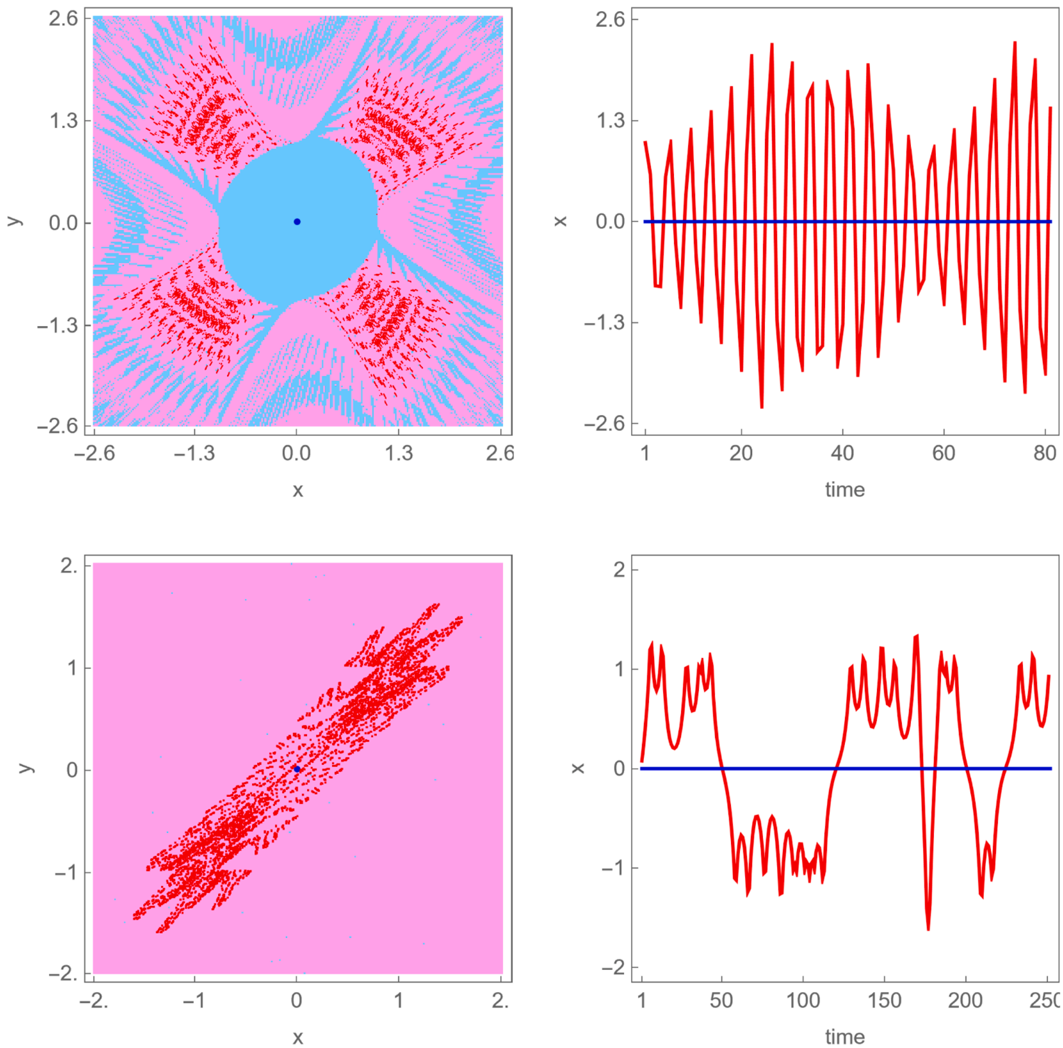

Fig. 13. Second set of examples for endogenous financial market dynamics. Left panels: The blue dots represent the fixed points, while the red dots indicate weird quasiperiodic attractors, with their basins of attraction shown in matching colors. Right panels: The blue and red lines show the corresponding dynamics in the time domain. Top panels: Map MF_3 with $a = 1.7,\ b = 0.985,\ c = 0.35$ and $h\ = 1$. Bottom panels: Map MF_3 with $a = -0.11,\ b = 0.95,\ c = 0.35$ and $h\ = 1$.

