

# On some hydrodynamical aspects of quantum mechanics

Research Article

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**Abstract:**

In this note we first set up an analogy between spin and vorticity of a perfect 2d-fluid flow, based on the complex polynomial (*i.e.* Borel-Weil) realization of the irreducible unitary representations of  $SU(2)$ , and looking at the Madelung-Bohm velocity attached to the ensuing spin wave functions. We also show that, in the framework of finite dimensional geometric quantum mechanics, the Schrödinger velocity field on projective Hilbert space is divergence-free (being Killing with respect to the Fubini-Study metric) and fulfils the stationary Euler equation, with pressure proportional to the Hamiltonian uncertainty (squared). We explicitly determine the critical points of the pressure of this “Schrödinger fluid”, together with its vorticity, which turns out to depend on the spacings of the energy levels. These results follow from hydrodynamical properties of Killing vector fields valid in any (finite dimensional) Riemannian manifold, of possible independent interest.

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## 1. Introduction

The present note can be viewed as a follow-up of [4] and [5] in that it explores geometric and more generally “classical” features of the *standard* quantum mechanical formalism, in the hope of shedding some light on delicate conceptual issues, such as entanglement or quantum measurement (see the above references), or, at least, to get an intuitive grip on traditionally elusive topics. So we first set up an analogy between spin and vorticity of a perfect 2d-fluid flow, based on the complex polynomial (*i.e.*

Borel-Weil) construction of the irreducible unitary representations of  $SU(2)$ , and looking at the (Madelung-Bohm) velocity attached to the ensuing spin wave functions. This is motivated by the algebro-geometric approach to 2d-superfluids devised in [26] (see e.g. [17] for physical background). The point is that, unlike Borel-Weil, we have a *configuration* space interpretation of spin, whereby putting standard and internal degrees of freedom on an equal footing. The vortex strength interpretation of spin is consistent with the fact that vorticity is related to angular momentum, in the superfluid context (see Section 2, Theorem 2.1).

In Section 4, pursuing a hydrodynamical thread, we also show that, in the framework of finite dimensional geometric quantum mechanics, the Schrödinger velocity field on projective Hilbert space is divergence-free (being Killing with

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respect to the Fubini–Study metric) and fulfils the stationary Euler equation, with the pressure being proportional to the Hamiltonian uncertainty (dispersion) – squared. We explicitly determine the critical points of the pressure of this “Schrödinger fluid”: in particular, the energy eigenstates appear as the minimal (*i.e.* zero) pressure states, an interpretation that could be relevant in the context of quantum measurement (collapse of the wave function). The vorticity of the fluid is determined as well and shown to depend on the spacings of the energy levels. These results are collected in Theorem 4.1 and follow directly from (possibly new, or at least differently formulated) “hydrodynamical” properties of Killing vector fields – of possible independent interest – valid in any (finite dimensional) Riemannian manifold, which we discuss in detail in Section 3 (see Theorem 3.1). It is perhaps useful to note at this point that, although complex projective spaces are dealt with in Sections 2 (the special case  $\mathbf{P}^1 \equiv \mathbf{P}(\mathbf{C}^2) \equiv S^2$ ) and 4, (the general finite dimensional case), their actual roles in their respective contexts are completely different. The paper ends with some final comments and outlook.

## 2. Spin = Vorticity

We begin by recalling that the (unitary)  $SU(2)$ -representation of spin  $s$  can be realized on the  $(2s + 1)$ -dimensional complex vector space consisting of all homogeneous complex polynomials of degree  $2s$  in  $z_0$  and  $z_1$  – homogeneous coordinates on the Riemann sphere  $S^2 \cong \mathbf{P}(\mathbf{C}^2) \equiv \mathbf{P}^1$ , with inhomogeneous coordinate  $\zeta = \frac{z_1}{z_0}$  (or the reciprocal) – whereupon  $SU(2)$  acts *via* Möbius transformations. This is of course in accordance with the fact that a particle of spin  $s$  corresponds to a symmetric spinor of rank  $2s$  (see [22], Ch. VIII). A spin wave function ultimately becomes a polynomial  $\chi = \chi(\zeta)$  of degree  $2s$ , and can be viewed as a meromorphic function on  $S^2$  (*i.e.* a rational function, in this case), with a pole of order  $2s$  at infinity (for a meromorphic function on a compact Riemann surface one has number of zeros = number of poles, both counted according to their multiplicity (see [18] or [25])). The functions  $\chi_k \equiv \zeta^k$ ,  $k = 0, \dots, 2s$  become, after suitable normalization, an orthonormal basis for the spin space. The spin operator “ $S_z$ ” reads, in the above basis,  $S_z \chi_k = (k - s) \chi_k$ . Therefore,  $S^2 \cong \mathbf{P}^1$  is the classical *phase space* attached to spin (*cf.* [27]).

