Symmetric ground states for doubly nonlocal equations with mass constraint

Silvia Cingolani
Dipartimento di Matematica
Università degli Studi di Bari Aldo Moro
Via E. Orabona 4, 70125 Bari, Italy
silvia.cingolani@uniba.it

Marco Gallo
Dipartimento di Matematica
Università degli Studi di Bari Aldo Moro
Via E. Orabona 4, 70125 Bari, Italy
marco.gallo@uniba.it

Kazunaga Tanaka
Department of Mathematics
School of Science and Engineering
Waseda University
3-4-1 Ohkubo, Shijuku-ku, Tokyo 169-8555, Japan
kazunaga@waseda.jp

Abstract

We prove the existence of a spherically symmetric solution for a Schrödinger equation with a nonlocal nonlinearity of Choquard type, i.e.

$$(-\Delta)^s u + \mu u = (I_\alpha * F(u))f(u) \quad \text{in } \mathbb{R}^N,$$

where $N \geq 2$, $s \in (0,1)$, $\alpha \in (0,N)$, $I_\alpha(x) = \frac{A_{N,\alpha}}{|x|^{N-\alpha}}$ is the Riesz potential, $\mu > 0$ is part of the unknowns, and $F \in C^1(\mathbb{R},\mathbb{R})$, $F' = f$ is assumed to be subcritical and to satisfy almost optimal assumptions. The mass of of the solution, described by its norm in the L^2-space, is prescribed in advance by $\int_{\mathbb{R}^N} u^2\,dx = c$ for some $c > 0$. The approach to this constrained problem relies on a Lagrange formulation and new deformation arguments. In addition, we prove that the obtained solution is also a ground state, which means that it realizes minimal energy among all the possible solutions to the problem.

Keywords: Double nonlocality, Choquard nonlinearity, Hartree term, Fractional Laplacian, Nonlinear Schrödinger equation, Normalized solutions, Symmetric solutions, Lagrange formulation, Pohozaev identity
AMS Subject Classification: 35B38, 35J20, 35Q40, 35Q55, 35R09, 35R11, 58E05

Contents

1 Introduction

In 1954, the nonlocal equation

$$-\Delta u+\mu u=\left(\frac{1}{4\pi|x|}*|u|^2\right)u \quad \text{in } \mathbb{R}^3 \tag{1}$$

was introduced by Pekar [63] in the framework of quantum mechanics, and in 1976 it arose in the work of Choquard on the modeling of an electron trapped in its own hole, in a certain approximation to the Hartree–Fock theory of one-component plasma [45] (see also [32, 29, 30]). Later, (1) was proposed by Penrose [64, 65, 66, 57] for modeling the self-gravitational collapse of a quantum mechanical wave-function (see also [69]). The first investigations of the existence and symmetry of the solutions for Equation (1) date back to [45]. Due to its physical relevance, the existence of an infinite number of standing wave solutions to (1) with prescribed L^2-norm was faced by Lions in [49].

Variational methods were also employed to derive the existence results of standing wave solutions for the nonlinear Choquard equation without prescribed mass [55, 14, 15, 19, 43, 58, 59, 60]; the existence of L^2-normalized solutions was also investigated when $F(t)=|t|^p$ in [72] and in [4, 18, 17] for generalized Choquard nonlinearities.

In this paper, we study the existence of solutions to the nonlocal problem

$$\begin{cases} (-\Delta)^s u+\mu u=(I_\alpha*F(u))f(u) \quad \text{in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} u^2\,dx=c, \\ (\mu,u)\in(0,+\infty)\times H^s_r(\mathbb{R}^N), \end{cases} \tag{2}$$

where $s\in(0,1)$, $N\geq 2$, $\alpha\in(0,N)$, $F\in C^1(\mathbb{R},\mathbb{R})$ with $f=F'$, $c>0$, and μ is a Lagrange multiplier, part of the unknowns. Here, $I_\alpha:\mathbb{R}^N\setminus\{0\}\to\mathbb{R}$ is the Riesz potential defined by

$$I_\alpha(x)=A_{N,\alpha}\frac{1}{|x|^{N-\alpha}},$$

with $A_{N,\alpha}=\frac{\Gamma(\frac{N-\alpha}{2})}{2^\alpha\pi^{N/2}\Gamma(\frac{\alpha}{2})}$, and the symbol

$$(-\Delta)^s u(x)=C_{N,s}\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy$$

denotes the fractional power of the Laplace operator, where $C_{N,s} = \frac{4^s \Gamma(\frac{N+2s}{2})}{\pi^{N/2}|\Gamma(-s)|}$ and the integral is meant in the principal value sense. Finally, we indicate by

$$H^s_r(\mathbb{R}^N) = \{u \in H^s(\mathbb{R}^N);\, u(x) = u(|x|)\}$$

the subspace composed of radially symmetric functions of the fractional Sobolev space

$$H^s(\mathbb{R}^N) = \left\{u \in L^2(\mathbb{R}^N);\, \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2\, dx < \infty \right\}.$$

When $s \in (0,1)$, the equation in (2) is a fractional PDE, as it involves derivatives and integrals of fractional order. The fractional Laplacian operator was introduced by Laskin [42] as an extension of the classical one ($s = 1$) in the study of NLS equations, replacing the path integral over Brownian motions with Lévy flights. This operator arises naturally in many contexts and concrete applications in various fields, such as optimization, finance, crystal dislocations, charge transport in biopolymers, flame propagation, minimal surfaces, water waves, geo-hydrology, anomalous diffusion, neural systems, phase transition and Bose–Einstein condensation (see [41, 7, 28, 22, 40, 51] and references therein). We refer to [56, 70] for a discussion of recent developments in the description of anomalous diffusion via fractional dynamics and to [3, 2] for some recent applications of fractional operators to different frameworks (analysis of the amount of bromsulphthalein in the human liver, study of thermostat systems and others). Mathematically, equations involving the fractional Laplacian, together with local nonlinearities, have been largely investigated, and some fundamental contributions can be found in [10, 9, 27]. In particular, the existence and qualitative properties of the solutions for more general classes of fractional NLS equations with local source were studied in [24, 11, 8, 36]. The mass-constrained case was, instead, recently considered in [54, 16].

In the case of Hartree type nonlinearities one of the most relevant applications arises in relativistic physics, when the nonlinearity describes the short time interactions between particles. The minimization related to the problem (2) plays a fundamental role in the mathematical description of the gravitational collapse of boson stars [48, 28]. Other applications can be found in quantum chemistry [1, 20] (see also [13] for some orbital stability results) and in the study of graphene [52].

In this work, we consider the problem (2), which presents some nonlocal characteristics in the source, as well as in the fractional diffusion.

In particular we assume

(f1) $f \in C(\mathbb{R}, \mathbb{R})$;

(f2) there exists $C > 0$ such that for every $t \in \mathbb{R}$,

$$|tf(t)| \le C(|t|^{\frac{N+\alpha}{N}} + |t|^{\frac{N+\alpha+2s}{N}});$$

(f3) $F(t) = \int_0^t f(\tau)\, d\tau$ satisfies

$$\lim_{t\to 0} \frac{F(t)}{|t|^{\frac{N+\alpha}{N}}} = 0, \quad \lim_{t\to +\infty} \frac{F(t)}{|t|^{\frac{N+\alpha+2s}{N}}} = 0;$$

(f4) there exists $t_0 \in \mathbb{R}$, $t_0 \neq 0$ such that $F(t_0) \neq 0$.

We remark that the exponent $\frac{N+\alpha+2s}{N}$ appears as an L^2-critical exponent for the fractional Choquard equations and the conditions (f1)–(f4) correspond to L^2-subcritical growths.

The unconstrained case was studied by [21] for a power nonlinearity and by [6] in the case of combined local and nonlocal power-type nonlinearities; see also [68, 53, 31].

For the general class of nonlinearities of the Berestycki–Lions type [5, 59], satisfying (f1)–(f4), we introduce a Lagrangian formulation in order to obtain L^2-normalized solutions of the nonlocal problem (2) in the spirit of [16], where it is applied for fractional NLS equations with a local source (see also [35]). Namely, set $\mathbb{R}_+ = (0, +\infty)$, a radially symmetric solution $(\mu, u) \in \mathbb{R}_+ \times H^s_r(\mathbb{R}^N)$ of (2) corresponds to a critical point of the functional $\mathcal{T}^c : \mathbb{R}_+ \times H^s_r(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\mathcal{T}^c(\mu, u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u) \, dx + \frac{\mu}{2} \big(\|u\|_2^2 - c \big). \tag{3}$$

Using a new variant of the Palais–Smale condition [35, 37], which takes into account the Pohozaev identity, we will prove a deformation theorem which enables us to detect mini–max structures in the product space $\mathbb{R}_+ \times H^s_r(\mathbb{R}^N)$, by means of a *Pohozaev mountain*. As stressed in [16], our deformation arguments show that solutions without Pohozaev identity are suitably deformable, and thus they *do not influence the topology* of the sublevels of the functional. This information could be relevant in a fractional framework since it is not known if the Pohozaev identity holds for general continuous f and general values of $s \in (0, 1)$.

We state our main results.