4.1. A business cycle setup

As is standard in Keynesian goods market models, income is determined by aggregate demand

$$X_t = Z_t. \tag{4.1}$$

Aggregate demand, in turn, consists of consumption, investment, and government expenditure

$$Z_t = C_t + I_t + G_t. \tag{4.2}$$

Consumption expenditures are proportional to past income

$$C_t = aX_{t-1}, \tag{4.3}$$

where the parameter $0 < a < 1$ represents the marginal propensity to consume. Investment expenditures are specified as

$$I_t = \bar{I} + b(C_t - C_{t-1}). \tag{4.4}$$

Parameter $\bar{I} > 0$ denotes autonomous investment, and parameter $b > 0$ captures the strength of the accelerator principle. Investment is higher when consumption increases and lower when consumption decreases. As a first step, we assume that government expenditures are equal to $\overline{G} > 0$, i.e.,

$$G_t = \overline{G}, \tag{4.5}$$

This yields the linear income dynamics

$$X_t = (a + ab)X_{t-1} - abX_{t-2} + \bar{I} + \overline{G}. \tag{4.6}$$

The fixed point of this model is given by the Keynesian multiplier solution

$$\overline{X} = \frac{1}{1-a}(\bar{I} + \overline{G}). \tag{4.7}$$

Defining income deviations from the fixed point as $x_t\ = X_t - \overline{X}$, and introducing the auxiliary variable $y_t\ =\ x_{t-1}$, we obtain the two-dimensional linear map

$$LR = \begin{cases} x' = (a + ab)x - aby \\ y' = x \end{cases}, \tag{4.8}$$

with the unique fixed point

$$\bar{x} = \bar{y} = 0. \tag{4.9}$$

The Jacobian matrix of map LR is

$$J(LR) = \begin{pmatrix} a+ab & -ab \\ 1 & 0 \end{pmatrix}, \tag{4.10}$$

with trace $tr(LR) = a + ab$ and determinant $det(LR) = ab$. Hence, the fixed point of map LG is globally stable when the stability condition (iii) $a < 1/b$ is satisfied.

4.2. A modified intervention rule CC

We now extend the model by letting government expenditures follow the piecewise linear rule

$$G_t = \begin{cases} \overline{G} - c(C_t - C_{t-1}) & \text{if } |C_t - C_{t-1}| > ah \\ \overline{G} & \text{if } |C_t - C_{t-1}| \leq ah \end{cases}. \tag{4.11}$$

Note that $|C_t - C_{t-1}| > ah$ is equivalent to $|X_{t-1} - X_{t-2}| > h$. The government seeks to weaken the destabilizing effect of the accelerator mechanism by increasing expenditures when income falls by more than h, and decreasing it when income rises by more than h.

For brevity, we only consider one intervention rule. Implementing and analyzing the CD, DD, and DC rules is straightforward. To limit the strength of the fiscal intervention, we furthermore assume that the government's policy parameter is a fraction of the accelerator parameter. Specifically, we set $c = bw$, with $0 < w < 1$. Expressed in deviations from its fixed point, the model dynamics are governed by

$$MR_1 : \begin{cases} MR_{E,1} = \begin{cases} x' = (a + ab(1-w))x - ab(1-w)y & \text{if } |x-y| > h \\ y' = x \end{cases} \\ MR_{I,1} = \begin{cases} x' = (a+ab)x - aby & \text{if } |x-y| \leq h \\ y' = x \end{cases} \end{cases}. \tag{4.12}$$

To be precise, the dynamics of map MR_1 follow the external sub-map $MR_{E,1}$ when the change in income exceeds threshold $h > 0$; otherwise, they follow the internal sub-map $MR_{I,1}$. Both sub-maps share the fixed point $\overline{x} = \overline{y} = 0$, which is either virtual or real for map MR_1.

The stability conditions for the fixed point of sub-map $MR_{I,1}$ are the same as those for map LR. The Jacobian matrix of the external sub-map $MR_{E,1}$ is

$$J(MR_{E,1}) = \begin{pmatrix} a + ab(1-w) & -ab(1-w) \\ 1 & 0 \end{pmatrix}, \tag{4.13}$$

with trace $tr(MR_{E,1}) = a + ab(1-w)$ and determinant $det(MR_{E,1}) = ab(1-w)$. The fixed point of $MR_{E,1}$ is globally stable when the condition (iii) $a < \frac{1}{b(1-w)}$ holds. This condition becomes less restrictive as $w \to 1$.

