Applied Mathematical Sciences, Vol. 10, 2016, no. 27, 1299 - 1306
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2016.63100

Buffon Type Problems in Archimedean Tilings II

Salvatore Vassallo

Università Cattolica del S. Cuore, Largo Gemelli 1, I-20123 Milan, Italy

Abstract

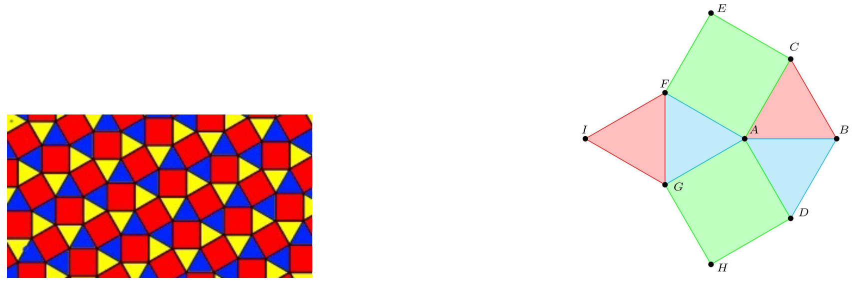
In this paper we consider the snub square tiling of the plane ($(3^2, 4, 3, 4)$ Archimedean tiling) and compute the probability that a random circle or a random segment intersects a side of the lattice. Moreover we compute the same probability if also the diameter of the circle or the length of the segment is a random variable.

Mathematcs Subject Classification: Primary 60D05; Secondary 52A22

Keywords: Geometric probability, stochastic geometry, random sets, random convex sets and integral geometry

1 Introduction

A *tiling* or *tessellation* in the plane is a collection of disjoint closed sets (the *tiles*) that can intersect only on the boundary, which cover the plane. A tiling is said to be polygonal if the tiles are polygon, a polygonal tiling is said to be *edge-to-edge* if two non disjoint tiles have in common or a vertex or a segment that is an edge for both the polygons. In this case we call any edge of a tile an *edge of the tiling*. An edge-to-edge tiling is called *regular* if it is composed of congruent copies of a single regular polygon. An *Archimedean tessellation* (semi-regular or uniform tessellation) is an edge-to-edge tessellation of the plane made of more than one type of regular polygon so that the same polygons surround each vertex. There are eight different Archimedean tilings and we can classify them giving the types of polygons as they come together at the vertex [5]. The *snub square tiling* is a tiling such that three triangles and two squares come together in any vertex in the order (triangle, triangle, square, triangle, square) so it can be called a $(3^2, 4, 3, 4)$ Archimedean tiling (see Figure 1a). Many



(a) Snub square tiling (b) The fundamental cell T_0

Figure 1: The tiling $\mathcal{R}$

authors studied Buffon type problems for different lattices of figures or tilings and different test bodies: See for example [8], [2], [3], [4], [1].In particular the case of the $(3^3, 4^2)$ Archimedean tiling (elongated triangular tiling) is studied in [9].

Let E_2 be the Euclidean plane and let $\mathcal{R}$ be the snub square tiling of E_2 given in Figure 1a. Let us denote by K a convex body (which here means a compact convex set) that we shall call *test body*. We will study this generalized problem of Buffon type: "Which is the probability $p_{K,\mathcal{R}}$ that the random convex body K, or more precisely, a random congruent copy of K, meets some edge of the tiling $\mathcal{R}$?"

We denote by T_0 the *fundamental cell* of $\mathcal{R}$ as in Figure 1b (in the following we will use the same notation for the vertices of the tiles): The probability $p_{K,\mathcal{R}}$ can be calculated as the probability that the body K meets an edge of T_0 supposing that a (fixed) point of K is in T_0. We will study Buffon type problems for two special test bodies: A circle of diameter D and a line segment of length l. In addition we will study the same problem if the diameter of K itself (i.e. the diameter of the circle or the length of the segment) is a random variable with first and second moment known.

We denote by $\mathcal{M}$ the set of all test bodies K whose centroid M is in the interior of T_0 and by $\mathcal{N}$ the set of all test bodies K that are completely contained in one of the triangles or in one of the squares that forms T_0. We also assume that the convex test bodies are uniformly distributed, i.e. that the coordinates of M are a bidimensional random variable with uniform distribution in T_0, and that the random variable φ is uniformly distributed in $[0, 2\pi]$, M and φ stochastically independent. We have

$$p_{K,\mathcal{R}} = 1 - \frac{\mu(\mathcal{N})}{\mu(\mathcal{M})} \tag{1}$$

where μ is the Lebesgue measure.

