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Consistency issues in finance: A state-dependent utility approach

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Abstract

This thesis investigates consistency issues in economic and financial decision-making, with particular focus on the coherence of preferences and optimality across time and information flow.

First, building upon Debreu's work on preference representations and its infinite-dimensional extension, we establish additional regularity properties for state-dependent utilities in a general setting.

From a dynamic decision-theoretic perspective, we examine the role of Savage's Sure-Thing Principle in the conditioning of preferences on intermediate information. For any terminal random outcome, we characterize the intermediate outcome such that the agent is indifferent between the two. Reformulating this in terms of preference-representing functionals, we identify conditional certainty equivalents as conditional Chisini means. We provide explicit representations of these conditional means, linking state-dependent utility theory with the theory of nonlinear expectations. In particular, we characterize consistent families of nonlinear functionals as conditional certainty equivalents with respect to state-dependent utilities, generalizing prior results on law-invariant functionals.

The second part of this Thesis addresses consistency in optimal portfolio choice. We study an agent maximizing terminal utility in a simple financial market and investigate how optimal strategies vary with investment horizons. Classical multi-period approaches rely on backward induction and the Bellman optimality principle, which presupposes time-consistency. However, practical problems often exhibit time-inconsistency.

We show that even in classical settings, optimal strategies corresponding to distinct horizons may not coincide over overlapping intervals, highlighting the critical role of the randomness of the market price of risk. Motivated by forward performance theory, we model the agent's preferences through a state-dependent utility function with risk aversion evolving under exponential stochastic dynamics. Our main result establishes that, for exponential-type utilities, consistency is equivalent to the risk-aversion process being a martingale with uniquely determined dynamics. We demonstrate that, while many dynamics can indeed generate forward performance utilities, only the martingale specification ensures consistency of the optimal strategy.

Overall, this Thesis provides a unified framework connecting state-dependent utility theory, consistent conditioning, and consistent dynamic optimization. Chapter 1 extends the theoretical foundations of state-dependent utilities and conditional preferences, while Chapter 2 applies these insights to dynamic portfolio choice, revealing the interplay between market information, state-dependent utilities, and consistent strate-

gies. The results contribute both to the formal understanding of state-dependent utilities and to practical methods for designing robust investment strategies in uncertain financial environments.

The research for this Thesis resulted in the following works:

1. E. Berton, A. Doldi, M. Maggis *On conditioning and consistency for nonlinear functionals*, SIAM Journal on Financial Mathematics, 2025.

2. E. Berton, M. De Donno, M.Maggis *On consistency of optimal portfolio choice for state-dependent exponential utilities*, Preprint: arXiv:2501.01748, 2025.

All of these articles have been submitted to a journal: the former is published on the *SIAM Journal on Financial Mathematics*, the latter is under review.

Contents

Introduction

The objective of this thesis is to study consistency issues arising in the field of economics and finance. The term *consistency* refers to different notions throughout the two chapters composing this thesis, although a conceptual common thread can be found in the idea of maintaining coherence between preferences and optimality across time and information flow.

Particular emphasis is placed on state-dependent utility functions, which generalize classical utility functions by allowing the utility of an outcome to depend on the state of nature with which it is associated. Wakker and Zank (1999) extend to an infinite state space the original result by Debreu (1960), characterizing the relationship between a preference ordering and its representing state-dependent expected utility functional. The first main contribution of this thesis is to establish natural regularity properties, namely joint measurability and continuity, for the class of state-dependent utility functions within this framework.

The decision-theoretical framework at hand is then considered from a dynamical viewpoint. Precisely, whenever the agent is provided with intermediate information we ask whether for any *terminal* random outcome it is possible to find a (suitably measurable) *intermediate* random outcome such that the agent is indifferent between the two, regardless of the realization that obtains. As preference orderings naturally induce preference representing functionals, the decision-theoretic problem can be equivalently reformulated as a functional problem. Switching perspectives we are then able to actually characterize such a *conditional certainty equivalent* as a conditional Chisini mean for the preference representing functional. Inspired by previous literature on generalized means we provide an explicit representation of the latter in terms of conditional certainty equivalents with respect to state-dependent utilities.

The final part of the first chapter of this thesis contributes to the literature on conditional nonlinear expectations. Indeed, building on previous results that rely on preference-induced functionals to identify consistent projections of random outcomes onto smaller subspaces, we characterize the class of nonlinear functionals that are compatible with consistent conditioning with respect to the flow of information.

The second chapter of the thesis is devoted to the study of a financial market-related inconsistency phenomenon. Considering an agent investing in a simple complete financial market we address the behaviour of the optimal strategies for different investment horizons. We provide a definition of *consistent optimization problem* adapting the seminal idea by Kydland and Prescott (1977). Even in the case of classical deterministic utility functions, in fact, a simple example reveals that the optimal strategies corresponding to distinct maturities do not coincide on the overlapping time period.

The cause of this inherent inconsistency appears to be found in the randomness of the market price of risk. Drawing inspiration from the theory of *forward performances*, a class of stochastic dynamic utility functions introduced in a series of works by Musiela and Zariphopoulou (2006, 2009, 2010), we propose an approach based on state-dependent utility functions. Specifically, the agent's unknown preferences are captured by a stochastic risk aversion parameter which evolves following exponential dynamics.

A notable implication of consistency of the optimization problem is that the risk-aversion process must be a martingale under the risk-neutral measure, with uniquely identified diffusion coefficient. Conversely, we show that there exist infinitely many risk-aversion process dynamics that qualify the stochastic utility function as a forward performance. We point out that the unique choice of parameters ensuring consistency guarantees that the stochastic utility is also a forward performance.
The novelty of this approach lies in the interaction between the information flowing from the market and the agent's risk aversion. This leads to a strong interplay between the time evolution of the agent's strategy and the updating of their preference order.

In the following sections, we give a more detailed introduction to each of the topics presented in this work and we present briefly the main results obtained. While rigorous definitions are deferred to Chapters 1 and 2, we introduce here, informally, a few key concepts that will play a central role in the following discussion.

I.1 On conditioning and consistency for nonlinear functionals

Chapter 1 of this thesis deals extensively with the representation of a preference ordering $\succeq$ on random variables in terms of state-dependent expected utility. The formulation of preferences (*over lotteries*[1]) in terms of expected utility traces back to the seminal idea of Bernoulli (1954). The first axiomatic treatment of preference orderings that admit a numerical representation of the form

$$U : \mu \mapsto \int_E u(x)\,\mathrm{d}\mu,$$

where $\mu : \mathcal{E} \to \mathbb{R}$ is a probability measure on $(E, \mathcal{E})$, was later developed by von Neumann and Morgenstern (1944).

By recasting the decision problem in terms of bounded random variables, commonly referred to as "acts" in economic literature, Savage (1954) introduces a framework where *subjective probability* emerges endogenously from the agent's preference ordering. Notably, the notion of subjective probability had been studied by de Finetti (1931b). Among the axioms introduced by Savage, particular emphasis shall be placed

[1] Elements of a convex subset of the space of probability measures on some measurable space $(E, \mathcal{E})$

on the so called "Sure-Thing Principle", or P2 (see **(ST)** on page 21 for a formal defin-tion). Put simply, the Sure-Thing principle states that if an agent prefers an act f to g conditional on the event A occurring, and the same holds whenever the event A^c (the complement) occurs, then they should prefer f to g regardless of whether A or A^c obtains.

On a finite state space, Debreu (1960) proposes a representation of preferences which is allowed to depend on the specific state of nature that attains, thereby introducing state-dependence in utility theory. As we shall discuss in detail in later sections, for $\Omega = \{\omega_1, ..., \omega_N\}$, under some technical assumptions a preference $\succeq$ on the space of bounded random variables can be represented by an additively decomposable functional U defined as

$$U(X) = \sum_{i=1}^{N} u_i(X(\omega_i)).$$

Additionally, such a representation is unique up to increasing linear affine transformations. As a consequence, each u_i can be seen equivalently as a product $p_i v_i$ with $\sum_{i=1}^{n} p_i = 1$ and such that $p_i v_i(X(\omega_i)) = u_i(X(\omega_i))$. Therefore a central issue in state-dependent expected utility, compared to classical expected utility theory, is the nonidentifiability of probability.

The extension of state-dependent expected utility to infinite dimensional spaces is developed by Wakker and Zank (1999). Under the assumption of *pointwise continuity* of the preference $\succeq$, they show the existence of a countably additive measure μ and of a strictly increasing stochastic utility function $\mathbb{u} : \Omega \times \mathbb{R} \to \mathbb{R}$ such that $\succeq$ is represented by

$$f \mapsto \int_{\Omega} \mathbb{u}(\omega, f(\omega))\, d\mu.$$

As noted by the authors, the state-dependent expected utility functional inherits pointwise continuity from the preference. However, whether the same pointwise continuity holds for $\mathbb{u}$ is not addressed. We give a first contribution in this direction: we show that it is possible to construct a specific version of the state-dependent utility such that for every $\omega \in \Omega$ the map $x \mapsto \mathbb{u}(\omega, x)$ is continuous, a desirable property in many applications.

The dynamic consistency of the agent's behavior naturally leads to the study of conditional preferences. Accordingly, we examine how a given preference relation can be conditioned on intermediate information, allowing for a coherent update of the associated preference-representing functional.
Conditional preferences might be assumed as primitives (see e.g. Luce and Krantz, 1971; Ghirardato, 2002; Karatzas and Shreve, 1991; Drapeau and Jamneshan, 2016) or rather deduced from the unconditional preference ordering as follows. Considering a preference ordering $\succeq$ and a sub-σ-algebra $\mathcal{G}$, the conditional decision problem of the agent is described by the collection $\{\succeq_A\}_{A \in \mathcal{G}}$ with

$$f \succeq_A g \quad \text{if and only if} \quad f\mathbb{1}_A \succeq g\mathbb{1}_A,$$

where $\mathbb{1}_A$ is the indicator function of the event A. Savage's Sure-Thing Principle plays a central role in ensuring the dynamic consistency of preferences. In this regard, (Gilboa, 2009, page 98) asserts "P2[2] is akin to requiring that you have conditional preferences, namely, preferences between f and g conditional on A occurring, and that these conditional preferences determine your choice between f and g if they are equal in case A does not occur". In Section 1.7 we explore how this principle is a strongly characterizing property for the existence of conditional (consistent) backward projections of the initial ordering relation.

In the theory of intertemporal decision-making, a central role is played by the *certainty equivalent*. The concept of aggregating present utility with uncertain future utility through a certainty equivalent is indeed a recurring theme throughout the economic literature in both discrete time Lucas and Stokey (1984); Epstein and Zin (1989) and continuous time models Duffie and Epstein (1992). A constant a is said to be the certainty equivalent of a random variable f if $f \sim a$. The idea of a dynamic generalization of the static certainty equivalent defined by Pratt (1964), referred to as *conditional certainty equivalent*, is studied in Frittelli and Maggis (2011); Maggis and Maran (2021). Let a sub-σ-algebra $\mathcal{G}$ represents the intermediate information available to the agent, we define the conditional certainty equivalent of f as the $\mathcal{G}$-measurable random variable g such that for any event $A \in \mathcal{G}$ it holds that $g \sim_A f$[3].

Let T be the preference representing functional for a preference $\succeq$, that is

$$T(f) \geq T(g) \quad \text{if and only if} \quad f \succeq g.$$

The decision-theoretical problem can equivalently be formulated as a functional problem, where the preference structure is captured through the functional T. For some bounded random variable f, the certainty equivalent is then the constant a that solves the functional equation $T(f) = T(a)$. Analogously, the conditional certainty equivalent of f given the intermediate information $\mathcal{G}$ is the $\mathcal{G}$-measurable random variable g such that $T(f\mathbb{1}_A) = T(g\mathbb{1}_A)$, for all $A \in \mathcal{G}$. Within this framework, in Theorem 1.7.1 we establish that the Sure-Thing Principle is equivalent to the existence of a conditional certainty equivalent for every sub-σ-algebra $\mathcal{G}$. Equivalently, we shall say that the (preference representing) functional T is conditionable for every sub-σ-algebra $\mathcal{G}$.

The preceding result draws on the theory of generalized means and relies on the characterization of g as the solution to a (conditional) functional equation. Building upon Chisini (1929) seminal idea of defining the expected value as the solution to a functional equation, Doldi and Maggis (2023) extend this concept to a dynamic framework introducing conditional Chisini means. Motivated by the classical results on generalized means in Nagumo (1931); de Finetti (1931a); Kolmogorov (1930), in Section 1.7 we provide a representation of the conditional Chisini mean in the form $\mathbb{u}_{\mathcal{G}}^{-1}(\mathbb{E}_{\mathbb{P}}[\mathbb{u}(f)|\mathcal{G}])$, where $\mathbb{u}_{\mathcal{G}}(\omega, x)$ is the $\mathcal{G}$-measurable projection of the state-dependent utility $\mathbb{u}$. This representation can be easily compared to the general formulation proved in Cerreia-Vioglio et al. (2011), Lemma 5.2, under additional law invariance assumptions, which extends the celebrated Nagumo-de Finetti-Kolmogorov Theorem (see Theorem 1.2.4).

[2] the "Sure-Thing Principle"

[3] $\sim_A$ represents the equivalence relation induced by the conditional preference $\succeq_A$, i.e. $g\mathbb{1}_A \sim f\mathbb{1}_A$ for some $A \in \mathcal{G}$

The final contributions of Chapter 1 pertains nonlinear expectations. The formulation of the conditional Chisini mean as the solution of a functional equation in fact provides a natural link between our theoretical results and the theory of nonlinear expectations.

Namely, Theorem 1.11.2 characterizes the families of *consistent* conditional nonlinear expectations as those that admit a representation in terms of a conditional certainty equivalent with respect to a state-dependent utility function. Indeed, we mention that Kupper and Schachermayer (2009) have shown that, under suitable assumptions on the probabilistic structure, every consistent family of strictly monotone, continuous and law invariant nonlinear maps can be represented as a conditional certainty equivalent $u^{-1}\left(\mathbb{E}_{\mathbb{P}}\left[u(X)|\mathcal{F}_t\right]\right)$, for a suitable deterministic utility function u. Hence our theorem can be seen as a version of this result relaxing the law-invariance and without particular attention on the underlying probability structure.
Notably, non law-invariant nonlinear expectations arise naturally in the theory of Backward Stochastic Differential Equations, through g-expectations. These nonlinear expectations, introduced and developed in Peng (1997); Coquet et al. (2002); Peng (2004), are typically limited to conditioning with respect to a Brownian filtration, therefore hindering the flexibility of information to be considered.

I.2 On consistency of optimal portfolio choice for state-dependent exponential utilities

In Chapter 2, we examine an inconsistency phenomenon arising in the context of optimal portfolio choice. In this framework, an agent maximizes their terminal utility by investing in a financial market. Numerous approaches to optimal investment have been proposed since the seminal contributions of Samuelson (1969) and Merton (1969), which extended this theory beyond Markowitz (1952) one-period model.

In a multi-period setting, an optimal strategy is an investment plan that ensures the agent maximizes a function of their terminal wealth. For an agent endowed with utility function $\mathbb{u} : \mathbb{R} \to \mathbb{R}$, the typical optimal portfolio problem reads

$$\text{maximize } \mathbb{E}_{\mathbb{P}}\left[\mathbb{u}(V_T^\alpha)\right] \text{ over } \alpha \in \mathcal{A}, \text{ constrained to } V_0^\alpha = x \in \mathbb{R},$$

where V_t^α denotes the value of the portfolio at time t resulting from the strategy α, x is the initial wealth of the agent and $\mathcal{A}$ is a suitable family of admissible investment strategies. These problems are typically approached through stochastic optimal control techniques, which presuppose a fixed terminal time T and determine optimal strategies via backward induction. This methodology fundamentally relies on the *time consistency* property ensured by the *Bellman optimality principle* (Bellman, 1957, Page 83):

"An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."

In other words, if some strategy α is optimal on (t, T), then it is optimal also on the subinterval (s, T) for $s > t$.

Ideally, such a strategy is determined at the beginning of the trading period and is intended to be followed "*consistently*" until the specified terminal date. However, the optimization problem may be *time inconsistent*, meaning that the currently devised optimal strategy may not remain optimal in subsequent periods. Consequently, it is not universally clear what optimality should signify in such contexts.

It is well-known that certain optimization problems exhibit inherent time inconsistency. Examples include investment problems involving non-exponential discounting (often referred to as "present bias") as in Ekeland and Pirvu (2008), or when the preferences of the agent are represented by a mean-variance utility (see for example Cerreia-Vioglio et al., 2024). A comprehensive treatment of time inconsistent optimization problems is given in Björk et al. (2021). From a microeconomic perspective, time inconsistent investment strategies may stem from behavioral biases (Clotfelter and Cook, 1993) or rational inattention (Sims, 2003; Abel et al., 2013), for instance.

The literature addresses time inconsistency in three primary approaches, which we briefly describe:

precommitment: At inception, say t_0, the agent commits to the optimal strategy α^* that maximizes the objective functional at the terminal time T. At any later point in time $t_0 < t < T$ the agent simply disregards the fact that the strategy α^* is not optimal anymore. This approach has been extensively studied in Li and Ng (2000); Zhou and Li (2000); Jin and Zhou (2008);

myopic behaviour: At any time $t \in (t_0, T)$ the myopic agent revises the strategy α^*, devising a new strategy which however is optimal for that specific point in time only. They thus roll over a continuously updated sequence of pre-committed optimal strategy. This approach, also knwon as "dynamically optimal", has been studied in Pedersen and Peskir (2016, 2017) and Pollak (1968);

consistent planning: This notion, introduced by Strotz (1955), defines the optimal strategy as *"the best plan among those that [the agent] will actually follow"*. In this framework, inconsistency is addressed through a game-theoretical approach, viewing the problem as an intrapersonal non-cooperative game. Each player represents a future incarnation of the preferences of the agent, and the resulting Nash subgame-perfect equilibrium points yield the optimal strategy. Some relevant works in this direction are Ekeland and Lazrak (2010); Basak and Chabakauri (2010); Björk et al. (2014, 2017).

The relevance of *consistent* strategies is effectively explained by the following scenario: Consider an agent devising their optimal strategy today with a certain investment horizon in mind. After some time, the market undergoes a downturn (e.g. dot-com bubble, subprime mortgage crisis, COVID-19 outbreak) with clear repercussions on the investment outlook. At this point, the precommitted agent would not be able to revise their strategy possibly facing huge losses. On the other hand, the consistent agent would assimilate the information arising from the market scenario and adjust the strategy ahead accordingly, efficiently navigating the varying economic outlook.

As we previously pointed out, classical studies focus on maximizing terminal expected utility for a prescribed terminal date T. Whenever the strategies are time-consistent, one can indeed consider the original problem the restriction to $[0, T]$ of an

analogous problem with terminal date $T' > T$. If this is not the case, however, the concept of optimality for a strategy is inherently tied to the specific terminal date. This feature is however not desirable in practice as the agent may decide to either quit investing earlier than previously envisaged or further extend the investment period. Such situations typically emerge in investment problems with rolling horizons or when the investment horizon is adjusted in response to new capital inflows or evolving market opportunities.
To account for this feature, more recent literature introduces an alternative, maturity-independent framework. Specifically, Henderson and Hobson (2007) focus on a (potentially) infinite-horizon mixed optimal investment/optimal stopping problem and provide conditions for the objective functional to be horizon-unbiased. A different approach, introduced by Musiela and Zariphopoulou (2006, 2008, 2009), models the utility function as a stochastic process that evolving in time. Known as *forward performance* theory, this framework mirrors the classical indirect utility function with a forward utility criterion derived from the financial market, but extends it beyond the paradigm of a fixed terminal date. The stochastic utility is deduced in order to satisfy specific (super)martingale conditions to recover the Dynamic Programming Principle. A recent contribution to the theory of forward performances in Källblad et al. (2014) incorporates ambiguity in model specification and develops the corresponding duality theory.

In Chapter 2, we approach portfolio choice in a simple market model from a "forward" perspective, focusing primarily on the issue of consistency. In contrast to the forward performance framework, however, our methodology focuses on identifying optimal strategies that can be consistently extended beyond a fixed terminal date without compromising the agent's ability to reach the optimum.
We adopt the seminal notion of consistency by Kydland and Prescott (1977), p. 475, adapting it to our setup (see Definition 2.2.4). A policy α^* is consistent if, "for any time period t, it optimizes the objective functional, taking as given previous decisions and that future policy decisions are similarly selected".

The contribution of Chapter 2 is twofold. In section 2.3 we argue that inconsistency may arise in the classical Merton problem whenever the market price of risk exhibits sufficient randomness. This result is shown for an agent whose preferences are of the exponential type and of the power type. Remarkably, for logarithmic utility functions, the Merton problem is always consistent.

In section 2.5, in order to mitigate the inconsistency rooted in the optimization problem, we assumed the agent's preferences are modeled through a state-dependent utility function. Motivated by the theory of forward performances, we assume that the agent preferences evolve over time in a forward manner. Specifically, the agent's risk aversion evolves under the risk-neutral measure according to the stochastic differential equation

$$\mathrm{d}\frac{1}{\gamma_t} = \frac{1}{\gamma_t}\left(\eta_t \mathrm{d}t + \beta_t \, \mathrm{d}W_t^{\mathbb{Q}}\right).$$

A similar perspective is adopted in Desmettre and Steffensen (2023), although in that setting the random risk-aversion is a single random variable. Theorem 2.5.2, the main result of this chapter, establishes that for exponential-type utilities consistency is equivalent to requiring that the agent's risk aversion evolves according to a unique martingale dynamic, matching in this way the form obtained in (Žitković, 2009, Theorem 4.4). Similarly to our paper, Žitković (2009) focuses on random fields with an exponential structure and provides necessary and sufficient conditions for the self-generation of preferences over time, as well as the related duality theory. Further, in Proposition 2.5.4, we show that, although the consistency condition restricts the dynamics of $1/\gamma_t$ to a single choice, even within a simplified market model an infinite class of forward performances of the exponential type can be readily constructed. In particular, we address the circular relationship between preferences and optimal strategy that this result suggests. Thus, consistency emerges as a crucial criterion for inferring the agent's preferences "adjusted" to the temporal evolution of market information.

Finally, in Section 2.8, we provide an alternative class of state-dependent utility functions. In this somewhat näive example, a deterministic utility function is perturbed by a stochastic martingale noise. As a result, the agent only has access to a distorted measurement of their underlying utility function. Although the specification of the diffusion coefficient in the martingale noise may reflect certain economic modeling considerations, imposing that the problem be consistent forces the noise process to suitably offset the randomness of the market. Moreover, the unique specification in which the state-dependent utility verifies the forward performance criterion results in the agent evaluating investment opportunities under the risk-neutral measure, thus investing only in the risk-less asset.

Chapter 1

On conditioning and consistency for nonlinear functionals

In this chapter we consider a family of nonlinear conditional expectations $\{\mathcal{E}_{\mathcal{G}}\}_{\mathcal{G}\in\Sigma}$ on the space of bounded random variables $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ indexed by the collection Σ of all sub-σ-algebras of $\mathcal{F}$. In light of the application of nonlinear expectations to, for example, risk measures (Cohen, 2010; Di Nunno and Rosazza Gianin, 2024) and g-expectations (Peng, 1997; Coquet et al., 2002), consistent conditioning is a remarkably important property. We show that whenever such family satisfies a natural consistency property, similar to the classical *tower* property of conditional expectations, then the family can be represented as a conditional certainty equivalent for a suitable choice of state-dependent utility function. The representation provided in Theorem 1.11.2 can be seen as a dynamic extension of the celebrated Nagumo-de Finetti-Kolmogorov Theorem. Similar results include the representation result of Kupper and Schachermayer (2009) for time-consistent and law-invariant functionals.