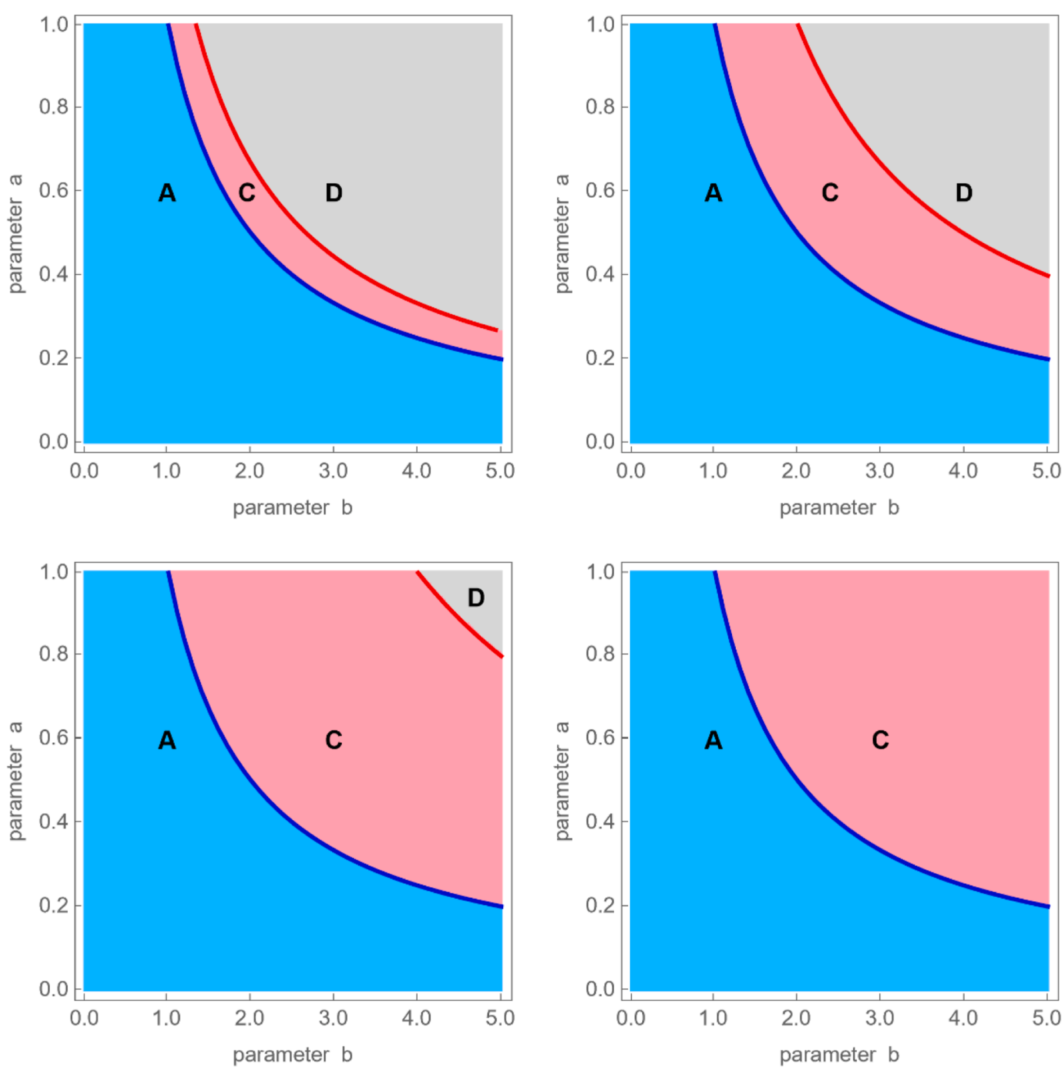

Fig. 14. Two-dimensional bifurcation diagrams for map MR_1 with $h = 1$. Light blue parameter combinations yield fixed-point dynamics, light red indicate weird quasiperiodic dynamics, and light gray correspond to divergent dynamics. Top left: $w = 0.25$. Top right: $w = 0.5$. Bottom left: $w = 0.75$. Bottom right: $w = 0.95$. The stability boundaries of the internal and external sub-maps $MR_{I,1}$ and $MR_{E,1}$ are shown in blue and red, respectively.