2 The test body is a circle

Let us suppose that the test body K is a circle of diameter D. Easy geometrical considerations lead us to distinguish between the cases $D < \frac{a}{\sqrt{3}}$, $\frac{a}{\sqrt{3}} \leq D < a$ and $D \geq a$. It is obvious that in the third case $p_{K,\mathcal{R}} = 1$, so we have to study only the first two cases.

Proposition 2.1. *The probability that the circle K of diameter D intersects the tiling $\mathcal{R}$ is given by*

$$p_{K,\mathcal{R}} = \begin{cases} \frac{D\left[10a-\left(2+3\sqrt{3}\right)D\right]}{\left(2+\sqrt{3}\right)a^2} & \text{if } D < \frac{a}{\sqrt{3}} \\ \frac{\sqrt{3}a^2+4aD-2D^2}{\left(2+\sqrt{3}\right)a^2} & \text{if } \frac{a}{\sqrt{3}} \leq D < a \end{cases} \tag{2}$$

Proof. We compute the measures $\mu(\mathcal{M})$ e $\mu(\mathcal{N})$ with the help of the elementary kinematic measure $dK = dx \wedge dy \wedge d\phi$ of E_2 (see [6], [7]) where x and y are the coordinates of the center of K (or the components of a translation), and ϕ is the angle of rotation. We have

$$\mu(\mathcal{M}) = \int_0^\pi d\phi \iint_{(x,y)\in T_0} dxdy = \pi \cdot \text{area}(T_0) = \pi a^2 \left(2+\sqrt{3}\right)$$

Let $\mathcal{N}_1$ be the set of circles of diameter D that are contained in the triangle ABC and $\mathcal{N}_2$ be the set of circles of diameter D that are contained in the square $ACEF$. From equation (1) we obtain

$$p_{K,\mathcal{R}} = 1 - \frac{4\mu(\mathcal{N}_1) + 2\mu(\mathcal{N}_2)}{\pi a^2 \left(2+\sqrt{3}\right)} \tag{3}$$

From Figure 2a it is easy to see that $\mu(\mathcal{N}_1)$ is π times the area of the triangle

(a) Triangle tile (b) Square tile

Figure 2: The case $K =$ circle with $D < \frac{a}{\sqrt{3}}$

$A'B'C'$ whose sides are parallel to the sides of the triangle ABC at distance $D/2$ from them (A' is the center of a disk interior to the triangle ABC and tangent to the sides AB and AC and so on). Since the side of the triangle is

$a - D\sqrt{3}$ we have $\mu(\mathcal{N}_1) = \frac{\pi\sqrt{3}}{4}\left(a - \sqrt{3}D\right)^2$ and in the same way we obtain that $\mu(\mathcal{N}_2) = \pi\left(a - D\right)^2$.

Then in the case $D < \frac{a}{\sqrt{3}}$ we have

$$\mu(\mathcal{N}) = \pi\left[\sqrt{3}\left(a - \sqrt{3}D\right)^2 + 2\left(a - D\right)^2\right]$$

and so:

$$p_{K,\mathcal{R}} = \frac{D\left[10a - \left(2 + 3\sqrt{3}\right)D\right]}{\left(2 + \sqrt{3}\right)a^2}$$

Let $\frac{a}{\sqrt{3}} \leq D \leq a$. If the center of the circle K is in the triangle ABC, the circle always intersects one of the side of the triangle so that $\mu(\mathcal{N}_1) = 0$. If the center of the circle is in the square $ACDF$ the circle does not intersect the side of the square if its center is in the square $A'C'E'F'$ (the figure is similar to Figure 2b with a smaller interior square); since the side of this square is $a - D$ we have $\mu(\mathcal{N}_2) = \pi\left(a - D\right)^2$ and so in this case:

$$p_{K,\mathcal{R}} = \frac{\sqrt{3}a^2 + 4aD - 2D^2}{\left(2 + \sqrt{3}\right)a^2}$$

□

Let us now consider the problem of the intersection with an edge of $\mathcal{R}$ of a random circle K whose diameter is a bounded random variable Δ with upper bound $D \leq \frac{a}{\sqrt{3}}$

Proposition 2.2. *The probability that a circle K whose diameter is a random variable Δ, with $\Delta \leq D \leq \frac{a}{\sqrt{3}}$, and known moments $E(\Delta)$ and $E(\Delta^2)$, intersects the tiling $\mathcal{R}$ is given by*

$$p_{K,\mathcal{R}} = \frac{10}{\left(2 + \sqrt{3}\right)a}E(\Delta) - \frac{2 + 3\sqrt{3}}{\left(2 + \sqrt{3}\right)a^2}E(\Delta^2) \tag{4}$$

