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A Norm-based Truthmaker Semantics for Modal Logic

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Abstract

In the last few decades, truthmaker semantics has attracted the attention of scholars from various areas, arguably due to its philosophical underpinning and its flexibility in addressing a number of problems that are difficult to solve using alternative semantic frameworks. Attention is now focused on developing truthmaker semantics for modal logic; this paper contributes to that direction. The basic idea we introduce is that truthmakers of modal sentences are norms. This allows us to derive two interesting results: first, we introduce a new modal semantics for minimal monotonic logic; second, we provide a truthmaker semantics for every modal logic characterized by a class of Kripke frames. This is, as far as we can tell, the first general result about truthmaker semantics for normal modal logics.

1 Introduction

This paper has three main aims. The first is to explore what a semantics for modal notions looks like when adopting a truthmaker semantics, wherein truthmakers for modal sentences are identified with specific kinds of norms. The second aim is to show that such a semantics is characterized by the basic system of monotonic minimal logic. The final aim is to prove that it is general enough to allow us to interpret all modal logics characterized by a class of Kripke frames. The last result is somewhat surprising, as it demonstrates that an intuitively appealing idea—namely, that every world is endowed with a set of norms determining which other worlds are possible relative to it—is sufficient for interpreting Kripke semantics within the framework of truthmaker semantics.

The structure of the paper is the following. In Section 2, we propose a brief introduction to truthmaker semantics. In Section 3, we put forward the basic assumptions and the framework for our norm-based semantics. In Section 4, we introduce the semantics and prove that the monotonic system of minimal logic is sound and complete with

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respect to it. In Section 5, we prove our main theorem: a norm-based semantics enables us to interpret all modal logics that are characterized by a class of Kripke frames. In the final section, we conclude by presenting some open lines of research.

2 A Sketch of Truthmaker Semantics

In this section, we provide an overview of standard truthmaker semantics (Fine [7, 8]) and outline the key theorems that we aim to generalize in the modal context in Section 5.

2.1 State Spaces

Truthmaker Semantics is based on the notion of a *state space*.

Definition 1 (State Space) A state space is a pair $(S, \sqsubseteq)$ where

1. $S \neq \varnothing$ is a set of states;
2. $\sqsubseteq$ is a partial ordering on S;
3. every $X \subseteq S$ has a least upper bound $\bigsqcup X$, so that $s \sqsubseteq \bigsqcup X$ for all $s \in X$; and $\bigsqcup X \sqsubseteq x$ for all $x \in S$ such that $s \sqsubseteq x$ for all $s \in X$.

Intuitively, $\sqsubseteq$ is a relation of part between states. Thus, when $x \sqsubseteq s$, we also say that x is a part of s, or that s extends x. In every complete state space, there exists a lowest element $\bigsqcup \varnothing$, the *null state*, and a greatest element $\bigsqcup S$, the *full state*. In fact, by definition, $\bigsqcup \varnothing \sqsubseteq s$ for every $s \in S$ and $s \sqsubseteq \bigsqcup S$ for every $s \in S$.

Definition 2 (State Set) For all $s \in S$, the *state set* $\mathbf{st}(s)$ of s is the set containing all parts of s: $\mathbf{st}(s) = \{x \in S : x \sqsubseteq s\}$.

Since we want to cash out Kripke's intuitions based on a state space semantics, we need frames featuring possible worlds.

Definition 3 (State Frame) A *state frame* is a triple $(S, W, \sqsubseteq)$ where $\mathcal{S} = (S, \sqsubseteq)$ is a complete state space, $\varnothing \neq W \subseteq S$, and for every $v, w \in W$, if $v \sqsubseteq w$, then $v = w$.

We say that $s \in S$ is a possible state precisely when $s \sqsubseteq w$ for some $w \in W$. The set $S^{\Diamond}$ of possible states in S is then such that:

1. $\varnothing \neq S^{\Diamond} \subseteq S$ and $W \subseteq S^{\Diamond}$, as $w \sqsubseteq w$;
2. if $s \in S^{\Diamond}$ and $x \sqsubseteq s$, then $x \in S^{\Diamond}$, as $\sqsubseteq$ is transitive.

We say that $s, x \in S$ are compatible precisely when $s, x \sqsubseteq w$ for some $w \in W$. Then a relation C of compatibility can be defined such that:

1. $C(s, s)$ if $s \in S^{\Diamond}$;
2. if $C(s, x)$, then $C(x, s)$, by the definition of compatibility.

Intuitively, a possible world is a complete state, that is a state that has as a part every state that is compatible with it, and indeed this can be proved without difficulty.

Proof Suppose that $C(s, w)$ for some $s \in S$ and $w \in W$. Then $s, w \sqsubseteq v$ for some $v \in W$. So, $w = v$, by the definition of a world, and therefore $s \sqsubseteq w$. □

As a consequence, we obtain that a world is incompatible with every state that is not a part of it and that the fusion of a world with a state that is not a part of it gives rise to an impossible state. So, $w \sqcup s$ is impossible for every $s \in S - \mathbf{st}(w)$.

Definition 4 (Incompatibility and essentiality) Let s, x be states and P, X be sets of states in S.

1. *Incompatibility*:
   $x \perp s$ iff x is not compatible s;
   $x \perp P$ iff x is not compatible with any state in P;
   $X \perp P$ iff no state in X is compatible with any state in P.
2. *Essentiality*:
   E is a set of essential states iff, for every $w \in W$,
   (i) $s \sqsubseteq w$, for some $s \in E$;
   (ii) if $s \in E$ and $w \perp s$, then $x \perp s$ for some $x \sqsubseteq w$ such that $x \in E$.

The definitions of the relations of compatibility and incompatibility between states and sets of states are relatively intuitive.[1] In contrast, the notion of essentiality, newly introduced in our account, will play a crucial role in developing our framework.[2] Let E be a set of essential states. Then E is characterized by two specific features: first, each world has one of its elements as a part, this captures the idea that the states in E are essential for a world to exist, and second, a world is incompatible with a state in E only if some essential part of that world is incompatible with that state, this captures the idea that the incompatibility between a world and one essential state is grounded on the essential states that are parts of that world. To illustrate this notion and see why it is related with essence, let us consider the relationship between a massive object, such as a cat, and the mass it possesses: first, it is essential for a cat to have a mass; second, when the cat is not characterized by a certain mass, this is due to the fact that it possesses a different mass. Thus, limiting the set of worlds to the worlds where a certain cat exists, the set of states of affairs concerning the mass of that cat is a set of essential states.

Armed with this, let us now see how state frames provide a semantics for a basic propositional language.

2.2 Truthmaker Semantics

The propositional language $\mathcal{L}$ is defined as follows:

$$\phi, \psi ::= p \mid \neg\phi \mid \phi \wedge \psi \mid \phi \vee \psi$$

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[1] In the present approach, the relations of compatibility and incompatibility are defined from the notion of possibility. For truthmaker-based accounts where these notions are defined based on a primitive relation of incompatibility, see Plebani et al. [22] and Champollion and Bernard [2].

[2] A set E of essential states is a kind of unilateral proposition: E is true at every world and, if a world in compatible with a truthmaker for E, then it contains a truthmaker for E that is incompatible with all the other truthmakers.

where $p \in Var$ and Var is a countable set of propositional letters. The connective of implication ($\rightarrow$) and double implication ($\leftrightarrow$) are introduced by the usual definitions.

Definition 5 (State Model) A *state model* $\mathcal{M}$ is a tuple $(S, W, \sqsubseteq, |\cdot|^+, |\cdot|^-)$ where

1. $(S, W, \sqsubseteq)$ is a state frame;
2. $|\cdot|^+ : Var \rightarrow \wp(S)$;
3. $|\cdot|^- : Var \rightarrow \wp(S)$.

Intuitively, $|\cdot|^+ : Var \rightarrow \wp(S)$ is a function mapping propositional letters to the states that make them true, while $|\cdot|^- : Var \rightarrow \wp(S)$ is a function mapping propositional letters to the states that make them false. So, $|p|^+$ is the set containing the verifiers of p and $|p|^-$ is the set containing the falsifiers of p. Finally, the notions of exact verification and falsification are introduced in their non-inclusive version (Fine and Jago [12]).

Definition 6 (Exact Verification) Let $(S, W, \sqsubseteq, |\cdot|^+, |\cdot|^-)$ be a state model. The notions of exact verification, $\Vdash^+$, and exact falsification, $\Vdash^-$, are recursively defined as follows.

$s \Vdash^+ p$ iff $s \in |p|^+$
$s \Vdash^- p$ iff $s \in |p|^-$
$s \Vdash^+ \neg\phi$ iff $s \Vdash^- \phi$
$s \Vdash^- \neg\phi$ iff $s \Vdash^+ \phi$
$s \Vdash^+ \phi \wedge \psi$ iff $s = s_1 \sqcup s_2$ for some $s_1 \Vdash^+ \phi$ and $s_2 \Vdash^+ \psi$
$s \Vdash^- \phi \wedge \psi$ iff $s \Vdash^- \phi$ or $s \Vdash^- \psi$
$s \Vdash^+ \phi \vee \psi$ iff $s \Vdash^+ \phi$ or $s \Vdash^+ \psi$
$s \Vdash^- \phi \vee \psi$ iff $s = s_1 \sqcup s_2$ for some $s_1 \Vdash^- \phi$ and $s_2 \Vdash^- \psi$

The sets of verifiers and falsifiers are defined accordingly, where the sum $+$ of two sets of states is defined so that $X + Y = \{x \sqcup y : x \in X, y \in Y\}$:

$$\begin{array}{ll}
\mathbf{Tm}(p) = |p|^+ & \mathbf{Fm}(p) = |p|^- \\
\mathbf{Tm}(\neg\phi) = \mathbf{Fm}(\phi) & \mathbf{Fm}(\neg\phi) = \mathbf{Tm}(\phi) \\
\mathbf{Tm}(\phi \wedge \psi) = \mathbf{Tm}(\phi) + \mathbf{Tm}(\psi) & \mathbf{Fm}(\phi \wedge \psi) = \mathbf{Fm}(\phi) \cup \mathbf{Fm}(\psi) \\
\mathbf{Tm}(\phi \vee \psi) = \mathbf{Tm}(\phi) \cup \mathbf{Tm}(\psi) & \mathbf{Fm}(\phi \vee \psi) = \mathbf{Fm}(\phi) + \mathbf{Fm}(\psi)
\end{array}$$

The notions of verification and falsification can be further specified by allowing states to make a formula true or false due to having as part a verifier or a falsifier of it. Let us do that by introducing the following notation.

Definition 7 (Inexact Verification and Falsification) Let $\mathcal{M} = (S, W, \sqsubseteq, |.|^+, |.|^-)$ be a state model and $s \in S$. Then

(i) s *inexactly verifies* ϕ iff s contains an exact verifier of ϕ.
$[\phi]^+ = \{s \in S : \textit{for some } t \sqsubseteq s, t \Vdash^+ \phi\}$ is the set of inexact verifiers of ϕ.
(ii) s *inexactly falsifies* ϕ iff s contains an exact falsifier of ϕ.
$[\phi]^- = \{s \in S : \textit{for some } t \sqsubseteq s, t \Vdash^- \phi\}$ is the set of inexact falsifiers of ϕ.

Definition 8 [Inexact Verification and Falsification at a world] Let $\mathcal{M} = (S, W, \sqsubseteq, |.|^+, |.|^-)$ be a state model and $w \in W$. Then

(i) $\mathcal{M}, w \models \phi$ iff $w \in [\phi]^+$.
$[\phi]^+_W$ is the set of worlds that inexactly verify ϕ.