Fig. 14 presents two-dimensional bifurcation diagrams for map MR_1 with $h = 1$. Parameter combinations in light blue yield fixed-point dynamics, those in light red exhibit weird quasiperiodic dynamics, and those in light gray result in divergent dynamics. The top-left, top-right, bottom-left, and bottom-right panels correspond to $w = 0.25,\ 0.5,\ 0.75$ and 0.95, respectively. Initial conditions in all simulations are chosen close to the origin. The stability boundaries of sub-maps $MR_{I,1}$ and $MR_{E,1}$ are depicted in blue and red, respectively. As in the previous two sections, labels A, C, and D indicate the parameter regions in which both the fixed points of the internal and external sub-maps $MR_{I,1}$ and $MR_{E,1}$ are stable, in which the fixed point of the internal sub-map is $MR_{I,1}$ unstable, while that of the external sub-map $MR_{E,1}$ is stable, and in which both the fixed points of the internal and external sub-maps $MR_{I,1}$ and $MR_{E,1}$ are unstable. Scenarios in which the fixed point of the internal sub-map $MR_{I,1}$ is stable while that of the external sub-map $MR_{E,1}$ is unstable do not occur under map MR_1; therefore, region B does not exist in this case.

Fig. 14 allows the following conclusions. Increasing parameter w, representing a more aggressive countercyclical fiscal policy, expands the region C associated with weird quasiperiodic dynamics, while shrinking the region D exhibiting divergent behavior. For instance, when $w = 0.95$, no divergent dynamics occurs for $b < 5$. Samuelson's (1939) multiplier-accelerator model is sometimes criticized for producing divergent income dynamics at realistic values of the accelerator parameter. The presence of countercyclical fiscal policy can mitigate such outcomes.

4.3. Emergence of business cycle dynamics

Fig. 15 illustrates two examples of the dynamics of map MR_1. In the left panels, blue and red dots represent the fixed point and the weird quasiperiodic attractor, respectively. The basins of attraction of the weird quasiperiodic attractors are shown in matching colors. In both scenarios, almost all initial conditions converge to the weird quasiperiodic attractor. The right panels display the corresponding income dynamics in the time domain. The top panels are based on $a = 0.5,\ b = 3,\ w = 0.5$ and $h = 1$, yielding fairly regular business cycle dynamics. The bottom panels use $a = 0.8,\ b = 4,\ w = 0.95$ and $h = 1$, resulting in more irregular business cycle dynamics. The emergence of weird quasiperiodic business cycle dynamics offers a compelling mechanism to explain endogenous fluctuations in economic activity. In this case, such dynamics may even be desirable: without government intervention, the national income would exhibit divergent behavior.