Proof. The upper boundary of Δ assures that K is always "small" when compared to the elementary tile of $\mathcal{R}$. Let $f(\delta)$ be the density of Δ and $p(\mathcal{R}|\delta)$ the probability that K intersects a cell of $\mathcal{R}$ given the condition $\Delta = \delta$. Then the probability $p(\mathcal{R}|\delta)$ that K intersects at least one of the edges of the tiling can be computed as

$$p_{K,\mathcal{R}} = \int_0^D p(\mathcal{R}|\delta)f(\delta)d\delta$$

From equation (2) we know that $p(\mathcal{R}|\delta) = \frac{\delta\left[10a-\left(2+3\sqrt{3}\right)\delta\right]}{\left(2+\sqrt{3}\right)a^2}$ and so

$$p_{K,\mathcal{R}} = \int_0^D \frac{10a\delta - \left(2+3\sqrt{3}\right)\delta^2}{\left(2+\sqrt{3}\right)a^2} f(\delta)d\delta =$$
$$\frac{10}{\left(2+\sqrt{3}\right)a}\int_0^D \delta f(\delta)d\delta - \frac{2+3\sqrt{3}}{\left(2+\sqrt{3}\right)a^2}\int_0^D \delta^2 f(\delta)d\delta =$$
$$\frac{10}{\left(2\sqrt{3}\right)a}E(\Delta) - \frac{2+3\sqrt{3}}{\left(2+\sqrt{3}\right)a^2}E(\Delta^2)$$

□

3 The test body is a line segment

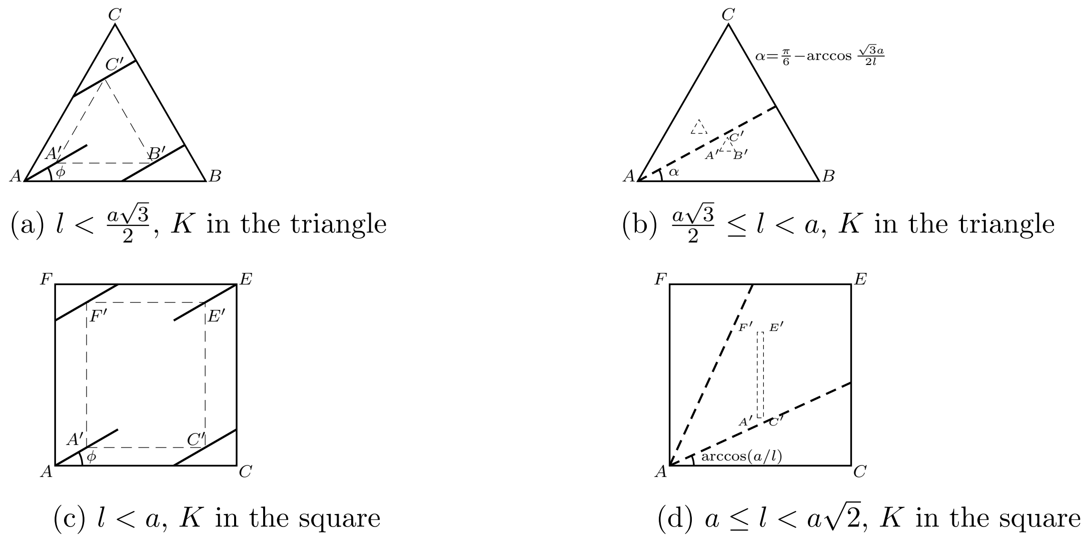
Let us now consider the case in which K is a line segment of length l. Also in this case easy geometrical considerations give us four cases: $l < \frac{a\sqrt{3}}{2}$, $\frac{a\sqrt{3}}{2} \leq l < a$, $a \leq l < a\sqrt{2}$ and $l \geq a\sqrt{2}$. In the last case the segment always intersects an edge of $\mathcal{R}$, so we have to study the other cases.