(ii) $\mathcal{M}, w \models \neg\phi$ iff $w \in [\phi]^-$.
$[\phi]^-_W$ is the set of worlds that inexactly falsify ϕ.

So, for all $w \in W$, w inexactly verifies ϕ iff $x \in \mathbf{Tm}(\phi)$ for some $x \sqsubseteq w$. Similarly, w inexactly falsifies ϕ iff $x \in \mathbf{Fm}(\phi)$ for some $x \sqsubseteq w$.[3]

Definition 9 (Classical Model) A model $(S, W, \sqsubseteq, |\cdot|^+, |\cdot|^-)$ is a *classical model* if and only if $|\cdot|^+$ and $|\cdot|^-$ satisfy the following conditions. For all $p \in Var$,

1. *Exclusivity*:
   if $w \in W$ and $s \in \mathbf{st}(w) \cap |p|^+$ for some s, then $t \in |p|^-$ for no $t \sqsubseteq w$.
2. *Exhaustivity*:
   if $w \in W$ and $s \in \mathbf{st}(w) \cap |p|^+$ for no s, then $t \in |p|^-$ for some $t \sqsubseteq w$.

Hence, exclusivity is equivalent to the principle that a world that extends a verifier of p is incompatible with each falsifier of p, while exhaustivity is equivalent to the principle that a world that is incompatible with each verifier of p extends a falsifier of p.

Theorem 1 *Let* $(S, W, \sqsubseteq, |\cdot|^+, |\cdot|^-)$ *be a classical model. Then, for all* $\phi \in \mathcal{L}$

1. Exclusivity*:*
   if $w \in W$ *and* $s \in \mathbf{st}(w) \cap \boldsymbol{Tm}(\phi)$*, then* $t \in \boldsymbol{Fm}(\phi)$ *for no* $t \sqsubseteq w$.
2. Exhaustivity*:*
   if $w \in W$ *and* $s \notin \mathbf{st}(w) \cap \boldsymbol{Tm}(\phi)$*, then* $t \in \boldsymbol{Fm}(\phi)$ *for some* $t \sqsubseteq w$.

Proof By a straightforward induction on the complexity of a formula: see Fine [7], p. 628. □

Definition 10 (Classical Consequence) Let $\Delta \subseteq \mathcal{L}$ and $\phi \in \mathcal{L}$. Then ϕ is a *classical consequence* of Δ, denoted with $\Delta \models \phi$, if and only if, for all $w \in W$ and for all classical models $\mathcal{M}$, if $\mathcal{M}, w \models \Delta$, then $\mathcal{M}, w \models \phi$.

Theorem 2 *Let* $(S, W, \sqsubseteq, |\cdot|^+, |\cdot|^-)$ *be a classical model. Then, for all* $\Delta \in \mathcal{L}$ *and* $\phi \in \mathcal{L}$*,* $\Delta \models \phi$ *if and only if* $\Delta \models_{CL} \phi$*, where* CL *indicated the classical logical consequence.*

Proof See Fine [7], p. 669. □

Before closing, let us say something more about why state models are defined based on a set W of possible worlds. The introduction of W offers an elegant way to capture Kripke's intuition concerning the relationship between modality and reality. On the one hand, we share the idea that there is an actual world, that possible worlds are ways the world could have been, and that the truth of a modal sentence like $\Box\phi$ is

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[3] Inexact verification at a world is equivalent to the notion of *loose verification* and, similarly, the notion of classical consequence in Definition 10 is equivalent to the notion of *loose consequence*, as defined in Fine [7, p. 669].

connected to the truth value of ϕ at a certain class of worlds. On the other hand, we acknowledge that the actual world is just a complete actual state, that possible worlds can be defined as complete states, and that the truth of a modal sentence like $\Box\phi$ can be grounded in specific states within a world. Intuitively, while Kripke's semantics relies on an external perspective on worlds—where worlds are unstructured entities connected by an accessibility relation and modal truths are grounded in terms of this relation—the present semantics relies on an internal perspective on worlds, where: worlds are entities having parts; their relations are grounded in their essential parts; modal truths are grounded in their essential parts as well. Hence, this framework admits possible worlds as a specific type of states, though it commits us to the somewhat controversial principle that every possible state can be extended to a maximal possible state.[4] In the present context, we don't find such principle problematic, as part of our project is to show how Kripke's semantics, which features possible worlds, can be constructed based on a state space semantics. However, we find of utmost interest the idea of eliminating the assumption of possible worlds, and thereby generalizing the framework.[5]

3 Modal State Frames

The following account rests on some key assumptions about the relations between a particular set N of *essential states* and the modal profile of a world, i.e. the modal propositions that are true at a world. A state $n \in N$ corresponds to the fact that a norm is in force, so that a norm is in force in a world w precisely when it is part of that world. The specific interpretation of a norm depends on the modality we want to model. So, metaphysical laws can be seen as norms grounding metaphysical necessities; physical laws as norms grounding physical necessities; epistemic laws as norms grounding a priori knowledge; legal norms as those grounding legal obligations, social norms as those grounding social obligations, etc.[6] In general, the states in N can be seen as a kind of modal states (see Kim [15]), as they have implications on the modal status of other states, but having in mind the intended interpretation of N as a set of norms help understand some of the constraints on N that we are going to introduce.

3.1 The Assumptions

We make three basic assumptions about norms and their relationships with worlds.

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[4] See Fine [6], where Fine develops a truthmaker semantics for intuitionistic logic and presents some concerns about the requirements of closure and completeness for states, driven by the aim to capture a metaphysics consistent with intuitionism.

[5] A philosophical exploration of this point lies beyond the scope of the present work, as even a simple framing of the problem would require considerable space. However, we are grateful to two anonymous reviewers for pushing us toward the discussion of this important issue.

[6] The relationship between norms and modalities is well-documented in different areas of philosophy. See, for instance, Lowe [19] and Maudlin [20] for accounts of metaphysical and physical modalities in terms of laws, Correia [4] and Fine [5], where the role of norms is played by essences, and Hall [13] for a general account, where the role of norms is played by what he calls constraints.

Assumption 1: *norms ground modal truths*, i.e., the modal profile of a world—what is possible or impossible according to a certain modality—is determined by a corresponding set of norms.

Assumption 2: *norms are essential*, i.e., each world has at least one norm as a part, since every world has a modal profile; and each world incompatible with a norm has another norm that grounds that incompatibility. Thus, the set of norms is a set of essential states.

Assumption 3: *norms are universal*, i.e., a norm in force in a world extends to every aspect of that world and grounds its entire modal profile, meaning that every part of the world is to conform to that norm. Thus, norms are assumed to be without exceptions.

Finally, it is important to note that *being compatible* with a norm and *being possible* according to a norm are very distinct notions: for instance, every world that is possible given a norm in w is incompatible with w, if it differs from it.

3.2 The Framework

Let us now make the preceding assumptions precise.

Definition 11 (Modal State Frame) A *modal state frame* is a tuple $(S, W, \sqsubseteq, f)$ where $(S, W, \sqsubseteq)$ is a state frame and

$$f : N \to \wp(W), \text{ where } N \subseteq S^{\Diamond} \text{ is a set of } \textit{essential states}.^{7}$$

According to basic Assumption 1, N is a set of norms that enables a specific modality. In addition, norms are essential states, in line with Assumption 2. f is the function that selects the worlds that comply with a norm, as norms are related to entire worlds, in line with Assumption 3. So, for each $n \in N$, $f(n)$ is the sets of worlds that comply with n.

Definition 12 (World's Modal Function) Let $(S, W, \sqsubseteq, f)$ be a modal state frame and N be the domain of f. Then, the *world's modal function* induced by N is the function $\sigma : W \to \wp(N)$ such that, for every $w \in W$, $\sigma(w) = N \cap \mathbf{st}(w)$.

Hence, σ assigns to each world w the set $\sigma(w)$ of norms that are part of w.

Definition 13 (Modal Code) Let $(S, W, \sqsubseteq, f)$ be a modal state frame and N be the domain of f. Then, the *modal code* of w is the state $\bigsqcup \sigma(w)$, where σ is the modal function induced by N.

The code of w is the fusion of the norms that are part of w, i.e. the least upper bound of all these norms. Therefore, it is necessarily a *part* of that very world, while it is not necessarily a *norm*. In what follows, we will say that a state $s \in S$ is a code just in case it is the code of some world, that is, just in case $s = \bigsqcup \sigma(w)$ for some $w \in W$. In addition, when $n \in N$ and $n \sqsubseteq \bigsqcup \sigma(w)$, we will say that n is a norm of the code $\bigsqcup \sigma(w)$.

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[7] Here we adopt the view that a function is specified by giving its domain, its codomain, and a law of correspondence that associates each entity in the domain with one and only one entity in the codomain. Therefore, we do not need to introduce N explicitly as an element of the modal frame.

Proposition 1 *The following conditions are derivable.*

N1: if $n \in \sigma(w)$, then $n \sqsubseteq w$
(all the norms in force at w are in w).
N2: if $n \sqsubseteq w$ and n is a norm, then $n \in \sigma(w)$
(all the norms in w are in force at w).
N3: if $\sigma(v) \subseteq \sigma(w)$, then $\sigma(v) = \sigma(w)$
(if all the norms in force at v are in w, then v and w have the same norms).
N4: if $\bigsqcup \sigma(v) \sqsubseteq \bigsqcup \sigma(w)$, then $\sigma(v) = \sigma(w)$
(this is just a different formulation of **N3**).
N5: if $\bigsqcup \sigma(v), \bigsqcup \sigma(w) \sqsubseteq u$, for some $u \in W$, then $\sigma(v) = \sigma(w)$
(this is just a different formulation of **N3**).

N1 and **N2** are straightforward consequences of the definition of σ. Here we prove **N3**: given **N3**, the proofs of **N4** and **N5** are not difficult and are left to the reader.

Proof Suppose that $v, w \in W$ and $\sigma(v) \subseteq \sigma(w)$. Then $N \cap \mathbf{st}(v) \subseteq N \cap \mathbf{st}(w)$. Suppose now that $\sigma(w) \nsubseteq \sigma(v)$, i.e. $n \notin N \cap \mathbf{st}(v)$ and $n \in N \cap \mathbf{st}(w)$ for some norm n. Since n is a norm, then $n \notin \mathbf{st}(v)$, that is n is a norm that is not a part of v. So, as N is an essential set, there exists some norm x in v such that $x \perp n$. Since $\sigma(v) \subseteq \sigma(w)$, it follows that $x \in N \cap \mathbf{st}(w)$. Given that $n \in N \cap \mathbf{st}(w)$ and $x \perp n$, this contradicts the fact that w is a possible world. □

Proposition 2 *It is impossible for a world to contain two different codes.*

Proof Suppose that s and s' are two different codes. Then, $s = \bigsqcup \sigma(v)$ and $s' = \bigsqcup \sigma(v')$ for some $v, v' \in W$. Suppose now that $s \sqsubseteq w$ and $s' \sqsubseteq w$ for some w. Then

1. for all $n \in N$, $n \in \sigma(v) \Rightarrow n \sqsubseteq s$, by the definition of s;
2. for all $n \in N$, $n \in \sigma(v') \Rightarrow n \sqsubseteq s'$, by the definition of s';
3. for all $n \in N$, $n \in \sigma(v) \Rightarrow n \sqsubseteq w$, from 1, as s is a part of w;
4. for all $n \in N$, $n \in \sigma(v') \Rightarrow n \sqsubseteq w$, from 2, as s' is a part of w;
5. for all $n \in N$, $n \in \sigma(v) \Rightarrow n \sqsubseteq \sigma(w)$, from 3, by the definition of σ;
6. for all $n \in N$, $n \in \sigma(v') \Rightarrow n \sqsubseteq \sigma(w)$, from 4, by the definition of σ;
7. $\sigma(v) \subseteq \sigma(w)$ and $\sigma(v') \subseteq \sigma(w)$, from 4 and 5;
8. $\sigma(v) = \sigma(v') = \sigma(w)$, from 6, by **N3**. □

Definition 14 Let $(S, W, \sqsubseteq, f)$ be a modal state frame and N be the domain of f. Then, the *basic modal relations* induced by f are the following. For all $P \subseteq S$,
– n *necessitates* P iff every world in $f(n)$ has a part in P;
– n *permits* P iff some world in $f(n)$ has a part in P.