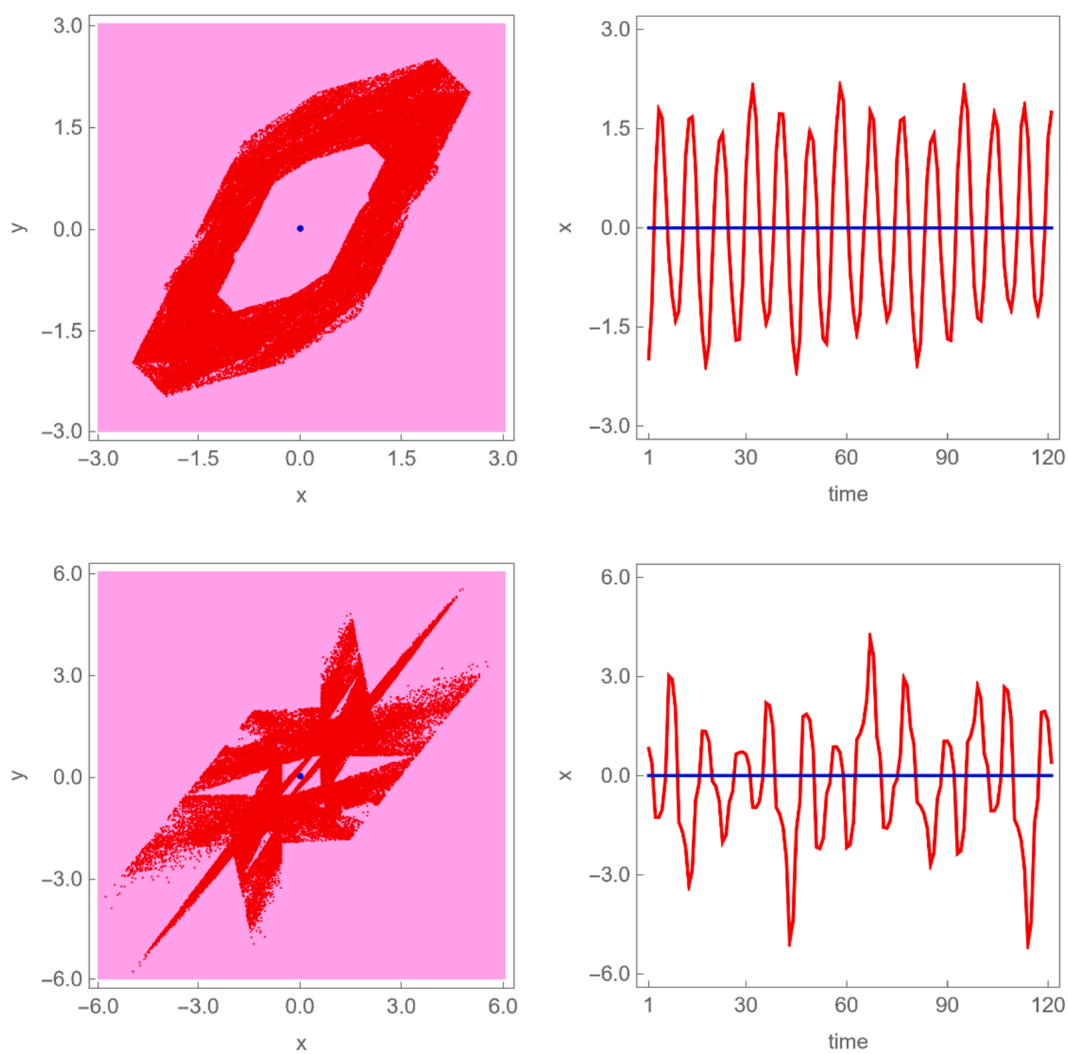

Fig. 15. Examples of the dynamics of map MR_1 for $h = 1$. Left panels: The blue and red dots represent the fixed point and the weird quasiperiodic attractor, respectively. The basins of attraction of the weird quasiperiodic attractors are shown in matching colors. Right panels: The blue and red lines show the corresponding dynamics in the time domain. Top panels: $a = 0.5,\ b = 3$ and $w = 0.5$. Bottom panels: $a = 0.8,\ b = 4$ and $w = 0.95$.

5. Conclusions

In this paper, we have demonstrated that regulators who employ piecewise linear feedback rules may induce weird quasiperiodic dynamics in otherwise linear discrete dynamical systems of dimension two (or higher). Such dynamics may be desirable when they prevent divergence, or undesirable when they disrupt convergence to a fixed point. The fact that weird quasiperiodic dynamics can emerge naturally in these systems underscores their potential relevance across a wide range of applied disciplines. As shown in this study, for example, regulators may generate boom-bust cycles in financial markets or trigger endogenous business cycle fluctuations simply by applying piecewise linear intervention rules. Our findings offer a novel explanation for the emergence of persistent endogenous fluctuations in dynamical systems.

Acknowldegements

Laura Gardini and Davide Radi acknowledge financial support from the Czech Science Foundation (Project No. 26–21034S) and from the European Union through the REFRESH Project, *"Research Excellence for Region Sustainability and High-Tech Industries"*, funded by the European Just Transition Fund (Grant No. CZ.10.03.01/00/22_003/000004). This paper was also supported by the Student Grant Competition at the Faculty of Economics, VSB – Technical University of Ostrava (Project No. SP2026/003). Davide Radi additionally acknowledges financial support from the Gruppo Nazionale di Fisica Matematica (GNFM-INdAM). The work of Davide Radi and Iryna Sushko was funded by the European Union – Next Generation EU, Mission 4 "Education and Research", Component 2 "From Research to Business", through PRIN 2022 under the Italian Ministry of University and Research (MUR), Project No. 2022JRY7EF (*Qnt4Green – Quantitative Approaches for Green Bond Market: Risk Assessment, Agency Problems and Policy Incentives*), CUP J53D23004700008. The work of Iryna Sushko was partially supported by a project of the National Academy of Sciences of Ukraine (No. 0125U000299).

Data availability

No data was used for the research described in the article.

References

Anufriev, M., Gardini, L., Radi, D., 2020. Chaos, border collisions and stylized empirical facts in an asset pricing model with heterogeneous agents. Nonlinear Dyn. 102, 993–1017.