Theorem 1.1 *Suppose $N \geq 2$, and* (f1)–(f4)*. Then there exists $c_0 \geq 0$ such that, for any $c > c_0$, the problem* (2) *has a radially symmetric solution.*

Theorem 1.2 *Suppose $N \geq 2$, and* (f1)–(f4) *together with an L^2-subcritical growth at zero, i.e.,*

(f5) $\lim_{t \to 0} \frac{F(t)}{|t|^{\frac{N+\alpha+2s}{N}}} = +\infty.$

Then, for any $c > 0$, the problem (2) *has a radially symmetric solution.*

We naively notice that (f5) automatically implies (f4). We remark that, as in the local unconstrained case [39], the Mountain Pass solutions obtained in the above theorems are ground state solutions, that is, they have the least energy among all solutions; see Section 6 for details. This fact gives a strong indication of the stability properties of the found solution [25, 13].

Remark 1.3 *We highlight that we assume* a priori *the positivity of the Lagrange multiplier μ in* (2)*. As a matter of fact, this condition seems to be quite natural: indeed, if u is a ground state on the sphere $\int_{\mathbb{R}^N} u^2 \, dx = c$, and its energy is negative, then* a posteriori *the corresponding Lagrange multiplier μ is strictly positive (see Proposition 6.1). In addition, from a physical perspective, in the study of standing waves the multiplier μ describes the frequency of the particle, and thus it is positive; moreover, this prescribed sign is characteristic also of chemical potentials in the description of ideal gases, see [47, 67].*

The paper is organized as follows. In Section 2, we introduce the functionals and the main features on the fractional setting and the nonlocal nonlinearity, while in Section 3, we introduce the Pohozaev mountain and gain some important asymptotic results on the Mountain Pass level. Then, in Section 4, a weaker version of the Palais–Smale condition, modeled on the Pohozaev identity, is obtained together with a deformation theorem. Section 5 is devoted to the proofs of Theorems 1.1 and 1.2. Finally, in Section 6, we show the equivalence of our Mountain Pass approach with the minimization approach.

2 Functional Setting

Let $N > 2s$. In what follows, we use the notation:

$$B(x_0, R) = \{x \in \mathbb{R}^N;\ |x - x_0| < R\},$$
$$\|u\|_r = \left(\int_{\mathbb{R}^N} |u|^r\, dx\right)^{1/r} \qquad \text{for } r \in [1, \infty).$$

We recall the following generalized Hardy–Littlewood–Sobolev inequality [46].

Proposition 2.1 *Let $\alpha \in (0, N)$, and let $r,\ h \in (1, +\infty)$ be such that $\frac{1}{r} - \frac{1}{h} = \frac{\alpha}{N}$. Then the map*

$$L^r(\mathbb{R}^N) \to L^h(\mathbb{R}^N);\ f \mapsto I_\alpha * f$$

is continuous. In particular, if $r,\ t \in (1, +\infty)$ verify $\frac{1}{r} + \frac{1}{t} = \frac{N+\alpha}{N}$, then there exists a constant $C = C(N, \alpha, r, t) > 0$ such that

$$\left|\int_{\mathbb{R}^N} (I_\alpha * g) h\, dx\right| \le C\|g\|_r \|h\|_t$$

for all $g \in L^r(\mathbb{R}^N)$ and $h \in L^t(\mathbb{R}^N)$.

We endow the space $H^s(\mathbb{R}^N)$ and its subspace $H^s_r(\mathbb{R}^N)$ of radially symmetric functions with the norm

$$\|u\|^2_{H^s_r} = \int_{\mathbb{R}^N} \left(|(-\Delta)^{s/2} u|^2 + u^2\right) dx.$$

Recall the fractional Sobolev critical exponent

$$2^*_s = \frac{2N}{N - 2s}.$$

In [50], Lions proved that

$$H^s_r(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^q(\mathbb{R}^N)$$

whenever $N \ge 2$ and $q \in (2, 2^*_s)$. However, as shown in [12] for general $0 < s \le \frac{1}{2}$, a result in the spirit of Radial Lemma by Strauss is not available in a fractional framework.

Moreover, we recall the fractional version of the Gagliardo–Nirenberg inequality [62]

$$\|u\|_r \le C\|(-\Delta)^{s/2} u\|_2^\beta\ \|u\|_2^{1-\beta} \tag{4}$$

for $u \in H^s(\mathbb{R}^N)$, $r \in [2, 2^*_s]$ and β satisfying

$$\frac{1}{r} = \frac{\beta}{2^*_s} + \frac{1 - \beta}{2}.$$

In what follows, we will often denote, by q, the lower Hardy–Littlewood–Sobolev critical exponent and, by p, the L^2-critical exponent appearing in (f2)–(f3), i.e.,

$$q = \frac{N + \alpha}{N}, \quad p = \frac{N + \alpha + 2s}{N}.$$

Remark 2.2 *We observe that, defining the Riesz potential by $x \mapsto \frac{A_{N,\beta}}{|x|^\beta}$, as some authors do, we have that the critical exponents $\frac{N+\alpha}{N} < \frac{N+\alpha+2s}{N} < \frac{N+\alpha}{N-2s}$ become respectively $\frac{2N-\beta}{N} < \frac{2N-\beta+2s}{N} < \frac{2N-\beta}{N-2s}$.*

We consider the functional $\mathcal{T}^c : \mathbb{R}_+ \times H^s_r(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\mathcal{T}^c(\mu, u) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2\,dx - \frac{1}{2}\mathcal{D}(u) + \frac{\mu}{2}\big(\|u\|_2^2 - c\big),$$

where we set

$$\mathcal{D}(u) = \int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)\,dx.$$

Using Proposition 2.1, it is easy to see that $\mathcal{T}^c \in C^1\big(\mathbb{R}_+ \times H^s_r(\mathbb{R}^N), \mathbb{R}\big)$. We notice that $(\mu, u) \in \mathbb{R}_+ \times H^s_r(\mathbb{R}^N)$ solves problem (2) if, and only if, $\partial_u \mathcal{T}^c(\mu, u) = 0$ and $\partial_\mu \mathcal{T}^c(\mu, u) = 0$.

Moreover we define the functional $\mathcal{J} : \mathbb{R}_+ \times H^s_r(\mathbb{R}^N) \to \mathbb{R}$ by setting

$$\mathcal{J}(\mu, u) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2\,dx - \frac{1}{2}\mathcal{D}(u) + \frac{\mu}{2}\|u\|_2^2. \tag{5}$$

For a fixed $\mu > 0$, u is the critical point of $\mathcal{J}(\mu, \cdot)$ means that u solves

$$\begin{cases} (-\Delta)^s u + \mu u = (I_\alpha * F(u))f(u) & \text{in } \mathbb{R}^N, \\ \qquad u \in H^s_r(\mathbb{R}^N), \end{cases} \tag{6}$$

in the weak sense. It is immediate that

$$\mathcal{T}^c(\mu, u) = \mathcal{J}(\mu, u) - \frac{\mu}{2}c.$$

Observing that

$$H^s_r(\mathbb{R}^N) = \mathrm{Fix}(\mathrm{O}(N)) = \{u \in H^s(\mathbb{R}^N) \mid \tau(Q, u) = u \ \text{ for each } Q \in \mathrm{O}(N)\},$$

where $\mathrm{O}(N)$ is the orthogonal group of rotation matrices and the isometric action is given by

$$\tau : (Q, \mu, u) \mapsto (\mu, u(Q\cdot));\ \mathrm{O}(N) \times (\mathbb{R}_+ \times H^s(\mathbb{R}^N)) \to \mathbb{R}_+ \times H^s(\mathbb{R}^N),$$

and observed that $\mathcal{T}^c$, as well as $\mathcal{J}$, is $\mathrm{O}(N)$-invariant, we have by the Principle of Symmetric Criticality of Palais [61] (see also [71]) that every critical point of $\mathcal{T}^c$ (resp. $\mathcal{J}$) restricted to $\mathbb{R}_+ \times H^s_r(\mathbb{R}^N)$ is actually a critical point of $\mathcal{T}^c$ (resp. $\mathcal{J}$) on the whole $\mathbb{R}_+ \times H^s(\mathbb{R}^N)$. This observation justifies our restriction onto the radial setting.

Finally, we recall that C^2-solutions to (6) satisfy the Pohozaev identity (see [68, Proposition 2] and [21, Equation (6.1)])

$$\frac{N-2s}{2}\|(-\Delta)^{s/2}u\|_2^2 + \frac{N}{2}\mu\|u\|_2^2 = \frac{N+\alpha}{2}\,\mathcal{D}(u). \tag{7}$$

Inspired by this identity, we also introduce the Pohozaev functional $\mathcal{P} : \mathbb{R}_+ \times H^s_r(\mathbb{R}^N) \to \mathbb{R}$ by setting

$$\mathcal{P}(\mu, u) = \frac{N-2s}{2}\|(-\Delta)^{s/2}u\|_2^2 - \frac{N+\alpha}{2}\mathcal{D}(u) + \frac{N}{2}\mu\|u\|_2^2 \tag{8}$$

which will be used to model both the Palais–Smale condition and the geometry of the problem. We also note that

$$\mathcal{J}(\mu, u(\cdot/t)) = \frac{1}{2}t^{N-2s}\|(-\Delta)^{s/2}u\|_2^2 - \frac{1}{2}t^{N+\alpha}\mathcal{D}(u) + \frac{1}{2}\mu t^N\|u\|_2^2$$

and $\frac{d}{dt}\big|_{t=1}\mathcal{J}(\mu, u(\cdot/t)) = \mathcal{P}(\mu, u)$.