We obtain such a representation by recasting the problem in a decision-theoretical setting. In particular, the $\mathcal{G}$-conditional nonlinear expectation of some $\mathcal{F}$-measurable function X can be thought of as the $\mathcal{G}$-equivalent of X for some preference ordering $\succeq$ on the space of bounded random variables. From this perspective, a new characterization of Savage (1954) "Sure-Thing Principle" is provided, formulated in terms of the *conditionability* of the preference representing functional.

We build on state-dependent expected utility theory, introduced by Debreu (1960) for finite state-space and extended by Wakker and Zank (1999) to the space of bounded random variables. A positive answer to a conjecture posed by Wakker and Zank (1999) regarding the regularity of a class of state-dependent utility functions completes the chapter and allows for the construction of the machinery for the following arguments. Such regularity pertains pointwise continuity and joint measurability, particularly relevant properties for applications.

Chapter 1 is structured as follows: in Section 1.1 we set the notation. In Section 1.2 we recall some results on generalized means and functional representation which are central to the representation results to be studied. Section 1.3 fixes the setup for preference orderings and introduces (conditional) certainty equivalents and their relationship with (conditional) Chisini means. Section 1.4 presents Debreu's Theorem, a fundamental result underpinning state-dependent utility theory, along with its extension to infinite dimensional spaces. In Section 1.5 we contribute additional regularity properties to a class of state-dependent utility functions and in Section 1.7 we derive an explicit representation of conditional Chisini means in terms of a conditional certainty equivalent for such state-dependent utilities (after a suitable conditioning). Finally in 1.11 we employ the tools developed in the previous sections to characterize nonlinear expectations that admit consistent conditioning with respect to a family of σ-algebras.

1.1 Notations and preliminaries

We consider a measurable space $(\Omega, \mathcal{F})$, where Ω is a general state space and $\mathcal{F}$ a σ-algebra of subsets of Ω, which we refer to as events. Throughout this chapter Σ denotes the collection of all sub-σ-algebras of $\mathcal{F}$. We denote by $\mathbb{R}$ the set of real numbers, which is always assumed to be a measurable space endowed with the usual Borel σ-algebra $\mathscr{B}_{\mathbb{R}}$. Whenever needed, we indicate by $\overline{\mathbb{R}}$ the extended real line $[-\infty, +\infty]$ and by $\mathscr{B}_{\overline{\mathbb{R}}}$ the associated σ-algebra. We use $<, >$ for strict inequalities between real numbers. Similarly, $\subset, \supset$ are used to indicate proper subsets. In case equalities are not excluded, we use $\leq, \geq$ and $\subseteq, \supseteq$, respectively. We consider, as conventional, $\inf \varnothing = +\infty$.

We denote by $\mathcal{L}(\Omega, \mathcal{F})$ the space of $\mathcal{F}$-measurable functions taking values in $\mathbb{R}$. We shall call an element $f \in \mathcal{L}(\Omega, \mathcal{F})$ a random variable and we denote by $\mathcal{L}^\infty(\Omega, \mathcal{F})$ the collection of random variables f such that $|f(\omega)| \leq k$ for every $\omega \in \Omega$, for some $k \geq 0$. On $\mathcal{L}(\Omega, \mathcal{F})$ and $\mathcal{L}^\infty(\Omega, \mathcal{F})$ we consider the natural pointwise order $f \geq g$ if and only if $f(\omega) \geq g(\omega)$ for all $\omega \in \Omega$. The space $\mathcal{L}^\infty(\Omega, \mathcal{F})$ is a Banach space with respect to the usual sup norm $\|f\|_\infty = \sup_{\omega \in \Omega} |f(\omega)|$. For $A \in \mathcal{F}$, we define $\mathbb{1}_A$ as the element of $\mathcal{L}^\infty(\Omega, \mathcal{F})$ that takes value 1 if $\omega \in A$ and 0 otherwise. We denote by $\sigma(A) = \{A, A^c, \Omega, \varnothing\}$ the σ-algebra generated by $A \in \mathcal{F}$. For any pair $f, g \in \mathcal{L}^\infty(\Omega, \mathcal{F})$, we indicate by $f\mathbb{1}_A + g\mathbb{1}_{A^c}$ the random variable that agrees with f whenever event A occurs and with g otherwise.

Working with the product space $\Omega \times \mathbb{R}$, we shall make use of the product σ-algebra $\mathcal{F} \times \mathscr{B}_{\mathbb{R}}$, i.e. the σ-algebra generated by sets of the form $F \times B$ with $F \in \mathcal{F}$ and $B \in \mathscr{B}_{\mathbb{R}}$.

Whenever we fix a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$, we call $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. An event $N \in \mathcal{F}$ is said to be $\mathbb{P}$ negligible if $\mathbb{P}(N) = 0$. We say that $B \in \mathcal{F}$ is an atom if for every non-empty $C \in \mathcal{F}$ with $C \subset B$ either C or $B \setminus C$ is $\mathbb{P}$ negligible. We say that two elements $f, g \in \mathcal{L}(\Omega, \mathcal{F}, \mathbb{P})$ are equivalent if there exists $N \in \mathcal{F}$ negligible such that $f(\omega) = g(\omega)$ for every $\omega \in N^c$. For such a probability $\mathbb{P}$ on $(\Omega, \mathcal{F})$ we shall denote the space of integrable functions by $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}) = \{f \in \mathcal{L}(\Omega, \mathcal{F}) : \mathbb{E}_{\mathbb{P}}[|f|] < \infty\}$. Finally, when appropriate we will deal with $L^1(\Omega, \mathcal{F}, \mathbb{P})$, the quotient space of $\mathcal{F}$-measurable random variables with respect to $\mathbb{P}$-a.s. equality such that $\mathbb{E}_{\mathbb{P}}[|f|] < \infty$, and $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, the set of bounded elements of $L^1(\Omega, \mathcal{F}, \mathbb{P})$. For $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ we denote by $X = [f]_{\mathbb{P}}$ the equivalence class of random variables $\mathbb{P}$-a.s. equal to f.

A (finite) signed measure ν on $(\Omega, \mathcal{F})$ is said to be dominated by $\mathbb{P}$ ($\nu \ll \mathbb{P}$) if, for any $A \in \mathcal{F}$, $\mathbb{P}(A) = 0$ implies $|\nu|(A) = 0$, where $|\nu|$ is the total variation measure of ν. By the Hahn decomposition theorem, $\nu \ll \mathbb{P}$ is equivalent to: $\mathbb{P}(A) = 0$ implies

$\nu(A) = 0$. Whenever $\nu \ll \mathbb{P}$, the Radon-Nikodym Theorem ensures that there exists $h \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ such that $\nu(A) = \int_A h \, \mathrm{d}\mathbb{P}$ for any $A \in \mathcal{F}$. We shall customarily refer to h as the Radon-Nikodym derivative of ν with respect to $\mathbb{P}$, and we shall often denote it as $\mathrm{d}\nu / \mathrm{d}\mathbb{P}$. Finally, we say ν is equivalent to $\mathbb{P}$ ($\nu \sim \mathbb{P}$) if $\nu \ll \mathbb{P}$ and $\mathbb{P} \ll \nu$. For any $f \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and a sub σ-algebra $\mathcal{G}$, $\mathbb{E}_{\mathbb{P}}[f|\mathcal{G}]$ is the conditional expectation i.e.

$$\mathbb{E}_{\mathbb{P}}[f|\mathcal{G}] = \{g \in \mathcal{L}^1(\Omega, \mathcal{G}, \mathbb{P}) : \mathbb{E}_{\mathbb{P}}[g \mathbb{1}_A] = \mathbb{E}_{\mathbb{P}}[f \mathbb{1}_A] \ \forall A \in \mathcal{G}\}. \tag{1.1}$$

In the following we shall often refer to the following properties of conditional expectation:

- **Tower property**: For $\mathcal{H}, \mathcal{G}, \mathcal{F}$ σ-algebras such that $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$ and $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ then
$$\mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}_{\mathbb{P}}[X|\mathcal{H}],$$
- **"Taking out what is known"**: For $Y \in L^\infty(\Omega, \mathcal{G}, \mathbb{P})$
$$\mathbb{E}_{\mathbb{P}}[X \cdot Y|\mathcal{G}] = \mathbb{E}_{\mathbb{P}}[X|\mathcal{G}] \cdot Y.$$

1.2 On generalized means and functional representations

In order to keep the discussion of the results of this thesis as self-contained as possible, we give here a brief overview of generalized means and their representation. In particular, we shall recall the concept of Chisini means and state (without proof) some results that will be of frequent use in later sections.

In the earlier works on generalized means it was customary to deal with distribution functions (or laws) rather than with random variables. For this reason, we formulate the Nagumo-de Finetti-Kolmogorov Theorem in terms of distribution functions. Note, however, that it is possible to reframe the result using spaces of random variables instead, as pointed out in Remark 1.2.5.

Definition 1.2.1 (Distribution of a random variable)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ a random variable. The* distribution *of X is the non-decreasing, right-continuous function $F : \mathbb{R} \to [0, 1]$ defined as*

$$F(x) := \mathbb{P}(X \leq x).$$

Whenever X has distribution function F we shall write $X \sim F$.

We define the space $\mathcal{D}(a, b)$ of distribution functions on a compact interval $[a, b] \subset \mathbb{R}$ as

$$\mathcal{D}(a, b) := \{F : \mathbb{R} \to [0, 1] : F(x) = 0 \ \forall x < a, \ F(x) = 1 \ \forall x > b\}.$$

In particular we will consider the the set $\mathcal{D}_f(a, b)$ of distribution functions with finite support on the compact interval $[a, b]$[1]:

$$\mathcal{D}_f(a, b) := \left\{ F \in \mathcal{D}(a, b) : F(x) = \sum_{i=1}^{n} y_i \mathbb{1}_{[x_i, +\infty)}(x), \text{ for some } n \in \mathbb{N} \right\}$$

[1] which corresponds to the space of bounded simple random variables taking values in $[a, b]$.

We can now provide the formal definition of *generalized means*:

Definition 1.2.2 (Generalized means). *A function* $\mathsf{m} : \mathcal{D}_f(a,b) \to \mathbb{R}$ *is a* generalized mean *if*

(i) $\mathsf{m}(\mathbb{1}_{[\overline{x},+\infty)}) = \overline{x}$ *for any* $\overline{x} \in (a,b)$,

(ii) Given $F_1, F_2 \in \mathcal{D}_f(a,b)$ *such that* $F_1(x) \leq F_2(x)$ *for all* $x \in \mathbb{R}$ *and* $F_1(\overline{x}) < F_2(\overline{x})$ *for some* $\overline{x} \in \mathbb{R}$, *then* $\mathsf{m}(F_1) > \mathsf{m}(F_2)$,

(iii) Given $F_1, F_2, F_3 \in \mathcal{D}_f(a,b)$ *such that* $\mathsf{m}(F_1) = \mathsf{m}(F_2)$ *then*

$$\mathsf{m}(\lambda F_1 + (1-\lambda)F_3) = \mathsf{m}(\lambda F_2 + (1-\lambda)F_3) \quad \forall \lambda \in [0,1].$$

Remark 1.2.3. Property (i) concerns the behaviour of m on degenerate distributions with mass in $\overline{x}$ and (ii) asserts the monotonicity of m with respect to first order stochastic dominance. On the other hand, (iii) states that, given F_1, F_2, G_1, G_2 such that $\mathsf{m}(F_1) = \mathsf{m}(F_2)$ and $\mathsf{m}(G_1) = \mathsf{m}(G_2)$ it follows that $\mathsf{m}(\lambda F_1 + (1-\lambda)G_1) = \mathsf{m}(\lambda F_2 + (1-\lambda)G_2)$ for all $\lambda \in [0,1]$. That is, $\mathsf{m}(\lambda F_1 + (1-\lambda)G_1)$ is determined uniquely by $\mathsf{m}(F_1), \mathsf{m}(G_1), \lambda$.

The celebrated Nagumo-de Finetti-Kolmogorov Theorem provides an integral characterization of generalized means on finitely supported distributions and represents a crucial building block for our ensuing results.

Theorem 1.2.4 (Nagumo (1931), de Finetti (1931a), Kolmogorov (1930)). *The following are equivalent:*

- $\mathsf{m} : \mathcal{D}_f(a,b) \to \mathbb{R}$ *satisfies (i), (ii), (iii),*
- *there exists a function* $\phi : [a,b] \to \mathbb{R}$ *continuous and strictly increasing such that*

$$\mathsf{m}(F) = \phi^{-1} \circ \int_{[a,b]} \phi \, \mathrm{d}F.$$

Remark 1.2.5. Extending the representation result of Theorem 1.2.4 to infinite dimensional spaces requires the problem to be reformulated in terms of random variables rather than distributions. It is well known that the space of distribution functions on the real line is in one-to-one correspondence with the space of probability measures on the real line (see Billingsley, 2012, pages 175-176). Moreover, *Skorohod's Theorem* (Billingsley, 2012, Theorem 25.6) ensures that for μ_n, μ probability measures on $(\mathbb{R}, \mathscr{B}_\mathbb{R})$ such that $\mu_n \to \mu$ weakly there exist random variables Y_n, Y on a common non atomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $Y_n \sim \mu_n$ and $Y \sim \mu$[2] and $Y_n(\omega) \to Y(\omega)$ for all $\omega \in \Omega$.

In the following sections, we shall often deal with functionals $T : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$, as they allow to project decision-theoretical problems on $\mathbb{R}$. We therefore provide the definition of "law-invariance", a property that a functional might enjoy and that will play a central role in the subsequent results.

[2]With a slight abuse of notation, here the symbol $\sim$ means "has law"

Definition 1.2.6. *We say $T : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ is* law-invariant *if for any pair $X, Y \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ such that $X \sim F, Y \sim F$ we have $T(X) = T(Y)$.*

Lemma 1.2.7 (Cerreia-Vioglio et al. (2011))**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be non atomic. A law-invariant functional $T : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ satisfies:*

- *$T(a\mathbb{1}_\Omega) = a$ for all $a \in \mathbb{R}$,*
- *$T(X) \leq T(Y)$ for any $X, Y \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ such that $X \leq Y$,*
- ***Sure-Thing Principle**: for $A \in \mathcal{F}$ and $X, Y, Z, \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ such that*

  $$T(X\mathbb{1}_A + Z\mathbb{1}_{A^c}) \geq T(Y\mathbb{1}_A + Z\mathbb{1}_{A^c}),$$

  we have

  $$T(X\mathbb{1}_A + \overline{Z}\mathbb{1}_{A^c}) \geq T(Y\mathbb{1}_A + \overline{Z}\mathbb{1}_{A^c})$$

  for any $\overline{Z} \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$,
- *for any uniformly bounded sequence $(X_n)_{n\in\mathbb{N}} \subseteq L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ such that $X_n \to X$ $\mathbb{P}$-a.s. then $T(X_n) \to T(X)$,*

if and only if there exists a strictly increasing and continuous $u : \mathbb{R} \to \mathbb{R}$ such that

$$T(X) = u^{-1}\, \mathbb{E}_\mathbb{P}\left[u(X)\right] \quad \forall\, X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$$

Remark 1.2.5 establishes the equivalence between dealing with random variables and with distributions. Accordingly, throughout the remainder of this section, we shall formulate our arguments in terms of random variables.

Chisini (1929) introduces the concept of mean as the solution of a functional equation induced by some functional T of choice. We formalize the concept below as it will regularly appear in the subsequent discussion. Note that, for the purposes of this thesis, the choice of functionals in the following definition is restricted to those defined on the space of bounded random variables. It can however be generalized to functionals defined on vector spaces of measurable random variables satisfying mild requirements[3].

Definition 1.2.8 (Chisini mean)**.** *Given a functional $T : \mathcal{L}^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$ the* Chisini mean *of $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$, denoted $\mathfrak{m}\,(f) \in \mathbb{R}$, is the solution to the functional equation*

$$T(f) = T(\mathfrak{m}\,(f)\, \mathbb{1}_\Omega) \tag{1.2}$$

Existence and uniqueness of such Chisini mean is provided by (Doldi and Maggis, 2023, Proposition 2.1) and depends only on the regularity properties of the restriction of the functional T to constant random variables.

Proposition 1.2.9. *Assume that $T : \mathcal{L}^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$ is such that $T(-\|f\|_\infty) \leq T(f) \leq T(\|f\|_\infty)$ and the mapping $\mathbb{R} \ni a \mapsto T(a\mathbb{1}_\Omega)$ is continuous and strictly increasing. Then for any $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ there exists a unique $\mathfrak{m}\,(f) \in \mathbb{R}$ such that $T(f) = T(\mathfrak{m}\,(f)\, \mathbb{1}_\Omega)$.*

[3] i.e. vector spaces $\mathcal{X}$ such that: for each $A \in \mathcal{F}$ $\mathbb{1}_A$ belongs to $\mathcal{X}$ and $f\mathbb{1}_A \in \mathcal{X}$ whenever $f \in \mathcal{X}$.

We conclude the section by mentioning a result by Kupper and Schachermayer (2009) which can be seen as the dynamic extension of the Nagumo-de Finetti- Kolmogorov Theorem and hints at the representation of time-consistent functionals in terms of conditional certainty equivalents. The reader will observe that the representation that we provide in Section 1.11 is in fact comparable in spirit to the one in Theorem 1.2.12, although the functional that we present relies on a state-dependent utility function and we need only assume that the sample space Ω has three disjoint non-null events.

Until the end of the section we denote by $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\in\mathbb{N}_0}, \mathbb{P})$ a standard filtered probability space, that is: $(\Omega, \mathcal{F}, \mathbb{P})$ is isomorphic to $[0,1]^{\mathbb{N}_0}$ equipped with its Borel σ-algebra $\mathscr{B}_{[0,1]^{\mathbb{N}_0}}$ and the product of Borel measures $\lambda^{\mathbb{N}_0}$. The filtration $\{\mathcal{F}\}_{t\in\mathbb{N}_0}$ is generated by the coordinate functions.
Additionally, we need the following notions of continuity and time-consistency. Let $b^{\varepsilon,p}$ denote a Bernoulli random variable taking the values $+\varepsilon$ and $-\varepsilon$ with probabilities p and $1-p$.

Definition 1.2.10. *A functional $f : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ satisfies the continuity condition (C) if for any $\varepsilon > 0$ and all $x \in (a,b)$ with $a+\varepsilon < x < b-\varepsilon$ there is $p = p(x) \in (0,1)$ such that*

$$f(x + b^{\varepsilon,p}) < x.$$

Consider now a (discrete) filtration $\{\mathcal{F}_t\}_{t\in\mathbb{N}}$ on the reference probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We point out that the following definition of *time-consistent* function anticipates Definition 1.11.1 for consistent nonlinear expectations. While closely related, these two definitions are somewhat different. While the latter pertains a general family of sub-σ-algebras, the time-consistency in the former is related to filtrations and is therefore inherently ordered.

Definition 1.2.11 (Time-consistent function)**.** *A functional $c_0 : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ is said to be* time-consistent *if for each $t \in \mathbb{N}$ there exists a mapping $c_t : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to L^\infty(\Omega, \mathcal{F}_t, \mathbb{P})$ which satisfies the local property*

$$\mathbb{1}_A X = \mathbb{1}_A Y \text{ implies } \mathbb{1}_A c_t(X) = \mathbb{1}_A c_t(Y) \text{ for all } A \in \mathcal{F}_t, X, Y \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$$

and

$$c_0(X) = c_0(c_t(X)) \text{ for all } X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}).$$

Theorem 1.2.12. *A functional $c_0 : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ is normalized on constants, strictly monotone, $\|\cdot\|_\infty$-continuous, law invariant, time-consistent and satisfies condition (C) if and only if*

$$c_0(X) = u^{-1} \circ \mathbb{E}_{\mathbb{P}}[u(X)],$$

for a strictly increasing, continuous function $u : \mathbb{R} \to \mathbb{R}$. In this case, the function u is uniquely defined up to affine transformations of the form $u \mapsto \alpha u + \beta$, $\alpha > 0, \beta \in \mathbb{R}$ and

$$c_t(X) = u^{-1} \circ \mathbb{E}_{\mathbb{P}}[u(X)|\mathcal{F}_t] \qquad \text{for all } t \in \mathbb{N}.$$

1.3 Conditioning of preferences and their numerical representation

We now introduce the decision theoretical setup adopted in Wakker and Zank (1999), which is the natural generalization of Debreu (1960) for infinite state spaces. As discussed further on, preference orderings naturally induce functionals and provide an intuitive application to Chisini means. We then extend the latter to a dynamic setting to allow for considerations in terms of conditional preferences.
In this thesis, a preference relation is a binary relation $\succeq$ on $\mathcal{L}^\infty(\Omega,\mathcal{F})$ which satisfies completeness and transitivity and induces a (strict) preference order $\succ$ by $f \succ g$ if and only if $f \succeq g$ but $g \not\succeq f$. In addition we set $g \sim f$ whenever $g \succeq f$ and $f \succeq g$. We denote by

$$\mathcal{N}_{\succeq} := \{A \in \mathcal{F} : f\mathbb{1}_A + g\mathbb{1}_{A^c} \sim g \ \ \forall f, g \in \mathcal{L}^\infty(\Omega,\mathcal{F})\} \tag{1.3}$$

the set of null events (induced by the preference $\succeq$).
We present a few additional properties that the preference relation might enjoy.

(SM) $\succeq$ is strictly monotone if $x\mathbb{1}_A + f\mathbb{1}_{A^c} \succ y\mathbb{1}_A + f\mathbb{1}_{A^c}$, for all non null events $A \in \mathcal{F}$, for all $f \in \mathcal{L}^\infty(\Omega,\mathcal{F})$ and constant outcomes $x, y \in \mathbb{R}$ with $x > y$;

(ST) $\succeq$ satisfies the Sure-Thing Principle whenever, considering arbitrary $f, g, h \in \mathcal{L}^\infty(\Omega,\mathcal{F})$ and $A \in \mathcal{F}$ such that $f\mathbb{1}_A + h\mathbb{1}_{A^c} \succeq g\mathbb{1}_A + h\mathbb{1}_{A^c}$, we have $f\mathbb{1}_A + \tilde{h}\mathbb{1}_{A^c} \succeq g\mathbb{1}_A + \tilde{h}\mathbb{1}_{A^c}$ for every $\tilde{h} \in \mathcal{L}^\infty(\Omega,\mathcal{F})$;

(NC) $\succeq$ is norm continuous if $\forall f \in \mathcal{L}^\infty(\Omega,\mathcal{F})$ the sets $\{g \in \mathcal{L}^\infty(\Omega,\mathcal{F}) : g \succeq f\}$ and $\{g \in \mathcal{L}^\infty(\Omega,\mathcal{F}) : f \succeq g\}$ are $\|\cdot\|_\infty$-closed;

(PC) $\succeq$ is pointwise continuous if, for any uniformly bounded sequence $(f_n)_n \subseteq \mathcal{L}^\infty(\Omega,\mathcal{F})$, such that $f_n(\omega) \to f(\omega)$ for any $\omega \in \Omega$ then for all $g \in \mathcal{L}^\infty(\Omega,\mathcal{F})$ such that $g \succ f$ (resp. $f \succ g$) there exists $N \in \mathbb{N}$ such that $g \succ f_n$ (resp. $f_n \succ g$) for every $n > N$.