Now recall the simplest example of Madelung–Bohm velocity, that for a wave function of a particle of mass one on the real line,  $\psi(x) = |\psi(x)| e^{i\phi(x)}$ : one has  $v^\psi \equiv \Im(d \log \psi / dx) = d\phi / dx$ . Similarly, we introduce, for a spin wave function  $\chi = \chi(\zeta) = \prod_k (\zeta - a_k)^{\mu_k}$ ,  $\sum_k \mu_k = 2s$  (obvious notation), the *Madelung–Bohm ve-*

*locity form*  $v^\chi = \Im d \log \chi \equiv d\varphi$  (with  $\varphi$  a local phase function), which is, in general, a closed form on  $S^2 \setminus \{Z_i, P_i\}$  (zeros and poles). In this way, we are looking at (punctured)  $S^2$  as a *configuration space*, hosting a 2-d perfect fluid with velocity form given by  $v^\chi$  (also *cf.* Eq. (3) below). Now, letting  $\gamma$  be a circuit encircling – once, counterclockwise – some of the roots  $a_k$ , the Residue Theorem immediately yields the following conclusion:

### Theorem 2.1 (Spin = Vorticity).

(i) *With the notation above*

$$\frac{1}{2\pi} \int_\gamma v^\chi = \sum \mu_k, \quad (1)$$

where the sum ranges over the roots encircled by  $\gamma$ .

(ii) *In particular, if the circuit  $\Gamma$  encircles once, counterclockwise, all the zeros of  $\chi$ , then*

$$\frac{1}{2\pi} \int_\Gamma v^\chi = 2s, \quad (2)$$

that is, the total spin is given by the circulation of the velocity field along a loop encircling once the zeros of the wave functions, so ultimately it can be looked upon as a (quantized) vorticity strength (Feynman–Onsager quantization condition).

### Remark 2.1.

The above result is technically straightforward, coming directly from the residue theorem. However, the main *physical* point is that upon using the above complex polynomial representation of spin wave functions we get the sought for hydrodynamical and configuration space interpretation of spin. It is also important to stress the fact that the above arrangement matches (in the genus zero case) exactly the algebro–geometric description of superfluids (more precisely, of their order parameters) devised, e.g. in [26]: indeed, the above result can be equivalently cast into the following distributional equation, working on  $\mathbf{C}$ :

$$\begin{aligned} dv^\chi &= 2\pi \sum_k \mu_k \delta_{a_k}(\zeta) \left( \frac{i}{2} d\zeta \wedge d\bar{\zeta} \right) \\ &= 2\pi \sum_k \mu_k \delta_{a_k}(\zeta) dx \wedge dy, \quad \delta v^\chi = 0 \end{aligned} \quad (3)$$

( $\delta$  denotes divergence). In the r.h.s. we have a *singular* ( $\delta$ -like) vorticity 2-form (viewed as a current (singular Poincaré dual), see [6, 16, 26, 29]) corresponding to the *vorticity divisor*  $D = \sum_k \mu_k a_k$ , which represents an assembly of point vortices located at  $a_k$ , with (quantized) strength  $\mu_k$ . See e.g. [17] for a physical discussion of the Feynman–Onsager condition.