Baumol, W., 1961. Pitfalls in contracyclical policies: some tools and results. Rev. Econ. Stat. 43, 21–26.

Bischi, G.I., 2025. From Samuelson's multiplier-accelerator to bifurcations and chaos in economic dynamics. Decis. Econ. Finance 48, 2193–2214.

Day, R., Huang, W., 1990. Bulls, bears and market sheep. J. Econ. Behav. Organ. 14, 299–329.

Dieci, R. and He, X.-Z. (2018): Heterogeneous agent models in finance. In: Hommes, C. and LeBaron, B. (eds.): Handbook of Computational economics: Heterogeneous Agent Modeling. North-Holland, Amsterdam, 257–328.

Dieci, R., Gardini, L., Westerhoff, F., 2022. On the destabilizing nature of capital gains taxes. Int. Rev. Financ. Anal. 83, 102258.

Gardini, L., Sushko, I., Naimzada, A., 2008. Growing through chaotic intervals. J. Econ. Theory 143, 541–557.

Gardini, L., Radi, D., Schmitt, N., Sushko, I., Westerhoff, F., 2022. Currency manipulation and currency wars: analyzing the dynamics of competitive central bank interventions. J. Econ. Dyn. Control 145, 104545.

Gardini, L., Radi, D., Schmitt, N., Sushko, I., Westerhoff, F., 2023. Sentiment-driven business cycle dynamics: an elementary macroeconomic model with animal spirits. J. Econ. Behav. Organ. 210, 342–359.

Gardini, L., Radi, D., Schmitt, N., Sushko, I., Westerhoff, F., 2025a. On the limits of informationally efficient stock markets: new insights from a chartist-fundamentalist model. Int. Rev. Financ. Anal. 105, 104436.

Gardini, L., Radi, D., Schmitt, N., Sushko, I., Westerhoff, F., 2025b. On the emergence and properties of weird quasiperiodic attractors. Chaos Solit. Fractals 199, 116916.

Gardini, L., Radi, D., Schmitt, N., Sushko, I., Westerhoff, F., 2025c. Abundance of weird quasiperiodic attractors in piecewise linear discontinuous maps. Int. J. Bifurc. Chaos 35, 2530030.

Jungeilges, J., Maklakova, E., Perevalova, T., 2021. Asset price dynamics in a "bull and bear market. Struct. Change Econ. Dyn. 56, 117–128.

Lü, J., Zhou, T., Chen, G., Yang, X., 2002. Generating chaos with a switching piecewise-linear controller. Chaos 12, 344–349.

Matsuyama, K., 2007. Credit traps and credit cycles. Am. Econ. Rev. 97, 503–516.

Matsuyama, K., Sushko, I., Gardini, L., 2016. Revisiting the model of credit cycles with good and bad projects. J. Econ. Theory. 163, 525–556.

Medio, A., Lines, M., 2001. Nonlinear dynamics: a Primer. Cambridge University Press, Cambridge.

Ott, E., Grebogi, C., Yorke, J., 1990. Controlling chaos. Phys, Rev, Lett 64, 1196–1199.

Panchuk, A., Sushko, I., Westerhoff, F., 2018. A financial market model with two discontinuities: bifurcation structures in the chaotic domain. Chaos 28, 055908.

Samuelson, P., 1939. Interactions between the multiplier analysis and the principle of acceleration. Rev. Econ. Stat. 21, 75–78.

Schöll, E., Schuster, H., 2007. Handbook of Chaos Control. Wiley, Berlin.

Sushko, I., Tramontana, F., 2025. Regular and chaotic dynamics in a 2D discontinuous financial market model with heterogeneous traders. Math. Comput. Simul. 230, 386–399.

Tramontana, F., Westerhoff, F., Gardini, L., 2013. The bull and bear market model of Huang and Day: some extensions and new results. J. Econ. Dyn. Control 37, 2351–2370.

Westerhoff, F., 2008. The use of agent-based financial market models to test the effectiveness of regulatory policies. Jahrb. Natl. Stat. 228, 195–227.

Westerhoff, F., Franke, R., 2018. Agent-based models for economic policy design: two illustrative examples. In: Chen, S.H., Kaboudan, M., Du, Y.R. (Eds.), The Oxford Handbook of Computational Economics and Finance. Oxford University Press, Oxford, pp. 520–558.