Given a preference order $\succeq$ on $\mathcal{L}^\infty(\Omega,\mathcal{F})$, then the certainty equivalent of $f \in \mathcal{L}^\infty(\Omega,\mathcal{F})$ is $a \in \mathbb{R}$ such that $a \sim f$. We say that $\succeq$ admits a numerical representation if there exists a functional $T : \mathcal{L}^\infty(\Omega,\mathcal{F}) \to \mathbb{R}$ such that for any given $f, g \in \mathcal{L}^\infty(\Omega,\mathcal{F})$

$$T(f) \geq T(g) \quad \text{ if and only if } \quad f \succeq g. \tag{1.4}$$

Remark 1.3.1. We observe that (SM) and (PC) of the preference $\succeq$ are sufficient to show the existence of a preference representing functional T. Indeed, (SM) and (PC) imply the pointwise monotonicity of $\succeq$ on $\mathcal{L}^\infty(\Omega,\mathcal{F})$ (see Lemma A.1). Consequently, for any $f \in \mathcal{L}^\infty(\Omega,\mathcal{F})$ the sets $\underline{F} := \{x \in \mathbb{R} : f \succeq x\}$ and $\overline{F} := \{x \in \mathbb{R} : x \succeq f\}$ are non empty closed halflines in $\mathbb{R}$ such that $\overline{F} \cup \underline{F} = \mathbb{R}$. Then $\overline{F} \cap \underline{F} = \{z\}$, with z being the (unique) certainty equivalent of f, i.e. $f \sim z$, and we can define $T(f) := z$.

Based on the previous remark, which is a version of Wakker and Zank (1999), Lemma 8, we can (essentially) always define a representing functional T from a preference relation $\succeq$, and conversely, we can introduce a preference relation $\succeq$ from T

via the relation in (1.4). Moreover, the properties of T (respectively, $\succeq$) are inherited by $\succeq$ (respectively, T), as will be clarified in the following sections. Considered the nature of the Chisini mean as a solution to the functional equation in (1.2), whenever T is a preference representing functional one can naturally make sense of the Chisini mean as a certainty equivalent. Then for $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$, $\mathfrak{m}(f) \in \mathbb{R}$ represents the certain amount such that $f \sim \mathfrak{m}(f)$.

The static decision problem described before naturally extends to a dynamic one whenever the agent is provided with additional intermediate information, represented by the sub-σ-algebra $\mathcal{G} \subseteq \mathcal{F}$. In particular, suppose the agent learns that an event $A \in \mathcal{G}$ has occurred and updates her preferences accordingly, resulting in a conditional preference $\succeq_A$ defined as

$$f \succeq_A g \text{ if and only if } f\mathbb{1}_A \succeq g\mathbb{1}_A.$$

The class of preferences $\{\succeq_A\}_{A\in\mathcal{G}}$ then describes the conditional decision problem. In particular, observe that if $\succeq$ satisfies (ST), for $f, g \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ and $A \in \mathcal{G}$

$$f \succeq_A g \text{ implies } f\mathbb{1}_A + \overline{h}\mathbb{1}_{A^c} \succeq g\mathbb{1}_A + \overline{h}\mathbb{1}_{A^c}$$

for any $\overline{h} \in \mathcal{L}^\infty(\Omega, \mathcal{F})$. That is, the choice of $\overline{h}$ is irrelevant for the conditional decision problem. We can introduce the notion of conditional certainty equivalent as follows:

Definition 1.3.2 (Conditional certainty equivalent)**.** *For any σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, the* $\mathcal{G}$-conditional certainty equivalent *of $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ is any $g \in \mathcal{L}^\infty(\Omega, \mathcal{G})$ such that*

$$f\mathbb{1}_A \sim g\mathbb{1}_A \quad \forall\, A \in \mathcal{G}. \tag{1.5}$$

Loosely speaking, $g \in \mathcal{L}^\infty(\Omega, \mathcal{G})$ can be considered as a $\mathcal{G}$-measurable counterpart of the $\mathcal{F}$-measurable function $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ (see also Frittelli and Maggis (2011) and Maggis and Maran (2021) for applications of the conditional certainty equivalent).

At this point, pivoting on the analogy between Chisini means and certainty equivalent, the definition in (1.2) can be extended to a dynamic setting. Doldi and Maggis (2023) introduces the notion of *conditional Chisini mean* as the solution of an infinite dimensional system of equations as we now recall in the following definition.

Definition 1.3.3 (Conditional Chisini mean)**.** *Consider $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ and $\mathcal{G} \subseteq \mathcal{F}$ a sub-σ-algebra. For a given functional $T : \mathcal{L}^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$ the* conditional Chisini mean*, denoted by $\mathfrak{m}(f|\mathcal{G})$, is the family of solutions $g \in \mathcal{L}^\infty(\Omega, \mathcal{G})$ to*

$$T(f\mathbb{1}_A) = T(g\mathbb{1}_A) \quad \forall\, A \in \mathcal{G}. \tag{1.6}$$

Definition 1.3.4. *For a given functional $T : \mathcal{L}^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$, we say that T is conditionable on a sub σ-algebra $\mathcal{G} \subseteq \mathcal{F}$ if for any $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ it holds that $\mathfrak{m}(f|\mathcal{G}) \neq \varnothing$. We say that T is conditionable if it is conditionable on any $\mathcal{G} \in \Sigma$*

Remark 1.3.5. Observe that the conditional certainty equivalent and the conditional Chisini mean produce the same class of solutions, namely

$$\mathfrak{m}(f|\mathcal{G}) = \{g \in \mathcal{L}^\infty(\Omega, \mathcal{G}) \mid f\mathbb{1}_A \sim g\mathbb{1}_A \,\forall\, A \in \mathcal{G}\}, \tag{1.7}$$

keeping in mind that Eq. (1.7) holds as soon as we characterize a preference $\succeq$ through a representing functional (as defined in (1.4)). In this way we can translate automatically the results of the present framework in those of Doldi and Maggis (2023).

If compared with (1.1), it evidently appears that the conditional Chisini mean $\mathfrak{m}(f|\mathcal{G})$ is the nonlinear generalization of the standard conditional expectation. Considering a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and letting $T : \mathcal{L}^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$ be the Lebesgue integral with respect to $\mathbb{P}$, that is $T : f \mapsto \mathbb{E}_{\mathbb{P}}[f]$, we have trivially that $\mathfrak{m}(f) = \mathbb{E}_{\mathbb{P}}[f]$. Analogously, for a sub-σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, the conditional Chisini mean $\mathfrak{m}(f|\mathcal{G})$ of $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ coincides with the conditional expectation $\mathbb{E}_{\mathbb{P}}[f|\mathcal{G}]$. Any element $g \in \mathbb{E}_{\mathbb{P}}[f|\mathcal{G}]$ is in fact by definition a solution to (1.6).

Although linearity is lost, nonlinear expectation still enjoys few useful properties of its linear counterpart such as, for instance, the "tower" and "taking out what is known" properties. Before the abstract formulation in Doldi and Maggis (2023), we remark that the problem of existence of nonlinear conditional expectation was already addressed in Coquet et al. (2002), yet the provided solution relies on a Brownian framework which in turn demands for more technical assumptions.

1.4 Preliminary results on state-dependent utility

This section provides some preliminary results in the theory of state-dependent utility. In particular, we recall the seminal contribution of Debreu (1960), who introduced the framework of state-dependent utility functions through his characterization of additively decomposable preference-representing functionals.

Theorem 1.4.1 (Debreu (1960), page 10)**.** *Let $\mathcal{L}^\infty(\Omega, \mathcal{F})$ be the set of acts and $\succeq$ a preference relation on it. Let the state space $\Omega = \{\omega_1, ..., \omega_N\}$, where at least three states are non-null. Then the following two statements are equivalent:*

1. *There exist n continuous functions $V_j : \mathbb{R} \to \mathbb{R}$, $j = 1, ..., N$, that are strictly increasing for all nonnull states and constant for all null states, and such that $\succeq$ is represented by*
$$V(f) = \sum_{j=1}^{N} V_j(f(\omega_j)). \tag{1.8}$$
2. *$\succeq$ is a preference order satisfying (NC), (SM) and (ST).*

The following uniqueness holds for (1.8): $W(f) = \sum_{j=1}^{N} W_j(f(\omega_j))$ represent $\succeq$ if and only if there exist $\tau_1, ..., \tau_n \in \mathbb{R}$ and $\sigma > 0$ such that $W_j = \tau_j + \sigma V_j$ $\forall j$, implying that $W = \tau + \sigma V$ for $\tau = \tau_1 + ... + \tau_N$.

For a general state space $(\Omega, \mathcal{F})$, Wakker and Zank (1999) provide a characterization of those preference relations $\succeq$ on the space of bounded random acts $\mathcal{L}^\infty(\Omega, \mathcal{F})$ which admit a numerical representation through an additively decomposable functional (Wakker and Zank, 1999, Theorem 11). This result extends Theorem 1.4.1 to an infinite dimensional setup. We report it below, in a slightly rephrased version.

Theorem 1.4.2 (Wakker and Zank (1999), Theorem 11)**.** *Let $\Pi(\mathcal{F})$ denote the class of finite partitions of Ω into $\mathcal{F}$-measurable sets such that at least three disjoint events are non-null. Assume that $\Pi(\mathcal{F}) \neq \varnothing$ and $\succeq$ satisfies (SM), (NC) and (ST). Then, there exist a functional $T : \mathcal{L}^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$ such that:*

(i) T is finitely additive, i.e. for each $\pi \in \Pi(\mathcal{F})$ there exist functionals $T_A : \mathcal{L}^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$, $A \in \pi$ such that

$$T(f) = \sum_{A \in \pi} T_A(f \mathbb{1}_A),$$

(ii) T is $\|\cdot\|_\infty$-continuous,

(iii) for each $A \in \pi, x \in \mathbb{R}$, $T_A(x\mathbb{1}_A)$ is either constant or strictly increasing in x,

and given $f, g \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ we have

$$f \succeq g \text{ if and only if } T(f) \geq T(g) \tag{1.9}$$

The proof of the previous theorem in its entirety requires a few auxiliary results and is outside the scope of this thesis. We provide a brief sketch of the proof and refer the interested reader to the aforementioned work (pages 23-26) for an exhaustive discussion.

Sketch of the proof. The proof articulates in the following steps:

1. For some partition $\pi = \{A_1, ..., A_n\}$ of Ω with at least three disjoint non-null events and a preference relation $\succeq$ satisfying (SM), (NC) and (ST), Theorem 1.4.1 guarantees the existence of continuous and strictly increasing (on non-null events) functions $T_{A_i} : \mathbb{R} \to \mathbb{R}$ such that the functional

$$T : \sum_{i=1}^{n} x_i \mathbb{1}_{A_i} \mapsto \sum_{i=1}^{n} T_{A_i}(x_i \mathbb{1}_{A_i})$$

   represents $\succeq$ on the space of $\mathcal{F}$-measurable simple functions.

2. The functional T derived in Item 1 can be shown to be additively decomposable for any choice of partition $\pi \in \Pi(\mathcal{F})$.

3. By (NC) and and (SM) of the preference $\succeq$, each $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ has a unique certainty equivalent $\phi \in \mathbb{R}$, i.e. $\phi \sim f$. The preference representing functional T is then extended to $\mathcal{L}^\infty(\Omega, \mathcal{F})$ by setting $T(f) := T(\phi)$.

4. The functional T extended as in Item 3 represents $\succeq$ on $\mathcal{L}^\infty(\Omega, \mathcal{F})$ and continuity of $\succeq$ ensures it is $\|\cdot\|_\infty$-continuous.

5. Finally, $\|\cdot\|_\infty$-continuity and additivity on simple functions imply the extended functional T is finitely additive for any partition $\pi \in \Pi(\mathcal{F})$ with each T_A strictly increasing in $x \in \mathbb{R}$ for $A \notin \mathcal{N}_\succeq$.

□

1.5 An open question in Wakker and Zank (1999)

The results that we are going to provide in the remainder of this chapter mostly rely on an enhanced version of the preference representation result due to Wakker and Zank (1999), which we are now ready to present in detail given the preliminary theorems of the previous section.
By replacing sup norm continuity with the stronger notion of pointwise continuity of $\succeq$, Theorem 11 in Wakker and Zank (1999) can be specialized to the following integral representation form.

Theorem 1.5.1 (Wakker and Zank (1999), Theorem 12 and Castagnoli and LiCalzi (2006), Theorem 5)**.** *Assume that $\mathcal{F}$ contains at least three disjoint non-null events and $\succeq$ satisfies (SM), (PC) and (ST). Then there exist:*

(i) a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ such that $\mathcal{N}_{\succeq} = \{A \in \mathcal{F} | \mathbb{P}(A) = 0\}$,

(ii) a function $\mathbb{u} : \Omega \times \mathbb{R} \to \mathbb{R}$ with $\mathbb{u}(\omega, \cdot)$ strictly increasing on $\mathbb{R}$ for every $\omega \in \Omega$ and $\mathbb{u}(\cdot, x) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ for every $x \in \mathbb{R}$,

such that given $f, g \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ we have

$$f \succeq g \text{ if and only if } \int_\Omega \mathbb{u}(\omega, f(\omega))\mathrm{d}\mathbb{P} \geq \int_\Omega \mathbb{u}(\omega, g(\omega))\mathrm{d}\mathbb{P} \tag{1.10}$$

Remark 1.5.2 (On the uniqueness of $\mathbb{u}$). We point out that, in its original form in Wakker and Zank (1999) Theorem 12, the preference representing functional is characterized by a couple $(\mathbb{u}, \mu)$ where $\mathbb{u}$ is the state-dependent utility function and μ a general countably additive measure on $(\Omega, \mathcal{F})$. In such a case $\mathbb{u}$ is unique up to transformations of the form $(\mathrm{d}\mu/\mathrm{d}\mu^*)\alpha\mathbb{u} + \sigma$, where $\sigma : \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable, $\alpha > 0$ and $\mathrm{d}\mu/\mathrm{d}\mu^*$ is the Radon-Nikodym density function of μ with respect to an equivalent measure μ^*. However, in order to obtain a representation of the form of (1.10) of an expected (state-dependent) utility with respect to a probability measure $\mathbb{P}$, one must restrict the class of admissible state-dependent utilities[4].

The previous theorem leaves an interesting issue unsolved as declared by the authors Wakker and Zank (1999), page 29:

"[...] Pointwise continuity is not overly restrictive because continuity of each state-dependent utility implies pointwise continuity. Whether the reversed implication holds, i.e., whether continuity of the functions $\mathbb{u}$, in Statement (i) of Theorem[5] 12 holds (after appropriate modification on null events), is an open question to us."

To the best of our knowledge this question did not find an answer in the subsequent literature. This drawback of the state-dependent utility functions $\mathbb{u}$ as constructed in Wakker and Zank (1999) hampers their employment in several practical applications. To mention one, in Appendix A of Maggis and Maran (2021) a somewhat cumbersome notion of continuity is introduced in order to overcome some technical hurdles

[4] see the proof of Theorem 1.5.4 for further details.

[5] Referred to the original statement in Wakker and Zank (1999)

in the proof of the main result. Guaranteeing the existence of a continuous $\mathbb{u}$ allows to easily circumvent issues of this kind. Among the principal aims of the present chapter, Theorem 1.5.4 stated below provides the construction of a pointwise continuous version of the state-dependent utility $\mathbb{u}$. Furthermore, we prove the desirable joint-measuraibility of such utility functions. The following definition of regularity is well-suited for our purposes.

Definition 1.5.3 ($\mathcal{G}$-regularity). *For any $\mathcal{G} \in \Sigma$ we say that $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ is a* $\mathcal{G}$-regular function *if*

- *$\varphi(\omega, \cdot)$ is continuous and strictly increasing on $\mathbb{R}$ for every $\omega \in \Omega$;*
- *$\varphi(\omega, 0) = 0$ for every $\omega \in \Omega$;*
- *φ is $\mathcal{G} \otimes \mathscr{B}_{\mathbb{R}}$ measurable.*

Theorem 1.5.4. *Assume that $\mathcal{F}$ contains at least three disjoint non-null events and $\succeq$ satisfies (SM), (PC) and (ST). Then $\mathbb{u}$ in Theorem 1.5.1 can be chosen to be $\mathcal{F}$-regular.*

1.6 Proof of Theorem 1.5.4

We devote this section entirely to the proof of Theorem 1.5.4, which is quite involved and requires some preliminary groundwork.
In the first part of this section we lay the basis for the main argument by rephrasing some known results from Wakker and Zank (1999); Doldi and Maggis (2023), obtaining in particular a functional that represents $\succeq$, which turns out to be additive on measurable sets. We provide some auxiliary lemmata: Lemma 1.6.6 proves that it is possible to construct an integrand function with the desired properties; this is based on Lemma 1.6.4 which firstly ensures the suitable measurability by a construction on rational numbers; Lemma 1.6.7 states that the integral representation is well defined and enjoys the desired properties; finally Lemma 1.6.8 shows continuity of $\mathbb{u}$. The concluding argument leads to Proposition 1.6.1 and illustrates how ultimately Theorem 1.5.4 follows.

This proof, along with those that follow, relies extensively on the framework developed in Doldi and Maggis (2023). To keep the discussion self-contained, we summarize below the key properties of such a functional T, which will be referenced frequently throughout.

Let $T : \mathcal{L}^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$ be a preference representing functional for some preference order $\succeq$ on $\mathcal{L}^\infty(\Omega, \mathcal{F})$ and $\mathcal{G}$ any sub-σ-algebra of $\mathcal{F}$. We denote by $\mathcal{N}_\mathcal{G} \subseteq \mathcal{N}_\succeq$ the set of null events of $\mathcal{G}$, i.e $\mathcal{N}_\mathcal{G} := \mathcal{G} \cap \mathcal{N}_\succeq$.

($\mathcal{G}$-Mo) T is $\mathcal{G}$-monotone if, for all $x, y \in \mathbb{R}$ with $x < y$, all $g \in \mathcal{L}^\infty(\Omega, \mathcal{G})$ and all $A \in \mathcal{G} \backslash \mathcal{N}_\mathcal{G}$, we have

$$T(x\mathbb{1}_A + g\mathbb{1}_{A^c}) < T(y\mathbb{1}_A + g\mathbb{1}_{A^c});$$

($\mathcal{G}$-QL) T is $\mathcal{G}$-quasilinear if, given any $g_1, g_2 \in \mathcal{L}^\infty(\Omega, \mathcal{G})$ and $A \in \mathcal{G}$, $T(g_1 \mathbb{1}_A + \overline{g}\mathbb{1}_{A^c}) \le T(g_2 \mathbb{1}_A + \overline{g}\mathbb{1}_{A^c})$ for some $\overline{g} \in \mathcal{L}^\infty(\Omega, \mathcal{G})$ implies $T(g_1 \mathbb{1}_A + g\mathbb{1}_{A^c}) \le T(g_2 \mathbb{1}_A + g\mathbb{1}_{A^c})$ for all $g \in \mathcal{L}^\infty(\Omega, \mathcal{G})$;

($\mathcal{G}$-PC) T is $\mathcal{G}$-pointwise continuous if, for every norm bounded sequence $\{g_n\}_n \subseteq \mathcal{L}^\infty(\Omega, \mathcal{G})$ such that $g_n(\omega) \to g(\omega)$ for all $\omega \in \Omega$ and some $g \in \mathcal{L}^\infty(\Omega, \mathcal{G})$ we have $\lim_n T(g_n) = T(g)$.

Furthermore, for a fixed $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$:

($\mathcal{G}$-PS) T is $\mathcal{G}$-pasting at f if, given $A_1, A_2 \in \mathcal{G}$ with $A_1 \cap A_2 = \varnothing$ and $x_1, x_2 \in \mathbb{R}$ satisfying $T(f_i \mathbb{1}_{A_i}) = x_i$ for $i = 1, 2$ then

$$T(f_1 \mathbb{1}_{A_1} + f_2 \mathbb{1}_{A_2}) = T(x_1 \mathbb{1}_{A_1} + x_2 \mathbb{1}_{A_2});$$

($\mathcal{G}$-NB) T is $\mathcal{G}$-norm bounded at f if $T(-\|f\|_\infty \mathbb{1}_A) \le T(f\mathbb{1}_A) \le T(\|f\|_\infty \mathbb{1}_A)$ for every $A \in \mathcal{G}$.

Consider a preference order $\succeq$ on $\mathcal{L}^\infty(\Omega, \mathcal{F})$ satisfying (SM), (PC) and (ST). Suppose additionally that $\mathcal{F}$ contains at least three disjoint events A_1, A_2, A_3 such that $A_i \notin \mathcal{N}_\succeq$ for $i = 1, 2, 3$. As proved in Remark 1.3.1, under these assumptions $\succeq$ admits a numerical representation, i.e. there exists a functional $T : \mathcal{L}^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$ such that for any given $f, g \in \mathcal{L}^\infty(\Omega, \mathcal{F})$

$$T(f) \ge T(g) \quad \text{if and only if} \quad f \succeq g.$$

In particular Wakker and Zank (1999), Theorem 12, or Castagnoli and LiCalzi (2006), Theorem 5, imply that the Assumptions in Doldi and Maggis (2023), Theorem 5.4[6], are met by the functional T and we can therefore provide a further refinement of the representing functional. Specifically one can find a $V : \mathcal{F} \times \mathcal{L}^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$, $(A, f) \mapsto V_A(f)$ such that for $f_1, f_2 \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ and $A \in \mathcal{F}$ we have $f_2 \mathbb{1}_A \succeq f_1 \mathbb{1}_A$ if and only if $V_A(f_1) \le V_A(f_2)$. Furthermore, V can be taken to satisfy the following properties:

(C1) for every $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ the map $A \mapsto V_A(f)$ is a signed measure on $(\Omega, \mathcal{F})$ with $V_A(0) = 0$ for every $A \in \mathcal{F}$. Moreover, $A \mapsto V_A(\mathbb{1}_\Omega) =: \mathbb{P}(A)$ is a probability measure on $(\Omega, \mathcal{F})$;

(C2) for every $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$, $A \in \mathcal{F}$ we have $V_A(f) = V_\Omega(f\mathbb{1}_A)$;

(C3) the functional V is strictly monotone in that for any $f, g \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ such that $f \le g$ then $V_A(f) \le V_A(g)$ for any $A \in \mathcal{F}$, and if A is such that $\mathbb{P}(A) > 0$ then $V_A(x) < V_A(y)$ for any $x, y \in \mathbb{R}$ with $x < y$;

(C4) for every $A \in \mathcal{F}$ the functional $f \mapsto V_A(f)$, defined on $\mathcal{L}^\infty(\Omega, \mathcal{F})$, is pointwise continuous, in that for any sequence $(f_n)_n \subset \mathcal{L}^\infty(\Omega, \mathcal{F})$, $\sup_n \|f_n\|_\infty < +\infty$, $f_n(\omega) \to f(\omega)$ for any $\omega \in \Omega$, then $V_A(f_n) \to V_A(f)$.