### Remark 2.2.

We notice in passing that the above representation provides the simplest instance of the Borel-Weil construction of unitary representation of simple Lie groups, and can be phrased in the language of (Kählerian) geometric quantization, see e.g. [15] and [27], Ch. 4, for details, and also [31, 32] for general background); here we just point out that the spin wave functions correspond to the holomorphic sections of the  $2s$ -th tensor power  $\mathcal{O}(2s) = \mathcal{O}(1)^{\otimes 2s}$  of the hyperplane section bundle  $\mathcal{O}(1) \rightarrow \mathbf{P}^1$ , dual to the tautological bundle.

## 3. Hydrodynamical properties of Killing vector fields

In this section we discuss some (possibly new or, at least, differently formulated) results valid for Killing vector fields on a (connected) Riemannian manifold  $(M, g)$  (i.e. those generating infinitesimal isometries; they always exist, at least locally). As general references we may quote [12, 14, 21]. For hydrodynamics we refer, among others, to [1, 2, 13, 23, 30].

The Levi-Civita connection of  $(M, g)$  will be denoted by  $\nabla$ . We shall employ the notation  $\langle X, Y \rangle \equiv g(X, Y)$ , for  $X, Y \in \Gamma(TM)$  (vector fields on  $M$ ). Upon freely using the musical isomorphism notation ( $\sharp =$  vector field,  $\flat =$  1-form, corresponding to index raising and lowering, respectively, so, for instance,  $\langle X^\flat, Y \rangle = \langle X, Y \rangle$ , with  $\langle \cdot, \cdot \rangle$  being the pairing between 1-forms and vector fields), we begin by recalling the following basic identity (cf. [1], 5.5.8, p. 474, or [2], Ch. IV, Theorem 1.17, p. 202):

$$\mathcal{L}_Y Y^\flat = (\nabla_Y Y)^\flat + \frac{1}{2} d\langle Y, Y \rangle \quad (4)$$

( $\mathcal{L}$  is the Lie derivative). The following result is crucial.

### Lemma 3.1.

Let  $X$  be a Killing vector field on a Riemannian manifold  $(M, g)$ . Then

$$\mathcal{L}_X X^\flat = 0. \quad (5)$$

**Proof.** If  $X$  is Killing, then for any vector field  $Y$ , one has

$$\mathcal{L}_X(Y^\flat) = (\mathcal{L}_X Y)^\flat \quad (6)$$

which yields immediately

$$\mathcal{L}_X(X^\flat) = (\mathcal{L}_X X)^\flat = [X, X]^\flat = 0 \quad (7)$$

Q.E.D.  $\square$

Recall that the *Euler equation* on a Riemannian manifold reads, among others, in the following equivalent guises, in terms of 1-forms:

$$\frac{\partial X^\flat}{\partial t} + (\nabla_X X)^\flat = -dp \quad (8)$$

or (cf. Lemma 3.1)

$$\frac{\partial X^\flat}{\partial t} + \mathcal{L}_X X^\flat = d\left(\frac{1}{2}\langle X, X \rangle - p\right) \quad (9)$$

( $p$  being the *pressure*) together with  $\operatorname{div}X = 0$  (see e.g. [2] or [30], Ch. 17, 1.15, p. 469). One immediately establishes the following

### Lemma 3.2.

A divergence-free vector field  $Y$  on a (finite dimensional, connected) Riemannian manifold  $(M, g)$  satisfies the stationary Euler equation, with pressure  $p = \frac{1}{2}\langle Y, Y \rangle$  (up to a constant) if and only if  $\mathcal{L}_Y Y^\flat = 0$ .

Let us also notice, for future use, the general identity, valid for a Killing vector field  $X$ ,

$$(\mathcal{L}_X g)(Y, Z) = \langle \nabla_Y X, Z \rangle + \langle \nabla_Z X, Y \rangle = 0, \quad (10)$$

which implies, setting  $Z = Y$ ,

$$\langle Y, \nabla_Y X \rangle = 0 \quad (11)$$

and, in particular,

$$\langle X, \nabla_X X \rangle = 0. \quad (12)$$

The main result of this section is the following

### Theorem 3.1.

Let  $X$  be a Killing vector field on a finite dimensional, connected Riemannian manifold  $(M, g)$ . Then:

(i) the (necessarily divergence-free) vector field  $X$  fulfils the stationary Euler equation, with pressure given by  $p = \frac{1}{2}\langle X, X \rangle$  (up to a constant);

(ii) the vorticity form of the (stationary) Euler equation reads (with  $w = dX^\flat$  the vorticity 2-form)

$$\mathcal{L}_X w = 0; \quad (13)$$

(iii) the (Riemannian) gradient of the pressure,  $(dp)^\sharp$ , is orthogonal to  $X$ ;

(iv) if  $\gamma$  is an integral curve of  $X$  starting from a point  $m \in M$ , then  $\gamma$  is a geodesic if and only if  $dp = 0$  (at  $m$  and hence along  $\gamma$ ).

**Proof.** Ad (i). The conclusion follows immediately from Lemmata 3.1 and 3.2.

Ad (ii). This is clear from Eq. (9) and the fact that  $\mathcal{L}d = d\mathcal{L}$ .

Ad (iii). This is straightforward from  $(dp)^\sharp = -\nabla_X X$  and from Eq. (12).

Ad (iv). Let  $\gamma : s \mapsto \gamma(s)$  denote the integral curve of  $X$  starting from a point  $m$ . Then, due to the stationary Euler equation fulfilled by  $X$ , one has

$$(\nabla_{\dot{\gamma}} \dot{\gamma})^\sharp = -dp|_{\gamma(s)}. \quad (14)$$

Thus  $\gamma$  is a geodesic if and only if  $dp|_{\gamma(s)} = 0$  for all  $s$ . On the other hand,  $p$ , and hence  $dp$  are invariant under the flow of  $X$ , by (iii), whence  $dp|_{\gamma(s)} = 0$  for all  $s$  if and only if it holds at  $m = \gamma(0)$ , this yielding (iv).  $\square$

Let us also recall and prove, for completeness, the following

**Proposition 3.1.**

(cf. [30]). *Along a geodesic  $\gamma$ , if the vector field  $Y$  restricts to its velocity field thereon, and  $X$  is Killing, we have*

$$Y\langle Y, X \rangle = 0. \quad (15)$$

**Proof.** One has, along  $\gamma$

$$Y\langle Y, X \rangle = \langle \nabla_Y Y, X \rangle + \langle Y, \nabla_Y X \rangle = \langle Y, \nabla_Y X \rangle \quad (16)$$

and, by Eq. (11), the conclusion. Q.E.D.  $\square$

**Remark 3.1.**

The above proposition says that the scalar product of the velocity field of a geodesic with a Killing field is conserved (see e.g. [30], Ch. 18, proposition 3.3, p. 546). For surfaces of revolution, it amounts to the classical Clairaut's Theorem.

**Remark 3.2.**

Assertion (iii) of Theorem 3.1 also appears in [21], Prop. 5.7, p. 252 (one direction, and the proof is different) and can be also proved by exploiting the variational characterization of geodesics as critical paths of the energy functional (together with the Killing condition). Application to a surface of revolution (along the  $z$ -axis, say) yields the standard characterization of geodesic parallels as extremals of the radial function (i.e. the profile curve viewed as a function of  $z$ ): indeed, using standard notation, one has  $X = \frac{\partial}{\partial \varphi}$  and  $g\left(\frac{\partial}{\partial \varphi}, \frac{\partial}{\partial \varphi}\right) = \varrho^2$ , where  $g = \left(1 + \varrho'^2\right) dz^2 + \varrho^2 d\varphi^2$ .

**Remark 3.3.**

Notice that for one-sided invariant metrics on Lie groups, even in the finite dimensional case, geodesics do not correspond to 1-parameter Lie subgroups (see e.g. [14]). Consequently - even ignoring the subtleties of the infinite dimensional situation - one cannot directly conclude that a divergence-free vector field on a (compact, say) Riemannian manifold, which is an element of the "Lie algebra" of the group  $Sdiff(M)$  of measure preserving diffeomorphisms of  $M$ , automatically yields a solution of the (stationary) Euler equation, namely, a geodesic of the natural right-invariant (but not bi-invariant) metric induced by the kinetic energy of the fluid ([2, 13]). But Killing vector fields do, as we have shown.