3 Asymptotic Geometry

For a fixed $\mu > 0$ we introduce the set of paths

$$\Gamma_\mu = \big\{\gamma \in C\big([0,1], H^s_r(\mathbb{R}^N)\big);\ \gamma(0) = 0,\ \mathcal{J}(\mu, \gamma(1)) < 0\big\}$$

and the Mountain Pass (MP for short) value

$$l(\mu) = \inf_{\gamma \in \Gamma_\mu} \max_{t \in [0,1]} \mathcal{J}(\mu, \gamma(t)). \tag{9}$$

Moreover we set

$$\Sigma = \big\{(\mu, u) \in \mathbb{R}_+ \times H^s_r(\mathbb{R}^N);\ \mathcal{P}(\mu, u) > 0\big\} \cup \big\{(\mu, 0);\ \mu > 0\big\},$$

and observe

$$\big\{(\mu, 0);\ \mu > 0\big\} \subset int(\Sigma), \tag{10}$$

so that the *Pohozaev mountain* is given by

$$\partial\Sigma = \big\{(\mu, u) \in \mathbb{R}_+ \times H^s_r(\mathbb{R}^N);\ \mathcal{P}(\mu, u) = 0,\ u \neq 0\big\}.$$

Here, the boundary of Σ is made with respect to the topology relative to the set $\mathbb{R}_+ \times H^s_r(\mathbb{R}^N)$.
We prove the following proposition.

Proposition 3.1 *Assume* (f1)–(f4). *Then* $\Gamma_\mu \neq \emptyset$ *and* $\partial\Sigma \neq \emptyset$. *Moreover*

(i) $\mathcal{J}(\mu, u) \geq 0$ *for all* $(\mu, u) \in \Sigma$.

(ii) $\mathcal{J}(\mu, u) \geq l(\mu) > 0$ *for all* $(\mu, u) \in \partial\Sigma$.

Proof. By exploiting (f4) and arguing as in ([59,], Proposition 2.1), we obtain the existence of a function $u \in H^s_r(\mathbb{R}^N)$ such that $\mathcal{D}(u) > 0$. Thus defined $\gamma(t) = u(\cdot/t)$ for $t > 0$ and $\gamma(0) = 0$ we have

- $\mathcal{J}(\mu, \gamma(t)) < 0$ for t large and $\mathcal{J}(\mu, \gamma(t)) > 0$ for t small;
- $\mathcal{P}(\mu, \gamma(t)) < 0$ for t large and $\mathcal{P}(\mu, \gamma(t)) > 0$ for t small.

The first claim ensures, after a suitable rescaling, that $\gamma \in \Gamma_\mu$, and, in particular, $l(\mu)$ is well defined. The second claim instead ensures, by the intermediate value theorem, that there exists a t^*, such that $\mathcal{P}(\mu, \gamma(t^*)) = 0$, and thus $(\mu, \gamma(t^*)) \in \partial\Sigma$.
(i) We notice that for all $(\mu, u) \in \Sigma$

$$\mathcal{J}(\mu, u) \geq \mathcal{J}(\mu, u) - \frac{\mathcal{P}(\mu, u)}{N + \alpha} = \frac{\alpha + 2s}{2(N + \alpha)} \|(-\Delta)^{s/2} u\|_2^2 + \frac{\alpha\mu}{2(N + \alpha)} \|u\|_2^2$$

and thus (i) follows.
(ii) Let $(\mu, u) \in \partial\Sigma$, and observe that $\mathcal{D}(u) > 0$. We define again $\gamma(t) = u(\cdot/t)$ so that $t \in (0, +\infty) \mapsto \mathcal{J}(\mu, \gamma(t))$ is negative for large values of t, and it attains the maximum in $t = 1$. After a suitable rescaling, we have $\gamma \in \Gamma_\mu$ and thus

$$\mathcal{J}(\mu, u) = \max_{t \in [0,1]} \mathcal{J}(\mu, \gamma(t)) \geq l(\mu) \tag{11}$$

which is the claim.

To show that $l(\mu) > 0$ we argue in this way. Let $\gamma \in \Gamma_\mu$; by definition of Γ_μ and by (i) there exists t^* such that $\gamma(t^*) \in \partial\Sigma$ and $\gamma(t^*) \neq 0$, thus $\mathcal{P}(\mu, \gamma(t^*)) = 0$. This means that

$$\mathcal{J}(\mu, \gamma(t^*)) = \frac{\alpha + 2s}{2(N+\alpha)} \|(-\Delta)^{s/2}\gamma(t^*)\|_2^2 + \frac{\alpha\mu}{2(N+\alpha)} \|\gamma(t^*)\|_2^2 \simeq \|\gamma(t^*)\|_{H^s}$$

thus

$$l(\mu) \gtrsim \inf_{u \in \partial\Sigma} \|u\|_{H^s}.$$

Since, by (10), $\partial\Sigma$ is far from the line $(\mu, 0)$, we obtain that the right-hand side is strictly positive. ∎

To see that $\mathcal{T}^c(\mu, u) = \mathcal{J}(\mu, u) - \frac{\mu}{2}c$ has a MP geometry in $\mathbb{R}_+ \times H_r^s(\mathbb{R}^N)$, it is crucial to analyze the behavior of $l(\mu)$ as $\mu \to +\infty$.

Lemma 3.2 *Assume* (f1)–(f4). *Then,*

$$\lim_{\mu \to +\infty} \frac{l(\mu)}{\mu} = +\infty.$$

Proof. We recall $p = \frac{N+\alpha+2s}{N}$ and $q = \frac{N+\alpha}{N}$. By (f3), for any $\delta > 0$, there exists $C_\delta > 0$ such that

$$|F(t)| \leq \delta |t|^p + C_\delta |t|^q \quad \text{for all } t \in \mathbb{R}.$$

For $v \in H_r^s(\mathbb{R}^N)$, setting $u_t = t^{N/2} v(t\cdot)$, we have

$$\begin{aligned}
\mathcal{D}(u_t) &= \mathcal{D}(t^{N/2} v(tx)) = t^{-N-\alpha} \mathcal{D}(t^{N/2} v(x)) \\
&\leq t^{-N-\alpha} \int_{\mathbb{R}^N} (I_\alpha * (\delta t^{\frac{N}{2}p} |v|^p + C_\delta t^{\frac{N}{2}q} |v|^q))(\delta t^{\frac{N}{2}p} |v|^p + C_\delta t^{\frac{N}{2}q} |v|^q)\, dx \\
&= t^{2s} \int_{\mathbb{R}^N} (I_\alpha * (\delta |v|^p + C_\delta t^{-s} |v|^q))(\delta |v|^p + C_\delta t^{-s} |v|^q)\, dx \\
&\equiv t^{2s} D_{\delta, C_\delta t^{-s}}(v).
\end{aligned} \tag{12}$$

Here, we write for $\delta > 0$ and $A \geq 0$,

$$\begin{aligned}
D_{\delta,A}(v) &= \int_{\mathbb{R}^N} (I_\alpha * (\delta |v|^p + A|v|^q))(\delta |v|^p + A|v|^q)\, dx, \\
\mathcal{J}_{\delta,A}(v) &= \frac{1}{2} \|(-\Delta)^{s/2} v\|_2^2 + \frac{1}{2}\|v\|_2^2 - \frac{1}{2} D_{\delta,A}(v).
\end{aligned}$$

We also denote by $b(\delta, A)$ the MP value of $\mathcal{J}_{\delta,A}$. Taking the continuity and monotonicity property of $b(\delta, A)$ into account, with respect to each variable δ and A, and noting that $\mathcal{J}_{\delta,A}$ satisfies the Palais–Smale condition, we have

$$b(\delta, A) \to b(\delta, 0) \quad \text{as } A \to 0^+,$$

$$b(\delta, 0) \to +\infty \quad \text{as } \delta \to 0^+.$$

Thus, we have from (12) that

$$\mathcal{J}(\mu, u_t) \geq t^{2s} \left(\frac{1}{2} \|(-\Delta)^{s/2} v\|_2^2 + \frac{1}{2} \mu t^{-2s} \|v\|_2^2 - \frac{1}{2} D_{\delta, C_\delta t^{-s}}(v) \right).$$

Setting $t = \mu^{1/2s}$, we have

$$\mathcal{J}(\mu, u_{\mu^{1/2s}}) \geq \mu \mathcal{J}_{\delta, C_\delta \mu^{-1/2}}(v)$$

and thus

$$\frac{l(\mu)}{\mu} \geq b(\delta, C_\delta \mu^{-1/2}).$$

Therefore, we gain

$$\liminf_{\mu \to +\infty} \frac{l(\mu)}{\mu} \geq \lim_{A \to 0^+} b(\delta, A) = b(\delta, 0).$$

Since $\delta > 0$ is arbitrary, we obtain the claim for $\delta \to 0^+$. ■

Corollary 3.3 *We have*

$$B_c \equiv \inf_{(\mu,u)\in\partial\Sigma} \mathcal{T}^c(\mu, u) > -\infty.$$

Proof. By (ii) of Proposition 3.1 and Lemma 3.2, we have

$$\inf_{(\mu,u)\in\partial\Sigma} \mathcal{T}^c(\mu, u) \geq \inf_{\mu>0} \left(l(\mu) - \frac{\mu}{2}c \right) > -\infty.$$

■

When the nonlinearity is L^2-subcritical in the origin, we are able to also prove the following behavior of $l(\mu)$ as $\mu \to 0^+$.