[6] Adopting $\mathcal{F}$ in place of $\mathcal{G}$

The following proposition provides a general representation result which stretches somewhat beyond the scope of the present work. A similar problem is in fact addressed within some relevant contributions in the stream of literature of Appell and Zabrejko (1990) and Buttazzo (1989), Chapter 2, under the name of Nonlinear Superposition Operator and exploiting the theory of Carathéodory functions. Another application of these representation results to subjective expected utility is presented in Stanca (2020).

Proposition 1.6.1. *Let $V : \mathcal{F} \times \mathcal{L}^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$ be a functional satisfying (C1), (C2), (C3)and (C4). Then there exists a $\mathcal{F}$-regular function $\mathbb{u} : \Omega \times \mathbb{R} \to \mathbb{R}$ with $\mathbb{u}(\cdot, x) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ for every $x \in \mathbb{R}$, such that*

$$V_A(f) = \int_A \mathbb{u}(\omega, f(\omega))\mathrm{d}\mathbb{P} \quad \textit{for any } f \in \mathcal{L}^\infty(\Omega, \mathcal{F}), A \in \mathcal{F}.$$

In the following,

Remark 1.6.2. Consider a functional $V : \mathcal{F} \times \mathcal{L}^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$ satisfying (C1), (C2), (C3) and (C4). For any $A \in \mathcal{F}, f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ we denote

$$\mu_f(A) \coloneqq V_{\Omega \cap A}(f) = V_A(f),$$

and observe that, for all $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$, μ_f is dominated by the probability $\mathbb{P}$ defined as in (C1).

We now construct a special version of the state-dependent utility, which enjoys the desired properties.

Definition 1.6.3. *For any $q \in \mathbb{Q}$, we let $u(\cdot, q) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ be a version of the Radon-Nikodym derivative $\mathrm{d}\mu_q/\mathrm{d}\mathbb{P}$. In particular, we set $u(\omega, 0) = 0$ for all $\omega \in \Omega$ by virtue of (C1).*

For any $A \in \mathcal{F}$ and $q \in \mathbb{Q}$, Definition 1.6.3 motivates the following notation

$$V_A(q) = \int_A \mathrm{d}\mu_q(\omega) = \int_\Omega u(\omega, q)\mathbb{1}_A(\omega)\mathrm{d}\mathbb{P} = \int_\Omega u(\omega, q\mathbb{1}_A(\omega))\mathrm{d}\mathbb{P} \tag{1.11}$$

Lemma 1.6.4. *Denote by*

$$A^{\mathbb{Q}} \coloneqq \{\omega \in \Omega : u(\omega, q_1) < u(\omega, q_2), \ \forall\, q_1 < q_2 \in \mathbb{Q}\}$$

and

$$B^{\mathbb{Q}} \coloneqq \{\omega \in \Omega : u(\omega, q) = \inf_{\mathbb{Q} \ni \tilde{q} > q} u(\omega, \tilde{q}), \ \forall\, q \in \mathbb{Q}\}.$$

Then, $A^{\mathbb{Q}}, B^{\mathbb{Q}} \in \mathcal{F}$ and $\mathbb{P}(A^{\mathbb{Q}}) = \mathbb{P}(B^{\mathbb{Q}}) = 1$.

Proof. We start fixing $q_1 < q_2 \in \mathbb{Q}$ and denoting by $C = \{\omega \in \Omega \mid u(\omega, q_1) \geq u(\omega, q_2)\}$. By (1.11) we can write

$$V_C(q_1) = \int_C u(\omega, q_1)\mathrm{d}\mathbb{P} \geq \int_C u(\omega, q_2)\mathrm{d}\mathbb{P} = V_C(q_2),$$

which contradicts (C3) unless $\mathbb{P}(C) = 0$. We conclude that $\mathbb{P}(\{\omega \in \Omega : u(\omega, q_1) < u(\omega, q_2)\}) = 1$. By (countable) intersection over the rational numbers we obtain

$$A^{\mathbb{Q}} = \bigcap_{\substack{q_1, q_2 \in \mathbb{Q}, \\ q_1 < q_2}} \{\omega \in \Omega : u(\omega, q_1) < u(\omega, q_2)\}$$

and clearly $A^{\mathbb{Q}} \in \mathcal{F}$, with $\mathbb{P}(A^{\mathbb{Q}}) = 1$.
We now turn our attention to $B^{\mathbb{Q}}$. We show that

$$\mathbb{P}(\{\omega \in \Omega : u(\omega, q) \leq \inf_{\tilde{q} > q} u(\omega, \tilde{q}),\ \forall\, q \in \mathbb{Q}\}) = 1 \tag{1.12}$$

and that

$$\mathbb{P}(\{\omega \in \Omega : u(\omega, q) \geq \inf_{\tilde{q} > q} u(\omega, \tilde{q}),\ \forall\, q \in \mathbb{Q}\}) = 1, \tag{1.13}$$

so that (1.12) and (1.13) together imply our claim. First of all we show that (1.12) holds. Fix $q, \tilde{q} \in \mathbb{Q}$ such that $\tilde{q} > q$, the previous argument shows that the event $\{\omega \in \Omega : u(\omega, q) \leq u(\omega, \tilde{q})\}$ has probability 1. Notice that for $q \in \mathbb{Q}$ fixed, we have

$$\bigcap_{\tilde{q} > q} \{\omega \in \Omega : u(\omega, q) \leq u(\omega, \tilde{q})\} = \{\omega \in \Omega : u(\omega, q) \leq \inf_{\tilde{q} > q} u(\omega, \tilde{q})\}.$$

We denote by E_q the event $\bigcap_{\tilde{q} > q} \{\omega \in \Omega : u(\omega, q) \leq u(\omega, \tilde{q})\}$. Naturally, the countable intersection is still measurable and in particular $\mathbb{P}(E_q) = 1$. Finally, since

$$\bigcap_{q \in \mathbb{Q}} E_q = \{\omega \in \Omega : u(\omega, q) \leq \inf_{\tilde{q} > q} u(\omega, \tilde{q}),\ \forall\, q \in \mathbb{Q}\},$$

we find $\mathbb{P}\left(\cap_{q \in \mathbb{Q}} E_q\right) = 1$ and (1.12) holds.
We proceed proving (1.13) by contradiction and set

$$A = \{\omega \in \Omega : u(\omega, q) < \inf_{\tilde{q} > q} u(\omega, \tilde{q})\}.$$

Suppose that $\mathbb{P}\left(A\right) > 0$, then, for some $n \in \mathbb{N}$,

$$A_n := \{\omega \in \Omega : u(\omega, q) + \frac{1}{n} < \inf_{\tilde{q} > q} u(\omega, \tilde{q})\}$$

is such that $\mathbb{P}(A_n) > 0$. Then it holds that

$$\int_{A_n} \left(u(\omega, q) + \frac{1}{n}\right) \mathrm{d}\mathbb{P} \leq \int_{A_n} \inf_{\tilde{q} > q} u(\omega, \tilde{q}) \mathrm{d}\mathbb{P} \leq \inf_{\tilde{q} > q} \int_{A_n} u(\omega, \tilde{q}) \mathrm{d}\mathbb{P}. \tag{1.14}$$

It follows from (1.14) that

$$V_{A_n}(q) + \frac{1}{n} \mathbb{P}(A_n) \leq \inf_{\tilde{q} > q} V_{A_n}(\tilde{q}). \tag{1.15}$$

Since by (C3) and (C4) V is strictly monotone and pointwise continuous we have that $\inf_{\tilde{q} > q} V_{A_n}(\tilde{q}) = V_{A_n}(q)$, which in turn yields the desired contradiction. To conclude, since both (1.12) and (1.13) hold with probability 1, we have

$$\mathbb{P}(B^{\mathbb{Q}}) = \mathbb{P}(\{\omega \in \Omega : u(\omega, q) = \inf_{\tilde{q} > q} u(\omega, \tilde{q}),\ \forall\, q \in \mathbb{Q}\}) = 1.$$

$\square$

We define the set

$$\Theta = A^{\mathbb{Q}} \cap B^{\mathbb{Q}} \in \mathcal{F} \text{ with } \mathbb{P}(\Theta) = 1, \tag{1.16}$$

and recall that for any $\omega \in \Theta$ we have $u(\omega, q_1) < u(\omega, q_2)$ for all $q_1 < q_2$, $q_1, q_2 \in \mathbb{Q}$ and $u(\omega, q) = \inf_{\tilde{q} > q} u(\omega, \tilde{q})$ for every $q \in \mathbb{Q}$.

Definition 1.6.5. *For any $x \in \mathbb{R}$ and $\omega \in \Omega$ we define*

$$u^+(\omega, x) = \inf_{\substack{q \in \mathbb{Q}, \\ q \geq x}} \{u(\omega, q)\mathbb{1}_{\Theta}(\omega) + x\mathbb{1}_{\Omega \setminus \Theta}(\omega)\}.$$

We observe that $u^+(\omega, x)$ is $\mathcal{F}$-measurable for any $x \in \mathbb{R}$, as it is the countable pointwise infimum of measurable functions. Notice that $u^+(\omega, 0) = 0$ for all $\omega \in \Omega$.

Lemma 1.6.6. *Let $u^+ : \Omega \times \mathbb{R} \to \mathbb{R}$ be as in Definition 1.6.5. Then the following hold:*

(i) $u^+(\omega, q) = u(\omega, q)$ for any $\omega \in \Theta$ (as defined in (1.16)) and $q \in \mathbb{Q}$;

(ii) if $x < y$, $u^+(\omega, x) < u^+(\omega, y)$ for every $\omega \in \Omega$;

(iii) $u^+(\cdot, x) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ for every $x \in \mathbb{R}$;

(iv) for every $\omega \in \Omega$ the function $u^+(\omega, \cdot)$ is RCLL[7] on $\mathbb{R}$;

Proof. Item (i) follows from the definition of Θ. Item (ii) holds trivially on $\Omega \setminus \Theta$. Take now any $\omega \in \Theta$: provided that $x < y$, there exist $q_1, q_2 \in \mathbb{Q}$ such that $x \leq q_1 < q_2 \leq y$. By definition of Θ we have $u(\omega, q_1) < u(\omega, q_2)$ and moreover by monotonicity of u and definition of u^+

$$u(\omega, q) \leq u^+(\omega, y), \quad \text{for every } q \in \mathbb{Q}, y \in \mathbb{R} \text{ s.t. } q \leq y.$$

Then, we see

$$u^+(\omega, x) = \inf_{\substack{q \in \mathbb{Q}, \\ q \geq x}} u(\omega, q) \leq u(\omega, q_1) < u(\omega, q_2) \leq \inf_{\substack{q \in \mathbb{Q}, \\ q \geq y}} u(\omega, q) = u^+(\omega, y).$$

As to item (iii), fix any $x \in \mathbb{R}$ and consider $q_1, q_2 \in \mathbb{Q}$ such that $q_1 \leq x \leq q_2$. Then by definition and item (i) $u(\omega, q_1) \leq u^+(\omega, x) \leq u(\omega, q_2)$ for any $\omega \in \Theta$. Since $\mathbb{P}(\Theta) = 1$ and $u(\cdot, q_1), u(\cdot, q_2)$ are versions of the Radon-Nikodym derivatives $d\mu_{q_1}/\mathrm{d}\mathbb{P}, d\mu_{q_2}/\mathrm{d}\mathbb{P} \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, integrability follows.

Finally, observe that item (iv) holds trivially for $\omega \in \Omega \setminus \Theta$. Fix now $\omega \in \Theta$, we wish to show that for any $\varepsilon > 0$ there exists δ such that $|u^+(\omega, x) - u^+(\omega, y)| < \varepsilon$ for any $y \in (x, x + \delta)$. Fix $\varepsilon > 0$ and let $(q_n)_n \subset \mathbb{Q}$ be such that $q_n \downarrow_n x$. Then there exists $\bar{n} \in \mathbb{N}$ such that, for all $n > \bar{n}$ we have $|u(\omega, q_n) - \inf_{q > x} u(\omega, q)| = u(\omega, q_n) - \inf_{q > x} u(\omega, q) < \varepsilon$. Recalling that, from item (i), $u(\omega, q_n) = u^+(\omega, q_n)$ and $\inf_{q > x} u(\omega, q) = u^+(\omega, x)$ by definition, monotonicity of u^+ implies that any $y \in (x, q_{\bar{n}}) \subset \mathbb{R}$ satisfies $|u^+(\omega, x) - u^+(\omega, y)| < \varepsilon$. Existence of the left limit follows again from monotonicity proved in item (ii). $\square$

[7] right continuous with left limits

Lemma 1.6.7. *Let $u^+ : \Omega \times \mathbb{R} \to \mathbb{R}$ be as in Definition 1.6.5. Then the map*

$$f \mapsto T_{u^+}(f) := \int_\Omega u^+(\omega, f(\omega))\mathrm{d}\mathbb{P} \tag{1.17}$$

is well defined, monotone with respect to the pointwise order in $\mathcal{L}^\infty(\Omega, \mathcal{F})$, pointwise continuous[8] *and it satisfies $V_A(f) = T_{u^+}(f\mathbb{1}_A)$ for every $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ and $A \in \mathcal{F}$.*

Proof. Observe that, since $u^+(\omega, \cdot)$ is RCLL for every $\omega \in \Omega$, by Lemma A.2 we have that T_{u^+} is well defined on $\mathcal{L}^\infty(\Omega, \mathcal{F})$ and monotone as $f(\omega) \le g(\omega)$ implies $u^+(\omega, f(\omega)) \le u^+(\omega, g(\omega))$. We now show that $V_\Omega(f) = T_{u^+}(f)$ for every $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$, then pointwise continuity of T_{u^+} will follow from (C4). Fix any $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$, we can find two sequences of simple functions $\underline{f_n} = \sum_{A \in \underline{\pi}_n} \underline{x}^n_A \mathbb{1}_A$, $\overline{f_n} = \sum_{B \in \overline{\pi}_n} \overline{x}^n_B \mathbb{1}_B$, with $\overline{\pi}_n, \underline{\pi}_n \subseteq \mathcal{F}$ finite partitions of Ω, such that $\underline{f_n} \uparrow_n f, \overline{f_n} \downarrow_n f$. Moreover by standard arguments we can assume for any n that $\underline{x}^n_A, \overline{x}^n_B \in \mathbb{Q}$ for any $A \in \underline{\pi}_n$ and $B \in \overline{\pi}_n$. We have as a consequence of (1.11) and Lemma 1.6.6

$$\begin{aligned} V_\Omega(\underline{f}_n) &= \sum_{A \in \underline{\pi}_n} \int_\Omega u(\omega, \underline{x}^A_n)\mathbb{1}_A \mathrm{d}\mathbb{P} = \sum_{A \in \underline{\pi}_n} \int_\Omega u^+(\omega, \underline{x}^A_n)\mathbb{1}_A \mathrm{d}\mathbb{P} \\ &= \int_\Omega u^+(\omega, \underline{f}_n(\omega))\mathrm{d}\mathbb{P} \le \int_\Omega u^+(\omega, f(\omega))\mathrm{d}\mathbb{P} \end{aligned}$$

and by a similar argument we can get

$$\int_\Omega u^+(\omega, f(\omega))\mathrm{d}\mathbb{P} \le \int_\Omega u(\omega, \overline{f}_n(\omega))\mathrm{d}\mathbb{P} = V_\Omega(\overline{f}_n)$$

By (C4) we get the desired equality $V_\Omega(f) = T_{u^+}(f)$ for every $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$. Finally, we see that (C2) yields $V_A(f) = V_\Omega(f\mathbb{1}_A) = T_{u^+}(f\mathbb{1}_A)$, which concludes the proof. □

Lemma 1.6.8. *There exists a function $\mathbb{u} : \Omega \times \mathbb{R} \to \mathbb{R}$ such that $\mathbb{u}(\omega, \cdot)$ is continuous and strictly increasing on $\mathbb{R}$ for every $\omega \in \Omega$, $\mathbb{u}(\cdot, x) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ for every $x \in \mathbb{R}$ and*

$$V_A(f) = \int_A \mathbb{u}(\omega, f(\omega))\mathrm{d}\mathbb{P} \quad \forall\, f \in \mathcal{L}^\infty(\Omega, \mathcal{F}) \text{ and } A \in \mathcal{F}. \tag{1.18}$$

Proof. We observe that, by Lemma 1.6.6, $u^+(\omega, \cdot)$ is strictly increasing, RCLL on $\mathbb{R}$ for all $\omega \in \Omega$ and $u^+(\cdot, x) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ for every $x \in \mathbb{R}$. Moreover, as argued in Lemma 1.6.7, T_{u^+} is pointwise continuous and in particular continuous from below. Consequently the assumptions of Corollary A.4 are verified and its application yields the function $\widehat{u^+}$ which we rename $\mathbb{u} := \widehat{u^+}$. Finally, $\mathbb{u}$ satisfies all the requirements since

$$T_{\mathbb{u}} \overset{\text{Def.}}{=} T_{\widehat{u^+}} \overset{\text{Cor. A.4}}{=} T_{u^+} \overset{\text{Lem. 1.6.7}}{=} V.$$

□

[8] as previously mentioned: for any uniformly bounded sequence $(f_n)_n \subset \mathcal{L}^\infty(\Omega, \mathcal{F})$, $f_n(\omega) \to f(\omega)$ for any $\omega \in \Omega$, then $T_{u^+}(f_n) \to T_{u^+}(f)$

Conclusions. We conclude the section by proving Proposition 1.6.1 and we show that Theorem 1.5.4 follows.

Let $V : \mathcal{F} \times \mathcal{L}^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$ be a functional satisfying (C1), (C2), (C3) and (C4). Lemma 1.6.4 and 1.6.6 provide the construction of a suitable integrand function u^+ and Lemma 1.6.7 ensures that the map $f \mapsto \int_\Omega u^+(\omega, f(\omega))\mathrm{d}\mathbb{P}$ is well defined. Lemma 1.6.8 concludes the proof by showing the existence of the desired version $\mathbb{u}$ with the required properties.

To prove Theorem 1.5.4, we exploit Proposition 1.6.1. Consider a preference $\succeq$ that is (SM), (PC) and (ST) and assume that $\mathcal{F}$ contains at least three disjoint events A_1, A_2, A_3 such that $A_i \notin \mathcal{N}_\succeq$ for $i = 1, 2, 3$. As argued in the beginning of Section 1.6, the preference relation $\succeq$ induces a functional $V : \mathcal{F} \times \mathcal{L}^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$ that satisfies (C1), (C2), (C3) and (C4) with $V_\Omega(\cdot)$ acting as numerical representation of the preference $\succeq$. Application of Proposition 1.6.1 provides a pair $(\mathbb{u}, \mathbb{P})$ such that $V_\Omega(f) = \int_\Omega \mathbb{u}(\omega, f(\omega))\mathrm{d}\mathbb{P}$ for any $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$.

1.7 Sure-Thing Principle and Conditionability

In the first result of this dissertation we contributed the desirable regularity properties of state-dependent utility functions, thus giving proper tractability to the integral representation of preferences introduced in Wakker and Zank (1999). In this section we establish how the Sure-Thing Principle (ST) property is related to the conditionability of preference representing functional. We repeat below the definition for simplicity.

Definition 1.3.4. *For a given functional $T : \mathcal{L}^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$, we say that T is conditionable on a sub σ-algebra $\mathcal{G} \subseteq \mathcal{F}$ if for any $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ there exists $g \in \mathcal{L}^\infty(\Omega, \mathcal{G})$ such that*

$$T(f\mathbb{1}_A) = T(g\mathbb{1}_A) \quad \forall A \in \mathcal{G}. \tag{1.6}$$

We say that T is conditionable if it is conditionable on any $\mathcal{G} \in \Sigma$ and we call conditional Chisini mean $\mathfrak{m}(f|\mathcal{G})$ the family of solutions g of equation (1.6).

In particular, (ST) is identified as a necessary and sufficient condition for a preference representing functional T to be conditionable. Furthermore, we take advantage of the results of Section 1.5 to explicitly characterize the conditioning family $\mathfrak{m}(f|\mathcal{G})$.

Theorem 1.7.1. *Let $\succeq$ be a preference order satisfying (SM) and (PC). Then the following are equivalent*

(i) $\succeq$ satisfies (ST);

(ii) any representing functional T for $\succeq$ is conditionable;

(iii) any representing functional T for $\succeq$ is conditionable on $\sigma(A)$ for any $A \in \mathcal{F}$.

Motivated by the literature on generalized means and the representation Theorems 1.2.4 and 1.2.7 that were concisely mentioned in Section 1.2, we provide the following definition that identifies *representable* conditional Chisini means as those admitting a representation in the form of a conditional certainty equivalent, for a suitable choice of state-dependent conditional utility function.

Definition 1.7.2 (Representability)**.** *We say that* representability *holds for conditional Chisini means if there exist an $\mathcal{F}$-regular function $\mathbb{u}$, a probability measure $\mathbb{P}$ on $\mathcal{F}$ and, for every σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, a $\mathcal{G}$-regular function $\mathbb{u}_{\mathcal{G}}$ such that $\mathbb{u}(\cdot, x)$ and $\mathbb{u}_{\mathcal{G}}(\cdot, x)$ are integrable for every $x \in \mathbb{R}$ and for any $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ and $g \in \mathfrak{m}(f|\mathcal{G})$ we can write*[9]

$$g(\omega) = \mathbb{u}_{\mathcal{G}}^{-1}(\omega, h(\omega)) \quad \forall\, \omega \in \Omega,$$

up to a choice of a suitable version h of $\mathbb{E}_{\mathbb{P}}[\mathbb{u}(\cdot, f)|\mathcal{G}]$.

We shall address the actual construction of the functions $\mathbb{u}, \mathbb{u}_{\mathcal{G}}$ together with the existence of the probability measure $\mathbb{P}$ in Theorem 1.7.4. However, we anticipate a result that, under the somewhat stronger requirement of the existence of three disjoint non-null events, couples together the representability of conditional Chisini means and the Sure-Thing Principle property of preferences.

Corollary 1.7.3. *Let $\succeq$ be a preference order satisfying (SM) and (PC). Whenever $\mathcal{F}$ contains at least three disjoint non-null events, condition (i) of Theorem 1.7.1 is equivalent to*

(iv) conditional Chisini means are representable.

In the following, we apply Theorem 1.5.4 to obtain an explicit formula for conditional Chisini means, which is the cornerstone of the proofs of Theorem 1.11.2 and of Corollary 1.7.3.