The definitions are thought to connect truthmaker semantics to standard possible world semantics: necessitation is defined so that a norm necessitates a proposition just in case *every world* that is possible according to the norm extends some truthmaker for that proposition, in line with the idea that what is necessary is what is true at every possible world. Similarly, permission is defined so that a norm permits a proposition just in case *some world* that is possible according to the norm extends some truthmaker for that proposition, in line with the idea that what is possible is what is true at some accessible possible world.

Definition 15 (Modal State Model) A *modal state model* $\mathcal{M}$ is a tuple $(\mathcal{S}, |.|^+, |.|^-)$ such that $\mathcal{S}$ is a modal state frame, and $|.|^+$ and $|.|^-$ are functions that map the propositional variables of $\mathcal{L}_\Box$ to subsets of S so that, for every propositional variable $p \in Var$, *Exclusivity* and *Exhaustivity* hold.

Hence, it is impossible for a truthmaker and a falsemaker of a propositional variable to be part of one world at once, and it is necessary for each world to extend one of either a truthmaker or a falsemaker of any propositional variable.

4 Modal Truthmaker Semantics

Let $\mathcal{M} = (S, W, \sqsubseteq, f, |\cdot|^+, |\cdot|^-)$ be a modal state model. The definition of exact verification and falsification for non-modal sentences is the same as in Definition 6, while the definition of exact verification and falsification for modal sentences is as follows:

$s \Vdash^+ \Box\phi$ iff $s \in N$ and $f(s) \subseteq [\phi]^+_W$;
$s \Vdash^- \Box\phi$ iff s is a code whose norms permit $[\phi]^-_W$.

The sets of verifiers and falsifiers are defined accordingly:

$\mathbf{Tm}(\Box\phi) = \{s \in N : f(s) \subseteq [\phi]^+_W\}$;
$\mathbf{Fm}(\Box\phi) = \{s : s$ is a code whose norms permit $[\phi]^-_W\}$.

The notions of inexact verifiers and falsifiers and the notions of verification and falsification at a world are defined as before. Furthermore, we extend the definition of classical consequence to the formulas of $\mathcal{L}_\Box$ in the obvious way.

4.1 Exclusivity and Exhaustivity

Let us go through the truth conditions for a modal formula. First, an exact verifier of a formula like $\Box\phi$ is a norm that necessitates ϕ. Hence, all the worlds that are compatible with that norm verifies ϕ. Second, an exact falsifier of $\Box\phi$ is a code that contains as parts only norms that permit $[\phi]^-$.

In light of this, let us consider the following notions.

1. n necessitates ϕ iff n necessitates $[\phi]^+_W$ iff $f(n) \subseteq [\phi]^+_W$.
2. n permits ϕ iff n permits $[\phi]^+_W$ iff $f(n) \cap [\phi]^+_W \neq \varnothing$.
3. n precludes ϕ iff n necessitates $[\phi]^-_W$ iff $f(n) \subseteq [\phi]^-_W$.
4. n exempts from ϕ iff n permits $[\phi]^-_W$ iff $f(n) \cap [\phi]^-_W \neq \varnothing$.

As usual in standard modal logic, the modality of possibility can be defined from the modality of necessity, by saying that the possibility of ϕ corresponds to the negation of the necessity of $\neg\phi$: $\Diamond\phi := \neg\Box\neg\phi$. The truth conditions of $\Diamond\phi$ support the intuitive idea that what makes ϕ possible is a code whose norms permit ϕ, while what makes ϕ impossible is a norm that precludes ϕ. Indeed

$$
\begin{array}{lll}
s \Vdash^{+} \Diamond\phi & \text{iff} & s \Vdash^{+} \neg\Box\neg\phi, \\
 & \text{iff} & s \Vdash^{-} \Box\neg\phi, \\
 & \text{iff} & s \text{ is a code whose norms permit } \neg\phi, \\
 & \text{iff} & s \text{ is a code such that, for every } n \sqsubseteq s,\ f(n) \cap [\neg\phi]^{-}_{W} \neq \varnothing, \\
 & \text{iff} & s \text{ is a code such that, for every } n \sqsubseteq s,\ f(n) \cap [\phi]^{+}_{W} \neq \varnothing, \\
 & \text{iff} & s \text{ is a code such that, for every } n \sqsubseteq s,\ n \text{ permits } \phi. \\
s \Vdash^{-} \Diamond\phi & \text{iff} & s \Vdash^{-} \neg\Box\neg\phi, \\
 & \text{iff} & s \Vdash^{+} \Box\neg\phi, \\
 & \text{iff} & s \text{ is a norm that necessitates } \neg\phi, \\
 & \text{iff} & s \text{ is a norm such that } f(n) \subseteq [\neg\phi]^{+}_{W}, \\
 & \text{iff} & s \text{ is a norm such that } f(n) \subseteq [\phi]^{-}_{W}, \\
 & \text{iff} & s \text{ is a norm that precludes } \phi.
\end{array}
$$

Asymmetry between verification and falsification The idea underlying the truth conditions for modal formulas is that a norm is sufficient for making a sentence like $\Box\phi$ true, but it is not sufficient for making such a sentence false. To understand why this asymmetry arises, let us consider the case of legal norms. To determine whether we are subject to a certain obligation, it is enough to check if there is a norm that prescribes the content of that obligation. For example, to determine whether we are obligated to drive on the right side of the road, it suffices to check for a norm stating that we ought to drive to the right. In contrast, determining whether we are not subject to an obligation requires reviewing the entire legal code to ensure that none of its norms prescribes the content of that obligation. Thus, to verify that we are not required to drive with our lights on, we need to review the entire code, ensuring that no norm dictates driving with the lights on. In sum, if a norm permits ϕ, nothing prevents another norm within the same world from precluding ϕ. Therefore, ϕ is possible with respect to a world if and only if all the norms of that world agree that there are accessible possible worlds where ϕ is true.

Logical Truths The universality of the norms is reflected by the fact that the content of a norm determines a set of possible worlds—the worlds that conform to the norm. An immediate consequence is that all logical truths share the same truthmakers, which some might view as an undesirable outcome. However, what makes a sentence true is different from what makes its necessitation true. Hence, what makes a logical truth like $p \vee \neg p$ true, i.e. a state that makes either p or its negation true, can differ from what makes its necessitation true, and therefore we suggest that sentences like $\Box(p \vee \neg p)$ and $\Box(q \vee \neg q)$ should not be grounded by specific norms, since the necessity of a logical truth depends on its logical form, not on the specific propositional variables involved—and this is why we typically require a system of logic to be closed under uniform substitution. In the present framework, which is based on an intensional approach, this implies that every norm serves as a truthmaker for the necessity of a sentence like $\Box(p \vee \neg p)$, an outcome that can be questioned—for example, by arguing that different necessary validities should have distinct subject matters. Still, given that norms are introduced as sources for modalities, and that the necessity of logical truths seems to be independent from their subject matter, the fact that no particular norm is needed to make any necessitated logical truth seems to be the right outcome. So, we acknowledge that there is a tension here, that could be solved by moving to a hyperintensional framework, where a unique norm grounds the necessity of all logical truths.[8]

The relation between necessity and possibility just described entails a good behavior of the modality with respect to possible states and possible worlds, as they satisfy the requirements of exclusivity and exhaustivity.

Theorem 3 (Exclusivity) *Let* $\mathcal{M} = (S, W, \sqsubseteq, f, |\cdot|^+, |\cdot|^-)$ *be a modal model and* $\phi \in \mathcal{L}_\Box$. *Then, no compatible states can both verify and falsify* ϕ.

The proof is by induction on the complexity of a formula, and we only prove the modal case: if $s \in \mathbf{Tm}(\Box\phi)$ and $s' \in \mathbf{Fm}(\Box\phi)$, then $s \bot s'$.

Proof The proof is a bit long, so we divided into three steps.
Step 1: $[\phi]^+_W$ and $[\phi]^-_W$ are disjoint propositions.

This is by the induction hypothesis.
Step 2: if n necessitate P, then n precludes any proposition X disjoint from P.

Suppose $f(n) \subseteq P$. Then, $f(n) \cap X = \varnothing$, as $P \cap X = \varnothing$.
Step 3: conclusion.

Suppose by contradiction that $s \in \mathbf{Tm}(\Box\phi)$ and $s' \in \mathbf{Fm}(\Box\phi)$. Then s is a verifier of $\Box\phi$ and s' is a code whose norms permit $[\phi]^-_W$. Suppose now, by contradiction, that $s, s' \sqsubseteq w$, for some $w \in W$. Then s' is the code of w, by Prop. 2, and s is a norm of w, by assumption. So, s is one of the norms of s', and therefore it permits $[\phi]^-_W$. Thus,

1. $[\phi]^+_W \cap [\phi]^-_W = \varnothing$, by Step 1;
2. $f(s) \subseteq [\phi]^+_W$, as s necessitates ϕ;
3. $f(s) \cap [\phi]^-_W = \varnothing$, from 1, by Step 2;
4. $f(s) \cap [\phi]^-_W \neq \varnothing$, as s permits $[\phi]^-_W$.

So, we get a contradiction, and this concludes the proof. □

Let us observe that the fact that N is an essential set plays a key role in this proof, as it allows us to derive that no two codes are part of the same world (Step 3). This result is not surprising: intuitively, the code of a world is to that world as the essence of a thing is to that thing; as no thing has two different essences, no world has two different codes.

Theorem 4 (Exhaustivity) *Let* $\mathcal{M} = (S, W, \sqsubseteq, f, |\cdot|^+, |\cdot|^-)$ *be a modal state model and* $\phi \in \mathcal{L}_\Box$. *Then, for all* $w \in W$, $\mathcal{M}, w \models \phi$ *or* $\mathcal{M}, w \models \neg\phi$.

The proof is again by induction on the complexity of a formula, and we prove the modal case: $\mathcal{M}, w \models \Box\phi$ or $\mathcal{M}, w \models \neg\Box\phi$.

Proof Suppose $\mathcal{M}, w \not\models \Box\phi$ and $\mathcal{M}, w \not\models \neg\Box\phi$. Then we have two hypothesis:

- for all $s \sqsubseteq w$, s is not a verifier of $\Box\phi$
  i.e., for all $s \in \sigma(w)$, $f(s) \nsubseteq [\phi]^+_W$;

---

[8] In fact, developing such a hyperintensional framework is one of the main task we plan to undertake.

- for all $s \sqsubseteq w$, s is not a falsifier of $\Box\phi$
  i.e., for all $s \sqsubseteq w$, s is not a code whose norms permit $[\phi]^-_W$.