## 4. A quantum mechanical application

In this section we wish to apply the results of the preceding section to the velocity vector field determined by the Schrödinger equation, for time independent Hamiltonians and in finite dimensional quantum Hilbert spaces (which is not so severe a limitation, in view of their occurrence in various contexts, from quantum chemistry to quantum computing). We begin by reviewing briefly the formalism of geometric quantum mechanics, referring to [4, 5, 28] for full details (but see also [3, 7-11]). This will provide a genuine higher (even) dimensional example of perfect fluid. We assume  $\hbar = 1$ . Let  $V$  be a complex Hilbert space of finite dimension  $n + 1$ , with scalar product  $\langle \cdot | \cdot \rangle$ , linear in the second variable. Let  $P(V) \cong \mathbf{P}(\mathbf{C}^{n+1}) \equiv \mathbf{P}^n$  denote its associated projective space, of complex dimension  $n$ . This is the space of (pure) states in quantum mechanics. Upon free employ of Dirac's bra-ket notation, we can identify a point in  $P(V)$ , which is, by definition, the ray (i.e. one-dimensional vector space)  $\langle v \rangle$  pertaining to (resp. generated by) a non zero vector  $v \equiv |v\rangle$  - and often conveniently denoted by  $[v]$  - with the projection operator onto that line, namely

$$[v] = \frac{|v\rangle\langle v|}{\|v\|^2} \quad (17)$$

(actually, the above identification can be interpreted in terms of a moment map, see [4]). If  $U(V)$  denotes the unitary group pertaining to  $V$ , with Lie algebra  $u(V)$ , consisting of all skew-hermitian endomorphisms of  $V$  - namely, the quantum observables, with a slight abuse of language - then the projective space  $P(V)$  is a  $U(V)$ -homogeneous Kähler manifold. The isotropy group (stabilizer) of a point  $[v] \in P(V)$  is isomorphic to  $U(V') \times U(1)$ , with  $V'$  the orthogonal complement of  $\langle v \rangle$  in  $V$ , the  $U(1)$  part coming

from phase invariance:  $[e^{i\alpha}v] = [v]$ . Hence

$$P(V) \cong \frac{U(V)}{U(V') \times U(1)} \cong \frac{U(n+1)}{U(n) \times U(1)}. \quad (18)$$

The fundamental vector field  $A^\#$  associated to  $A \in \mathfrak{u}(V)$  reads (evaluated at  $[v] \in P(V)$ ,  $\|v\| = 1$ )

$$A^\#|_{[v]} = |v\rangle\langle Av| + |Av\rangle\langle v|. \quad (19)$$

One finds, for the *dispersion* (or variance, or uncertainty) squared of the observable  $A$  in the state  $[v]$ :

$$(\Delta_{[v]}A)^2 \equiv \|Av - \langle Av|v\rangle v\|^2 = g_{FS}(A^\#, A^\#), \quad (20)$$

with  $g_{FS}$  the Fubini-Study metric on  $P(V)$  (see the references given above). We deal with a non degenerate Hamiltonian ( $\lambda_i < \lambda_j$  for  $i < j$ )

$$H = \sum_{i=0}^n \lambda_i |e_i\rangle\langle e_i| \quad (21)$$

(in terms of an orthonormal basis  $(e_i)$ ,  $i = 0, \dots, n$  of  $V$ ). We write, for a generic state vector (of norm one)

$$v = \sum_{i=0}^n \alpha_i e_i, \quad \sum_{i=0}^n |\alpha_i|^2 = 1. \quad (22)$$

The dispersion (squared) of the hamiltonian  $H$  in the state  $[v]$  is easily computed:

$$\begin{aligned} (\Delta H)^2 &\equiv (\Delta_{[v]}H)^2 = \langle v|H^2v\rangle - \langle v|Hv\rangle^2 \\ &= g_{FS}((-iH)^\#, (-iH)^\#). \end{aligned} \quad (23)$$

The vector field  $X =: (-iH)^\#$  is called the *Schrödinger vector field* on  $\mathbf{P}^n$  (the Schrödinger equation reads, of course,  $\partial_t |v\rangle = -iH|v\rangle$ ) and is *Killing* thereon (hence divergence-free). It is also stationary since the Hamiltonian  $H$  is time independent.