Theorem 1.7.4. *Assume that the hypotheses of Theorem 1.5.4 are satisfied and consider a resulting pair $(\mathbb{u}, \mathbb{P})$ with $\mathbb{u}$ being $\mathcal{F}$-regular. Then, for any given sub-σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, $\mathfrak{m}(f|\mathcal{G}) \neq \varnothing$ and there exist*

- *a $\mathcal{G}$-regular function $\mathbb{u}_{\mathcal{G}} : \Omega \times \mathbb{R} \to \mathbb{R}$, with*
  - *(i) $\mathbb{u}_{\mathcal{G}}(\cdot, x) \in \mathcal{L}^1(\Omega, \mathcal{G}, \mathbb{P})$ for every $x \in \mathbb{R}$,*
  - *(ii) $\mathbb{u}_{\mathcal{G}}(\cdot, x)$ is a version of $\mathbb{E}_{\mathbb{P}}[\mathbb{u}(\cdot, x)|\mathcal{G}]$ for every $x \in \mathbb{R}$.*
- *a suitable version g of $\mathbb{E}_{\mathbb{P}}[\mathbb{u}(\cdot, f(\cdot))|\mathcal{G}]$ such that*

$$\Phi_{\mathcal{G}}(\cdot, g(\cdot)) \in \mathfrak{m}(f|\mathcal{G}), \tag{1.19}$$

where $\Phi_{\mathcal{G}} : \Omega \times \mathbb{R} \to [-\infty, +\infty]$ defined by

$$\Phi_{\mathcal{G}}(\omega, x) := \inf\{y \in \mathbb{R} \mid \mathbb{u}_{\mathcal{G}}(\omega, y) > x\}$$

is $\mathcal{G} \otimes \mathscr{B}_{\mathbb{R}}$ measurable.

We write informally (1.19) as

$$\mathfrak{m}(f|\mathcal{G}) = \mathbb{u}_{\mathcal{G}}^{-1}(\mathbb{E}_{\mathbb{P}}[\mathbb{u}(f)|\mathcal{G}]),$$

[9] More details can be found in the statement of Theorem 1.7.4.

and recall that, under the assumptions of Theorem 1.7.4, $\mathfrak{m}(f|\mathcal{G})$ is unique up to $\succeq$-null events (see Doldi and Maggis, 2023, Theorem 3.2 and Proposition 1.8.1 in the following section).

The representation of the conditional Chisini mean of Theorem 1.7.4 conceals few technical subtleties. To begin with, the random nature of the conditional state-dependent utility $\mathbb{u}_{\mathcal{G}}$ implies that the domain of $\Phi_{\mathcal{G}}$ is itself ω-dependent and is to be handled properly in order for $\Phi_{\mathcal{G}}(\omega, x)$ to be finite. Additionally, for some bounded function $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ and $\mathcal{G} \subseteq \mathcal{F}$, $g \in \mathbb{E}_{\mathbb{P}}[\mathbb{u}(f)|\mathcal{G}]$ is not in general an element of $\mathcal{L}^\infty(\Omega, \mathcal{G})$. While the case of a finitely generated σ-algebra is rather simple (as we shall depict in Example 1.7.5 below), we shall dedicate a comprehensive treatment to the issue for an arbitrary σ-algebra $\mathcal{G}$ in the proof postponed to Section 1.9.

Example 1.7.5. Consider $\{A_1, ..., A_n\}$ being a finite partition of Ω such that $\mathbb{P}(A_i) > 0$ for every $i \leq n$ and let $\mathcal{G}$ be the σ-algebra generated by this partition. Then every $\mathcal{G}$-measurable function g can be written as a finite linear combination of indicator functions of the atoms A_i. In particular, for $\tilde{\omega} \in A_i$, $\mathbb{u}_{\mathcal{G}}(\tilde{\omega}, x)$ coincides with

$$u_i(x) = \frac{1}{\mathbb{P}(A_i)} \int_{A_i} \mathbb{u}(\omega, x) \mathrm{d}\mathbb{P},$$

u_i being finite since $\mathbb{u}(\cdot, x) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Clearly $\mathbb{u}_{\mathcal{G}}(\cdot, x) \in \mathcal{L}^1(\Omega, \mathcal{G}, \mathbb{P})$ for all $x \in \mathbb{R}$, $\mathbb{u}_{\mathcal{G}}(\omega, x) = \mathbb{E}_{\mathbb{P}}[\mathbb{u}(\cdot, x)|\mathcal{G}](\omega)$ and $x \mapsto \mathbb{u}_{\mathcal{G}}(\omega, x)$ is increasing and continuous for all $\omega \in \Omega$.
For any fixed A_i we notice that the image of $\mathbb{u}_{\mathcal{G}}(\omega, \cdot)$, namely $R_i \coloneqq \mathrm{Im}(\mathbb{u}_{\mathcal{G}}(\omega, \cdot))$ does not depend on the specific ω as long as $\omega \in A_i$. Given the generalized inverse $\Phi_{\mathcal{G}} : \Omega \times \mathbb{R} \to \mathbb{R}$ defined in Theorem 1.7.4, it is crucial to observe that the map $x \mapsto \Phi_{\mathcal{G}}(\omega, x) = \mathbb{u}_{\mathcal{G}}^{-1}(\omega, x)$ is well defined as a standard inverse function for $\omega \in A_i$ and $x \in R_i$ (as a result of strict monotonicity of $\mathbb{u}_{\mathcal{G}}(\omega, \cdot)$).

In particular if we now choose any $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ and set $h \coloneqq \mathbb{E}_{\mathbb{P}}[\mathbb{u}(\cdot, f(\cdot))|\mathcal{G}]$ then $h \in \mathcal{L}^\infty(\Omega, \mathcal{G})$ by definition of $\mathcal{G}$ and $h(\omega) \in R_i$ for $\omega \in A_i$ and indeed

$$(\mathbb{u}_{\mathcal{G}} \circ \Phi_{\mathcal{G}})(\omega, h(\omega)) = h(\omega).$$

For $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ denote $\mathfrak{m}_i = u_i^{-1}\left(\frac{1}{\mathbb{P}(A_i)} \int_{A_i} \mathbb{u}(\omega, f(\omega)) \mathrm{d}\mathbb{P}\right)$ so that for any $\omega \in A_i$ we have

$$\mathbb{u}_{\mathcal{G}}(\omega, \mathfrak{m}_i) = \mathbb{E}_{\mathbb{P}}[\mathbb{u}(\cdot, f(\cdot))|\mathcal{G}](\omega).$$

We conclude the example noting that the Chisini mean $\mathfrak{m}(f|\mathcal{G})$ for the functional $T(f) = \int_\Omega \mathbb{u}(\omega, f(\omega)) \mathrm{d}\mathbb{P}$ consists, in the current setup, of a single random variable $\mathfrak{m}(f|\mathcal{G})$ which assumes the value

$$u_i^{-1}\left(\frac{1}{\mathbb{P}(A_i)} \int_{A_i} \mathbb{u}(\omega, f(\omega)) \mathrm{d}\mathbb{P}\right)$$

on the event A_i. This can be readily verified following the procedure in Section 1.9, step 4.

1.8 Proof of Theorem 1.7.1

This section is dedicated to the proof of Theorem 1.7.1. We first show that the conditional Chisini mean exists for a preference satisfying (SM), (PC) and (ST) and that it is unique up to $\succeq$-null events. Then we provide a "taking out what is known" property for the conditional Chisini mean before focusing on the proof of the main result.

Proposition 1.8.1. *Let $\succeq$ be a preference order satisfying (SM), (PC), (ST) and consider a sub-σ-algebra $\mathcal{G} \subseteq \mathcal{F}$ and $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$. Then $\mathfrak{m}(f|\mathcal{G}) \neq \varnothing$ and for any $g, \tilde{g} \in \mathfrak{m}(f|\mathcal{G})$ it holds that $\{g \neq \tilde{g}\} \in \mathcal{N}_\succeq$.*

Proof. First, recall from Remark 1.3.1 that for any preference order satisfying (SM) and (PC) there exists a representing functional $T : \mathcal{L}^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$.
We show that the hypotheses of Doldi and Maggis (2023), Theorem 3.2 hold for such a functional T. Indeed, (SM), (PC) and (ST) of $\succeq$ imply that T satisfies[10] ($\mathcal{G}$-Mo), ($\mathcal{G}$-PC), ($\mathcal{G}$-QL) and ($\mathcal{G}$-NB) for any σ-algebra $\mathcal{G} \subseteq \mathcal{F}$. It remains to show that T satisfies ($\mathcal{G}$-PS). Consider $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$, $A_1, A_2 \in \mathcal{G}$ with $A_1 \cap A_2 = \varnothing$ and let $x_1, x_2 \in \mathbb{R}$ be such that $T(f\mathbb{1}_{A_i}) = T(x_i \mathbb{1}_{A_i})$ for $i = 1, 2$. Define $g_1 := x_1 \mathbb{1}_{A_1}$ and $g_2 := f\mathbb{1}_{A_2}$. By (ST) we obtain $T(f\mathbb{1}_{A_1} + g_2 \mathbb{1}_{A_1^c}) = T(x_1 \mathbb{1}_{A_1} + g_2 \mathbb{1}_{A_1^c})$ and similarly we find $T(f\mathbb{1}_{A_2} + g_1 \mathbb{1}_{A_2^c}) = T(x_2 \mathbb{1}_{A_2} + g_1 \mathbb{1}_{A_2^c})$. Since $A_2^c \cap A_1 = A_1$ and $A_1^c \cap A_2 = A_2$, it follows that $T(f\mathbb{1}_{A_1} + f\mathbb{1}_{A_2}) = T(x_1 \mathbb{1}_{A_1} + f\mathbb{1}_{A_2})$ and $T(f\mathbb{1}_{A_2} + x_1 \mathbb{1}_{A_1}) = T(x_2 \mathbb{1}_{A_2} + x_1 \mathbb{1}_{A_1})$, whence $T(f\mathbb{1}_{A_1} + f\mathbb{1}_{A_2}) = T(x_1 \mathbb{1}_{A_1} + x_2 \mathbb{1}_{A_2})$. We can therefore apply Doldi and Maggis (2023), Theorem 3.2 which in turn provides $\mathfrak{m}(f|\mathcal{G}) \neq \varnothing$ and the desired uniqueness. □

Lemma 1.8.2. *Let $\succeq$ be a preference order satisfying (SM), (PC), (ST) and consider a sub-σ-algebra $\mathcal{G} \subseteq \mathcal{F}$ and $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$. For $A \in \mathcal{G}$, we have*

$$\mathfrak{m}(f\mathbb{1}_A|\mathcal{G}) = \mathfrak{m}(f|\mathcal{G}) \mathbb{1}_A \tag{1.20}$$

meaning that $g \in \mathfrak{m}(f\mathbb{1}_A|\mathcal{G})$ if and only if $g = \overline{g}\mathbb{1}_A$ for some $\overline{g} \in \mathfrak{m}(f|\mathcal{G})$.

Proof. Consider $g \in \mathfrak{m}(f\mathbb{1}_A|\mathcal{G})$. By definition we have that $T(g\mathbb{1}_B) = T((f\mathbb{1}_A)\mathbb{1}_B)$ for all $B \in \mathcal{G}$. As $\mathbb{1}_A \mathbb{1}_B = \mathbb{1}_{A\cap B}$ and $A \cap B \in \mathcal{G}$, we can write $T((f\mathbb{1}_A)\mathbb{1}_B) = T(f\mathbb{1}_{A\cap B}) = T(\overline{g}\mathbb{1}_{A\cap B}) = T((\overline{g}\mathbb{1}_A)\mathbb{1}_B)$, with $\overline{g} \in \mathfrak{m}(f|\mathcal{G})$, and hence it follows that $g = \overline{g}\mathbb{1}_A$.
Now let $\overline{g} \in \mathfrak{m}(f|\mathcal{G})$ and fix $A \in \mathcal{G}$. For all $G \in \mathcal{G}$ it holds that $A \cap G \in \mathcal{G}$, so that in particular $T(f\mathbb{1}_{A\cap G}) = T(\overline{g}\mathbb{1}_{A\cap G})$. It then follows for all $G \in \mathcal{G}$ that $T(\overline{g}\mathbb{1}_{A\cap G}) = T(f\mathbb{1}_{A\cap G}) = T((f\mathbb{1}_A)\mathbb{1}_G) = T(g\mathbb{1}_G)$, with $g \in \mathfrak{m}(f\mathbb{1}_A|\mathcal{G})$. We conclude that $\overline{g}\mathbb{1}_A = g$. □

We further observe that, as $\mathfrak{m}(0|\mathcal{G}) = 0$ by definition, for any $A \in \mathcal{G}$ property (1.20) is equivalent to

$$\mathfrak{m}(f\mathbb{1}_A + h\mathbb{1}_{A^c}|\mathcal{G}) = \mathfrak{m}(f|\mathcal{G}) \mathbb{1}_A + \mathfrak{m}(h|\mathcal{G}) \mathbb{1}_{A^c}.$$

Proof of Theorem 1.7.1. We begin by showing that Item (i) implies (ii). To this aim, note that the hypotheses of Proposition 1.8.1 are satisfied and hence for any $\mathcal{G} \subseteq \mathcal{F}$

[10] We adopt the notation of Doldi and Maggis (2023)

there exists a function $g \in \mathcal{L}^\infty(\Omega, \mathcal{G})$ that satisfies Eq. (1.5).
Item (iii) follows directly from Item (ii).
We conclude the proof showing that (iii) implies (i). Consider $f_1, f_2, f, h \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ and observe that, by definition of $\mathcal{N}_\succeq$, whenever $A \in \mathcal{N}_\succeq$ then $f\mathbb{1}_A + h\mathbb{1}_{A^c} \sim h$, so that $f_1\mathbb{1}_A + h\mathbb{1}_{A^c} \succeq f_2\mathbb{1}_A + h\mathbb{1}_{A^c}$ implies $f_1\mathbb{1}_A + \overline{h}\mathbb{1}_{A^c} \succeq f_2\mathbb{1}_A + \overline{h}\mathbb{1}_A^c$ for any $\overline{h} \in \mathcal{L}^\infty(\Omega, \mathcal{F})$. Similarly whenever $A^c \in \mathcal{N}_\succeq$. Suppose $A, A^c \in \mathcal{F} \setminus \mathcal{N}_\succeq$ and consider $\mathcal{G} := \sigma(A)$. First notice that, by construction of $\mathcal{G}$ and uniqueness, for any $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ the conditional Chisini mean $\mathfrak{m}\,(f|\mathcal{G})$ collapses to a single $\mathcal{G}$-measurable random variable. In particular, $\mathfrak{m}\,(f|\mathcal{G}) = x\mathbb{1}_A + y\mathbb{1}_{A^c}$ for some $x, y \in \mathbb{R}$.
Let $T : \mathcal{L}^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$ be a representing functional for $\succeq$ conditionable on $\mathcal{G} = \sigma(A)$ and take $f_1, f_2, h \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ such that $T(f_1\mathbb{1}_A + h\mathbb{1}_{A^c}) \geq T(f_2\mathbb{1}_A + h\mathbb{1}_{A^c})$. Conditionability of T ensures that there exists $g_i \in \mathcal{L}^\infty(\Omega, \mathcal{G})$ such that $T(g_i\mathbb{1}_G) = T((f_i\mathbb{1}_A + h\mathbb{1}_{A^c})\mathbb{1}_G)$ for any $G \in \mathcal{G}$ and $i = 1, 2$. As we previously pointed out, for this choice of $\mathcal{G}$ the conditional Chisini mean consists of a single random variable. Hence we denote with $\mathfrak{m}\,(f_i\mathbb{1}_A + h\mathbb{1}_{A^c}|\mathcal{G})$ the solution g_i to the previous equation. By Lemma 1.8.2, we have that for $i = 1, 2$

$$\mathfrak{m}\,(f_i\mathbb{1}_A + h\mathbb{1}_{A^c}|\mathcal{G}) = \mathfrak{m}\,(f_i|\mathcal{G})\,\mathbb{1}_A + \mathfrak{m}\,(h|\mathcal{G})\,\mathbb{1}_{A^c},$$

and recalling that any $\mathcal{G}$-measurable function is of the form $x\mathbb{1}_A + y\mathbb{1}_{A^c}$ for some $x, y \in \mathbb{R}$, we obtain that $\mathfrak{m}\,(f_i|\mathcal{G}) = x_i$ on A and analogously $\mathfrak{m}\,(h|\mathcal{G}) = \overline{y}$ on A^c for some $x_i, \overline{y} \in \mathbb{R}$. Now observe that $T(f_1\mathbb{1}_A + h\mathbb{1}_{A^c}) \geq T(f_2\mathbb{1}_A + h\mathbb{1}_{A^c})$ implies $x_1 \geq x_2$. Assume, by way of contradiction, that $x_1 < x_2$, then (SM) yields $T(x_1\mathbb{1}_A + \overline{y}\mathbb{1}_{A^c}) < T(x_2\mathbb{1}_A + \overline{y}\mathbb{1}_{A^c})$. However,

$$T(f_1\mathbb{1}_A + h\mathbb{1}_{A^c}) = T(x_1\mathbb{1}_A + \overline{y}\mathbb{1}_{A^c}) < T(x_2\mathbb{1}_A + \overline{y}\mathbb{1}_{A^c}) = T(f_2\mathbb{1}_A + h\mathbb{1}_{A^c}),$$

hence a contradiction.
Keeping f_1 and f_2 constant and replacing h with an arbitrary $k \in \mathcal{L}^\infty(\Omega, \mathcal{F})$, we can see that $\mathfrak{m}\,(f_i\mathbb{1}_A + k\mathbb{1}_{A^c}|\mathcal{G}) = x_i\mathbb{1}_A + z\mathbb{1}_{A^c}$ for $i = 1, 2$, with $z = \mathfrak{m}(k|\mathcal{G})$ on A^c. As $x_1 \geq x_2$, (SM) implies

$$T(f_1\mathbb{1}_A + k\mathbb{1}_{A^c}) = T(x_1\mathbb{1}_A + z\mathbb{1}_{A^c}) \geq T(x_2\mathbb{1}_A + z\mathbb{1}_{A^c}) = T(f_2\mathbb{1}_A + k\mathbb{1}_{A^c}).$$

Since k is arbitrary, we get in fact that $T(f_1\mathbb{1}_A + h\mathbb{1}_{A^c}) \geq T(f_2\mathbb{1}_A + h\mathbb{1}_{A^c})$ for one $h \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ entails $T(f_1\mathbb{1}_A + \overline{h}\mathbb{1}_{A^c}) \geq T(f_2\mathbb{1}_A + \overline{h}\mathbb{1}_{A^c})$ for all $\overline{h} \in \mathcal{L}^\infty(\Omega, \mathcal{F})$, which in turn implies that $\succeq$ satisfies (ST). □

1.9 Proof of Theorem 1.7.4

We begin by recalling that we work under the hypotheses of Theorem 1.5.4. In analogy to the previous section, it is possible to construct a functional $V : \mathcal{F} \times \mathcal{L}^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$ that satisfies (C1), (C2), (C3) and (C4). We also point out that properties (C1-4) are in fact stronger than those required in Doldi and Maggis (2023), Theorem 3.2[11]. Therefore, $\mathfrak{m}\,(f|\mathcal{G}) \neq \emptyset$ and for any $g_1, g_2 \in \mathfrak{m}\,(f|\mathcal{G})$ we have $\{g_1 \neq g_2\} \in \mathcal{N}_\succeq$.

[11] Indeed, the fact that $V_\Omega : \mathcal{L}^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$ is ($\mathcal{G}$-Mo), ($\mathcal{G}$-PC) and ($\mathcal{G}$-NB) follows trivially from (C3) and (C4), for any sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$. Furthermore, recalling that, by (C1), $V_A(0) = 0$ we are able to recover also ($\mathcal{G}$-QL) and ($\mathcal{G}$-PS). Notice that, for $g_1, g_2 \in \mathcal{L}^\infty(\Omega, \mathcal{G})$, $A \in \mathcal{G}$, it is possible to write $V_\Omega(g_1\mathbb{1}_A + g_2\mathbb{1}_{A^c}) = V_A(g_1) + V_{A^c}(g_2)$.

Step 1: We first show the existence of a state-dependent utility on $\mathcal{L}^\infty(\Omega,\mathcal{G})$ for any σ-algebra $\mathcal{G}\subseteq\mathcal{F}$. Consider the functional $V:\mathcal{F}\times\mathcal{L}^\infty(\Omega,\mathcal{F})\to\mathbb{R}$ defined above. By Proposition 1.6.1 it follows that V admits a representation of the kind $V_A(f)=\int_A \mathbb{u}(\omega,f(\omega))\mathrm{d}\mathbb{P}$. Let $\mathcal{G}\subseteq\mathcal{F}$ be an arbitrary sub-σ-algebra on Ω and denote by $V^{\mathcal{G}}:\mathcal{G}\times\mathcal{L}^\infty(\Omega,\mathcal{G})\to\mathbb{R}$ the restriction of the functional V to $\mathcal{G}\subseteq\mathcal{F}$ and $\mathcal{L}^\infty(\Omega,\mathcal{G})$.

Remark 1.9.1. We stress that it is sufficient that $\mathcal{F}$ contains at least three disjoint events of positive probability. The σ-algebra $\mathcal{G}$ can thus be any sub-σ-algebra of $\mathcal{F}$, including trivially σ-algebra $\mathcal{G}=\{\Omega,\varnothing\}$.

Analogously to V, $V^{\mathcal{G}}$ defines a probability measure $\mathbb{P}_{\mathcal{G}}$ on $\mathcal{G}$, which is the restriction of the probability measure $\mathbb{P}$ induced by V to the sub-σ-algebra $\mathcal{G}$. We will hence use only $\mathbb{P}$ in the following, to avoid impractical notation.
Applying Proposition 1.6.1 to $V^{\mathcal{G}}$ it is possible to construct a $\mathcal{G}\otimes\mathscr{B}_{\mathbb{R}}$-measurable function $\mathbb{u}_{\mathcal{G}}:\Omega\times\mathbb{R}\to\mathbb{R}$ such that

(i) $\mathbb{u}_{\mathcal{G}}(\omega,\cdot)$ is continuous and strictly increasing on $\mathbb{R}$ for all $\omega\in\Omega$,

(ii) $\mathbb{u}_{\mathcal{G}}(\cdot,x)\in\mathcal{L}^1(\Omega,\mathcal{G},\mathbb{P})$ for all $x\in\mathbb{R}$,

(iii) $V^{\mathcal{G}}_A(g)=\int_A \mathbb{u}_{\mathcal{G}}(\omega,g(\omega))\mathrm{d}\mathbb{P}$, $\forall A\in\mathcal{G}, g\in\mathcal{L}^\infty(\Omega,\mathcal{G})$.

Notice that for all $A\in\mathcal{G}$ and $g\in\mathcal{L}^\infty(\Omega,\mathcal{G})$

$$V^{\mathcal{G}}_A(g)=\int_A \mathbb{u}_{\mathcal{G}}(\omega,g(\omega))\mathrm{d}\mathbb{P}=\int_A \mathbb{u}(\omega,g(\omega))\mathrm{d}\mathbb{P}=V_A(g).$$

It is then immediate to verify by definition of conditional expectation that the random variable $\mathbb{u}_{\mathcal{G}}(\cdot,x)$ is an element of the class $\mathbb{E}_{\mathbb{P}}\left[\mathbb{u}(\cdot,x)|\mathcal{G}\right]$.