Proposition 3.4 *Assume* (f5) *in addition to* (f1)–(f4). *Then,*

$$\lim_{\mu \to 0^+} \frac{l(\mu)}{\mu} = 0. \tag{13}$$

Proof. We fix $u \in H^s_r(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\|u\|_\infty = 1$. We note that there exists $L_\sigma > 0$, such that

$$F(\sigma u(x)) \geq \sqrt{L_\sigma}\sigma^p |u(x)|^p \quad \text{for all } \sigma \in (0, 1] \text{ and } x \in \mathbb{R}^N,$$
$$L_\sigma \to \infty \quad \text{as } \sigma \to 0.$$

Recalling $D_{1,0}(u) = \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \, dx$, we have for $t > 0$

$$\begin{aligned}
\mathcal{J}(\mu, \sigma u(x/t)) &\leq \frac{1}{2}\sigma^2 t^{N-2s} \|(-\Delta)^{s/2} u\|_2^2 + \frac{\mu}{2}\sigma^2 t^N \|u\|_2^2 - \frac{1}{2} L_\sigma \sigma^{2p} t^{N+\alpha} D_{1,0}(u) \\
&= \mu^{-\frac{N-2s}{2s}} \left(\frac{1}{2}\sigma^2 \tau^{N-2s} \|\nabla u\|_2^2 + \frac{1}{2}\sigma^2 \tau^N \|u\|_2^2 - \frac{1}{2} L_\sigma \mu^{\frac{N-2s}{2s}} \mu^{-\frac{N+\alpha}{2s}} \sigma^{2p} \tau^{N+\alpha} D_{1,0}(u) \right) \\
&= \mu^{-\frac{N-2s}{2s}} \sigma^2 \left(\frac{1}{2} \|\nabla u\|_2^2 \tau^{N-2s} + \frac{1}{2} \|u\|_2^2 \tau^N - \frac{1}{2} L_\sigma \mu^{-\frac{2+\alpha}{2s}} \sigma^{2p-2} D_{1,0}(u) \tau^{N+\alpha} \right),
\end{aligned}$$

where we set $t = \mu^{-\frac{1}{2s}}\tau$. Moreover setting $\sigma = \mu^{\frac{N}{4s}}$, we have

$$\mathcal{J}(\mu, \mu^{\frac{N}{4s}} u(x/(\mu^{-\frac{1}{2s}}\tau))) \leq \mu \left(\frac{1}{2} \|(-\Delta)^{s/2} u\|_2^2 \tau^{N-2s} + \frac{1}{2} \|u\|_2^2 \tau^N - \frac{1}{2} L_{\mu^{N/4s}} D_{1,0}(u) \tau^{N+\alpha} \right).$$

For $\mu \in (0, 1)$, the map

$$\tau \mapsto \mu^{\frac{N}{4s}} u(x/\mu^{-\frac{1}{2s}}\tau); \; (0, +\infty) \to H^s_r(\mathbb{R}^N)$$

can be regarded as a path in Γ_μ. Thus

$$\frac{l(\mu)}{\mu} \leq \max_{\tau \in (0,+\infty)} \left(\frac{1}{2} \|(-\Delta)^{s/2} u\|_2^2 \tau^{N-2s} + \frac{1}{2} \|u\|_2^2 \tau^N - \frac{1}{2} L_{\mu^{N/4s}} D_{1,0}(u) \tau^{N+\alpha} \right).$$

Since $L_{\mu^{N/4s}} \to +\infty$ as $\mu \to 0^+$, we have

$$R.H.S. \to 0 \quad \text{as } \mu \to 0^+.$$

Thus, we have the conclusion. ■

4 Palais-Smale-Pohozaev Condition

Under the assumptions (f1)–(f4), it seems hard to verify the standard Palais–Smale condition for the functional $\mathcal{T}^c$. As in [16], we introduce a compactness condition which is weaker than the standard Palais–Smale one (see also [35, 37, 34, 38]). Namely, we provide the following definition, recalled that $\mathbb{R}_+ \equiv (0, +\infty)$.

Definition 4.1 *For $b \in \mathbb{R}$, we say that $\mathcal{T}^c$ satisfies the Palais–Smale–Pohozaev condition at level b (in short, the $(PSP)_b$ condition), if for any sequence $(\mu_j, u_j) \subset \mathbb{R}_+ \times H^s_r(\mathbb{R}^N)$, such that*

$$\mathcal{T}^c(\mu_j, u_j) \to b, \tag{14}$$

$$\mu_j \cdot \partial_\mu \mathcal{T}^c(\mu_j, u_j) \to 0, \tag{15}$$

$$\partial_u \mathcal{T}^c(\mu_j, u_j) \to 0 \quad \textit{strongly in } (H^s_r(\mathbb{R}^N))^*, \tag{16}$$

$$\mathcal{P}(\mu_j, u_j) \to 0, \tag{17}$$

it happens that (μ_j, u_j) has a strongly convergent subsequence in $\mathbb{R}_+ \times H^s_r(\mathbb{R}^N)$.

Remark 4.2 *The $(PSP)_b$ condition introduced in Definition 4.1 looks slightly different from the one in [16]. We emphasize that $\mathbb{R}_+ \times H^s_r(\mathbb{R}^N)$ with the standard metric induced by $\mathbb{R} \times H^s_r(\mathbb{R}^N)$ is not complete, and is not suitable for the deformation argument. Since $(\mathbb{R}_+, \frac{1}{x^2}dx^2)$ is complete, it is natural to introduce a related metric on $\mathbb{R}_+ \times H^s_r(\mathbb{R}^N)$, that is, we regard $M = \mathbb{R}_+ \times H^s_r(\mathbb{R}^N)$ as a Riemannian manifold with the metric*

$$((\nu_1, w_1), (\nu_2, w_2))_{T_{(\mu,u)}M} = \frac{1}{\mu^2}\nu_1\nu_2 + (w_1, w_2)_{H^s_r}$$

for (ν_1, w_1), $(\nu_2, w_2) \in T_{(\mu,u)}M$, $(\mu, u) \in M$. It is easy to see that $(M, (\cdot,\cdot)_{TM})$ is a complete Riemannian manifold. We regard $\mathcal{T}^c$ as a functional defined on M. We have

$$\|(\partial_\mu \mathcal{T}^c(\mu, u), \partial_u \mathcal{T}^c(\mu, u))\|^2_{(T_{(\mu,u)}M)^*} = \mu^2 |\partial_\mu \mathcal{T}^c(\mu, u)|^2 + \|\partial_u \mathcal{T}^c(\mu, u)\|^2_{(H^s_r)^*}.$$

Thus (15)–(16) *are equivalent to*

$$\|(\partial_\mu \mathcal{T}^c(\mu, u), \partial_u \mathcal{T}^c(\mu, u))\|_{(T_{(\mu,u)}M)^*} \to 0. \tag{18}$$

Moreover, setting

$$\widetilde{\mathcal{T}}^c(\lambda, u) = \mathcal{T}^c(e^\lambda, u) : \mathbb{R} \times H^s_r(\mathbb{R}^N) \to \mathbb{R},$$

we can observe that $(\lambda_j, u_j) \subset \mathbb{R} \times H^s_r(\mathbb{R}^N)$ satisfies

$$\partial_\lambda \widetilde{\mathcal{T}}^c(\lambda_j, u_j) \to 0,$$
$$\|\partial_u \widetilde{\mathcal{T}}^c(\lambda_j, u_j)\|_{(H^s_r)^*} \to 0$$

if and only if $(\mu_j, u_j) = (e^{\lambda_j}, u_j)$ satisfies (18).

We remark that this compactness condition takes the scaling properties of $\mathcal{T}^c$ into consideration through the Pohozaev functional $\mathcal{P}$. We now show the following crucial result.

Theorem 4.3 *Assume* (f1)–(f3). *Let $b < 0$. Then, $\mathcal{T}^c$ satisfies the $(PSP)_b$ condition on $\mathbb{R}_+ \times H^s_r(\mathbb{R}^N)$.*

Proof. Let $b < 0$ and $(\mu_j, u_j) \subset \mathbb{R} \times H^s_r(\mathbb{R}^N)$ be a sequence satisfying (14)–(17). First we note that, by (15), we have

$$\mu_j\big(\|u_j\|_2^2 - c\big) \to 0. \tag{19}$$

Step 1: $\liminf_{j\to\infty} \mu_j > 0$ and $\|u_j\|_2^2 \to c$.

By (17) and (14), we have

$$\begin{aligned}
o(1) &= \mathcal{P}(\mu_j, u_j) = \frac{N-2s}{2}\|(-\Delta)^{s/2}u_j\|_2^2 + \\
&\quad +(N+\alpha)\Big(\mathcal{T}^c(\mu_j,u_j) - \frac{1}{2}\|(-\Delta)^{s/2}u_j\|_2^2 - \frac{\mu_j}{2}\big(\|u_j\|_2^2 - c\big)\Big) + \frac{N}{2}\mu_j\|u_j\|_2^2 \\
&= -\frac{\alpha+2s}{2}\|(-\Delta)^{s/2}u_j\|_2^2 + (N+\alpha)(b+o(1)) + \frac{N}{2}\mu_j c + o(1),
\end{aligned}$$

Here, we have used (19). Since $b < 0$, we have $\liminf_{j\to\infty}\mu_j > 0$. Thus (19) implies $\|u_j\|_2^2 \to c$.