We shall use the representation  $\mathbf{P}^n \equiv \mathbf{P}(\mathbf{C}^{n+1}) \cong S^{2n+1}/S^1$ , where  $S^{2n+1}$  is the  $2n+1$ -dimensional sphere in  $\mathbf{C}^{n+1} \cong \mathbf{R}^{2(n+1)}$ .

Then Theorem 3.1 immediately implies part of the following

#### Theorem 4.1.

(i) If  $(M, g) = (\mathbf{P}^n, g_{FS})$ , and  $X$  is the Schrödinger vector field pertaining to the Hamiltonian  $H$ , then  $X$  fulfils the stationary Euler equation with  $2p = (\Delta H)^2$ .

(ii) The critical points of the pressure, in the Schrödinger case, are given by the energy eigenstates (minima, zero pressure) and by the equal probability superpositions of pairs thereof.

(iii) The vorticity 2-form  $w = dX^\flat$ , evaluated on the geodesic sphere  $S_{ij}$  - with area 2-form  $d\sigma$  and colatitude  $\vartheta$  - determined by the superpositions of two energy eigenstates, reads (see below for details):

$$w|_{S_{ij}} = 2(\delta h)_{ij} \cos \vartheta d\sigma. \quad (24)$$

**Proof.** Ad (i). This is just an application of Theorem 3.1, (i). Of course, the remaining assertions of that result hold in the present case. As a consistency check (see also Remark 3.3 in the preceding section) observe that, in the projective line (Riemann sphere) case, on the equator one has critical (actually maximal) uncertainty and the Schrödinger trajectory is a geodesic.

Ad (ii). In order to determine the critical points of the quantum mechanical pressure field explicitly, we proceed as follows.

Set  $q_i^2 = |\alpha_i|^2$  and  $f = (\Delta H)^2$  as a function of the  $q_i$ , namely

$$f = \sum_{i=0}^n \lambda_i^2 q_i^2 - \left( \sum_{i=0}^n \lambda_i q_i^2 \right)^2 \quad (25)$$

and introduce the constraint  $g = \sum_{i=0}^n q_i^2 - 1 = 0$ . Then the critical points of  $f$ , subject to  $g = 0$ , are given by the solutions of the (Lagrange) system

$$df = \mu dg, \quad g = 0, \quad (26)$$

namely

$$(\lambda_i^2 - 2\langle v|Hv\rangle \lambda_i - \mu)q_i = 0 \quad \forall i = 0, \dots, n. \quad (27)$$

Upon defining  $P(\lambda) = \lambda^2 - 2\langle v|Hv\rangle - \mu$ , we see that, if we have a solution with  $q_k \neq 0$ , then  $\lambda_k$  must be a root of  $P$ . Therefore, since the eigenvalues are all distinct, there are at most two indices  $i_1, i_2$  for which  $q_i \neq 0$ , and this leads to  $\langle v|Hv\rangle = \lambda_{i_1} q_{i_1}^2 + \lambda_{i_2} q_{i_2}^2 = \frac{1}{2}(\lambda_{i_1} + \lambda_{i_2})$ , whencefrom it follows that  $q_{i_1}^2 = q_{i_2}^2 = \frac{1}{2}$ , and  $\mu = -\lambda_{i_1} \lambda_{i_2}$ . The remaining possibility, that only one  $q_i \neq 0$ , yields the eigenstates of  $H$ .