As no $s \sqsubseteq w$ is a code whose norms permit $[\phi]^-_W$, the code of w is such that some of its norms preclude $[\phi]^-_W$. Thus, some of the norms in $\sigma(w)$ preclude $[\phi]^-_W$, and therefore they necessitate $[\phi]^+_W$. Still, according to the first hypothesis, no norm of w, and so no norm in $\sigma(w)$, necessitates $[\phi]^+_W$, and this is a contradiction. □

Corollary 1 *Let $\mathcal{M} = (S, W, \sqsubseteq, f, |\cdot|^+, |\cdot|^-)$ be a modal state model and $\phi \in \mathcal{L}_\Box$. Then, for all $w \in W$, $\mathcal{M}, w \models \phi$ iff $\mathcal{M}, w \not\models \neg\phi$.*

Straightforward given the previous theorems.

4.2 The Logic of Modal State Frames

In this section we prove that the logic of the class of modal state frames is the monotonic version of minimal modal logic, that is called **EM**, which is in turn sound and complete with respect to the class of multi-relational Kripke frames. To do that, we prove (i) that every modal state model can be translated into an equivalent multi-relational model, thus proving that **EM** is sound relative to a corresponding class of modal state models; and (ii) that every multi-relational model can be translated into an equivalent modal state model, thus proving that **EM** is complete relative to a corresponding class of multi-relational models.

Definition 16 A *multi-relational frame* is a pair $(W, \{R_i\}_{i\in I})$ where

- W is a non-empty set,;
- $\{R_i\}_{i\in I}$ is a non-empty set of relations, so that $I \neq \varnothing$.

A *multi-relational model* is a triple $(W, \{R_i\}_{i\in I}, V)$ where $(W, \{R_i\}_{i\in I})$ is a multi-relational frame and $V : Var \to \wp(W)$ is a valuation function.

A multi-relational frame is just a generalization of a relational frame, where more than a relation of accessibility is allowed. In what follows, we define $R_i(w) = \{v : R_i(w, v)\}$.

Definition 17 (Monotonic truth) The *monotonic* truth conditions for non-modal formulas are defined as usual, and for modal formulas are defined as follows. Let us denote the proposition that ϕ as $[\phi]^{mon} = \{w \in W \mid \mathcal{M}_{MK}, w \models_{mon} \phi\}$. Then:

$$(W, \{R_i\}_{i\in I}, V), w \models_{mon} \Box\phi \text{ iff } R_i(w) \subseteq [\phi]^{mon} \text{ for some } i \in I.$$

Consider the following inference rules:

RE From $\phi \leftrightarrow \psi$, infer $\Box\phi \leftrightarrow \Box\psi$
RM From $\phi \rightarrow \psi$, infer $\Box\phi \rightarrow \Box\psi$

The basic system of *minimal logic*, **E**, is the smallest propositional modal logic that contains all instances of classical validities and is closed under **RE** and *Modus Ponens*. The system of *monotonic minimal logic*, **EM**, is the smallest propositional modal logic that contains all instances of classical validities and is closed under **RE**, **RM** and *Modus Ponens* (See Chellas [3], Part III).

Theorem 5 *EM is sound and complete with respect to the class of multi-relational frames.*

The proof of soundness is by a straightforward induction on the length of a proof, while completeness follows from the fact that **EM** is complete with respect to the class of neighborhood models under the monotonic truth conditions.[9]

From Modal State Models to Multi-relational Models Let us start by showing that any modal state model induces an equivalent multi-relational model.

Theorem 6 *Let $\mathcal{M} = (S, W, \sqsubseteq, f, |\cdot|^+, |\cdot|^-)$ be a modal state model. Then, there exists a corresponding multi-relational model $\mathcal{M}_{MK}$ such that, for all $\phi \in \mathcal{L}_\Box$,*

$$\mathcal{M}_{MK}, w \models_{mon} \phi \text{ iff } \mathcal{M}, w \models \phi.$$

Proof We build a multi-relational model starting from $\mathcal{M} = (S, W, \sqsubseteq, f, |\cdot|^+, |\cdot|^-)$. Let $\mathcal{M}_{MK} = (W, \{R_i\}_{i \in I}, V)$ be a model where W is the set of worlds in $\mathcal{M}$, $I = N$, and for every $i \in I$, R_i is such that:

$$R_i(w) = \begin{cases} f(i) & \text{if } i \in \sigma(w) \\ \bigcup_{n \in \sigma(w)} f(n) & \text{if } i \notin \sigma(w) \end{cases}$$

$V : Var \to \wp(W)$ is defined so that $V(p) = [p]^+_W$, for all $p \in Var$.
It is clear that $\mathcal{M}_{MK}$ is a multi-relational model. Now we show that $\mathcal{M}$ and $\mathcal{M}_{MK}$ are equivalent. The proof proceeds by induction on the complexity of ϕ. As usual, we only check the modal case, by assuming the inductive hypothesis that $[\psi]^{mon} = [\psi]^+_W$.
From right-to left:

$$\begin{array}{llll} \mathcal{M}, w \models \Box\psi & \Rightarrow & f(n) \subseteq [\psi]^+_W \text{ for some } n \in \sigma(w) & \text{by inexact verification;} \\ & \Rightarrow & f(i) \subseteq [\psi]^+_W \text{ for some } i \in \sigma(w) & \text{by def. of } N; \\ & \Rightarrow & R_i(w) \subseteq [\psi]^+_W \text{ for some } i \in I & \text{by definition of } R_i; \\ & \Rightarrow & R_i(w) \subseteq [\psi]^{mon} \text{ for some } i \in I & \text{by inductive hypothesis;} \\ & \Rightarrow & \mathcal{M}_{MK}, w \models_{mon} \Box\psi & \text{by monotonic truth;} \end{array}$$

From left-to-right: $\mathcal{M}_{MK}, w \models_{mon} \Box\psi \Rightarrow R_i(w) \subseteq [\psi]^{mon}$ for some $i \in I$, by the definition of truth. Then, we consider two cases.

Case 1: $i \in \sigma(w)$, so that $R_i(w) = f(i)$. Then,

$$\begin{array}{llll} \mathcal{M}_{MK}, w \models_{mon} \Box\psi & \Rightarrow & f(i) \subseteq [\psi]^{mon}, \text{ with } i \in \sigma(w) & \text{by assumption;} \\ & \Rightarrow & f(i) \subseteq [\psi]^+_W, \text{ with } i \in \sigma(w) & \text{by inductive hypothesis;} \\ & \Rightarrow & \mathcal{M}, w \models \Box\psi & \text{by inexact semantics.} \end{array}$$

Case 2: $i \notin \sigma(w)$, so that $f(i) = \bigcup_{n \in \sigma(w)} f(n)$. Then,

$$\begin{array}{llll} \mathcal{M}_{MK}, w \models_{mon} \Box\psi & \Rightarrow & \bigcup_{n \in \sigma(w)} f(n) \subseteq [\psi]^{mon}, & \text{by assumption;} \\ & \Rightarrow & f(i) \subseteq \bigcup_{n \in \sigma(w)} f(n), \text{ for some } i \in \sigma(w) & \text{by essentiality of } N; \\ & \Rightarrow & f(i) \subseteq [\psi]^+_W, \text{ with } i \in \sigma(w) & \text{by IH;} \\ & \Rightarrow & \mathcal{M}, w \models \Box\psi & \text{by inexact semantics.} \end{array}$$

---

[9] Multi-relational frames were first proposed in Chellas [3, pp. 74–75], and the connection between them and neighborhood frames is mentioned in Pacuit [21, sec. 2.2.3]. A proof of the fact that every multi-relational model induces an equivalent monotonic neighborhood model and every monotonic neighborhood model induces an equivalent multi-relational model can be found in Leuenberger and Smith [17].

In both cases, we get that $\mathcal{M}, w \models \Box\psi$ and this concludes the proof. $\square$

From Multi-relational Models to Modal State Models We now move on to the second part of the proof.

Theorem 7 *Let $\mathcal{M}_{MK} = (W, \{R_i\}_{i \in I}, V)$ be a multi-relational Kripke model. Then, there exists a corresponding modal state model $\mathcal{M}$ such that, for all $\phi \in \mathcal{L}_\Box$,*

$$\mathcal{M}_{MK}, w \models_{mon} \phi \textit{ iff } \mathcal{M}, w \models \phi.$$

$\mathcal{M} = (S^*, W^*, \sqsubseteq, f, |\cdot|^+, |\cdot|^-)$ is defined as follows.

- $S^* = \{\{w\} \cup J : w \in W, J \subseteq I\}$;
- $W^* = \{\{w\} : w \in W\}$;
- $N = \{\{w, i\} : w \in W, i \in I\}$;
- $\sqsubseteq$ is such that $X \sqsubseteq Y$ iff $Y \subseteq X$;
- $f : N \to \wp(W^*)$ such that $f(\{w, i\}) = \{\{v\} : v \in R_i(w)\}$;
- $|p|^+ = \{\{w\} : w \in V(p)\}$ and $|p|^- = \{\{w\} : w \in V(\neg p)\}$.

$\mathcal{M}$ is a state model such that $w^* \in W^*$ iff $w^* = \{w\}$ for some $w \in W$. Furthermore, a possible state is always of the form $\{w, i, \ldots, j\}$ for some $w \in W$ and $\{i, \ldots, j\} \subseteq I$.

Proof $(S^*, W^*, \sqsubseteq)$ is a modal state frame. W is a nonempty subset of S and $\sqsubseteq$ is a complete partial order, by definition. $f : N \to \wp(W^*)$ is such that $N \subseteq S^\Diamond$, as a norm is a set of two elements, namely a world and an index, and thus it is always part of a world, since $\{w\} \subseteq \{w, i\}$. N is a set of essential states. First, N is nonempty, as $I \neq \varnothing$. Therefore, every world $w^* = \{w\}$ is a subset of some element of N, and so it has that norm as a part. Next, consider an arbitrary $\{w\}$ and a norm $\{v, i\}$ incompatible with $\{w\}$. Then $\{w\} \cap \{v, i\} \notin S^{*\Diamond}$. It follows that $w \neq v$, as otherwise $\{w\} \cap \{v, i\} = \{w\} \in S^{*\Diamond}$. As $\{w, i\}$ is a norm, by the definition of N, we conclude that $\{w\} \subseteq \{w, i\}$ and $\{v, i\} \cap \{w, i\} = \{i\}$. However, $\{i\} \notin S^{*\Diamond}$, and therefore there is a norm in $\{w\}$ that is incompatible with $\{v, i\}$. Hence, N is an essential set. Finally, $|.|^+$ and $|.|^-$ satisfy exclusivity and exhaustivity by definition. So, $\mathcal{M}$ is a modal state model. We now show that it is equivalent to the original multi-relational Kripke model. The proof proceeds by induction on the complexity of ϕ and we only check the modal case, by assuming the inductive hypothesis that $[\phi]^{mon} = [\phi]^+_W$, that is by assuming that $v \in [\phi]^{mon} \Leftrightarrow \{v\} \in [\phi]^+_W$.

$$\begin{array}{lll}
\multicolumn{2}{l}{\mathcal{M}, \{w\} \models \Box\psi} & \\
\Leftrightarrow & n \sqsubseteq \{w\} \text{ for some } n \in \mathbf{Tm}(\Box\phi), & \text{by inexact semantics;} \\
\Leftrightarrow & \{w\} \subseteq n \text{ and } f(n) \subseteq [\phi]^+_W, \text{ for some } n \in N, & \text{by exact semantics;} \\
\Leftrightarrow & \{w\} \subseteq \{w, i\} \text{ and } f(\{w, i\}) \subseteq [\phi]^+_W \text{ for } \{w, i\} \in N, & \text{by def. of } N; \\
\Leftrightarrow & \{w\} \subseteq \{w, i\} \text{ and } \{\{v\} : v \in R_i(w)\} \subseteq [\phi]^+_W, & \text{by def. of } f; \\
\Leftrightarrow & R_i(w) \subseteq [\phi]^{mon}, \text{ for some } i \in I, & \text{by inductive hypothesis;} \\
\Leftrightarrow & \mathcal{M}_{MK}, w \models_{mon} \Box\phi, & \text{by monotonic semantics.}
\end{array}$$

The crucial equivalence, $\{\{v\} : v \in R_i(w)\} \subseteq [\phi]^+_W$ iff $R_i(w) \subseteq [\phi]^{mon}$, is based on the inductive hypothesis. All the other equivalences are straightforward consequences of the definition of truth and the definition of $\mathcal{M}$. $\square$

Hence, we conclude that the modal logic of modal state frames is precisely the modal logic of multi-relational frames, that is **EM**.