To avoid overloading notation, in the following step 2 we shall write $\mathbb{E}_{\mathbb{P}}\left[\mathbb{u}(f)|\mathcal{G}\right]$ in place of $\mathbb{E}_{\mathbb{P}}\left[\mathbb{u}(\cdot,f(\cdot))|\mathcal{G}\right]$ and $\mathbb{u}_{\mathcal{G}}(g)$ instead of $\mathbb{u}_{\mathcal{G}}(\cdot,g(\cdot))$. Moreover, we shall often make the following standard abuse of notation: for any $f\in\mathcal{L}^\infty(\Omega,\mathcal{F})$ we write $g=\mathbb{E}_{\mathbb{P}}\left[\mathbb{u}(f)|\mathcal{G}\right]$ to indicate that $g\in\mathbb{E}_{\mathbb{P}}\left[\mathbb{u}(f)|\mathcal{G}\right]$ i.e. g is a version of the conditional expectation $\mathbb{E}_{\mathbb{P}}\left[\mathbb{u}(f)|\mathcal{G}\right]$.

Step 2: We show that:

$$\mathbb{u}_{\mathcal{G}}(g) \text{ belongs to } \mathbb{E}_{\mathbb{P}}\left[\mathbb{u}(g)|\mathcal{G}\right]\quad\forall\, g\in\mathcal{L}^\infty(\Omega,\mathcal{G}). \tag{1.21}$$

Consider indeed a simple function $g=\sum_{A\in\pi}y_A\mathbb{1}_A$, with π being a finite $\mathcal{G}$-measurable partition of Ω. Then we have

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left[\mathbb{u}(g)|\mathcal{G}\right]&=\sum_{A\in\pi}\mathbb{E}_{\mathbb{P}}\left[\mathbb{u}(g)\mathbb{1}_A|\mathcal{G}\right]=\sum_{A\in\pi}\mathbb{E}_{\mathbb{P}}\left[\mathbb{u}(y_A)\mathbb{1}_A|\mathcal{G}\right]\\&=\sum_{A\in\pi}\mathbb{E}_{\mathbb{P}}\left[\mathbb{u}(y_A)|\mathcal{G}\right]\mathbb{1}_A=\sum_{A\in\pi}\mathbb{u}_{\mathcal{G}}(y_A)\mathbb{1}_A=\mathbb{u}_{\mathcal{G}}(\sum_{A\in\pi}y_A\mathbb{1}_A),\end{aligned}$$

and therefore (1.21) is verified for $\mathcal{G}$-measurable simple functions. We proceed by showing that the property also holds for general functions $g\in\mathcal{L}^\infty(\Omega,\mathcal{G})$. Take any sequence of $\mathcal{G}$-measurable simple functions $(\underline{g}_n)_n\in\mathcal{L}^\infty(\Omega,\mathcal{G})$ such that $\underline{g}_n\uparrow_n g$ (again, the existence of such functions is guaranteed by standard arguments). Monotonicity of $\mathbb{u}_{\mathcal{G}}(\omega,\cdot)$ ensures that $(\mathbb{u}_{\mathcal{G}}(\omega,\underline{g}_n(\omega)))_n$ is an increasing sequence for all $\omega \in \Omega$. Finally, pointwise continuity of $\mathbb{u}_{\mathcal{G}}$ gives

$$\mathbb{u}_{\mathcal{G}}(g) = \lim_n \mathbb{u}_{\mathcal{G}}(\underline{g}_n),$$

and monotone convergence theorem yields (1.21) for general $g \in \mathcal{L}^\infty(\Omega,\mathcal{G})$ as

$$\lim_n \mathbb{E}_{\mathbb{P}}\left[\mathbb{u}(\underline{g}_n)\middle|\mathcal{G}\right] = \mathbb{E}_{\mathbb{P}}\left[\mathbb{u}(g)|\mathcal{G}\right].$$

Step 3: we now study the generalized inverse of the state-dependent utility $\mathbb{u}_{\mathcal{G}}$. If on the one hand $(\omega,x)\mapsto \mathbb{u}_{\mathcal{G}}(\omega,x)$ is $\mathcal{G}\otimes\mathscr{B}_{\mathbb{R}}$ measurable (by Proposition 1.6.1), on the other we would like the generalized inverse $\Phi_{\mathcal{G}}$ to enjoy the same measurability as $\mathbb{u}_{\mathcal{G}}$. We first extend the domain of $\mathbb{u}_{\mathcal{G}}:\Omega\times\mathbb{R}\to\mathbb{R}$ to $\Omega\times\overline{\mathbb{R}}$ by denoting

$$\mathbb{u}_{\mathcal{G}}(\omega,+\infty) := \lim_{x\to+\infty}\mathbb{u}_{\mathcal{G}}(\omega,x), \qquad \mathbb{u}_{\mathcal{G}}(\omega,-\infty) := \lim_{x\to-\infty}\mathbb{u}_{\mathcal{G}}(\omega,x).$$

Monotonicity and continuity of $\mathbb{u}_{\mathcal{G}}(\omega,\cdot)$ for all $\omega\in\Omega$ ensure that the limits exist. Fixed $\omega\in\Omega$, we denote with $\mathrm{Im}_\omega(\mathbb{u}_{\mathcal{G}}) := \mathrm{Im}(\mathbb{u}_{\mathcal{G}}(\omega,\cdot))$ the image of $\mathbb{u}_{\mathcal{G}}$, that is the interval $(\mathbb{u}_{\mathcal{G}}(\omega,-\infty),\mathbb{u}_{\mathcal{G}}(\omega,+\infty))$, and we stress that it is ω-dependent and open by strict monotonicity.

Let $\Phi_{\mathcal{G}}:\Omega\times\mathbb{R}\to\overline{\mathbb{R}}$ be the ω-wise right-continuous generalized inverse:

$$\Phi_{\mathcal{G}}(\omega,x) := \inf\{y\in\mathbb{R} \mid \mathbb{u}_{\mathcal{G}}(\omega,y) > x\}.$$

Observe that strict monotonicity and continuity of $\mathbb{u}_{\mathcal{G}}(\omega,\cdot)$ imply $\Phi_{\mathcal{G}}(\omega,\cdot)$ is strictly increasing and continuous over $\mathrm{Im}_\omega(\mathbb{u}_{\mathcal{G}})$ for all $\omega\in\Omega$. Then, in particular, $\mathbb{u}_{\mathcal{G}}(\omega,\cdot):\mathbb{R}\to\mathrm{Im}_\omega(\mathbb{u}_{\mathcal{G}})$ is a bijection and we can write

$$\Phi_{\mathcal{G}}(\omega,x) = \begin{cases} +\infty & \text{for } x \geq \mathbb{u}_{\mathcal{G}}(\omega,+\infty), \\ \mathbb{u}_{\mathcal{G}}^{-1}(\omega,x) & \text{for } x\in\mathrm{Im}_\omega(\mathbb{u}_{\mathcal{G}}), \\ -\infty & \text{for } x \leq \mathbb{u}_{\mathcal{G}}(\omega,-\infty). \end{cases} \tag{1.22}$$

Notice that, fixed $\omega\in\Omega$, whenever $x\in\mathrm{Im}_\omega(\mathbb{u}_{\mathcal{G}})$ then $\Phi_{\mathcal{G}}(\omega,x)$ is finite.

We claim that $\Phi_{\mathcal{G}}(\cdot,x):\Omega\to\overline{\mathbb{R}}$ is $\mathcal{G}$-measurable for all $x\in\mathbb{R}$. To see this, let $x\in\mathbb{R}$ be fixed and consider the set $A=(a,+\infty]\in\mathscr{B}_{\overline{\mathbb{R}}}$ with $a>-\infty$. We show that $\{\omega\in\Omega : \Phi_{\mathcal{G}}(\omega,x)\in A\}\in\mathcal{G}$. Observe that, by strict monotonicity of $\mathbb{u}_{\mathcal{G}}$ the latter coincides with $\{\omega\in\Omega : x > \mathbb{u}_{\mathcal{G}}(\omega,a)\}$ which is an element of $\mathcal{G}$ as $\mathbb{u}_{\mathcal{G}}$ is $\mathcal{G}$-measurable. Additionally, we wish to emphasize that $\Phi_{\mathcal{G}}(\omega,\cdot):\mathbb{R}\to\overline{\mathbb{R}}$ is continuous for all $\omega\in\Omega$, and therefore $\Phi_{\mathcal{G}}:\Omega\times\mathbb{R}\to\overline{\mathbb{R}}$ is a Carathéodory function (see Aliprantis and Border, 2006, Definition 4.50). In particular, we observe that $\mathbb{R}$ with the usual euclidean topology is a separable metrizable space and $\overline{\mathbb{R}}=[-\infty,+\infty]$ with the standard topology on the extended real numbers is metrizable. Then, by virtue of Aliprantis and Border (2006), Lemma 4.51, $\Phi_{\mathcal{G}}$ is $\mathcal{G}\otimes\mathscr{B}_{\mathbb{R}}$ measurable.

Remark 1.9.2. By (1.22), for any $\omega\in\Omega$ and $x\in\mathrm{Im}_\omega(\mathbb{u}_{\mathcal{G}})$ we have

$$\mathbb{u}_{\mathcal{G}}\Big(\omega,\Phi_{\mathcal{G}}(\omega,x)\Big) = x. \tag{1.23}$$

Step 4: we conclude proving the desired representation of the conditional Chisini mean. Chosen $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$, by strict monotonicity of $\mathbb{u}(\omega, \cdot)$ we have that

$$\mathbb{u}(\omega, -\|f\|_\infty) \leq \mathbb{u}(\omega, f(\omega)) \leq \mathbb{u}(\omega, \|f\|_\infty), \qquad \forall\, \omega \in \Omega.$$

Taking a version g of $\mathbb{E}_\mathbb{P}[\mathbb{u}(\cdot, f)|\mathcal{G}]$ we can find $A \in \mathcal{G}$ such that $\mathbb{P}(A) = 1$ and exploiting (1.21) we may write

$$\mathbb{u}_\mathcal{G}(\omega, -\|f\|_\infty) < g(\omega) < \mathbb{u}_\mathcal{G}(\omega, \|f\|_\infty) \quad \text{ for all } \omega \in A. \tag{1.24}$$

This particular version g of $\mathbb{E}_\mathbb{P}[\mathbb{u}(\cdot, f)|\mathcal{G}]$ is at this point only defined up to events of probability 1, and thus it is not in general an element of $\mathcal{L}^\infty(\Omega, \mathcal{G})$. Let us define $\tilde{g} = g\mathbb{1}_A + 0\mathbb{1}_{A^c}$ and observe that now $\tilde{g}(\omega) \in \mathrm{Im}_\omega(\mathbb{u}_\mathcal{G})$ for all $\omega \in \Omega$. Moreover, $\tilde{g}$ is still a version of the conditional expectation of f. Recalling that the mapping $\Phi_\mathcal{G}(\omega, \cdot) : \mathbb{R} \to \overline{\mathbb{R}}$ is strictly increasing on $\mathrm{Im}_\omega(\mathbb{u}_\mathcal{G}) \subseteq \mathbb{R}$ for all $\omega \in \Omega$, an application to (1.24) yields

$$\Phi_\mathcal{G}(\omega, \mathbb{u}_\mathcal{G}(\omega, -\|f\|_\infty)) < \Phi_\mathcal{G}(\omega, \tilde{g}(\omega)) < \Phi_\mathcal{G}(\omega, \mathbb{u}_\mathcal{G}(\omega, \|f\|_\infty)).$$

Consequently, by (1.23) $\Phi_\mathcal{G}(\omega, \mathbb{u}_\mathcal{G}(\omega, x)) = x$ for all $\omega \in \Omega$ and all $x \in \mathbb{R}$ and it follows that that $-\|f\|_\infty < \Phi_\mathcal{G}(\omega, \tilde{g}) < \|f\|_\infty$, for every $\omega \in \Omega$. Hence $\Phi_\mathcal{G}(\cdot, \tilde{g}) \in \mathcal{L}^\infty(\Omega, \mathcal{G})$. In particular we have

$$\mathbb{u}_\mathcal{G}\Big(\omega, \Phi_\mathcal{G}\big(\omega, \tilde{g}(\omega)\big)\Big) = \tilde{g}(\omega) \quad \text{ for every } \omega \in \Omega. \tag{1.25}$$

We prove that $\Phi_\mathcal{G}(\omega, \tilde{g}(\omega))$ is the conditional Chisini mean in that it solves the infinite dimensional system of equations given in (1.7), for the preference representing functional $V_\Omega(f) = \int_\Omega \mathbb{u}(\omega, f(\omega))\mathrm{d}\mathbb{P}$ defined in Theorem 1.5.4. Considering an arbitrary set $A \in \mathcal{G} \subseteq \mathcal{F}$, we have

$$\begin{aligned}
V_\Omega\big(\Phi_\mathcal{G}(\cdot, \tilde{g})\mathbb{1}_A\big) &= \mathbb{E}_\mathbb{P}\left[\mathbb{u}\big(\cdot, \Phi_\mathcal{G}(\cdot, \tilde{g})\,\mathbb{1}_A\big)\right] = \mathbb{E}_\mathbb{P}\left[\mathbb{E}_\mathbb{P}\left[\mathbb{u}\big(\cdot, \Phi_\mathcal{G}(\cdot, \tilde{g})\big)\mathbb{1}_A \middle| \mathcal{G}\right]\right] \\
&= \mathbb{E}_\mathbb{P}\left[\mathbb{E}_\mathbb{P}\left[\mathbb{u}\big(\cdot, \Phi_\mathcal{G}(\cdot, \tilde{g})\big)\middle|\mathcal{G}\right]\mathbb{1}_A\right] \overset{(1.21)}{=} \mathbb{E}_\mathbb{P}\left[\mathbb{u}_\mathcal{G}\big(\cdot, \Phi_\mathcal{G}(\cdot, \tilde{g})\big)\mathbb{1}_A\right] \\
&\overset{(1.23)}{=} \mathbb{E}_\mathbb{P}[\tilde{g}\mathbb{1}_A] = \mathbb{E}_\mathbb{P}[\mathbb{E}_\mathbb{P}[\mathbb{u}(\cdot, f)\mathbb{1}_A|\mathcal{G}]] = \mathbb{E}_\mathbb{P}[\mathbb{u}(\cdot, f)\mathbb{1}_A] \\
&= \mathbb{E}_\mathbb{P}[\mathbb{u}(\cdot, f\mathbb{1}_A)] = V_\Omega(f\mathbb{1}_A),
\end{aligned}$$

observing that since $\mathbb{u}(\omega, 0) = 0$ for all $\omega \in \Omega$ by construction (whence also $\Phi_\mathcal{G}(\omega, 0) = 0$ for all $\omega \in \Omega$) we have

$$\begin{cases} \mathbb{u}(\omega, f(\omega))\mathbb{1}_A(\omega) = \mathbb{u}\big(\omega, f(\omega)\mathbb{1}_A(\omega)\big) \\ \Phi_\mathcal{G}\left(\omega, \tilde{g}(\omega)\right)\mathbb{1}_A(\omega) = \Phi_\mathcal{G}\big(\omega, \tilde{g}(\omega)\mathbb{1}_A(\omega)\big) \end{cases} \qquad \text{for every } \omega \in \Omega.$$

1.10 Proof of Corollary 1.7.3

The proof of the corollary is merely an application of the results in the following sections, we briefly outline here the main steps. To show that (i) implies (iv), consider a preference order $\succeq$ that satisfies (SM), (PC) and (ST) and suppose $\mathcal{F}$ contains at least three disjoint non-null events. The existence of a conditional Chisini mean under

these assumptions is guaranteed by Doldi and Maggis (2023), Theorem 3.2 (See the proof of Proposition 1.8.1 for more details). Representability of conditional Chisini means is then obtained from Theorem 1.5.4 and Theorem 1.7.4 as follows. The former provides a pair $(\mathbb{u}, \mathbb{P})$ such that $\mathbb{u} : \Omega \times \mathbb{R} \to \mathbb{R}$ is $\mathcal{F}$-regular and $\mathbb{P}$-integrable. The latter allows the construction for any $\mathcal{G} \subseteq \mathcal{F}$ of a $\mathcal{G}$-regular function $\mathbb{u}_{\mathcal{G}} : \Omega \times \mathbb{R} \to \mathbb{R}$ that is $\mathbb{P}$-integrable and such that, for all $f \in \mathcal{L}^{\infty}(\Omega, \mathcal{F})$ and for a suitable version h of $\mathbb{E}_{\mathbb{P}}[\mathbb{u}(\cdot, f)|\mathcal{G}]$, we have $\mathbb{u}_{\mathcal{G}}^{-1}(\cdot, h(\cdot)) \in \mathfrak{m}(f|\mathcal{G})$.
Regarding the converse implication, observe that Item (iv) actually implies that any representing functional T for $\succeq$ is conditionable for every choice of $\mathcal{G} \subseteq \mathcal{F}$. Therefore, the exact same argument used in the proof of Theorem 1.7.1 (in particular Item (iii) $\Rightarrow$ Item (i)) shows that $\succeq$ satisfies (ST).

1.11 Representation of consistent nonlinear expectations

The representation of the conditional Chisini mean depicted so far allows to embed our theoretical results in the framework of nonlinear expectations. This deepens the understanding of the role of the Sure-Thing Principle in consistent conditioning.
Kupper and Schachermayer (2009) proved the noteworthy property that, under suitable structural assumptions, every consistent family of strictly monotone, continuous and law invariant nonlinear maps take the form of a conditional certainty equivalent

$$u^{-1}\left(\mathbb{E}_{\mathbb{P}}\left[u(X) \mid \mathcal{F}_t\right]\right), \tag{1.26}$$

where u is a strictly increasing function on a real interval. This result can be seen as a dynamic version of the aforementioned Nagumo-de Finetti-Kolmogorov Theorem.
A similar perspective can be observed in the recent stream of literature regarding the phenomenon of "collapse to the mean" for certain classes of functionals as in Bellini et al. (2021) for the law invariant case. Indeed our findings relate significantly to Theorem 6.1 in Delbaen (2021), where the assumption of law invariance is replaced with commonotonicity. In the realm of nonlinear expectations which are possibly not law invariant, the most important contribution comes from the theory of g-expectations related to Backward Stochastic Differential Equations, initiated and developed by Peng (1997); Coquet et al. (2002); Peng (2004). The main drawback of such theory is that the notion of "conditionability" is constrained to the use of the Brownian filtration. Therefore in general it is not possible to find a conditional nonlinear expectation for finitely generated σ-algebras, that is also consistent with the considered g-expectation.
We overcome this issue by taking into consideration a non atomic[12] probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the collection of sub-$\sigma$-algebras of $\mathcal{F}$

$$\Sigma \coloneqq \{\mathcal{G} | \mathcal{G} \text{ is a } \sigma\text{-algebra, } \mathcal{G} \subseteq \mathcal{F}\},$$

and a family of conditional nonlinear maps $\{\mathcal{E}_{\mathcal{G}}\}_{\mathcal{G}\in\Sigma}$, where $\mathcal{E}_{\mathcal{G}} : L^{\infty}(\Omega, \mathcal{F}, \mathbb{P}) \to L^{\infty}(\Omega, \mathcal{G}, \mathbb{P})$. We denote with $\mathcal{E}_0$ the map obtained choosing $\mathcal{G} = \sigma(\varnothing)$, the trivial

[12]Actually, requiring the existence of three disjoint sets in $\mathcal{F}$ with positive probability is sufficient to prove Theorem 1.11.2

σ-algebra. Moreover we indicate by capital letters the elements in $X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, and by $\mathbb{1}_A$ the indicator functions[13].

Definition 1.11.1. *We say that $\mathcal{E}_\mathcal{G} : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to L^\infty(\Omega, \mathcal{G}, \mathbb{P})$ is a conditional nonlinear expectation if for any $X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, $A \in \mathcal{G}$ we have $\mathcal{E}_\mathcal{G}(X\mathbb{1}_A) = \mathcal{E}_\mathcal{G}(X)\mathbb{1}_A$.*
A family of conditional nonlinear expectations $\{\mathcal{E}_\mathcal{G}\}_{\mathcal{G}\in\Sigma}$ is consistent if

$$\mathcal{E}_0(\mathcal{E}_\mathcal{G}(X)) = \mathcal{E}_0(X) \text{ for any } \mathcal{G} \in \Sigma \text{ and } X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}). \tag{1.27}$$

It is worth mentioning that a notion of consistency similar to (1.27), yet embedded in the framework of finite-dimensional marginal distributions, appears also in Denk et al. (2018), Definition 4.2. However, while the above definition pertains consistency relative to intermediate information, in Denk et al. (2018) the concept of consistency is associated with projection on smaller subspaces.

Assumption 1. *We assume that $\mathcal{E}_0$ is strictly monotone (on dichotomic random variables), i.e. for any $x, y \in \mathbb{R}$, $Z \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ and $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$*

$$x < y \text{ implies } \mathcal{E}_0(x\mathbb{1}_A + Z\mathbb{1}_{A^c}) < \mathcal{E}_0(y\mathbb{1}_A + Z\mathbb{1}_{A^c}), \tag{1.28}$$

and pointwise continuous, i.e. for any bounded sequence X_n converging $\mathbb{P}$-almost surely to X

$$\lim_{n\to\infty} \mathcal{E}_0(X_n) = \mathcal{E}_0(X). \tag{1.29}$$

The following representation result can be seen as an enhanced version of (1.26) dropping the law invariance.

Theorem 1.11.2. *$\{\mathcal{E}_\mathcal{G}\}_{\mathcal{G}\in\Sigma}$ is a family of consistent conditional nonlinear expectations such that $\mathcal{E}_0$ satisfies Assumption 1 if and only if*

$$\mathcal{E}_\mathcal{G}(X) = \mathbb{u}_\mathcal{G}^{-1}(\mathbb{E}_\mathbb{P}[\mathbb{u}(X)|\mathcal{G}]) \quad \text{for every } X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}),$$

for some $\mathbb{u} : \Omega \times \mathbb{R} \to \mathbb{R}$ which is jointly measurable, with $\mathbb{u}(\omega, 0) = 0$ and $\mathbb{u}(\omega, \cdot)$ continuous and strictly increasing for every $\omega \in \Omega$, $\mathbb{u}(\cdot, x)$ integrable for any $x \in \mathbb{R}$ and $\mathbb{u}_\mathcal{G}(\cdot, x) = \mathbb{E}_\mathbb{P}[\mathbb{u}(x)|\mathcal{G}]$.

Remark 1.11.3 (On uniqueness of $\mathbb{u}$, *continued*)*.* In the context of Theorem 1.11.2, we are ruling out the possibility of any change of measure, as we aim at a representation in terms of the reference probability measure $\mathbb{P}$, and by virtue of the "taking out what is known" property we necessitate $\mathbb{u}(\omega, 0) = 0$ so that σ needs to be null. Therefore in Theorem 1.11.2 any other representing function $\tilde{\mathbb{u}}$ is equal to $\alpha \cdot \mathbb{u}$ for some $\alpha > 0$.

Similarly to the representation in Theorem 1.2.12 (Theorem 1.4 in Kupper and Schachermayer, 2009) and differently from Lemma 5.2 in Cerreia-Vioglio et al. (2011), one of the most interesting aspects of our result is that the Sure-Thing Principle needs not to be explicitly assumed, but is rather deduced from the consistency of the maps involved.

[13] Differently to previous section, here $\mathbb{1}_A$ denotes the equivalence class of such indicator functions

1.12 Proof of Theorem 1.11.2

In this section we deal with equivalence classes of random variables, and therefore for any $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ we denote by $X = [f]_{\mathbb{P}}$ the element in $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ such that $f \in X$.