Proposition 3.1. *The probability that the line segment K of length l intersects the tiling $\mathcal{R}$ is given by*

$$p_{K,\mathcal{R}} = \begin{cases} \frac{l\left[60a-\left(15+2\pi\sqrt{3}\right)l\right]}{3\pi\left(2+\sqrt{3}\right)a^2} & \text{if } l < \frac{a\sqrt{3}}{2} \\ \frac{60al-27a\sqrt{4l^2-3a^2}-15l^2-2\sqrt{3}\pi l^2+6(3a^2+2l^2)\sqrt{3}\arccos\frac{a\sqrt{3}}{2l}}{3\left(2+\sqrt{3}\right)\pi a^2} & \text{if } \frac{a\sqrt{3}}{2} \leq l < a \\ \frac{4a^2-8a\sqrt{l^2-a^2}+2l^2+\sqrt{3}\pi a^2+8a^2\arccos(a/l)}{\left(2+\sqrt{3}\right)\pi a^2} & \text{if } a \leq l < a\sqrt{2} \end{cases} \quad (5)$$

Proof. We use the same notation as in proof of Proposition 3.1.

i) Let us consider the case $l < \frac{a\sqrt{3}}{2}$. First we compute the measure $\mu(\mathcal{N}_1)$ of the set $\mathcal{N}_1$ of all line segments of length l contained in the triangle ABC. With reference to Figure 3a, for a fixed angle $\phi \in [0, \frac{\pi}{3}[$, we denote by A' the midpoint of the line segment of length l with one endpoint in A and $\phi = \widehat{A'AB}$, B' the midpoint of the line segment of length l with endpoints on AB and BC that makes an angle ϕ with AB, and C' the midpoint of the line segment of length l with endpoints on AC and BC that makes an angle ϕ with the direction of AB.



(a) $l < \frac{a\sqrt{3}}{2}$, K in the triangle

(b) $\frac{a\sqrt{3}}{2} \leq l < a$, K in the triangle

(c) $l < a$, K in the square

(d) $a \leq l < a\sqrt{2}$, K in the square

Figure 3: The case $K =$ line segment

Since $\text{area}(A'B'C') = \frac{\sqrt{3}}{4}\left[a - \frac{2l}{\sqrt{3}}\sin\left(\frac{2}{3}\pi - \phi\right)\right]^2$ we obtain, by symmetry,

$$\mu(\mathcal{N}_1) = 3\int_0^{\pi/3} \text{area}(A'B'C')d\phi = \int_0^{\pi/3} \frac{\sqrt{3}}{4}\left[a - \frac{2l}{\sqrt{3}}\sin\left(\frac{2}{3}\pi - \phi\right)\right]^2 d\phi$$
$$= \frac{3\sqrt{3}\pi a^2 - 36al + \left(9 + 2\sqrt{3}\pi\right)l^2}{12} \quad (6)$$

In the same way, if $\psi \in [0, \frac{\pi}{2}[$, we obtain for the set $\mathcal{N}_2$ of the line segments contained in the square $ACEF$ (see Figure 3c) $\text{area}(A'C'E'F') = (a - l\sin\psi)(a - l\cos\psi)$, and so, by symmetry, we have

$$\mu(\mathcal{N}_2) = \int_0^{\pi/2} (a - l\cos\psi)(a - l\sin\psi)\,d\psi = \pi a^2 - 4al + l^2$$

Then $\mu(\mathcal{N}) = \left(2 + \sqrt{3}\right)\pi a^2 - 20al + \left(5 + \frac{2\pi}{\sqrt{3}}\right)l^2$ and hence if $l < \frac{a\sqrt{3}}{2}$

$$p_{K,\mathcal{R}} = \frac{l\left[60a - \left(15 + 2\pi\sqrt{3}\right)l\right]}{3\pi\left(2 + \sqrt{3}\right)a^2} \quad (7)$$

ii) Let now $\frac{a\sqrt{3}}{2} \leq l < a$. With reference to Figure 3b it is easy to see that the line segment can be contained in the triangle ABC only if the angle $\phi \in [0, \pi/3[$ between the line segment and the side AB satisfies $0 \leq \phi < \frac{\pi}{6} - \arccos\frac{\sqrt{3}a}{2l}$ or $\frac{\pi}{6} + \arccos\frac{\sqrt{3}a}{2l} < \phi < \frac{\pi}{3}$ (see Figure 3b).