Step 2: $\|(-\Delta)^{s/2}u_j\|_2^2$ and μ_j are bounded.

Since $\varepsilon_j \equiv \|\partial_u \mathcal{T}^c(\mu_j,u_j)\|_{(H^s_r(\mathbb{R}^N))^*} \to 0$, we have

$$\|(-\Delta)^{s/2}u_j\|_2^2 - \int_{\mathbb{R}^N}(I_\alpha * F(u_j))f(u_j)u_j\,dx + \mu_j\|u_j\|_2^2 \le \varepsilon_j\|u_j\|_{H^s_r}. \tag{20}$$

Note that $\frac{2Np}{N+\alpha} \in (2, 2^*_s)$. Moreover, we observe that, by (f3), for $\delta > 0$ fixed, there exists $C_\delta > 0$ such that

$$|F(t)| \le \delta|t|^p + C_\delta|t|^{\frac{N+\alpha}{N}}, \quad t \in \mathbb{R},$$

where $p = \frac{N+\alpha+2s}{N}$, and thus

$$\|F(u_j)\|_{\frac{2N}{N+\alpha}} \le \delta\||u_j|^p\|_{\frac{2N}{N+\alpha}} + C_\delta\||u_j|^{\frac{N+\alpha}{N}}\|_{\frac{2N}{N+\alpha}} = \delta\|u_j\|^p_{\frac{2Np}{N+\alpha}} + C_\delta\|u_j\|_2^{\frac{N+\alpha}{N}}.$$

Therefore, by (f2) we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}(I_\alpha * |F(u_j)|)|f(u_j)u_j|\,dx \\
&\le C\|F(u_j)\|_{\frac{2N}{N+\alpha}}\|f(u_j)u_j\|_{\frac{2N}{N+\alpha}} \\
&\le C\left(\delta\|u_j\|^p_{\frac{2Np}{N+\alpha}} + C_\delta\|u_j\|_2^{\frac{N+\alpha}{N}}\right)\cdot C'\left(\|u_j\|^p_{\frac{2Np}{N+\alpha}} + \|u_j\|_2^{\frac{N+\alpha}{N}}\right) \\
&= CC'\delta\|u_j\|^{2p}_{\frac{2Np}{N+\alpha}} + CC'(\delta+C_\delta)\|u_j\|^p_{\frac{2Np}{N+\alpha}}\|u_j\|_2^{\frac{N+\alpha}{N}} + CC'C_\delta\|u_j\|_2^{\frac{2(N+\alpha)}{N}} \\
&= CC'\delta\|u_j\|^{2p}_{\frac{2Np}{N+\alpha}} + CC'(\delta+C_\delta)\left(\frac{\delta}{2}\|u_j\|^{2p}_{\frac{2Np}{N+\alpha}} + \frac{1}{2\delta}\|u_j\|_2^{\frac{2(N+\alpha)}{N}}\right) + CC'C_\delta\|u_j\|_2^{\frac{2(N+\alpha)}{N}} \\
&\le C''\delta\|u_j\|^{2p}_{\frac{2Np}{N+\alpha}} + C''_\delta\|u_j\|_2^{\frac{2(N+\alpha)}{N}}
\end{aligned}$$

and thus, by Gagliardo–Nirenberg inequality (4), with $r = \frac{2Np}{N+\alpha}$ and $\beta = \frac{1}{p}$, we derive

$$\begin{aligned}
\|(-\Delta)^{s/2}u_j\|_2^2 + \mu_j\|u_j\|_2^2 &\le \int_{\mathbb{R}^N}(I_\alpha * |F(u_j)|)|f(u_j)u_j|\,dx + \varepsilon_j\|u_j\|_{H^s_r} \\
&\le C''\delta\|(-\Delta)^{s/2}u_j\|_2^2\|u_j\|_2^{2(p-1)} + C''_\delta\|u_j\|_2^{\frac{2(N+\alpha)}{N}} + \varepsilon_j\|u_j\|_{H^s_r}.
\end{aligned}$$

Since $\|u_j\|_2^2 = c + o(1)$, we have

$$\begin{aligned}&\big(1 - C''\delta(c+o(1))^{p-1}\big)\|(-\Delta)^{s/2}u_j\|_2^2 + \mu_j\big(c+o(1)\big)\\ &\quad\le\ C''_\delta\big(c+o(1)\big)^{\frac{N+\alpha}{N}} + \varepsilon_j\big(\|(-\Delta)^{s/2}u_j\|_2^2 + c + o(1)\big)^{1/2}.\end{aligned}$$

For a small enough δ, we have a boundedness of $\|(-\Delta)^{s/2}u_j\|_2$ and μ_j.

Step 3: Convergence in $\mathbb{R}_+ \times H_r^s(\mathbb{R}^N)$.

By Steps 1–2, the sequence (μ_j, u_j) is bounded in $\mathbb{R}_+ \times H_r^s(\mathbb{R}^N)$ and thus, after extracting a subsequence denoted in the same way, we may assume that $\mu_j \to \mu_0 > 0$ and $u_j \rightharpoonup u_0$ weakly in $H_r^s(\mathbb{R}^N)$ for some $(\mu_0, u_0) \in \mathbb{R}_+ \times H_r^s(\mathbb{R}^N)$.

Step 4: Conclusion.

Taking into account the assumptions (f1)–(f4), we obtain

$$\int_{\mathbb{R}^N} (I_\alpha * F(u_j))f(u_j)u_0\,dx \to \int_{\mathbb{R}^N} (I_\alpha * F(u_0))f(u_0)u_0\,dx$$

and

$$\int_{\mathbb{R}^N} (I_\alpha * F(u_j))f(u_j)u_j\,dx \to \int_{\mathbb{R}^N} (I_\alpha * F(u_0))f(u_0)u_0\,dx.$$

Thus, we derive that $\langle \partial_u \mathcal{T}^c(\mu_j, u_j), u_j\rangle \to 0$ and $\langle \partial_u \mathcal{T}^c(\mu_j, u_j), u_0\rangle \to 0$, and hence

$$\|(-\Delta)^{s/2}u_j\|_2^2 + \mu_0\|u_j\|_2^2 \to \|(-\Delta)^{s/2}u_0\|_2^2 + \mu_0\|u_0\|_2^2$$

which implies a $u_j \to u_0$ strongly in $H_r^s(\mathbb{R}^N)$. ∎

Remark 4.4 *We emphasize that the $(PSP)_b$ condition does not hold at level $b = 0$; it is sufficient to consider an infinitesimal sequence $(\mu_j, 0)$ with $\mu_j \to 0$.*

Now, we introduce the set of critical points of $\mathcal{T}^c$ satisfying the Pohozaev identity, that is

$$K_b = \big\{(\mu, u) \in \mathbb{R}_+ \times H_r^s(\mathbb{R}^N) \mid \mathcal{T}^c(\mu, u) = b,\ \partial_\mu \mathcal{T}^c(\mu, u) = 0,\ \partial_u \mathcal{T}^c(\mu, u) = 0,\ \mathcal{P}(\mu, u) = 0\big\}.$$

Notice that in the definition of K_b the condition $\mathcal{P}(\mu, u) = 0$ is not trivial, since it is not known if the equality is satisfied by every solution, for general continuous f and $s \in (0,1)$ (see [21, 68] and Proposition 6.1 for some particular cases). In fact, we can not generally recognize that the standard set of critical points at level b is compact, and thus we restrict this to the set K_b of critical points satisfying the Pohozaev identity.

Moreover, for each $b \in \mathbb{R}$, we introduce the following notation for sublevels

$$[\mathcal{T}^c \le b] = \big\{(\mu, u) \in \mathbb{R}_+ \times H_r^s(\mathbb{R}^N) \mid \mathcal{T}^c(\mu, u) \le b\big\}.$$

Following arguments in ([37,], Proposition 4.5) (see also [35,], Proposition 3.1 and Corollary 4.3), we can establish the following deformation theorem for the functional $\mathcal{T}^c$.

Theorem 4.5 *Assume* (f1)–(f3) *and $b < 0$. Then, K_b is compact in $\mathbb{R}_+ \times H_r^s(\mathbb{R}^N)$ and $K_b \cap (\mathbb{R}_+ \times \{0\}) = \emptyset$. Moreover, for any open neighborhood U of K_b and $\bar{\varepsilon} > 0$, there exists an $\varepsilon \in (0, \bar{\varepsilon})$ and a continuous map*

$$\eta(t, \mu, u) : [0,1] \times \mathbb{R}_+ \times H_r^s(\mathbb{R}^N) \to \mathbb{R}_+ \times H_r^s(\mathbb{R}^N)$$

such that

(1^o) $\eta(0,\mu,u)=(\mu,u) \quad \forall(\mu,u)\in\mathbb{R}_+\times H^s_r(\mathbb{R}^N)$;

(2^o) $\eta(t,\mu,u)=(\mu,u) \quad \forall(t,\mu,u)\in[0,1]\times[\mathcal{T}^c\le b-\bar{\varepsilon}]$;

(3^o) $\mathcal{T}^c(\eta(t,\mu,u))\le\mathcal{T}^c(\mu,u) \qquad \forall(t,\mu,u)\in[0,1]\times\mathbb{R}_+\times H^s_r(\mathbb{R}^N)$;

(4^o) $\eta(1,[\mathcal{T}^c\le b+\varepsilon]\setminus U)\subset[\mathcal{T}^c\le b-\varepsilon]$;

(5^o) $\eta(1,[\mathcal{T}^c\le b+\varepsilon])\subset[\mathcal{T}^c\le b-\varepsilon]\cup U$;

(6^o) *if* $K_b=\emptyset$, *we have* $\eta(1,[\mathcal{T}^c\le b+\varepsilon])\subset[\mathcal{T}^c\le b-\varepsilon]$.