We first prove the converse implication of the statement. To this aim consider an $\mathcal{F}$-regular function $\mathbb{u} : \Omega \times \mathbb{R} \to \mathbb{R}$ such that $\mathbb{u}(\cdot, x)$ is integrable for all $x \in \mathbb{R}$ and let Σ be a collection of sub-σ-algebras of $\mathcal{F}$. We define a preference relation on $\mathcal{L}^\infty(\Omega, \mathcal{F})$ by

$$f_1 \succeq f_2 \text{ if and only if } \int_\Omega \mathbb{u}(\omega, f_1(\omega))\mathrm{d}\mathbb{P} \geq \int_\Omega \mathbb{u}(\omega, f_2(\omega))\mathrm{d}\mathbb{P}.$$

For any $\mathcal{G} \in \Sigma$ we can directly apply Theorem 1.7.4 and find a $\mathcal{G}$-regular function $\mathbb{u}_\mathcal{G} : \Omega \times \mathbb{R} \to \mathbb{R}$ such that $\mathbb{u}_\mathcal{G}(\cdot, x)$ is a version of $\mathbb{E}_\mathbb{P}[\mathbb{u}(x)|\mathcal{G}]$. Now fix $X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ and let f be any representative element of the equivalence class X. Analogously to the argument in Section 1.9, steps 3 and 4, we take a suitable version $h \in \mathbb{E}_\mathbb{P}[\mathbb{u}(f)|\mathcal{G}]$ and compute $\Phi_\mathcal{G}(\cdot, h(\cdot))$ where $\Phi_\mathcal{G} : \Omega \times \mathbb{R} \to \overline{\mathbb{R}}$ is the right inverse of $\mathbb{u}_\mathcal{G}$, that is

$$\Phi_\mathcal{G}(\omega, x) := \inf\{y \in \mathbb{R} \mid \mathbb{u}_\mathcal{G}(\omega, y) > x\}.$$

Observe that the equivalence class $[\Phi_\mathcal{G}(\cdot, h(\cdot))]_\mathbb{P}$ generated by $\Phi_\mathcal{G}(\cdot, h(\cdot))$ does not depend on the choice of the representative $f \in X$, hence the definition

$$\mathbb{u}_\mathcal{G}^{-1}\left(\mathbb{E}_\mathbb{P}[\mathbb{u}(X)|\mathcal{G}]\right) := [\Phi_\mathcal{G}(\cdot, h(\cdot))]_\mathbb{P}$$

is well posed. In particular, the previous argument formalizes the definition of the mapping $\mathcal{E}_\mathcal{G} : X \mapsto \mathbb{u}_\mathcal{G}^{-1}\left(\mathbb{E}_\mathbb{P}[\mathbb{u}(X)|\mathcal{G}]\right)$ provided in the statement of Theorem 1.11.2. It remains to show that $\{\mathcal{E}_\mathcal{G}\}_{\mathcal{G}\in\Sigma}$ defined above is a family of consistent conditional nonlinear expectations such that $\mathcal{E}_0$ satisfies Assumption 1. It is just a simple checking that for any $X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, $A \in \mathcal{G}$ we have $\mathcal{E}_\mathcal{G}(X\mathbb{1}_A) = \mathcal{E}_\mathcal{G}(X)\mathbb{1}_A$. Similarly, $\mathcal{E}_0(\mathcal{E}_\mathcal{G}(X)) = \mathcal{E}_0(X)$ for any $\mathcal{G} \in \Sigma$ and $X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. Finally we observe that $\mathcal{E}_0(X) = \mathbb{u}_0^{-1}\left(\int_\Omega \mathbb{u}(\omega, f(\omega))\mathrm{d}\mathbb{P}\right)$ for $\mathbb{u}_0 : x \mapsto \mathbb{E}_\mathbb{P}[\mathbb{u}(x)]$ and some $f \in X$. Therefore from strict monotonicity and continuity of $\mathbb{u}$ we get: (a) for any $x, y \in \mathbb{R}$, $Z \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ and $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$, if $x < y$ then $\mathcal{E}_0(x\mathbb{1}_A + Z\mathbb{1}_{A^c}) < \mathcal{E}_0(y\mathbb{1}_A + Z\mathbb{1}_{A^c})$; (b) for any bounded sequence X_n converging $\mathbb{P}$-almost surely to X we have $\lim_{n\to\infty} \mathcal{E}_0(X_n) = \mathcal{E}_0(X)$. This concludes the first implication.
Now suppose $\{\mathcal{E}_\mathcal{G}\}_{\mathcal{G}\in\Sigma}$ is a family of consistent nonlinear conditional expectations such that $\mathcal{E}_0$ satisfies Assumption 1. We show that there exists an $\mathcal{F}$-regular function $\mathbb{u} : \Omega \times \mathbb{R} \to \mathbb{R}$ such that $\mathbb{u}(\cdot, x)$ is integrable for all $x \in \mathbb{R}$ and, for any $X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G} \in \Sigma$, $\mathcal{E}_\mathcal{G}$ is of the form

$$\mathcal{E}_\mathcal{G}(X) = \mathbb{u}_\mathcal{G}^{-1}(\mathbb{E}_\mathbb{P}[\mathbb{u}(X)|\mathcal{G}]),$$

with $\mathbb{u}_\mathcal{G}(\cdot, x) = \mathbb{E}_\mathbb{P}[\mathbb{u}(x)|\mathcal{G}]$. For any $f_1, f_2 \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ we define the preference order $\succeq$ as follows

$$f_1 \succeq f_2 \text{ if and only if } \mathcal{E}_0(X_1) \geq \mathcal{E}_0(X_2),$$

where $X_1 = [f_1]_\mathbb{P}, X_2 = [f_2]_\mathbb{P} \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ are the equivalence classes generated by f_1, f_2, and notice that the preference relation $\succeq$ is well defined.

We start by showing that the null sets of the preference as defined in (1.3) correspond to the null sets of the reference probability measure $\mathbb{P}$ i.e. $\mathcal{N}_{\succeq} = \{A \in \mathcal{F} \mid \mathbb{P}(A) = 0\}$. Let $A \in \mathcal{F}$ be such that $\mathbb{P}(A) = 0$ and consider $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ with f being a representative element in $X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. Take any $g \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ being a representative of any $Y \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. Then since $\mathbb{P}(A) = 0$, we have that $f\mathbb{1}_{A^c} + g\mathbb{1}_A$ is still a representative of X, thus $\mathcal{E}_0(X\mathbb{1}_{A^c} + Y\mathbb{1}_A) = \mathcal{E}_0(X)$. This yields $A \in \mathcal{N}_{\succeq}$. Now suppose $A \in \mathcal{F}$ is such that $\mathbb{P}(A) > 0$ and consider $x, y \in \mathbb{R}$ with $x < y$. For any choice of $Z \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, monotonicity of $\mathcal{E}_0$ implies that

$$\mathcal{E}_0(x\mathbb{1}_A + Z\mathbb{1}_{A^c}) < \mathcal{E}_0(y\mathbb{1}_A + Z\mathbb{1}_{A^c}),$$

which in turn yields $A \notin \mathcal{N}_{\succeq}$.
Secondly, we observe that $\succeq$ satisfies (SM) and (PC) as an immediate consequence of Assumption 1.11.1. Finally, notice that for any $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ the functional $T(f) = \mathcal{E}_0(X)$, with $X = [f]_{\mathbb{P}} \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, represents $\succeq$. By the properties of the family $\{\mathcal{E}_\mathcal{G}\}_{\mathcal{G}\in\Sigma}$, for any sub-$\sigma$-algebra $\mathcal{G}$ we have $\mathcal{E}_0(X\mathbb{1}_A) = \mathcal{E}_0(\mathcal{E}_\mathcal{G}(X)\mathbb{1}_A)$ so that $T(f\mathbb{1}_A) = T(g\mathbb{1}_A)$ for any $A \in \mathcal{G}$ and $g \in \mathcal{E}_\mathcal{G}(X)$. Hence T is conditionable and in particular, by Theorem 1.7.1, $\succeq$ satisfies (ST). By Theorem 1.5.4 we can find a measure $\mathbb{Q}$ and a state-dependent utility $\tilde{\mathbb{u}}$ such that Eq. (1.10) holds, adopting the pair $(\tilde{\mathbb{u}}, \mathbb{Q})$. We observe that $\mathbb{Q} \sim \mathbb{P}$ and hence we can redefine $\mathbb{u}(\omega, x) = (d\mathbb{Q}/d\mathbb{P})(\omega)\,\tilde{\mathbb{u}}(\omega, x)$, where $(d\mathbb{Q}/d\mathbb{P})$ is a version of the Radon-Nikodym derivative, and observe that Eq. (1.10) still holds true for $(\mathbb{u}, \mathbb{P})$, with $\mathbb{u}$ being $\mathcal{F}$-regular.

Applying Theorem 1.7.4 we indeed can define a $\mathcal{G}$-regular function $\mathbb{u}_\mathcal{G} : \Omega \times \mathbb{R} \to \mathbb{R}$, with $\mathbb{u}_\mathcal{G}(\cdot, x) \in \mathcal{L}^1(\Omega, \mathcal{G}, \mathbb{P})$ for every $x \in \mathbb{R}$, as a version of $\mathbb{E}_\mathbb{P}[\mathbb{u}(x)|\mathcal{G}]$ for every $x \in \mathbb{R}$. In this way, for an appropriate choice of $h \in \mathbb{E}_\mathbb{P}[\mathbb{u}(f)|\mathcal{G}]$, $\Phi_\mathcal{G}(\cdot, h(\cdot)) \in \mathfrak{m}(f|\mathcal{G})$ where $\Phi_\mathcal{G} : \Omega \times \mathbb{R} \to \overline{\mathbb{R}}$ defined by

$$\Phi_\mathcal{G}(\omega, x) := \inf\{y \in \mathbb{R} | \mathbb{u}_\mathcal{G}(\omega, y) > x\}$$

is $\mathcal{G} \otimes \mathscr{B}_\mathbb{R}$ measurable. Note that the map $X \mapsto [\Phi_\mathcal{G}(\cdot, h(\cdot))]_\mathbb{P}$ is well defined, since the procedure above does not depend (up to $\mathbb{P}$-null events) on the particular choice of a representative of X, and we shall write $\mathbb{u}_\mathcal{G}^{-1}(\mathbb{E}_\mathbb{P}[\mathbb{u}(X)|\mathcal{G}]) = [\Phi_\mathcal{G}(\cdot, h(\cdot))]_\mathbb{P}$. By the previous argument any element $g \in \mathcal{E}_\mathcal{G}(X)$ is also an element of $\mathfrak{m}(f|\mathcal{G})$ and therefore $\mathbb{u}_\mathcal{G}^{-1}(\mathbb{E}_\mathbb{P}[\mathbb{u}(X)|\mathcal{G}]) = \mathcal{E}_\mathcal{G}(X)$.

Remark 1.12.1. It is worthy of note that the "taking out what is known" property in Definition 1.11.1 jointly with Assumption 1 imply that for any $Y \in L^\infty(\Omega, \mathcal{G}, \mathbb{P})$ we have $\mathcal{E}_\mathcal{G}(Y) = Y$. To see this, first observe that by strict monotonicity and consistency one can show $\mathcal{E}_\mathcal{G}(x) = x$ for $x \in \mathbb{R}$. Consequently, applying the "taking out what is known" property, it follows that $\mathcal{E}_\mathcal{G}(Z) = Z$ for any $\mathcal{G}$-measurable simple function Z. To conclude, for a general $Y \in L^\infty(\Omega, \mathcal{G}, \mathbb{P})$ we can choose a sequence of $\mathcal{G}$-measurable simple functions $(\hat{Y}_N)_N$ such that $\hat{Y}_N \to_N Y$ with respect to the sup norm, then it can be shown that $||\mathcal{E}_\mathcal{G}(Y) - \hat{Y}_N||_\infty \to_N 0$ which in turn yields $\mathcal{E}_\mathcal{G}(Y) = Y$.

Chapter 2

On consistency of optimal portfolio choice for state-dependent exponential utilities

In this chapter we investigate a consistency issue arising in optimal portfolio problems of the Merton (1969) type. In a multi-period setting, an optimal strategy is an investment plan that ensures the agent maximizes a function of their terminal wealth resulting from trading in the market. Ideally, such a strategy is determined at the beginning of the trading period and is intended to be followed "*consistently*" until the specified terminal date. However, the optimization problem may be *time-inconsistent*, meaning that the strategy currently considered optimal may not remain optimal in subsequent periods. Consequently, it is not universally clear what optimality should signify in such contexts.

The literature on time-inconsistent utility maximization traces back to the foundational contribution of Strotz (1955), who formalizes the conflict between present and future selves in intertemporal decision-making. Subsequent works can be distinguished into three main approaches: *precommitment*, *myopic behaviour* and *consistent planning*. The first approach assumes that the agent fully commits to the strategy determined at the outset of the investment period and never deviates from it. In contrast, a myopic agent continuously updates their strategy at each point in time, selecting the instantaneous optimum, regardless of the potential long-term inefficiencies. Finally, the consistent planner adopts a game-theoretic perspective, treating future incentives to deviate as constraints to the current optimization problem. This perspective, pioneered by Ekeland and Lazrak (2010) and further developed by Basak and Chabakauri (2010) and Björk et al. (2014, 2017), often addresses time inconsistency arising from mean-variance utility formulations.

Classical approaches to expected utility maximization typically specify a terminal objective functional, and then derive the optimal strategy through backward induction. In contrast, more recent research (Musiela and Zariphopoulou, 2006, 2008, 2009; Henderson and Hobson, 2007) introduces an alternative, *maturity-independent* framework. In this setting, the utility function evolves stochastically and forward in time. This *forward performance* theory aligns with the classical notion of indirect utility function with a forward utility criterion derived from the financial market. The stochastic utility is deduced in order to satisfy specific (super)martingale conditions to recover the Dynamic Programming Principle.

In this chapter, we approach portfolio choice in a simple market model from a "forward" perspective, focusing primarily on the issue of consistency. Our approach centers on deriving optimal strategies that can be consistently extended beyond a fixed terminal date in a self-financing manner.
Building on the seminal definition in Kydland and Prescott (1977), p. 475, we adapt and refine their notion of consistency to our setup (see Definition 2.2.4). A policy α^* is consistent if, "for any time period t, it optimizes the objective functional, taking as given previous decisions and that future policy decisions are similarly selected".
In Proposition 2.3.1, we demonstrate that inconsistency may arise in the classical Merton problem when the market price of risk exhibits sufficient stochasticity. However, this drawback is mitigated when uncertainty about the agent's future risk aversion is introduced. Our main result, stated in Theorem 2.5.2, establishes that consistency for exponential-type utilities is achieved if and only if the agent's risk aversion evolves according to a unique dynamic. However, Proposition 2.5.4 demonstrates that, even within a simplified market model, an infinite class of forward performances can be readily constructed. However, we discuss how consistency emerges as a crucial criterion for inferring the agent's preferences "adjusted" to the temporal evolution of market information.

The structure of Chapter 2 is as follows: in Section 2.1 we set the notation. The portfolio choice problem at hand is formulated in Section 2.2. In Section 2.3, we show that inconsistency phenomena may occur even in the framework of optimization problems involving standard deterministic utility functions. In Section 2.5 we present our main results on consistent optimization with state-dependent utility functions and finally Section 2.8 presents an alternative (naive) state-dependent utility framework.

2.1 Notation and preliminaries

We collect here some recurrent notations we will adopt in this chapter.

- $\mathbb{R}^n$: the space of n-dimensional column vectors;
- $(\mathbb{R}^n)'$ the space of n-dimensional row vectors;
- $\mathbf{M}^{n\times d}(\mathbb{R})$: the space of $n \times d$ dimensional matrices of real numbers;

Let $u, v \in \mathbb{R}^n$:

- v': the transpose vector of v;
- $u \cdot v$: the scalar product of u and v;
- $\|v\|^2 := v_1^2 + ... + v_2^2$ for $v = (v_1, ..., v_n)'$. In particular, unless otherwise specified, the operator $u \mapsto \|u\|$ always refers to the Euclidean norm;
- $\mathbf{1} \in \mathbb{R}^n$: the n-dimensional vector where each entry takes value 1;

- with a slight abuse of notation we shall write 0 to indicate both $0 \in \mathbb{R}$ and $\mathbf{0} \in \mathbb{R}^n$ whenever the context allows for it.

Let $M \in \mathbf{M}^{n\times d}$:

- M': the $d \times n$-dimensional transpose matrix of M;

The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is fixed throughout the chapter and we shall denote by $L^0(\Omega, \mathcal{F}, \mathbb{P})$ the space of (equivalence classes of) $\mathcal{F}$-measurable random variables that are $\mathbb{P}$ a.s. finite. Similarly $L^1(\Omega, \mathcal{F}, \mathbb{P})$ will denote the space of integrable random variables. In this paper a filtration $(\mathcal{F}_t)_{t\geq 0}$ will describe the information available to an economic agent at any time $t \in [0, \infty)$ and we shall always assume that the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, satisfies the usual conditions. Analogously to the previous chapter, we denote by $\mathscr{B}_A$ the Borel σ-algebra on a subset A of the extended real line. Let E be a Borel set, we shall say that a property holds for Lebesgue almost every (a.e.) $t \in E$ if such property holds for every $t \in E\backslash N$, with N a Borel set with null Lebesgue measure.

2.2 Problem formulation

A decision maker aims at maximizing their utility by investing an initial endowment $x \in \mathbb{R}$ in a market composed by $d > 0$ risky assets $S^1, S^2, ..., S^d$ and a bond B. We denote by S_t the column vector of asset prices at time $t > 0$, that is

$$S_t := \begin{pmatrix} S_t^1 \\ \vdots \\ S_t^d \end{pmatrix}$$

For each $i = 1, ..., d$ the dynamics of $(S_t^i)_{t\geq 0}$ are as follows

$$\mathrm{d}S_t^i = S_t^i \mu_t^i \mathrm{d}t + S_t^i \sum_{j=1}^{n} \sigma_t^{ij}\, \mathrm{d}W_t^j, \qquad S_0^i > 0, \tag{2.1}$$

where μ_t^i, $(\sigma_t^{ij})_{j=1,...,n}$ are progressively measurable processes and $W_t = (W_t^j)_{j=1,...,n}$ is an n-dimensional Brownian motion. $(B_t)_{t\geq 0}$ denotes the price of the riskless asset at time t and its dynamics are given by

$$\mathrm{d}B_t = B_t r \mathrm{d}t, \qquad B_0 = 1 \tag{2.2}$$

with $r \in \mathbb{R}$.
To simplify the notation we write

$$\mu_t := \begin{pmatrix} \mu_t^1 \\ \vdots \\ \mu_t^d \end{pmatrix}, \qquad \sigma_t = \begin{pmatrix} \sigma_t^1 \\ \vdots \\ \sigma_t^n \end{pmatrix} = \begin{pmatrix} \sigma_t^{11} & \cdots & \sigma_t^{1n} \\ \vdots & \ddots & \vdots \\ \sigma_t^{d1} & \cdots & \sigma_t^{dn} \end{pmatrix},$$

and by denoting by $\sigma_t^i = (\sigma_t^{i1}, ..., \sigma_t^{in})$ the i-th row of the volatility matrix σ_t we can write (2.1) in a more compact form as

$$\mathrm{d}S_t^i = S_t^i \left(\mu_t^i \mathrm{d}t + \sigma_t^i\, \mathrm{d}W_t\right).$$

Finally, we can express the market dynamics as

$$\mathrm{d}S_t = D(S_t)\mu_t\mathrm{d}t + D(S_t)\sigma_t\,\mathrm{d}W_t \tag{2.3}$$

where $D(S_t)$ indicates the $d \times d$ diagonal matrix with entries $S_t^1, \ldots, S_t^d$.

Standing Assumption 2.2.1. *The market model described in* (2.2)-(2.3) *is complete.*

Remark 2.2.2. Recall that, for a d-dimensional market model to be complete, we have to require that the number of assets in the market is equal to the dimension of the underlying Brownian motion W_t, that is $n = d$, and that the filtration is the completion of the natural filtration generated by W. Finally, we have to impose that the volatility matrix σ_t is invertible for Lebesgue a.e. $t \in [0,T]$ $\mathbb{P}$-a.s..

We set

$$\theta_t := -\sigma_t^{-1}\left(\mu_t - r\mathbf{1}\right)$$

and impose that the positive local martingale

$$Z_t := \exp\left\{-\frac{1}{2}\int_0^t \|\theta_s\|^2\mathrm{d}s + \int_0^t \theta_s'\,\mathrm{d}W_s\right\} \tag{2.4}$$

is actually a $\mathbb{P}$-martingale.
A well-known sufficient condition for Z_t to be a martingale, due to Novikov (see Karatzas and Shreve, 1991, section 3.5.D), is that

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left\{\frac{1}{2}\int_0^T \|\theta_t\|^2\mathrm{d}t\right\}\right] < \infty$$

for a suitable $T < \infty$. However, for the sake of the exposition of this decision problem, we impose the following stronger condition.

Standing Assumption 2.2.3. *We assume that*

$$\mathbb{P}\left(\theta_t \neq 0 \text{ and } \sup_{0\le u\le t}\|\theta_u\| < K_t\right) = 1 \text{ for any } t > 0 \text{ and some } K_t \in [0,\infty), \tag{2.5}$$

As a consequence, at any $t > 0$ the probability measure $\mathbb{Q}_t$, with $\mathrm{d}\mathbb{Q}_t = Z_t\mathrm{d}\mathbb{P}$, defines a martingale measure for the discounted price process $(e^{-ru}S_u)_{u\in[0,t]}$. We observe that, for any choice of t, s such that $t > s$, the restriction of $\mathbb{Q}_t$ to $\mathcal{F}_s$ coincides with $\mathbb{Q}_s$. Hence, to avoid burdensome notation, we omit the subscript and denote by $\mathbb{Q}$ the family of measures induced by the process $(Z_t)_{t\geq 0}$. We shall denote by $W^{\mathbb{Q}}$ the d-dimensional $\mathbb{Q}$-Brownian motion defined as $\mathrm{d}W_t^{\mathbb{Q}} = \mathrm{d}W_t - \theta_t\mathrm{d}t$.