5 Kripkean Truthmaker Semantics

In the present section, we introduce a class of modal state models—the *normal state models*—that are equivalent to the class of Kripke frames with respect to which the logic **K** is sound and complete. We then show (i) that every normal state model can be translated into an equivalent Kripke model, thus proving that the logic **K** is sound with respect to the corresponding class of normal state models; and (ii) that every Kripke model can be translated into an equivalent normal state model, thus proving that the logic **K** is complete with respect to the class of normal state models. In addition, we show that the previous results can be generalized to every class of Kripke models. This implies that possible world semantics can be viewed as a specification of our truthmaker semantics.

Kripke frames are introduced here in their functional variation, as a pair (W, R) where

1. $W \neq \varnothing$ is a set of possible worlds;
2. $R : W \to \wp(W)$ is a function that assigns a set $R(w)$ of worlds to each $w \in W$.

Intuitively, $R(w)$ contains the worlds that are accessible from w, so that the usual accessibility relation can be recovered by setting $R(w, v)$ iff $v \in R(w)$.

5.1 From Modal State Models to Kripke Models

There is a precise connection between the accessibility relations involved in Kripke frames and the norms of our modal state frames, which is grounded on the content function f. In this section, we explore this connection in more detail and discuss new elements of the semantics that are crucial for the construction of Kripke models from modal state models. First, we introduce a general condition, **Normality**, that allows us to generate an accessibility relation based on the content function. This condition defines a class of modal state frames that we call normal and constitutes the most general requirement to capture Kripke frames in terms of our semantics. As a consequence, we can conclude that there is a precise sense in which Kripke frames are just modal state frames satisfying **Normality**. Next, we continue our analysis by introducing three classes of frames that capture different philosophical intuitions about the notion of a norm and its connection with the notion of an accessibility relation. We do not intend to take a stand from a philosophical perspective; instead, we present the new proposals as interesting specifications of normal frames, each offering a distinct norm-based characterization of what it is for a world to be accessible from another world.

5.1.1 Normal State Models

In any modal state space, f takes norms and returns sets of possible worlds, namely the worlds that are in conformity with that norm.

$$\begin{tikzcd}
N \arrow[r, "f"] & \wp(W)
\end{tikzcd}$$

It is not difficult to see that this function can be lifted to a function f^*, taking sets of norms and returning sets of possible worlds, by defining for each $X \subseteq N$

$$f^*(X) = \bigcap_{n \in X} f(n)$$

So, what is made accessible by all the norms in a set conjunctively coincides with the intersection of what is made accessible by each of the norms in that set individually.

$$\begin{tikzcd}
\wp(N) \arrow[r, "f^*"] & \wp(W)
\end{tikzcd}$$

By combining σ and f^*, we get the following diagram:

$$\begin{tikzcd}
& \wp(N) \arrow[dr, "f^*"] & \\
W \arrow[ur, "\sigma"] \arrow[rr, "f^* \circ \sigma"'] & & \wp(W)
\end{tikzcd}$$

Hence, the composite function $f^* \circ \sigma : W \to \wp(W)$ plays the same role of the accessibility function in a Kripke frame. In light of this, we are set now to prove our main theorems.

Definition 18 (Normal State Frame) A *normal state frame* $(S, W, \sqsubseteq, f)$ is a modal state frame satisfying

Normality: for all $w \in W$, there is a norm $n \in \sigma(w)$, such that $f(n) = f^*(\sigma(w))$.

A *normal state model* is a modal state model based on a normal state frame.

This allows us to prove the following lemma.

Lemma 1 *For all $w \in W$, $f^*(\sigma(w)) \subseteq X$ iff there is $n \in \sigma(w)$, such that $f(n) \subseteq X$.*

Proof Suppose $f^*(\sigma(w)) \subseteq X$. Then there is a norm $n \in \sigma(w)$ such that $f(n) = \bigcap_{n \in \sigma(w)} f(n)$ by **Normality**. Therefore $f(n) \subseteq X$. Suppose now there is a norm $n \in \sigma(w)$ such that $f(n) \subseteq X$. Then $f^*(\sigma(w)) \subseteq f(n) \subseteq X$, by the definition of $f^*(\sigma(w))$, and therefore, $f^*(\sigma(w)) \subseteq X$. $\square$

The notion of truth of ϕ at a world w in a Kripke models $\mathcal{M}_K$, in symbols $\mathcal{M}_K, w \models_K \phi$, for non-modal formulas is defined as usual, and truth for modal formulas is defined as follows:

$$\mathcal{M}_K, w \models_K \Box\phi \text{ iff } R(w) \subseteq [\phi]^K,$$

where $[\phi]^K$ is the set $\{w \in W : \mathcal{M}_K, w \models_K \phi\}$. We are now in a position to prove the first part of the main result of this section.

Theorem 8 *Let $\mathcal{M} = (S, W, \sqsubseteq, f, |.|^+, |.|^-)$ be a normal state model. Then, there exists a corresponding Kripke model $\mathcal{M}_K$ such that, for all $\phi \in \mathcal{L}_\Box$,*

$$\mathcal{M}_K, w \models_K \phi \textit{ iff } \mathcal{M}, w \models \phi.$$

Proof Let $\mathcal{M}_K = (W, R, V)$ be defined so that

- W is the set of worlds in $\mathcal{M}$;
- $R : W \to \wp(W)$ is such that $R(w) = f^*(\sigma(w))$, for all $w \in W$;
- $V : Var \to \wp(W)$ is such that $V(p) = [p]^+_W$, for all $p \in Var$.

It is not difficult to see that $\mathcal{M}_K$ is indeed a Kripke model.
We prove that $\mathcal{M}$ and $\mathcal{M}_K$ are modally equivalent. The proof proceeds by induction on the complexity of ϕ. The only non-trivial case is the one where $\phi = \Box\psi$. Assume the inductive hypothesis that $[\psi]^K = [\psi]^+_W$. Then,

$$\begin{array}{llll}
\mathcal{M}_K, w \models_K \Box\psi & \text{iff} & R(w) \subseteq [\psi]^K & \text{by semantics;} \\
& \text{iff} & f^*(\sigma(w)) \subseteq [\psi]^K & \text{by the def. of } R; \\
& \text{iff} & f^*(\sigma(w)) \subseteq [\psi]^+_W & \text{by the inductive hypothesis;} \\
& \text{iff} & f(n) \subseteq [\psi]^+_W \text{ for some } n \in \sigma(w) & \text{by Lemma 1;} \\
& \text{iff} & \mathcal{M}, w \models \Box\psi & \text{by semantics.}
\end{array}$$

This concludes the proof. □

The previous theorem shows that **Normality** is indeed what we need in order to link Kripke frames with modal state frames, and in a sense this is all we need. In fact, one could argue that a Kripke frame is just the result of forgetting the norms and focusing on the accessibility relation which is induced by them. Still, one could also aim at studying normal modal frames in more depth, trying to specify that condition. In the next sections, we provide three different accounts for **Normality**, that correspond to different intuitions on what is a truthmaker for a necessary sentence.

5.1.2 The Global Account

A first proposal to specify **Normality** is to admit one and only one norm in each possible world. This norm determines the whole modal profile of the world and it is completely relevant to exactly verify every necessity and possibility. This position is a sort of monism of norms, according to which there exists just one giant modal fact that is part of a possible world. The frames that correspond to this picture are called *global*.

Definition 19 (Global State Frame) A *global state frame* is a modal state frame satisfying

G: For all $w \in W$ and $n, n' \in N$, if $n, n' \in \sigma(w)$, then $n = n'$.

A *global state model* is a modal state model based on a global state frame.

It is not hard to see that this class of frames is a class of *normal* frames, as the only norm in a world trivially satisfies **Normality**, as well as the assumption of essentiality of norms. In fact, we have the following result, where n_w is the unique norm that is in force at w.

Lemma 2 *For all* $w \in W$, $f^*(\sigma(w)) = f(n_w)$.

Proof Straightforward, as $f^*(\sigma(w)) = \bigcap_{n \in \sigma(w)} f(n) = f(n_w)$, given that $\sigma(w) = \{n_w\}$. □

An immediate consequence of having just a norm in each world is that the world's modal function σ can now be defined as a function $\sigma : W \to N$, which assigns to each world the unique global norm that is in force in it.

Definition 20 (World Essence Function) Let $(S, W, \sqsubseteq, f)$ be a global frame and N be the domain of f. Then, the *world essence function* induced by N is the function $\sigma : W \to N$ such that, for every $w \in W$, $\sigma(w) = n_w$.

We call this σ function the *world's essence function*, as $\sigma(w)$ plays the same role as the essence of a thing: as the essence of a thing determines the modal profile of that thing, i.e. what is possible and impossible for that thing, so $\sigma(w)$ determines the modal profile of w. Accordingly, the diagram introduced before can be simplified is this way:

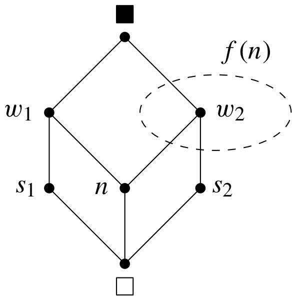
$$\begin{tikzcd}
 & N \arrow[dr, "f"] & \\
W \arrow[ur, "\sigma"] \arrow[rr, "f \circ \sigma"'] & & \wp(W)
\end{tikzcd}$$

Since $\sigma(w) = n_w$ is the modal code of that world, there is no asymmetry between exact verification and falsification of a modal proposition. Hence, the truth conditions for a modal sentence can be simplified as follows.

- $s \in \mathbf{Tm}(\Box\phi)$ iff $s \in N$ and $f(s) \subseteq [\phi]^+ \cap W$
- $s \in \mathbf{Fm}(\Box\phi)$ iff $s \in N$ and $f(s) \cap [\phi]^- \neq \varnothing$

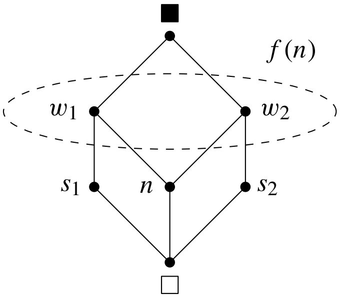
The resulting semantics preserves the properties of exclusivity with respect to possible states and exhaustivity with respect to possible worlds, and a specific version of Theorem 1 can be proved by assuming that the accessibility relation of the induced Kripke model is defined by setting $R(w) = f(\sigma(w))$ for all $w \in W$.