5 Minimax Theorem

For any $c>0$, let B_c be the constant defined in Corollary 3.3. As a minimax class for $\mathcal{T}^c$, we define

$$\begin{aligned}\Lambda_c \quad = \quad & \big\{\xi\in C\big([0,1],\mathbb{R}_+\times H^s_r(\mathbb{R}^N)\big);\, \xi(0)\in\mathbb{R}_+\times\{0\},\\ & \mathcal{T}^c(\xi(0))\le B_c-1,\ \xi(1)\notin\Sigma \text{ and } \mathcal{T}^c(\xi(1))\le B_c-1\big\}.\end{aligned}$$

In the following Proposition 5.1 we prove that $\Lambda_c\neq\emptyset$; moreover, by (10) we have $\xi([0,1])\cap\partial\Sigma\neq\emptyset$ for each $\xi\in\Lambda_c$. Therefore from Corollary 3.3, the mini–max value

$$\beta_c=\inf_{\xi\in\Lambda_c}\max_{t\in[0,1]}\mathcal{T}^c(\xi(t)) \tag{21}$$

is well-defined and finite. Since the Palais–Smale–Pohozaev condition holds for $b\in(-\infty,0)$, it is important to estimate β_c. We have the following proposition.

Proposition 5.1 *Assume* (f1)–(f4). *For each* $c>0$ *we have* $\Lambda_c\neq\emptyset$ *and*

$$\beta_c\le l(\mu)-\frac{\mu}{2}c, \quad \textit{for any } \mu>0. \tag{22}$$

As a consequence

(i) for a sufficiently large c, *that is*

$$c>c_0=2\inf_{\mu>0}\frac{l(\mu)}{\mu}$$

we have $\beta_c<0$;

(ii) if (f5) *holds, then* $\beta_c<0$ *for all* $c>0$;

(iii) $\lim_{c\to+\infty}\frac{\beta_c}{c}=-\infty$.

Proof. (i) Let $\mu>0$ and $\gamma\in\Gamma_\mu\neq\emptyset$. By definition we have $\mathcal{D}(\gamma(1))>0$, thus $\mathcal{J}(\mu,\gamma(1)(\cdot/t))\to-\infty$ and $\mathcal{P}(\mu,\gamma(1)(\cdot/t))\to-\infty$ as $t\to+\infty$. Thus for $L\gg0$

$$\mathcal{T}^c(\mu,\gamma(1)(\cdot/L))\le B_c-1, \quad \gamma(1)(\cdot/L)\notin\Omega.$$

We also note that $\mathcal{T}^c(t,0)\to-\infty$ as $t\to+\infty$. Therefore, joining the path γ_μ with the map $t\mapsto(\mu+Lt,0)$ with a large enough $L\gg0$, we can find a path $\xi\in\Lambda_c$ such that

$$\max_{t\in[0,1]}\mathcal{T}^c(\xi(t))\le\max_{t\in[0,1]}\mathcal{J}(\mu,\gamma(t))-\frac{\mu}{2}c,$$

and thus (22). For, $c \gg 0$ we have the claim:

(ii) If (f5) holds, we can apply Proposition 3.4 to obtain

$$\lim_{\mu \to 0} \frac{l(\mu) - \frac{c}{2}\mu}{\mu} = -\frac{c}{2} < 0, \tag{23}$$

and, thus, the claim by (22).

(iii) Finally for any $\mu > 0$ we have, by (22),

$$\limsup_{c \to +\infty} \frac{\beta_c}{c} \leq \lim_{c \to +\infty} \left(\frac{l(\mu)}{c} - \frac{\mu}{2} \right) = -\frac{\mu}{2}.$$

Since μ is arbitrary, we have (iii). ∎

Proof of Theorem 1.1 and Theorem 1.2. Using Theorem 4.5, we derive that the level $\beta_c < 0$, defined in (21), is critical, and thus Theorem 1.1 and Theorem 1.2 hold. ∎

6 L^2-Ground States

We introduce the functional $\mathcal{L} : S_c \to \mathbb{R}$ defined by

$$\mathcal{L}(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u) \, dx \tag{24}$$

on the sphere

$$S_c = \left\{ u \in H^s_r(\mathbb{R}^N);\ \|u\|_2^2 = c \right\}.$$

We consider the ground state level

$$\kappa_c = \inf_{u \in S_c} \mathcal{L}(u).$$

We pass this to recognize that the Mountain Pass solution found in Theorems 1.1 and 1.2 is a ground state solution, namely, a minimizer of $\mathcal{L}$ on the sphere.

Proposition 6.1 *Assume* (f1)–(f4), *and let $c \geq c_0$, where c_0 is introduced in Proposition 5.1. If* (f5) *holds, then take $c_0 = 0$. We have that the following statements hold.*

(i) Every Mountain Pass solution at level β_c is a Pohozaev minimum on the product space, that is

$$\beta_c = B_c$$

where B_c is defined in Corollary 3.3.

(ii) The Mountain Pass level and the ground state level coincide, i.e.,

$$\kappa_c = \beta_c. \tag{25}$$

In particular, thanks to Theorem 1.1, there exists a ground state of $\mathcal{L}_{|S_c}$.

(iii) Every ground state of $\mathcal{L}_{|S_c}$ is a solution of problem (2), *i.e., the associated Lagrange multiplier is positive.*

(iv) Every ground state of $\mathcal{L}_{|S_c}$ satisfies the Pohozaev identity (7) *with μ the associated (positive) Lagrange multiplier. Thanks to* (25), *the same conclusion holds for every Mountain Pass solution at level β_c.*

Proof. (i) From (22) and (ii) of Proposition 3.1 we have

$$\beta_c \le \inf_{\mu>0}\Big(l(\mu) - \frac{\mu}{2}c\Big) \le B_c.$$

On the other hand, each path of Λ_c passes through $\partial\Sigma$ by definition, and thus

$$\beta_c \ge B_c$$

which gives the claim (i).

(ii) Let u_* be the Mountain Pass solution obtained in Theorems 1.1 and 1.2, which verifies $\|u_*\|_2^2 = c$. Thus,

$$\kappa_c \le \mathcal{L}(u_*) = \beta_c < 0. \tag{26}$$

In particular, by (26) we can find a minimizing sequence $(u_n)_n \subset S_c$ for κ_c satisfying $\mathcal{L}(u_n) < 0$, and thus we can set

$$\mu_n = \frac{2}{Nc}\left(\frac{\alpha+2s}{2}\|(-\Delta)^{s/2}u_n\|_2^2 - (N+\alpha)\mathcal{L}(u_n)\right) > 0$$

so that $\mathcal{P}(\mu_n, u_n) = 0$, i.e., $(\mu_n, u_n) \in \partial\Sigma$. At this point Corollary 3.3 and (i) imply

$$\kappa_c + o(1) = \mathcal{L}(u_n) = \mathcal{T}^c(\mu_n, u_n) \ge B_c = \beta_c.$$

Passing to the limit, together with (26), we have (25).

(iii)–(iv) Let u_0 be a minimizer of $\mathcal{L}$ on S_c. Corresponding to u_0, there exists a Lagrange multiplier $\mu_0 \in \mathbb{R}$ such that

$$(-\Delta)^{s/2}u_0 + \mu_0 u_0 = (I_\alpha * F(u_0))f(u_0),$$

and thus, in particular,

$$\|(-\Delta)^{s/2}u_0\|_2^2 + \mu_0\|u_0\|_2^2 = \int_{\mathbb{R}^N}(I_\alpha * F(u_0))f(u_0)u_0\,dx. \tag{27}$$

We show first that u_0 satisfies the Pohozaev identity. In fact, we consider the $\mathbb{R}$-action $\Phi : \mathbb{R}\times S_c \to S_c$ defined by

$$(\Phi_\theta v)(x) = e^{\frac{N}{2}\theta}v(e^\theta x), \tag{28}$$

since $\|\Phi_\theta v\|_2^2 = \|v\|_2^2$. Then we have

$$\mathcal{L}(\Phi_\theta u_0) = \frac{1}{2}e^{2s\theta}\|(-\Delta)^{s/2}u_0\|_2^2 - \frac{1}{2}e^{-(N+\alpha)\theta}\int_{\mathbb{R}^N}(I_\alpha * F(e^{\frac{N}{2}\theta}u_0))F(e^{\frac{N}{2}\theta}u_0)\,dx.$$

Since u_0 is a minimizer, we have $\frac{d}{d\theta}\big|_{\theta=0}\mathcal{L}(\Phi_\theta u_0) = 0$, that is,

$$s\|(-\Delta)^{s/2}u_0\|_2^2 + \frac{N+\alpha}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u_0))F(u_0)\,dx - \frac{N}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u_0))f(u_0)u_0\,dx = 0. \tag{29}$$