Hence the measure of the line segments completely contained in the triangle ABC is, by symmetry,

$$\mu(\mathcal{N}_1) = 6\int_0^{\frac{\pi}{6}-\arccos\frac{\sqrt{3}a}{2l}} \frac{\sqrt{3}}{4}\left[a - \frac{2l}{\sqrt{3}}\sin\left(\frac{2}{3}\pi - \phi\right)\right]^2 d\phi =$$

$$\frac{9a\left(3\sqrt{4l^2-3a^2}-4l\right)+3\sqrt{3}\pi a^2+\left(9+2\sqrt{3}\pi\right)l^2-6\sqrt{3}\left(3a^2+2l^2\right)\arccos\frac{a\sqrt{3}}{2l}}{12}$$

The measure of the line segment completely contained in the square $ACEF$ is the same as in the case above: $\mu(\mathcal{N}_2) = \pi a^2 - 4al + l^2$. Hence if $\frac{a\sqrt{3}}{2} \leq l < a$ we have

$$p_{K,\mathcal{R}} = \frac{60al - 27a\sqrt{4l^2-3a^2} - 15l^2 - 2\sqrt{3}\pi l^2 + 6(3a^2+2l^2)\sqrt{3}\arccos\frac{a\sqrt{3}}{2l}}{3\left(2+\sqrt{3}\right)\pi a^2}$$

iii) Let now $a \leq l < a\sqrt{2}$. With reference to Figure 3d it is easy to see that if the centroid of the line segment is in the triangle ABC the line segment always meets one of the side of the triangle and can be contained in the square $ACEF$ only if the angle $\phi \in [0, \pi/2[$ between the line segment and the side AC satisfies $\arccos\frac{a}{l} < \phi \leq \frac{\pi}{2} - \arccos\frac{a}{l}$.

Therefore the measure of the line segments completely contained in the square $ACEF$ is given by:

$$\mu(\mathcal{N}_2) = 2\int_{\arccos(a/l)}^{\frac{\pi}{2}-\arccos\frac{a}{l}} (a - l\cos\phi)(a - l\sin\phi)\, d\phi =$$
$$= 4a\sqrt{l^2-a^2} - l^2 + (\pi-2)a^2 - 4a^2\arccos(a/l)$$

Hence we have if $a \leq l < a\sqrt{2}$

$$p_{K,\mathcal{R}} = \frac{4a^2 - 8a\sqrt{l^2-a^2} + 2l^2 + \sqrt{3}\pi a^2 + 8a^2\arccos(a/l)}{\left(2+\sqrt{3}\right)\pi a^2}$$

□

Let now K be a line segment whose length is a bounded random variable Δ with $\Delta \leq D \leq \frac{\sqrt{3}}{2}a$. With a proof similar to the proof of Proposition 2.2 we obtain easily

Proposition 3.2. *The probability that a segment K whose length is a random variable Δ, with $\Delta \leq D \leq \frac{\sqrt{3}}{2}a$, and known moments $E(\Delta)$ and $E(\Delta^2)$, intersects the tiling $\mathcal{R}$ is given by*

$$p_{K,\mathcal{R}} = \frac{20}{\pi\left(2+\sqrt{3}\right)a}E(\Delta) - \frac{15+2\pi\sqrt{3}}{3\pi\left(2+\sqrt{3}\right)a^2}E(\Delta^2) \tag{8}$$

References

[1] G. Caristi and M. I. Stoka, A Laplace type problem for a regular lattices with irregular hexagonal cell, *Far East J. Math. Sci.*, **50** (2011), no. 1, 23–36.

[2] A. Duma and M. I. Stoka, Geometric probabilities for non-regular lattices. I, 5th Italian Conference on Integral Geometry, Geometric Probability Theory and Convex Bodies, Italian, Milano, 1995, *Rend. Circ. Mat. Palermo (2) Suppl.*, **41** (1996), 81–92.

[3] A. Duma and M. I. Stoka, Problems of geometric probability for non-regular lattices. II, 2nd International Conference in "Stochastic Geometry, Convex Bodies and Empirical Measures" Agrigento, 1996, *Rend. Circ. Mat. Palermo (2) Suppl.*, **50** (1997), 143–151.

[4] A. Duma and M. I. Stoka, Problems of geometric probability for non-regular lattices. III, 2nd International Conference in "Stochastic Geometry, Convex Bodies and Empirical Measures" Agrigento, 1996, *Rend. Circ. Mat. Palermo (2) Suppl.*, **50** (1997), 153–157.

[5] G. C. Grünbaum, B. Shephard, *Tilings and Patterns*, Freeman, 1986.

[6] L. A. Santaló, *Integral Geometry and Geometric Probability*, Addison-Wesley, 1976.

[7] M. I. Stoka, *Geometria Integrala*, Ed. Academice Rep. Soc. Romania, 1967.

[8] M. I. Stoka, Probabilités géométriques de type "Buffon" dans le plan euclidien, *Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur.*, **110** (1976), 53–59.

[9] S. Vassallo, Buffon type problems in archimedean tilings I, *Universal Journal of Mathematics and Mathematical Sciences*, **4** (2013), no. 2, 201–219.

Received: February 1, 2016; Published: April 4, 2016