To illustrate how norms operate under the global account consider the following modal state model $\mathcal{M} = (S, W, \sqsubseteq, f, |\cdot|^+, |\cdot|^-)$, where $W = \{w_1, w_2\}$, $N = \{n\}$, and $f(n) = \{w_2\}$.



Suppose that $|p|^+ = \{s_2\}$ and $|p|^- = \{s_1\}$. Since there is only one norm n, that is part of both possible worlds and has content $f(n) = \{w_2\}$, this model is a *global* model. So, given the definition of the verification and falsification functions, we have $n \Vdash^+ \Box p$, and therefore $w_1 \models \Box p$ and $w_2 \models \Box p$. Hence, $\Box p$ is true at both w_1 and w_2, because both worlds share the same global norm n, which only allows for worlds where p is true, that is w_2. We see that having a unique global norm as part of two worlds implies that these worlds are indistinguishable as to the modal sentences that are true at them.

Let us consider now a small variation of the model, where $f(n) = \{w_1, w_2\}$. In this case, both worlds share a global norm n that permits the existence of worlds where p is true, namely w_2 and worlds where p is false, namely w_1. Hence, $\Box p$ is false at both w_1 and w_2.



In the corresponding Kripke frame, the same situation can be reproduced by stipulating that the accessibility relation satisfies $R(w_1) = R(w_2) = \{w_1, w_2\}$, so that each world accesses both itself and the other. What is worth noticing is that, in the modal state framework, this condition is not imposed externally via an accessibility relation, but is instead grounded internally, based on the normative structure of the world. Actually, it is the fact that all worlds are governed by the same global norm that directly account for their modal profile, in particular for the fact that p is not necessarily true.

The global account also finds an interesting application in the study of canonical modal models. When constructing the canonical Kripke frame (W, R) for a normal modal logic, the accessibility relation R is typically defined so that $R(w, v)$ holds if and only if $w_\Box \subseteq v$, where $w_\Box = \{\phi : \Box\phi \in w\}$. Interpreting sets of formulas as states, we see that $w_\Box$ corresponds to a global norm. Specifically, if we define $f(w_\Box)$ as the set of worlds that are in conformity to $w_\Box$, we get that all the worlds that contain the state $w_\Box$ as a part have access to the same set of worlds. In this respect, canonical models can be naturally interpreted as global states models.

Let us close by commenting on the pros and cons of this first account. The main advantage is given by its elegance and effectivity in treating Kripke semantics. Indeed, if propositions are interpreted as sets of possible worlds, then $R(w)$ is the strongest necessary proposition that is true in w, that is, it is precisely the giant modal fact that determines the entire modal profile of a world. The global account is the best way of cashing out this intuition in our framework. However, this might also be seen as the main disadvantage of the present position, as all the necessary propositions turn out to be true based on one common truthmaker. Hence, they are the same not only as sets of possible worlds, but also as sets of states, and this brings us to consider a more fine grained account.

5.1.3 The Witnessed Account

A second proposal to specify **Normality** is to impose a one to one correspondence between sets of worlds that are accessible in terms of f^* and sets of worlds that are accessible in terms of f. The idea is that what is made accessible by all the norms in a set conjunctively coincides with what is made accessible by one of the norms in that set individually.

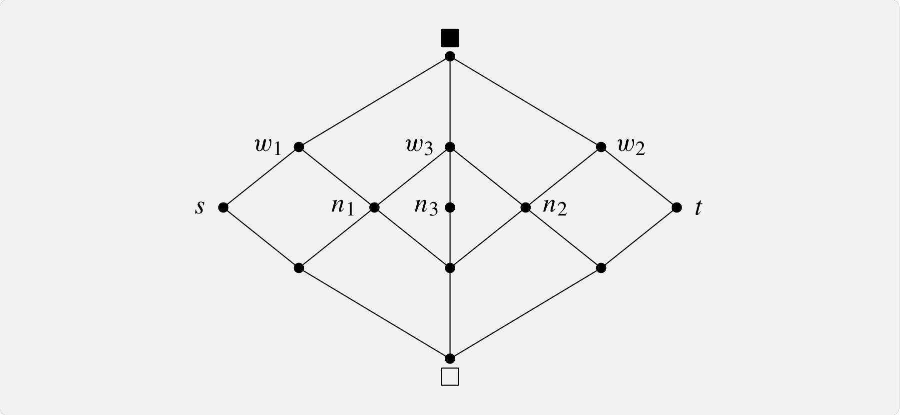
Definition 21 (Witnessed State Frame) A *witnessed state frame* $(S, W, \sqsubseteq, f)$ is a modal state frame satisfying:

$$\textbf{W}: \quad \text{for all } w \in W \text{ and } X \subseteq \sigma(w), \text{ there is some } n \in \sigma(w), \text{ such that } f(n) = f^*(X).$$

A *witnessed state model* is a modal state model based on a witnessed state frame.

So, when the norms in a set X are in force in a world, there is always a norm n in force in that world that is compatible with the worlds compatible with all the norms in X. Accordingly, n acts as a witness of the accessibility of the worlds in $f^*(X)$.

All witnessed state frames are normal, as $\sigma(w)$ is a set of norms in w, and so $f^*(\sigma(w))$ coincides with $f(n)$ for some $n \in \sigma(w)$. Hence, the resulting semantics preserves the properties of exclusivity with respect to possible states and exhaustivity with respect to possible worlds, and a specific version of Theorem 1 can be proved by assuming that the accessibility relation of the induced Kripke model is defined as $R(w) = f^*(\sigma(w))$. The advantage of this account over the global one is that it allows us to distinguish modal sentences in terms of the states that make them exactly true or false. In fact, let $\Box\phi$ and $\Box\psi$ be true at w. Then, there are norms n and n' in $\sigma(w)$ such that $f(n) \subseteq [\phi]^+_W$ and $f(n') \subseteq [\psi]^+_W$, and nothing prevents the fact that n and n' are different norms. To see how this is possible, consider a model $(S, W, N, \sqsubseteq, f, |.|^+, |.|^-)$ with three possible worlds, $W = \{w_1, w_2, w_3\}$, and three norms, $N = \{n_1, n_2, n_3\}$, structured as shown in the following diagram.



Let f be such that $f(n_1) = \{w_1, w_3\}$; $f(n_2) = \{w_2, w_3\}$; $f(n_3) = \{w_3\}$.

As we can see, the worlds w_1 and w_2 are characterized by having a global norm as a part, but the model is not global, since the code of w_3 contains n_1, n_2, n_3. Still, it is witnessed, because every set X of norms in a world is such that $f^*(X) = f(n)$ for some n in that world, the only non-trivial case being given by the norms n_1, n_3 in w_3, which are such that $f(n_1) \cap f(n_2) = f(n_3)$.

Let us define the exact verifiers and falsifiers for $Var = \{p, q\}$ so that

$|p|^+ = \{n_1\}$ and $|p|^- = \{t\}$, which implies that $f(n_1) \subseteq [p]^+_W$;

$|q|^+ = \{n_2\}$ and $|q|^- = \{s\}$, which implies that $f(n_2) \subseteq [q]^+_W$;

Finally, $n_3 \Vdash^+ \Box(p \wedge q)$, given that $f(n_3) = [p \wedge q]^+_W$.

It is not difficult to see that the conditions of exclusivity and exhaustivity are satisfied, so that this is a classical model. Now, it follows from the definition of f that $n_1 \Vdash^+ \Box p$ and $n_2 \Vdash^+ \Box q$; however, neither is a verifier of $\Box(p \wedge q)$. On the other hand, $n_3 \Vdash^+ \Box(p \wedge q)$. This shows that a witnessed frame is able to distinguish modal truths in terms of what grounds those truths—something that is not available in standard Kripke frames, where a counterpart of world w would simply have an accessibility relation such that $R(w) = f(n_3)$, and this would make $\Box p$, $\Box q$, and $\Box(p \wedge q)$ true without further differentiation.

However, and this is the main disadvantage of the present position, we also find that n_3 is an exact verifier of $\Box p$, $\Box q$, and $\Box(p \wedge q)$. The problem, one could argue, is that whenever two norms make two necessary sentences true, there is always another norm that necessitates those two sentences as well as their conjunction. As this is not the best instance of the notion of exact truthmaking, we are brought to consider a more exact account.

5.1.4 The Grounded Account

A final proposal to specify **Normality** is based on a specific conception of the notion of norm.

Definition 22 (Grounded State Frame) A *grounded state frame* $(S, W, \sqsubseteq, f)$ is a modal state frame satisfying

Gr1: for all $w \in W$ and $X \subseteq \sigma(w)$, $\bigsqcup X \in \sigma(w)$;
Gr2: for all $X \subseteq N$, $f(\bigsqcup X) = \bigcap_{x \in X} f(x)$;
Gr3: there is no infinite descending chain of norms.
A *grounded state model* is a modal state model based on a grounded state frame.

The first condition, **Gr1**, states that the fusion of every subset of norms in a world is in turn a norm, while condition **Gr2** states that the content of a norm which is the result of fusing together a set of norms is the intersection of the contents of the original norms. These two conditions together deliver a specific instance of a witnessed frame, and therefore of a normal frame. Recall that **W** requires that, for each set of norms in a world, there is always a norm whose content is the intersection of the contents of the original norms. In grounded frames we identify such a norm with the fusion of the original norms, which is required to be a norm itself. So, these conditions impose a form of upward closure with respect to the fusion of norms. Finally, condition **Gr3** states that there is no infinite descending chain of norms, that is a set $C \subseteq N$ such that

1. for all $n, n' \in C$, $n \sqsubseteq n'$ or $n' \sqsubseteq n$;
2. for all $n \in C$, there is a norm $n' \in C$ such that $n' \neq n$ and $n' \sqsubseteq n$.

This condition captures the intuitive idea that there are fundamental norms: metaphysically basic laws for the notion of metaphysical necessity; physically basic laws for the notion of physical necessity; basic laws of thought for the notion of epistemic necessity; basic legal norms for the notion of deontic necessity, and so on. In fact, we are unaware of any attempts to defend the view that there are laws all the way down, and it is certainly impossible to find legal codes where infinite chains of norms exist.

It is worth noting that **Gr3** implies the existence of minimal norms in N, and this allows us to adopt a new exact semantics for modal sentences, where a state exactly verifies $\Box\phi$ if and only if it is a minimal norm that necessitates ϕ. Let us first prove the following lemma, which will be essential for obtaining the main theorem of this section.

Lemma 3 *All grounded state frames are normal. Let* $(S, W, \sqsubseteq, f)$ *be a grounded state frame. Then, for every* $P \subseteq S$ *and* $w \in W$, $f(\bigsqcup \sigma(w)) \subseteq P$ *iff* $f(n) \subseteq P$ *for some* $n \in \sigma(w)$.

Proof We have to prove that, for all $P \subseteq S$ and $w \in W$, $f(\bigsqcup \sigma(w)) \subseteq P$ iff $f(n) \subseteq P$ for some $n \in \sigma(w)$. Suppose $f(\bigsqcup \sigma(w)) \subseteq P$. Then $f(n) \subseteq P$ for some $n \in \sigma(w)$, as $\bigsqcup \sigma(w)$ is a norm in $\sigma(w)$, by condition **Gr1**. Suppose now $f(n) \subseteq P$ for some $n \in \sigma(w)$. Then $f(\bigsqcup \sigma(w)) \subseteq P$, as $f(\bigsqcup \sigma(w)) \subseteq f(n)$ for every $n \in \sigma(w)$, by condition **Gr2**. □

The definition of exact verification and falsification is like the one provided before, with the following requirement of minimality for modal sentences:

$s \Vdash^+ \Box\phi$ iff s is a $\sqsubseteq$-*minimal* norm such that $f(s) \subseteq [\phi]^+_W$;
$s \Vdash^- \Box\phi$ iff s is a code whose norms permit $[\phi]^-_W$.