From (27) and (29), the Pohozaev identity follows

$$\frac{N-2s}{2}\|(-\Delta)^{s/2}u_0\|_2^2 + \frac{N}{2}\mu_0\|u_0\|_2^2 = \frac{N+\alpha}{2}\,\mathcal{D}(u_0). \tag{30}$$

Finally, from (26) we have $\mathcal{L}(u_0) = \kappa_c < 0$, that is

$$\frac{1}{2}\|(-\Delta)^{s/2}u_0\|_2^2 - \frac{1}{2}\mathcal{D}(u_0) = \kappa_c < 0, \tag{31}$$

which joined to (30) gives $\mu_0 > 0$. This concludes the proof. ∎

Remark 6.2 *We notice that, by using only*

$$0 > \beta_c \geq \kappa_c \geq B_c > -\infty$$

and the $(PSP)_{\kappa_c}$ condition, we can directly obtain the existence of a minimizer $\tilde{u}$ of $\mathcal{L}$ on S_c. We provide only an outline of the proof, and refer to [16] for details. Indeed, consider again the action Φ_θ on S_c defined in (28). *Set*

$$\widehat{\mathcal{L}}(\theta, u) = \mathcal{L}(\Phi_\theta u) : \mathbb{R} \times S_c \to \mathbb{R},$$

we observe

$$\kappa_c = \inf_{(\theta,u)\in\mathbb{R}\times S_c} \widehat{\mathcal{L}}(\theta, u).$$

Since $\kappa_c \in \mathbb{R}$, applying Ekeland Principle to $\widehat{\mathcal{L}} : \mathbb{R} \times S_c \to \mathbb{R}$, we find a sequence $(\theta_j, u_j) \subset \mathbb{R} \times S_c$ such that

$$\widehat{\mathcal{L}}(\theta_j, u_j) \to \kappa_c,\ \partial_\theta \widehat{\mathcal{L}}(\theta_j, u_j) \to 0,\ \partial_u \widehat{\mathcal{L}}(\theta_j, u_j) \to 0.$$

Setting $\hat{u}_j = \Phi_{\theta_j} u_j$, we observe that, for a suitable $\mu_j > 0$, $(\mu_j, \hat{u}_j)$ is a $(PSP)_{\kappa_c}$-sequence for $\mathcal{T}^c$.

Thus, provided that $\kappa_c < 0$, thanks to Theorem 4.3, $\hat{u}_j$ converges up to a subsequence, to a minimizer of $\mathcal{L}$ on S_c, and, therefore, the claim.

7 Conclusions

In this paper, we prove the existence of a radially symmetric solution to the nonlocal problem

$$(-\Delta)^s u + \mu u = (I_\alpha * F(u)) f(u) \quad \text{in } \mathbb{R}^N$$

coupled with the mass constraint $\int_{\mathbb{R}^N} u^2\, dx = c$. The result is obtained for general values of $s \in (0,1)$ and $\alpha \in (0, N)$, and by assuming quite general assumptions on the function f, which are almost optimal and include some particular cases such as pure powers $f(t) \sim t^r$, cooperating powers $f(t) \sim t^r + t^h$, competing powers $f(t) \sim t^r - t^h$ and saturable functions $f(t) \sim \frac{t^3}{1+t^2}$ (which arise, for instance, in nonlinear optics [23]).

The existence is obtained through a Lagrange formulation of the constrained problem, performing a minimax argument on a product space by means of a Pohozaev mountain. The use of the Pohozaev identity in the definition of the Palais–Smale condition allows to overcome the problem of the lack of a Pohozaev identity for general solutions, typical of the fractional framework.

Finally, we show that the Mountain Pass solution assumes the minimal energy among all the possible solutions with the same mass, which points out the physical relevance of the found solution.

When $s = \frac{1}{2}$, $N = 3$, $\alpha = 2$ and $f(t) = |t|^{r-2}t$, we obtain

$$\sqrt{-\Delta} u + \mu u = \left(\frac{1}{|x|} * |u|^r \right) |u|^{r-2} u \quad \text{in } \mathbb{R}^3.$$

In the L^2-critical case $r = 2$, the equation is the well-known *massless boson stars equation* [26, 44, 33]; in this case, the quantity $\int_{\mathbb{R}^3} u^2\, dx = c$ represents the total L^2-mass of the body and plays a fundamental role in the study of the gravitational collapse of boson stars, where a critical value is given by the Chandrasekhar limiting mass. In this paper, we address the subcritical case $r \in (\frac{5}{3}, 2)$, but we believe that this result, together with the developed minimax tools, can be a first step towards the study of the L^2-mass critical case, since, in this case, the minimization approch is not well posed. Moreover, the high generality assumed on the function f could be useful in the study of different physical problems.

Acknowledgments. The first and second authors are supported by PRIN 2017JPCAPN "Qualitative and quantitative aspects of nonlinear PDEs" and by INdAM-GNAMPA. The second author is supported in part by Grant-in-Aid for Scientific Research (19H00644, 18KK0073, 17H02855, 16K13771) of Japan Society for the Promotion of Science.

References

[1] C. Argaez, M. Melgaard, *Solutions to quasi-relativistic multi-configurative Hartree–Fock equations in quantum chemistry*, Nonlinear Anal. **75** (2012), no. 1, 384–404.

[2] D. Baleanu, S. Etemad, S. Rezapour, *A hybrid Caputo fractional modeling for thermostat with hybrid boundary value conditions*, Bound. Value Probl. **2020** (2020), no. 64, pp. 16.

[3] D. Baleanu, A. Jajarmi, H. Mohammadi, S. Rezapour, *A new study on the mathematical modelling of human liver with Caputo–Fabrizio fractional derivative*, Chaos Solitons Fractals **134** (2020), article ID 109705, pp. 7.

[4] T. Bartsch, Y. Liu, Z. Liu, *Normalized solutions for a class of nonlinear Choquard equations*, SN Partial Differential Equations and Applications **1** (2020), no. 34, pp. 25.

[5] H. Berestycki, P.-L. Lions, *Nonlinear scalar field equations. I. Existence of a ground state*, Arch. Rational Mech. Anal. **82** (1983), 313–345.

[6] S. Bhattara, *On fractional Schrödinger systems of Choquard type*, J. Diff. Equations **263** (2017), no. 6, 3197–3229.

[7] C. Bucur, E. Valdinoci, "Nonlocal Diffusion and Applications", vol. 20, Springer Cham Nature Switzerland AG, 2016.

[8] J. Byeon, O. Kwon, J. Seok, *Nonlinear scalar field equations involving the fractional Laplacian*, Nonlinearity **30** (2017), no. 4, 1659–1681.

[9] X. Cabré, Y. Sire, *Nonlinear equations for fractional Laplacians, I: regularity, maximum principles, and Hamiltonian estimates*, Ann. Inst. H. Poincaré Anal. Non Linéaire **31** (2014), no. 1, 23–53.

[10] L. Caffarelli, L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differ. Equ. **32** (2007), no. 8, 1245–1260.

[11] X. Chang, Z.-Q. Wang, *Ground state of scalar field equations involving a fractional Laplacian with general nonlinearities*, Nonlinearity **26** (2013), no. 2, 479–494.

[12] Y. Cho, T. Ozawa, *Sobolev inequalities with symmetry*, Commun. Contemp. Math. **11** (2009), no. 3, 355–365.

[13] Y. Cho, M.M. Fall, H. Hajaiej, P.A. Markowich, S. Trabelsi, *Orbital stability of standing waves of a class of fractional Schrödinger equations with Hartree-type nonlinearity*, Anal. Appl. (Singap.) **15** (2017), no. 5, 699–729.

[14] S. Cingolani, M. Clapp, S. Secchi, *Multiple solutions to a magnetic nonlinear Choquard equation*, Z. Angew. Math. Phys. **63** (2012), 233–248.

[15] S. Cingolani, M. Clapp, S. Secchi, *Intertwining semiclassical solutions to a Schrödinger-Newton system*, Discrete Contin. Dyn. Syst. Ser. S **6** (2013), no. 4, 891–908.

[16] S. Cingolani, M. Gallo, K. Tanaka, *Normalized solutions for fractional nonlinear scalar field equation via Lagrangian formulation*, Nonlinearity **34** (2021), no. 6, 4017–4056.

[17] S. Cingolani, L. Jeanjean, *Stationary solutions with prescribed L^2-norm for the planar Schrödinger-Poisson system*, SIAM J. Math. Anal. **51** (2019), no. 4, 3533–3568.

[18] S. Cingolani, K. Tanaka, *Ground state solutions for the nonlinear Choquard equation with prescribed mass*, in: V. Ferone, T. Kawakami, P. Salani, F. Takahashi (eds) "Geometric Properties for Parabolic and Elliptic PDE's" (2021), Springer INdAM Series **47**, Cortona 2019, 23–41.

[19] M. Clapp, D. Salazar, *Positive and sign changing solutions to a nonlinear Choquard equation*, J. Math. Anal. Appl. **407** (2013), no. 1, 1–15.

[20] A. Dall'Acqua, T. Østergaard Sørensen, E. Stockmeyer, *Hartree–Fock Theory for Pseudorelativistic Atoms*, Ann. Henri Poincaré **9** (2008), no. 4, 711–742.

[21] P. D'Avenia, G. Siciliano, M. Squassina, *On the fractional Choquard equations*, Math. Models Methods Appl. Sci. **25** (2015), no. 8, 1447–1476.