Ad (iii). In computing the vorticity 2-form  $dX^\flat$  pertaining to the Schrödinger velocity 1-form  $X^\flat$ , we first notice that in view of the previous discussion, it is enough, in order to grasp its physical meaning, to restrict to the (totally) geodesic spheres  $S_{ij}$ , say, determined by superpositions of two energy eigenstates. The Schrödinger motion is just a

uniform rotation around the axis whose poles are given by the eigenstates in question (see also [4, 5]); the angular velocity  $\omega \equiv (\delta h)_{ij}$  equals  $\lambda_i - \lambda_j$  ( $i > j$ ), the difference of the energy levels. We find ( $\vartheta$  is the colatitude, measured appropriately, and  $d\sigma$  is area 2-form; also recall that the radius  $R = \frac{1}{2}$ , cf. [5, 8]):

$$\begin{aligned} w|_{S_{ij}} &= dX^b|_{S_{ij}} = d(\omega R^2 \sin^2 \vartheta d\varphi) \\ &= 2\omega \cos \vartheta (R^2 \sin \vartheta d\vartheta \wedge d\varphi) = 2\omega \cos \vartheta d\sigma, \end{aligned} \quad (28)$$

whence the vorticity vanishes on the equator (maximal uncertainty) and it is maximal (with opposite signs) at the poles (zero uncertainty). Notice, as a further check, that the *scalar* vorticity function  $\tilde{w} \equiv 2\omega \cos \vartheta$  does indeed satisfy the 2d-vorticity equation on  $S_{ij}$  (obvious notation, cf. [30], Ch. 17, (1.27) p. 470)

$$\frac{\partial \tilde{w}}{\partial t} + \text{grad } \tilde{w} \cdot X = 0 \quad (29)$$

Q.E.D. □

#### Remark 4.1.

In geometric terms, the critical points are given by the vertices and the midpoints of the Atiyah - Guillemin-Sternberg convex polytope arising from the standard moment map (cf. [4, 24] for background).

#### Remark 4.2.

In essence, we provided an “Eulerian” counterpart to the “Lagrangian” portrait inherent to the geometric interpretation of the Schrödinger flow.

#### Remark 4.3.

We may depict the following picture of the “collapse of the wave function”: performing an energy measurement on a quantum system causes a perturbation of the Schrödinger fluid, forcing the quantum state to reach to a minimal (indeed, zero) pressure, *i.e.* an eigenstate (see also [4] for a complementary discussion of this issue).

#### Remark 4.4.

The geometrical and hydrodynamical set up may be useful in “visualising” the Quantum Zeno Effect (see e.g. [20], 3.3.1, p. 110): continual measurement “freezes” the motion: the rate of decay of a pure state (as a function of  $t$ ) goes as  $(\Delta H)^2 t^2$ , the “distance” (squared) travelled by the state under the Schrödinger motion (Lagrangian portrait), and related in turn to the fluid pressure. Upon repeating the measurement  $N$  times within the time interval  $t$  one finds  $(\Delta H)^2 \frac{t^2}{N}$ , tending to zero as  $N$  goes to infinity.

#### Remark 4.5.

Let us remark on the similarity between the general geometric quantum mechanical picture and that of an assembly of harmonic oscillators (also cf. [19] and the general discussion about integrability in [4]. Indeed, projective space comes from a Marsden-Weinstein reduction of the phase space of latter (see e.g. [15]). Following this path one again arrives at the conclusion that the Schrödinger field fulfils the stationary Euler equation. However, the general argument we gave is by no means more complicated and it is more intrinsic, allowing the extra consequences of Theorem 4.1 to be drawn.

## 5. Concluding remarks

We close the present note with the following additional observations.

1. The geodesic interpretation of Euler’s equation entails that the Schrödinger equation can be viewed as coming from a hydrodynamical variational principle in projective space, equipped with the Fubini-Study volume, *via* the Killing condition (yielding the natural  $U(n+1)$ -symmetry of  $\mathbf{P}^n$ ).
2. Notice that the Schrödinger motion itself can be viewed as a coadjoint orbit motion for the group  $U(n+1)$  (see e.g. [4] for full details). On the other hand, the vorticity form of the Euler equation is a manifestation of a coadjoint orbit motion relative to the group of measure preserving diffeomorphisms ([2, 23]). In our case we deal with a stationary fluid, and we arrive at Eq. (13).
3. Since the quantum state space  $(\mathbf{P}^n, g_{FS})$  is a Kähler-Einstein manifold (the “cosmological” constant is indeed a pressure term) the Schrödinger equation appears to be a (Killing) symmetry for a fictitious (Riemannian) “general relativity” (cf. Proposition 3.1) thereon, ultimately governed by uncertainty.

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