As usual, to say that s is a $\sqsubseteq$-minimal norm such that $f(s) \subseteq [\phi]^+_W$ just amounts to saying that s is a norm such that $f(s) \subseteq [\phi]^+_W$ and that there is no other norm $n \in N$

such that $n \neq s$ and $n \sqsubseteq s$ and $f(n) \subseteq [\phi]^+_W$. In this case, we also say that s is a *minimal verifier* for $\Box\phi$ or $\Box\phi$*-minimal*. The sets of verifiers and falsifiers are defined accordingly:

$$\begin{aligned}
&\mathbf{Tm}(\Box\phi) = \{s \in N : s \text{ is a } \sqsubseteq\text{-minimal norm such that } f(s) \subseteq X\};\\
&\mathbf{Fm}(\Box\phi) = \{s : s \text{ is a code whose norms permit } [\phi]^-_W\}.
\end{aligned}$$

The results on the duality between necessity and possibility are still provable when the new definition of exact verification and falsification is assumed. Similarly, the properties of exclusivity and exhaustivity are preserved.

Theorem 9 (Grounded Exclusivity) *Let $\mathcal{M} = (S, W, \sqsubseteq, f, |\cdot|^+, |\cdot|^-)$ be a grounded modal model and $\phi \in \mathcal{L}_\Box$. Then, no compatible states can verify and falsify ϕ.*

Proof The proof is a straightforward adaptation of the proof of Theorem 3. □

Lemma 4 *Let $\mathcal{M} = (S, W, \sqsubseteq, f, |\cdot|^+, |\cdot|^-)$ be a grounded model and $w \in W$. Then, a norm $n \in \sigma(w)$ necessitates P if and only if a $\sqsubseteq$-minimal norm $n \in \sigma(w)$ necessitates P.*

Proof The right-to-left direction is trivial. For the left-to-right direction, consider a grounded model $\mathcal{M}$ and let $n \in \sigma(w)$ and $f(n) \subseteq P$. Since $\mathcal{M}$ is grounded, there are no infinite descending chains of norms. Therefore, either n is P-minimal, or there is a $n' \sqsubseteq n$, which is P-minimal, that is $n' \in \sigma(w)$ (by **N2**) and is a $\sqsubseteq$-minimal norm which necessitates P, as $f(n) = f(n \sqcup n') = f(n) \cap f(n')$ (by **Gr2**) and, therefore, $f(n') \subseteq f(n) \subseteq P$. □

Theorem 10 (Grounded Exhaustivity) *Let $\mathcal{M} = (S, W, \sqsubseteq, f, |\cdot|^+, |\cdot|^-)$ be a grounded modal model and $\phi \in \mathcal{L}_\Box$. Then, for all $w \in W$, $\mathcal{M}, w \models \phi$ or $\mathcal{M}, w \models \neg\phi$.*

Proof The proof is again by induction on the length of a formula, and we prove the modal case: $\mathcal{M}, w \models \Box\phi$ or $\mathcal{M}, w \models \neg\Box\phi$. Suppose this is not the case, that is $\mathcal{M}, w \not\models \Box\phi$ and $\mathcal{M}, w \not\models \neg\Box\phi$. Then we have two hypothesis:

- for all $s \sqsubseteq w$, s is not a minimal verifier of $\Box\phi$,
  i.e., for all $\sqsubseteq$-minimal norm $s \in N \cap \mathbf{st}(w)$, $f(s) \not\subseteq [\phi]^+$;
- for all $s \sqsubseteq w$, s is not a falsifier of $\Box\phi$,
  i.e., for all $s \sqsubseteq w$, s is not a code whose norms permit $[\phi]^-_W$.

As no $s \sqsubseteq w$ is a code whose norms permit $[\phi]^-_W$, the code of w is such that some of its norms preclude $[\phi]^-_W$. Thus, some of the norms in $\sigma(w)$ preclude $[\phi]^-_W$, and therefore they necessitates $[\phi]^+_W$. It follows by Lemma 4 that there is a minimal norm in $\sigma(w)$ that necessitate $[\phi]^+_W$. Still, according to the first hypothesis, no σ-minimal norms of w, and so no norms in $\sigma(w)$, necessitates $[\phi]^+_W$, and this is a contradiction. □

We can now prove the main theorem of this section, namely the fact that for every grounded model we can build a Kripke model which satisfies the same formulas.

Theorem 11 *Let $\mathcal{M} = (S, W, \sqsubseteq, f, |.|^+, |.|^-)$ be a grounded state model. Then, there exists a corresponding Kripke model $\mathcal{M}_K$ such that, for all $\phi \in \mathcal{L}_\Box$,*

$$\mathcal{M}_K, w \models_K \phi \textit{ iff } \mathcal{M}, w \models \phi.$$

Proof Let $\mathcal{M}_K = (W, R, V)$ be defined so that

1. W is the set of worlds in $\mathcal{M}$;
2. $R : W \to \wp(W)$ is such that $R(w) = f^*(\sigma(w))$, for all $w \in W$;
3. $V : Var \to \wp(W)$ is such that $V(p) = [p]^+_W$, for all $p \in Var$.

$\mathcal{M}_K$ is a Kripke model. The proof proceeds by induction on the complexity of ϕ. The only non-trivial case is the one where $\phi = \Box\psi$. Assume the inductive hypothesis that $[\psi]^K = [\psi]^+_W$. Then

$$\begin{array}{llll}
\mathcal{M}_K, w \models_K \Box\psi & \text{iff} & R(w) \subseteq [\psi]^K & \text{by the definition of truth;} \\
& \text{iff} & f^*(\sigma(w)) \subseteq [\psi]^K & \text{by the definition of } R; \\
& \text{iff} & f^*(\sigma(w)) \subseteq [\psi]^+_W & \text{by the inductive hypothesis;} \\
& \text{iff} & f(\bigsqcup \sigma(w)) \subseteq [\psi]^+_W & \text{by } \mathbf{Gr3}; \\
& \text{iff} & f(n) \subseteq [\psi]^+_W, \text{ for some } n \in \sigma(w) & \text{by Lemmas 3 and 4;} \\
& \text{iff} & \mathcal{M}, w \models \Box\psi & \text{by exact verification.}
\end{array}$$

This concludes the proof. □

The advantages of this account are evident: different modal sentences can be made true by different verifiers, and the exactness of the truthmaking relation is ensured by the minimality requirement. Therefore, the only disadvantages of this grounded semantics are related to this very requirement, which is reminiscent of the proposal made by Yablo [24, p. 61]. In the literature, we find two main objections against assuming minimality, both championed by Fine. According to the first, see Fine [7, p. 670], exact verifiers are not to be identified with minimal inexact verifiers, as some sentences would appear to have no minimal verifiers even though they have exact verifiers. For example, any infinitude of stars will presumably be an exact verifier for the sentence "there are infinitely many stars", even though the removal of any star from such a verifier will still leave us with a verifier. These kinds of example are not problematic in our framework, as minimality is only involved in the definition of the truth conditions of modal sentences.

The second objection is based on the following case Fine [8, p. 9].

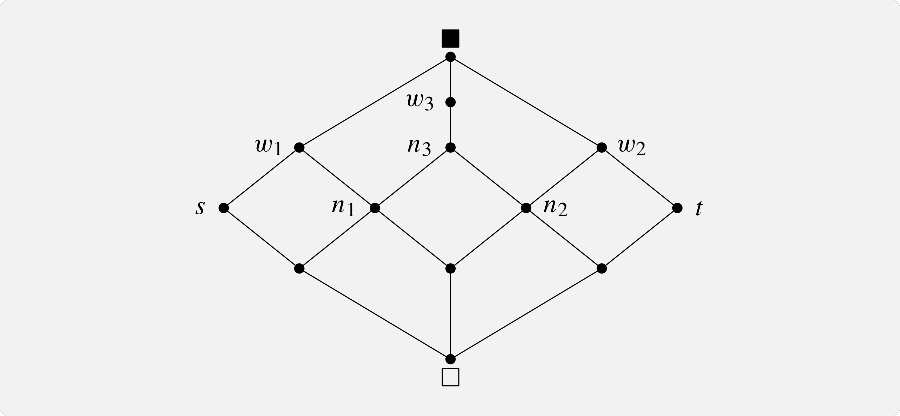
Example 1 Suppose that s is the sole verifier of p, s' the sole verifier of q, and $s' \not\sqsubseteq s$. Intuitively s and $s \sqcup s'$ should be both verifiers of $p \vee (p \wedge q)$, but only s is a minimal one.

Again, this is not a problem for the present account, as it concerns non-modal formulas. Still, let us consider the corresponding modal case.

Example 2 Suppose that n is the sole verifier of $\Box p$, n' the sole verifier of $\Box q$, and $n' \not\sqsubseteq n$. Intuitively n and $n \sqcup n'$ should be both verifiers of $\Box(p \vee (p \wedge q))$, as $n \sqcup n'$ is a verifier of $\Box(\phi \wedge \psi)$, by **Gr2** and **Gr3**, but only n is a minimal one.

We are of the opinion that in the present framework, where the semantics of $\Box$ is given in terms of the content function f, this argument can be resisted. Indeed, by the definition of truth, we know that $\Box(p \vee (p \wedge q))$ is true at w precisely when $\Box p$ is true at w, and therefore any verifier that is not exact with respect to $\Box p$ must turn out to be also not exact with respect to $\Box(p \vee (p \wedge q))$. This is a consequence of the assumption that norms are universal, that is, that they rule a world in its entirety, and we want to stick to this assumption here. In any case, for those who support a different conception, we still obtain a conditional conclusion: if norms are universal, then the minimality requirement is an effective way to select the exact verifiers of a modal sentence.

The grounded account can be further clarified by reconsidering the modal state model introduced in the context of the witness account, where $f(n_1) = \{w_1, w_3\}$, $f(n_2) = \{w_2, w_3\}$, $f(n_3) = \{w_3\}$.



Let us define the exact verification and falsification functions as before:
$|p|^+ = \{n_1\}$; $|p|^- = \{t\}$; $|q|^+ = \{n_2\}$; $|q|^- = \{s\}$.

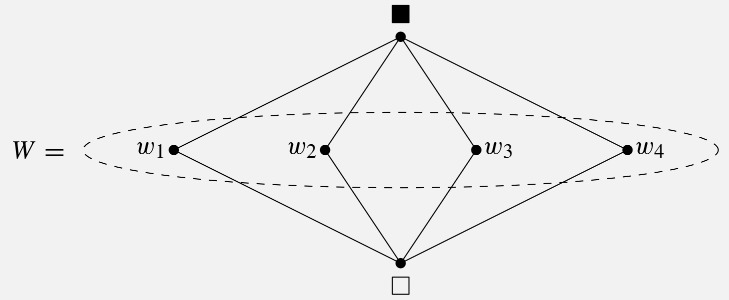
We are now in a position to see that n_1 is the only verifier of $\Box p$ and n_2 is the only verifier of $\Box q$. In fact, while n_3 is such that $f(n_3) \subseteq [p]^+_W \cap [q]^+_W$, it is not a minimal norm that respects that condition, and therefore does not count as a verifier for $\Box p$ or $\Box q$.