[22] E. Di Nezza, G. Palatucci, E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), no. 5, 521–573.

[23] L. Dong, D. Liu, W. Qi, L. Wang, H. Zhou, P. Peng, C. Huang, *Necklace beams carrying fractional angular momentum in fractional systems with a saturable nonlinearity*, Commun. Nonlinear Sci. Numer. Simul. **99** (2021), article ID 105840, pp. 8.

[24] P. Felmer, A. Quaas, J. Tan, *Positive solutions of the nonlinear Schrödinger equations with the fractional Laplacian*, Proc. Roy. Soc. Edinburgh Sect. A **142** (2012), no. 6, 1237–1262.

[25] B. Feng, H. Zhang, *Stability of standing waves for the fractional Schrödinger-Hartree equation*, J. Math. Anal. Appl. **460** (2018), no. 1, 352–364.

[26] R. L. Frank, E. Lenzmann, *On ground states for the L^2-critical boson star equation*, arXiv:0910.2721 (2010), pp. 16.

[27] R. L. Frank, E. Lenzmann, L. Silvestre, *Uniqueness of radial solutions for the fractional Laplacian*, Comm. Pure Appl. Math. **69** (2016), no. 9, 1671–1726.

[28] J. Fröhlich, B.L.G. Jonsson, E. Lenzmann, *Boson stars as solitary waves*, Comm. Math. Phys. **274** (2007), no. 1, 1–30.

[29] J. Fröhlich, E. Lenzmann, *Mean-field limit of quantum Bose gases and nonlinear Hartree equation*, in: Sémin. Équ. Dériv. Partielles, École Polytech., Palaiseau (2004), talk no. 18, pp. 26.

[30] J. Fröhlich, T.-P. Tsai, H.-T. Yau, *On the point-particle (Newtonian) limit of the non-linear Hartree equation*, Comm. Math. Phys. **225** (2002), no. 2, 223–274.

[31] Z. Gao, X. Tang, S. Chen, *Ground state solutions of fractional Choquard equations with general potentials and nonlinearities*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113** (2019), no. 1, 2037–2057.

[32] D. Hartree, *The wave mechanics of an atom with a non-Coulomb central field. Part I. Theory and methods*, Math. Proc. Cambridge Philos. Soc. **24** (1928), no. 1, 89–110.

[33] S. Herr, E. Lenzmann, *The Boson star equation with initial data of low regularity*, Nonlinear Anal. **97** (2014), 125–137.

[34] J. Hirata, N. Ikoma, K. Tanaka, *Nonlinear scalar field equations in $\mathbb{R}^N$: mountain pass and symmetric mountain pass approaches*, Topol. Methods Nonlinear Anal. **35** (2010), no. 2, 253–276.

[35] J. Hirata, K. Tanaka, *Nonlinear scalar field equations with L^2 constraint: mountain pass and symmetric mountain pass approaches*, Adv. Nonlinear Stud. **19** (2019), no. 2, 263–290.

[36] N. Ikoma, *Existence of solutions of scalar field equations with fractional operator*, J. Fixed Point Theory Appl. **19** (2017), no. 2, 649–690. *Erratum*: J. Fixed Point Theory Appl. **19** (2017), no. 2, 1649–1652.

[37] N. Ikoma, K. Tanaka, *A note on deformation argument for L^2-normalized solutions of nonlinear Schrödinger equations and systems*, Adv. Differential Equations **24** (2019), no. 11-12, 609–646.

[38] L. Jeanjean, *Existence of solutions with prescribed norm for semilinear elliptic equations*, Nonlinear Anal. **28** (1997), no. 10, 1633–1659.

[39] L. Jeanjean, K. Tanaka, *A remark on least energy solutions in $\mathbb{R}^N$*, Proc. Amer. Math. Soc. **131** (2003), no. 8, 2399–2408.

[40] K. Kirkpatrick, E. Lenzmann, G. Staffilani, *On the continuum limit for discrete NLS with long-range lattice interactions*, Comm. Math. Phys. **317** (2013), no. 3, 563–591.

[41] C. Klein, C. Sparber, P. Markowich, *Numerical study of fractional nonlinear Schrödinger equations*, Proc. Royal Soc. A **470** (2014), artilce ID 20140364, pp. 26.

[42] N. Laskin, *Fractional quantum mechanics and Lévy path integrals*, Phys. Rev. A **268** (2000), no. 4–6, 56–108.

[43] E. Lenzmann, *Uniqueness of ground states for pseudorelativistic Hartree equations*, Anal. PDE **2** (2009), no. 1, 1–27.

[44] E. Lenzmann, M. Lewin, *On singularity formation for the L^2-critical Boson star equation*, Nonlinearity **24** (2011), no. 12, 3515–3540.

[45] E. H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation*, Stud. Appl. Math., **57** (1977), no. 2, 93–105.

[46] E. H. Lieb, *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Annals of Mathematics, **118** (1983), no. 2, 349–374.

[47] E. H. Lieb, R. Seiringer, J. P. Solovej, J. Yngvason, "The mathematics of the Bose gas and its condensation", in: Oberwolfach Seminars **34**, Birkäuser Basel, 2005.

[48] E. H. Lieb, H.-T. Yau, *The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics*, Comm. Math. Phys. **112** (1987), no. 1, 147–174.

[49] P.-L. Lions, *The Choquard equation and related questions*, Nonlinear Anal. **4** (1980), no. 6, 1063–1073.

[50] P.-L. Lions, *Symétrie et compacité dans les espaces de Sobolev*, J. Funct. Anal. **49** (1982), no. 3, 315–334.

[51] S. Longhi, *Fractional Schrödinger equation in optics*, Optics Lett. **40** (2015), no. 6, 1117–1120.

[52] J. Lu, V. Moroz, C. B. Muratov, *Orbital-free density functional theory of out-of-plane charge screening in graphene*, J. Nonlinear. Sci. **25** (2015), no. 6, 1391–1430.

[53] H. Luo, *Ground state solutions of Pohozaev type for fractional Choquard equations with general nonlinearities*, Comput. Math. Appl. **77** (2019), no. 3, 877–887.

[54] H. Luo, Z. Zhang, *Normalized solutions to the fractional Schrödinger equations with combined nonlinearities*, Calc. Var. Partial Differential Equations **59** (2020), no. 143, pp. 35.

[55] L. Ma, L. Zhao, *Classification of positive solitary solutions of the nonlinear Choquard equation*, Arch. Rational Mech. Anal. **195** (2010), no. 2, 455–467.

[56] R. Metzler, J. Klafter, *The random walks guide to anomalous diffusion: A fractional dynamics approach*, Phys. Rep. **339** (2000), no. 1, 1–77.

[57] I. M. Moroz, R. Penrose, P. Tod, *Spherically-symmetric solutions of the Schrödinger-Newton equations*, Topology of the Universe Conference (Cleveland, OH, USA, 17–19 October 1997); IOP Publishing: Bristol, UK, Classical Quantum Gravity **15** (1998), no. 9, 2733–2742.

[58] V. Moroz, J. Van Schaftingen, *Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics*, J. Funct. Anal. **265** (2013), no. 2, 153–184.

[59] V. Moroz, J. Van Schaftingen, *Existence of groundstates for a class of nonlinear Choquard equations*, Trans. Amer. Math. Soc., **367** (2015), no. 9, 6557–6579.

[60] V. Moroz, J. Van Schaftingen, *A guide to the Choquard equation*, J. Fixed Point Theory Appl. **19** (2017), no. 1, 773–813.

[61] R. S. Palais, *The principle of symmetric criticality*, Comm. Math. Phys. **69** (1979), no. 1, 19–30.

[62] Y. J. Park, *Fractional Gagliardo-Nirenberg inequality*, J. Chungcheong Math. Soc. **24** (2011), no. 3, 583–586.

[63] S. Pekar, "Untersuchung über die Elektronentheorie der Kristalle", Akademie Verlag, Berlin, 1954.

[64] R. Penrose, *On gravity's role in quantum state reduction*, Gen. Rel. Grav., **28** (1996), no. 5, 581–600.

[65] R. Penrose, *Quantum computation, entanglement and state reduction*, R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci. **356** (1998), no. 1743, 1927–1939.

[66] R. Penrose, "The road to reality. A complete guide to the laws of the universe", Alfred A. Knopf Inc., New York (2005).

[67] L. Pitaevskii, S. Stringari, "Bose-Einstein condensation", Oxford University Press, Cambridge, MA, USA, 2003.

[68] Z. Shen, F. Gao, M. Yin, *Ground state for nonlinear fractional Choquard equations with general nonlinearities*, Mathematical Methods in the Applied Sciences **39** (2016), no. 14, 4082–4098.

[69] P. Tod, *The ground state energy of the Schrödinger-Newton equation*, Physics Letters A **280** (2001), no. 4, 173–176.

[70] J.L. Vázquez, *The mathematical theories of diffusion: nonlinear and fractional diffusion*, in: Nonlocal and nonlinear diffusions and Interactions: new methods and directions, Springer, Cham, Switzerland 2017, 205–278.

[71] M. Willem, "Minimax theorems", in: Progress in nonlinear differential equations and their applications **24**, Birkäuser Boston, 1996.

[72] H. Ye, *Existence and stability waves for nonlinear fractional Schrödinger equation with Hartree type nonlinearity*, J. Math. Anal. Appl. **411** (2014), no. 2, 530–542.