Finally, the grounded approach finds an important application in the study of canonical modal models, where each sentence of the form $\Box\phi$ can be interpreted as playing the role of a norm. Indeed, if we assume that $f(\Box\phi)$ is the set of worlds where ϕ is true, that is, the set of worlds containing ϕ, then we obtain $\Box\phi \Vdash^+ \Box\phi$, which allows us to draw the natural conclusion that the canonical verifier of a sentence like $\Box\phi$ is the sentence itself.

In conclusion, we have introduced **Normality** as a condition that enables us to interpret Kripke's possible world semantics within our framework. This condition can be supported in various ways, each with its own advantages and disadvantages. We have proposed three distinct accounts to substantiate it: the first is closely aligned with Kripke's original intuition, while the last one more closely approximates the approach of truthmaker semantics. While we like the grounded account, it is possible that other interpretations of the relationship between norms and necessities exist, potentially offering more effective solutions.

5.2 From Kripke Models to Modal State Models

In every Kripke frame, every possible world can be seen as a possible state. So, a Kripke frame can be transformed into a very specific instance of a grounded frame, one consisting only of possible worlds together with two new entities, standing for the full state and the null state. The partial ordering is then defined so that the null state is part of itself and all the other states, each world is part of itself and of the full state, and the full state part only of itself. In a picture, where $W = \{w_1, w_2, w_3, w_4\}$:



This allows us to prove that, for each Kripke model, there exists a corresponding grounded state model validating the same formulas. As grounded state models are normal state models, we conclude that, for each Kripke model, there exists a normal state model validating the same formulas.

Theorem 12 *Let $\mathcal{M}_K = (W, R, V)$ be a Kripke model. Then, there exists a corresponding grounded state model $\mathcal{M}$ such that, for all $\phi \in \mathcal{L}_\Box$,*

$$\mathcal{M}_K, w \models_K \phi \textit{ iff } \mathcal{M}, w \models \phi.$$

Proof Let $\mathcal{M} = (S, W, \sqsubseteq, f, |.|^+, |.|^-)$ be a model where

- $S = W \cup \{\top, \bot\}$;
- W is the set of worlds in $\mathcal{M}_K$;
- $N = W$, so that norms are worlds;
- $\sqsubseteq = \{(\top, s) : s \in S\} \cup \{(s, s) : s \in S\}$;
- $f : N \to \wp(W)$ maps $w \in W$ to $f(w) = R(w)$;
- $|p|^+ = V(p)$ and $|p|^- = V(\neg p)$ for all $p \in Var$.

Let us first prove that $\mathcal{M}$ is a grounded state model. $(S, W, \sqsubseteq)$ is indeed a modal state frame, as W is a non-empty subset of S. Furthermore, $f : N \to \wp(W)$ is such that $N \subseteq S^\Diamond$, as $N = W$, and N is a set of essential states, given that every world has one of the elements of N as a part, namely itself, and every element of N is incompatible with any other element in N. Furthermore, the frame is grounded, as norms are worlds, and therefore no norm is a proper part of another norm. Also, as fusion corresponds to set-theoretical union, conditions **Gr1** and **Gr2** are satisfied: as to **Gr1**, if X is a set of norms in w, then $X = \{w\}$ and its union is just w, which is a norm in w; as to **Gr2**, since every world has only one norm, that is itself. Finally, $|.|^+$ and $|.|^-$ satisfy exclusivity and exhaustivity by definition, as the valuation function of a Kripke model is classical.

The proof that $\mathcal{M}$ and $\mathcal{M}_K$ are modally equivalent proceeds again by induction on the complexity of ϕ. As before, we only check the modal case: $\phi = \Box\psi$. Let us assume that $[\psi]^+_W = [\psi]^K$. Then,

$$\begin{array}{llll} \mathcal{M}, w \models \Box\psi & \text{iff} & w \in \mathbf{Tm}(\Box\psi) & \text{as } w \text{ is the only norm in } w; \\ & \text{iff} & f(w) \subseteq [\psi]^+_W & \text{by the definition of truth and of } f; \\ & \text{iff} & f(w) \subseteq [\psi]^K & \text{by the inductive hypothesis } f; \\ & \text{iff} & R(w) \subseteq [\psi]^K & \text{by the definition of } f; \\ & \text{iff} & \mathcal{M}_K, w \models_K \Box\psi & \text{by the definition of truth.} \end{array}$$

This concludes the proof. □

5.3 A General Theorem

The possibility of moving back and forth between Kripke models and normal state models allows us to obtain a very general result about our semantics for modal logic. First, note that the theorems we just proved imply that the basic system **K** of standard modal logic is sound and complete with respect to our modal semantics. To be sure, **K** is the logic of the class of all Kripke frames, and therefore it is also the logic of the class of all normal state frames since, given theorems above, a formula can be validated in the first class if and only if it can be validated in the second one. Next, note that the relationship between the class of all Kripke frames and the class of all normal state frames can be generalized to any class of Kripke frames that is defined in terms of a condition on R. This is due to two crucial facts

1. R can be defined as $f^* \circ \sigma$,
   when moving from normal state frames to Kripke frames;
2. f can be substituted for $f^* \circ \sigma$,
   when moving from Kripke frames to normal state frames.

The last fact follows from the definition of f in a normal state frame induced by a Kripke frame. Since the domain of f is $N = W$ and $\mathbf{st}(w) = \{w, \top\}$, for every $w \in W, \sigma(w) = N \cap \mathbf{st}(w)$ is just $\{w\}$, for every $w \in W$. This implies that the domain of $f^* \circ \sigma$ is isomorphic to W, as it is just $\{\{w\} : w \in W\}$. This in turn implies that the operation performed by $f^* \circ \sigma$ is essentially that performed by f.

As a consequence, we obtain that:

1. any condition on R can be derived from a corresponding condition on $f^* \circ \sigma$, when moving from normal state frames to Kripke frames;
2. any condition on $f^* \circ \sigma$ can be derived from a corresponding condition on $f = R$, when moving from Kripke frames to normal state frames.

Hence, we get that there is a precise correspondence between classes of Kripke frames defined in terms of a condition on R and classes of normal state frames defined in terms of the same condition on $f^* \circ \sigma$. And this proves the following.

Theorem 13 *Every standard modal logic which is characterized by a class of Kripke frames defined by a condition on R is also characterized by a corresponding class of normal state frames.*

6 Conclusion

Let us conclude by giving a few comments on the framework we introduced and compare it with some alternative approaches providing a truthmaker semantics for modal logic.

6.1 Comments on the Present Account

The framework introduced in this work rests on three assumptions.

1. norms are grounds;
2. norms are essential;
3. norms are universal.

We favor this norm-based approach because, by linking norms and modal truths, it offers a distinctive understanding of the source of what is possible and necessary. Still, those who find the notion of a norm in need of further clarification may simply take N as a set of basic modal facts.[10] This means that the framework does not require the existence of norms as primitive entities but only presumes some basic, essential, and universal elements grounding modal truths.

As to the first two assumptions, whether formulated in terms of norms or modal facts, they constitute the building blocks of our framework, and therefore we will adhere to them. In contrast, the possibility of dropping Assumption 3, that is, the idea that norms affect worlds in their entirety, opens an interesting line of inquiry for future work, by allowing for the content of a norm to be a set of states or a set of propositions instead of a set of worlds.[11] Nevertheless, we maintain that universality should be accepted: we don’t think that the compliance of a world with a norm can be defined based on the compliance of its parts, as worlds are not sets of states but states themselves. Thus, the idea that norms select compliant worlds strikes us as a primitive fact. In addition, under the natural assumptions that (i) every part of what is compliant with a norm is also compliant with that norm and (ii) everything that is compliant with a norm is extended by at least a world which is also compliant with that norm, selecting worlds turns out to coincide with selecting states.

6.2 Comparison and Future Directions

There are a variety of works concerning the extension of truthmaker semantics with different modal operators. While our aim was to capture the intuitions underlying

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[10] See Shalkowski’s chapter in Bueno and Shalkowski [1] for a general introduction to modalism, i.e. the idea that there are basic modal fact that grounds the modal status of sentences.

[11] This is a line that fits at best with the spirit of truthmaker semantics and is well represented in Kim [15], where modal states are positively or negatively related to sets of states, and Litland [18], where modal states are related to sets of propositions, thus obtaining an even more flexible framework.

Kripke's semantics in its generality, truthmaker semantics have been mostly applied to address hyperintensional contexts and model a number of modal operators.[12]

The application of this kind of semantics to necessity and possibility has been explored in Fine [11], Plebani et al. [22], Litland [18], which however differ considerably from the present approach. Fine [11] focuses on the construction of impossible states starting from the fusion of possible states. Plebani et al. [22] give a truthmaker account where the notion of compatibility is primitive and is used to define possible states and a unilateral relation of verification. Finally, Litland [18] shows how to develop Fine's truthmaker semantics for intuitionistic logic to an exact truthmaker semantics for intuitionistic modal logic.

Norms and codes of conduct play an important role in the deontic logic in Fine [10], where deontic statements of obligation and permissions are true and false relative to a code of conduct. However, the notion of code is interpreted as a set of actions and modeled as a set of states associated to a model itself. In fact, the semantics of deontic operators is not exact but model-dependent. In contrast, in our framework, a code of conduct can be viewed as a set of norms that are in force in a world, so that norms characterize possible worlds, by being part of them, and ground what is permissible or obligatory through their content.

The approach to modalities that is more close to ours is proposed in Kim [15], where an intriguing bilateral semantics for normal modal logic is developed. This framework is, in a sense, complementary to our own. From a logical point of view, our account is more comprehensive, allowing for the interpretation of every modal system characterized semantically by a class of Kripke frames. In contrast, Kim's semantics, while effective in modeling a specific subset of all normal modal logics, is less general. However, it offers greater flexibility, since the concept of exact truthmaking does not rely on the minimality of a truthmaker. This trade-off between generality and flexibility opens a first important line of research, to be pursued in future works.[13]

In this paper, we did not comment on the differences between our approaches and Fine's, as we consider this question to lie outside the scope of our project, which is to provide a truthmaker semantics for Kripkean modal logics based on the idea that norms ground modalities. The aim of the project was precisely to interpret intensional systems of modal logic. Therefore, we did not use all the resources available to a truthmaker semanticist to make more fine-grained distinctions but limited ourselves to taking a natural step toward capturing Kripke's semantics in a setting that is compatible with truthmaker semantics—an outcome interesting both in itself and as a foundation for developing a more general, hyperintensional framework. In particular, since the intensionality of the account depends on two choices—namely, modeling the content of a norm as a set of possible worlds and modeling necessity as strict entailment—the framework can be refined in at least two directions: first, by relaxing the universality

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[12] See, for instance, Fine [9], for a semantics of imperatives; Hawke and Özgün [14], Krämer [16], Saitta [23], for an interpretation of epistemic notions; Fine [10], where a logic of obligations and permissions is developed.

[13] In this respect, a natural step is to follow the approach found in Litland [18] and Kim [15], introducing modal states that permit or exclude the possibility of other states, or that are compatible or incompatible with them. Indeed, referring to states rather than worlds will enable us to make a smooth transition to a hyperintensional framework.

constraint, thus admitting as elements of the norm's content not only possible worlds but also states; second, by changing the truth conditions of a necessary sentence, for instance, interpreting what it means for a norm to *necessitate* a sentence in terms of analytic entailment rather than strict entailment. This would allow for different degrees of granularity with respect to hyperintensionality within a uniform framework, and developing these lines of refinement is the expected focus of our further research.

Finally, from a formal perspective, the use of modal state spaces has proven to be insightful for interpreting minimal modal logic. This interpretation is based on a precise relationship between the notions of neighborhood and code, which we are currently in the process of developing.

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