Throughout this chapter, all the financial quantities of interest are to be considered in discounted terms. Hence, V^α denotes the discounted value of a self-financing portfolio and α_t^i represents the proportion of V_t^α invested in the i-th risky asset at time t. We collect the proportion α^i into the vector $\alpha' = (\alpha^1, ..., \alpha^d)$, which we shall refer to as the strategy of the agent, clearly $1 - \sum_{i=1}^d \alpha_t^i$ identifies the proportion of V_t^α invested in the riskless asset. The dynamics of V_t^α are given by

$$\mathrm{d}V_t^\alpha = V_t^\alpha\left[\alpha_t'(\mu_t - r\mathbf{1})\mathrm{d}t + \alpha_t'\sigma_t\,\mathrm{d}W_t\right], \tag{2.6}$$

or in scalar notation

$$\mathrm{d}V_t^\alpha = V_t^\alpha \left[\sum_{i=1}^d \alpha_t^i (\mu_t^i - r)\mathrm{d}t + \sum_{i=1}^d \left(\alpha_t^i \sum_{j=1}^d \sigma_t^{ij}\, \mathrm{d}W_t^j \right) \right]. \tag{2.7}$$

For a fixed time horizon $t \in (0, +\infty)$, the agent is endowed with a preference relation $\succeq^t$ that admits a numerical representation through a state-dependent utility $\mathbb{u}_t : \Omega \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ and the subjective probability of the agent coincides with the reference[1] measure $\mathbb{P}$. With an abuse of notation we shall always denote by $\mathbb{u}_t(X)$ the random variable $\omega \mapsto \mathbb{u}_t(\omega, X(\omega))$ for any $X \in L^0(\Omega, \mathcal{F}_t, \mathbb{P})$. Given $X, Y \in L^0(\Omega, \mathcal{F}_t, \mathbb{P})$ such that $\mathbb{u}_t(X), \mathbb{u}_t(Y) \in L^1(\Omega, \mathcal{F}_t, \mathbb{P})$, the preference ordering is given by

$$X \succeq^t Y \quad \text{if and only if} \quad \mathbb{E}_{\mathbb{P}}\left[\mathbb{u}_t(X)\right] \geq \mathbb{E}_{\mathbb{P}}\left[\mathbb{u}_t(Y)\right].$$

Wakker and Zank (1999) proved that this representation holds if the preference $\succeq^t$ is monotone, pointwise continuous, and satisfies the Sure-Thing principle. Additionally, in Chapter 1 we showed that $\mathbb{u}_t$ can be chosen to be $\mathcal{F}_t \otimes \mathbb{B}_{\mathbb{R}}$ measurable with $x \mapsto \mathbb{u}_t(\omega, x)$ strictly increasing and continuous for every $\omega \in \Omega$.
In the decision framework just depicted, the classical utility maximization problem thus takes the form

$$\text{maximize } \mathbb{E}_{\mathbb{P}}\left[\mathbb{u}_t(V_t^\alpha)\right] \text{ constrained to } V_0^\alpha = x, \tag{2.8}$$

where α belongs to a suitable family $\mathcal{A}$ of admissible controls and $x \in \mathbb{R}$ represents the initial wealth of the agent. In particular, we will always suppose that any admissible control $\alpha \in \mathcal{A}$ guarantees the existence and uniqueness of a (weak) solution of (2.6). We shall denote by $(\alpha_u^*)_{u\in[0,t]}$ the strategy that maximizes the expected utility of the agent in the time interval $[0, t]$ and, therefore, $V_t^{\alpha^*}$ denotes the maximizer of the objective functional $\mathbb{E}_{\mathbb{P}}\left[\mathbb{u}_t(\cdot)\right]$.

As illustrated in the introduction, we shall adopt throughout the paper the following notion of consistency, which rephrases the seminal notion in Kydland and Prescott (1977).

Definition 2.2.4. *For $s < t$, let $V_s^{\alpha^*}$ be the maximizer of $\mathbb{E}_{\mathbb{P}}\left[\mathbb{u}_s(\cdot)\right]$ and $V_t^{\alpha^*}$ be the maximizer[2] of $\mathbb{E}_{\mathbb{P}}\left[\mathbb{u}_t(\cdot)\right]$ both selected by solving (2.8). An optimization problem is* ***consistent*** *with respect to any time horizon if for any $s < t$ it holds that $\mathbb{E}_{\mathbb{Q}}\left[V_t^{\alpha^*} \middle| \mathcal{F}_s\right] = V_s^{\alpha^*}$.*

This type of consistency naturally leads to strategies which are not affected by a fixed time horizon, which is the basis of the well-known theory of forward performances, which we now recall as it will play a crucial role in the discussion.

[1] This can be assumed without loss of generality if the subjective probability is equivalent to the reference measure, as the function $\mathbb{u}_t$ is unique up to a change of measure, (see Wakker and Zank (1999))

[2] We make an abuse of notation, as a priori the strategies $(\alpha_u^*)_{u\in[0,s]}$, $(\alpha_u^*)_{u\in[0,t]}$ which define $V_s^{\alpha^*}$, $V_t^{\alpha^*}$ might be different. Nevertheless, in the chosen market model, and assuming consistency holds, they must coincide over the common time interval.

Definition 2.2.5 (Musiela and Zariphopoulou (2009))**.** *Given $u_0 : \mathbb{R} \to \mathbb{R}$ a concave and increasing function, an $\mathcal{F}_t$-adapted process $(\mathbb{u}_t)_{t\geq 0}$ with $\mathbb{u}_t : \Omega \times \mathbb{R} \to \mathbb{R}$ is a* forward performance *if*

- *the mapping $x \mapsto \mathbb{u}_t(\omega, x)$ is increasing and concave for each $\omega \in \Omega$ and $t \geq 0$;*
- *it satisfies $\mathbb{u}_0(\omega, x) = u_0(x)$;*
- *for all $t, s \in [0, \infty)$ such that $t > s$ and for any $\alpha \in \mathcal{A}$ it holds $\mathbb{E}_{\mathbb{P}}\left[\mathbb{u}_t(V_t^\alpha)|\mathcal{F}_s\right] \leq \mathbb{u}_s(V_s^\alpha)$;*
- *for all $t, s \in [0, \infty)$ such that $t > s$ there exists $\alpha^* \in \mathcal{A}$ such that $\mathbb{E}_{\mathbb{P}}\left[\mathbb{u}_t(V_t^{\alpha^*})\big|\mathcal{F}_s\right] = \mathbb{u}_s(V_s^{\alpha^*})$.*

Remark 2.2.6. Observe that Definition 2.2.4 is concerned with a notion of consistency which can be effectively interpreted as a self-financing constraint for the optimal portfolio, in order to guarantee optimality at any date t for a time dependent preference structure. Conversely, the concept of consistency that motivates Definition 2.2.5 pertains a natural martingale condition that is analogous to that of the Dynamic Programming Principle.

While both approaches offer the advantage of providing *horizon-less* optimizers, Proposition 2.5.4 will point out that the two notions differ significantly. In the forward performance framework, the optimal strategy $(\alpha_t^*)_{t\geq 0}$ plays a central role in determining the *a posteriori* dynamic utility $(\mathbb{u}_t)_{t\geq 0}$, so that the two processes are, in effect, jointly specified.
On the other hand, the consistency approach that we adopt imposes an *a priori* preference structure and derives conditions for it to produce consistent optima.

2.3 On consistency for Von Neumann-Morgenstern type preferences

Consistency may fail already in classical optimization problems, when utilities are not state dependent, as described in the following proposition. In complete markets, the classical Merton problem can be solved by computing explicitly at time t the optimal discounted wealth ξ_t^* and then deriving backwardly optimal strategies via a perfect hedging procedure, which intrinsically links them to the terminal date. This approach, pioneered in Karatzas et al. (1987), allows to recast the dynamic optimization problem at hand as an equivalent static problem.
Before turning to the main result, we consider a simple motivating example. In fact the following proposition implies that if the market price of risk fails to be deterministic, an agent with classical preferences may not be able to devise consistent optimal plans at all.

Proposition 2.3.1. *For*

$$u(x) = -\frac{1}{\gamma}e^{-\gamma x}, \quad \gamma > 0 \tag{2.9}$$

problem (2.8) is consistent if and only if $Var(\|\theta_t\|^2) = 0$ for Lebesgue a.e. $t \in [0, \infty)$.

Remark 2.3.2. The issue of inconsistency arises not only for utilities of the exponential type. Simple inspections show that the Merton problem is always consistent for $u(x) = \ln(x)$. This follows directly from the form of the optimal terminal wealth: $\xi_t^* = xZ_t^{-1}$. As Z_t^{-1} is a $\mathbb{Q}$-martingale by construction, the family of optima $\{\xi_t^*\}_{t\geq 0}$ is itself naturally a $\mathbb{Q}$-martingale.
Instead, for power utilities $u(x) = x^\gamma/\gamma$ with $\gamma \in (0, 1)$, problem (2.8) is consistent if and only if $Var(\|\theta_t\|^2) = 0$ for Lebesgue a.e. $t \in [0, \infty)$ (see the Section 2.4 for the proof), in analogy with the case treated in Proposition 2.3.1.

Remark 2.3.3 (What if consistency fails?). Consider two investment horizons $s < t$ and the respective optima $V_s^{\alpha^*}, V_t^{\alpha^*}$. Since the market is assumed to be complete, the unique price at time s of $V_t^{\alpha^*}$ is given by $\Pi_s := \mathbb{E}_{\mathbb{Q}}\left[V_t^{\alpha^*} \middle| \mathcal{F}_s\right]$. Whenever consistency does not hold, situations where $\mathbb{P}\left(\Pi_s \neq V_s^{\alpha^*}\right) > 0$ may arise. In other words, the agent that achieved their optimum at time s and wishes to roll over their strategy up to time t, might not be able to reach the (a priori) optimum $V_t^{\alpha^*}$. This issue is of little relevance whenever the agent cannot change their **initially** specified investment horizon. However, we stress that often in practice an agent chooses to optimize up to time s with the option of reconsidering their initially set horizon, either with $t_1 < s$ or $t_2 > s$. Usually short time horizons are considered and after that (and depending on the performance of the portfolio), the agent decides either to quit investing or to extend the final date.
In this scenario depicted above, once the optimum is determined, the agent follows the strategy α^* up to time s reaching the *optimal* wealth $V_s^{\alpha^*}$. At this point, as the event $A = \{V_s^{\alpha^*} < \Pi_s\} \in \mathcal{F}_s$ may have positive probability, if A occurs the agent will not be able to roll over their strategy from s to t and hence cannot achieve the time t optimum $V_t^{\alpha^*}$.

2.4 Proofs of Proposition 2.3.1 and Remark 2.3.2

This section is dedicated to the proofs of the result of the previous section. While Proposition 2.3.1 offers a useful starting point for developments in a later section of this thesis, Remark 2.3.2 highlights that inconsistency is not unique to exponential utility functions. The proof of Proposition 2.3.1 will be useful in proving consistency conditions for a wider family of "exponential-type" stochastic utility functions, which we shall introduce in Section 2.8. The study of consistent optimization problems with power utility functions however is not pursued further in this thesis.

Proof of Proposition 2.3.1. Consider $t > 0$ fixed. Since the market is assumed to be complete, we can recast Problem (2.8) as the following infinite dimensional problem

$$\sup \left\{ \mathbb{E}_{\mathbb{P}}\left[u(\xi)\right] \mid \xi \in L^1(\Omega, \mathcal{F}_t, \mathbb{Q}) \text{ and } \mathbb{E}_{\mathbb{Q}}\left[\xi\right] = x \right\}. \tag{2.10}$$

The application of Lemma B.1 then yields the optimum

$$\xi_t^* = x + \frac{1}{\gamma}\mathbb{E}_{\mathbb{Q}}\left[\ln(Z_t)\right] - \frac{1}{\gamma}\ln(Z_t),$$

with $\mathbb{P}(|\xi_t^*| < k) = 1$ for some $k \in \mathbb{R}$, as a consequence of Standing Assumption 2.2.3.

Considering some $s < t$, we compute the time s price of ξ_t^* as

$$\mathbb{E}_{\mathbb{Q}}\left[\xi_t^*|\mathcal{F}_s\right] = x + \frac{1}{\gamma}\,\mathbb{E}_{\mathbb{Q}}\left[\ln(Z_t)\right] - \frac{1}{\gamma}\,\mathbb{E}_{\mathbb{Q}}\left[\ln(Z_t)|\mathcal{F}_s\right]. \tag{2.11}$$

Observe that

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[\ln(Z_t)|\mathcal{F}_s\right] &= \underbrace{-\frac{1}{2}\int_0^s \|\theta_u\|^2 \mathrm{d}u + \int_0^s \theta_u'\,\mathrm{d}W_u}_{\ln(Z_s)} + \mathbb{E}_{\mathbb{Q}}\left[-\frac{1}{2}\int_s^t \|\theta_u\|^2 \mathrm{d}u + \int_s^t \theta_u'\,\mathrm{d}W_u \middle| \mathcal{F}_s\right] \\
&= \ln(Z_s) + \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{2}\int_s^t \|\theta_u\|^2 \mathrm{d}u + \int_s^t \theta_u'\,\mathrm{d}W_u^{\mathbb{Q}} \middle| \mathcal{F}_s\right] \\
&= \ln(Z_s) + \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{2}\int_s^t \|\theta_u\|^2 \mathrm{d}u \middle| \mathcal{F}_s\right],
\end{aligned} \tag{2.12}$$

where the first equality follows from the fact that the integrals up to time s are $\mathcal{F}_s$-measurable, the second is a mere change from the $\mathbb{P}$-brownian motion to the $\mathbb{Q}$-brownian motion and the last is a consequence of the stochastic integral being independent of $\mathcal{F}_s$. A similar argument yields

$$\mathbb{E}_{\mathbb{Q}}\left[\ln(Z_t)\right] = \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{2}\int_0^t \|\theta_u\|^2 \mathrm{d}u\right]. \tag{2.13}$$

Finally, plugging (2.12) and (2.13) into (2.11) results in

$$\mathbb{E}_{\mathbb{Q}}\left[\xi_t^*|\mathcal{F}_s\right] = x + \frac{1}{\gamma}\left(E_0^t - E_s - \ln(Z_s)\right),$$

where $E_0^t = \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{2}\int_0^t \|\theta_u\|^2\,\mathrm{d}u\right]$ and $E_s = \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{2}\int_s^t \|\theta_u\|^2\,\mathrm{d}u \middle| \mathcal{F}_s\right]$. In order to ensure the problem is time consistent we require $\mathbb{E}_{\mathbb{Q}}\left[\xi_t^*|\mathcal{F}_s\right] = \xi_s^*$, i.e.

$$x + \frac{1}{\gamma}\left[E_0^t - E_s - \ln(Z_s)\right] = x + \frac{1}{\gamma}\,\mathbb{E}_{\mathbb{Q}}\left[\ln(Z_s)\right] - \frac{1}{\gamma}\ln(Z_s), \tag{2.14}$$

where the dynamics of process $\ln(Z_t)$ under $\mathbb{Q}$ is given by $\mathrm{d}\ln(Z_t) = \frac{1}{2}\theta_t^2\mathrm{d}t + \theta_t\,\mathrm{d}W_t^{\mathbb{Q}}$. Observing that

$$E_0^t = \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{2}\int_0^t \|\theta_u\|^2 \mathrm{d}u\right] = \underbrace{\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{2}\int_0^s \|\theta_u\|^2 \mathrm{d}u\right]}_{\mathbb{E}_{\mathbb{Q}}[\ln(Z_s)]} + \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{2}\int_s^t \|\theta_u\|^2 \mathrm{d}u\right],$$

Eq. (2.14) boils down to

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{2}\int_s^t \|\theta_u\|^2 \mathrm{d}u \middle| \mathcal{F}_s\right] = \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{2}\int_s^t \|\theta_u\|^2 \mathrm{d}u\right]. \tag{2.15}$$

Because the choice of s is arbitrary, condition (2.15) must hold for all $s < t$. Whenever $\|\theta_t\|^2$ is deterministic the latter is clearly satisfied. We conclude by showing the converse implication. First, applying Fubini-Tonelli Theorem we can rewrite (2.15) as

$$\frac{1}{2}\int_s^t \mathbb{E}_{\mathbb{Q}}\left[\|\theta_u\|^2\middle|\mathcal{F}_s\right]\mathrm{d}u = \frac{1}{2}\int_s^t \mathbb{E}_{\mathbb{Q}}\left[\|\theta_u\|^2\right]\mathrm{d}u,$$

which in turn implies that $\mathbb{E}_{\mathbb{Q}}[\|\theta_u\|^2|\mathcal{F}_s] = \mathbb{E}_{\mathbb{Q}}[\|\theta_u\|^2]$ for Lebesgue a.e. $u \in (s,t)$. Furthermore, for $u < t$ fixed we also have that $\mathbb{E}_{\mathbb{Q}}[\|\theta_u\|^2|\mathcal{F}_s] = \mathbb{E}_{\mathbb{Q}}[\|\theta_u\|^2]$ for Lebesgue almost every $0 \le s < u$. Define the stochastic process $I(s) := \mathbb{E}_{\mathbb{Q}}[\|\theta_u\|^2|\mathcal{F}_s]$ for $0 \le s < u$. Since (2.5) holds true, $\|\theta_u\|^2 \in L^1(\Omega,\mathcal{F}_u,\mathbb{Q})$, then $I(s)$ is a martingale that admits a representation of the form $I(s) = I_0 + \int_0^s \psi_t dW_t^{\mathbb{Q}}$ for a suitable ψ_t and $I_0 \in \mathbb{R}$. Without loss of generality we consider the continuous version of $I(s)$ and hence, letting $s \to u$ we obtain $I(s) \to I(u) = \mathbb{E}_{\mathbb{Q}}[\|\theta_u\|^2|\mathcal{F}_u] = \|\theta_u\|^2$. Recalling that $\mathbb{E}_{\mathbb{Q}}[\|\theta_u\|^2|\mathcal{F}_s] = \mathbb{E}_{\mathbb{Q}}[\|\theta_u\|^2]$ for all $s < u$ we conclude that $\mathbb{E}_{\mathbb{Q}}[\|\theta_u\|^2] = \|\theta_u\|^2$ and, since the choice of $u \in (s,t)$ was arbitrary, $\|\theta_t\|^2$ must be deterministic. □

Proof of Remark 2.3.2. Fix $t > 0$. Analogously to the previous proof we reformulate Problem (2.8) as its infinite dimensional counterpart as in Eq. (2.10). Considering the power utility function

$$u(x) = \begin{cases} \frac{x^\gamma}{\gamma} & x \ge 0 \\ -\infty & x < 0 \end{cases} \tag{2.16}$$

with $\gamma \in (0,1)$, the optimal profile ξ_t^* takes the form

$$\xi_t^* = \frac{x \cdot Z_t^\beta}{\mathbb{E}_{\mathbb{Q}}\left[Z_t^\beta\right]},$$

where $\beta = -(1-\gamma)^{-1}$ and $Z_t = \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}$. Following a standard argument (see e.g. Björk, 2009) let us define

$$Z_t^0 := \exp\left\{-\frac{1}{2}\int_0^t \|\theta_u\|^2(\beta+1)^2\mathrm{d}u + \int_0^t (\beta+1)\theta_u'\,\mathrm{d}W_u\right\}$$

and observe that $Z_t^{\beta+1} = Z_t^0 \exp\left\{\frac{1}{2}\int_0^t \|\theta_u\|^2\beta^2\gamma\mathrm{d}u\right\}$. Clearly, Standing Assumption 2.2.3 ensures that Z_t^0 is a true martingale. Hence we denote by $\mathbb{Q}_0$ the probability measure induced by $Z_t^0 = \mathrm{d}\mathbb{Q}_0/\,\mathrm{d}\mathbb{P}$ and we write

$$H_t := \mathbb{E}_{\mathbb{Q}}\left[Z_t^\beta\right] = \mathbb{E}_0\left[\exp\left\{\frac{1}{2}\int_0^t \|\theta_u\|^2\beta^2\gamma\mathrm{d}u\right\}\right].$$

For some time $s < t$ let Π_s denote the discounted no-arbitrage price of the optimum ξ_t^*. The problem is thus time-consistent whenever $\Pi_s = \xi_s^*$ that is, whenever

$$\frac{x}{H_t}\mathbb{E}_{\mathbb{Q}}\left[Z_t^\beta\middle|\mathcal{F}_s\right] = \frac{x \cdot Z_s^\beta}{H_s}. \tag{2.17}$$

On the left hand side of (2.17) we have that

$$\mathbb{E}_{\mathbb{Q}}\left[Z_t^\beta\middle|\mathcal{F}_s\right] = \frac{\mathbb{E}_{\mathbb{P}}\left[Z_t^{\beta+1}\middle|\mathcal{F}_s\right]}{\mathbb{E}_{\mathbb{P}}\left[Z_t|\mathcal{F}_s\right]} = \frac{\mathbb{E}_{\mathbb{P}}\left[Z_t^{\beta+1}\middle|\mathcal{F}_s\right]}{Z_s},$$

Further simple computations leads to

$$\mathbb{E}_{\mathbb{P}}\left[Z_t^{\beta+1}\Big|\mathcal{F}_s\right]=\underbrace{Z_s^0\exp\left\{\frac{1}{2}\int_0^s\|\theta_u\|^2\beta^2\gamma\mathrm{d}u\right\}}_{Z_s^{\beta+1}}\mathbb{E}_0\left[\exp\left\{\frac{1}{2}\int_s^t\|\theta_u\|^2\beta^2\gamma\mathrm{d}u\right\}\Big|\mathcal{F}_s\right],$$

and therefore (2.17) becomes

$$\frac{H_t}{H_s}=\mathbb{E}_0\left[\exp\left\{\frac{1}{2}\int_s^t\|\theta_u\|^2\beta^2\gamma du\right\}\Big|\mathcal{F}_s\right]. \tag{2.18}$$

We define the following process

$$M_t:=\frac{\exp\left\{\frac{1}{2}\int_0^t\|\theta_u\|^2\beta^2\gamma\mathrm{d}u\right\}}{\mathbb{E}_0\left[\exp\left\{\frac{1}{2}\int_0^t\|\theta_u\|^2\beta^2\gamma\mathrm{d}u\right\}\right]},$$

and observe that

$$\begin{aligned}\mathbb{E}_0\left[M_t|\mathcal{F}_s\right]&=\frac{\mathbb{E}_0\left[\exp\left\{\frac{1}{2}\int_0^t\|\theta_u\|^2\beta^2\gamma\mathrm{d}u\right\}\Big|\mathcal{F}_s\right]}{\mathbb{E}_0\left[\exp\left\{\frac{1}{2}\int_0^t\|\theta_u\|^2\beta^2\gamma\mathrm{d}u\right\}\right]}\\&=\frac{\mathbb{E}_0\left[\exp\left\{\frac{1}{2}\int_s^t\|\theta_u\|^2\beta^2\gamma\mathrm{d}u\right\}\Big|\mathcal{F}_s\right]\exp\left\{\frac{1}{2}\int_0^s\|\theta_u\|^2\beta^2\gamma\mathrm{d}u\right\}}{\mathbb{E}_0\left[\exp\left\{\frac{1}{2}\int_0^s\|\theta_u\|^2\beta^2\gamma\mathrm{d}u\right\}\mathbb{E}_0\left[\exp\left\{\frac{1}{2}\int_s^t\|\theta_u\|^2\beta^2\gamma\mathrm{d}u\right\}\Big|\mathcal{F}_s\right]\right]}.\end{aligned}$$

Assuming Eq. (2.18) holds we have that $\mathbb{E}_0\left[\exp\left\{\frac{1}{2}\int_s^t\|\theta_u\|^2\beta^2\gamma\mathrm{d}u\right\}\Big|\mathcal{F}_s\right]\in\mathbb{R}$ and hence we can take it out of the expectation to obtain

$$\mathbb{E}_0\left[M_t|\mathcal{F}_s\right]=\frac{\exp\left\{\frac{1}{2}\int_0^s\|\theta_u\|^2\beta^2\gamma\mathrm{d}u\right\}}{\mathbb{E}_0\left[\exp\left\{\frac{1}{2}\int_0^s\|\theta_u\|^2\beta^2\gamma\mathrm{d}u\right\}\right]}=M_s$$

and conclude M_t is a $\mathbb{Q}_0$-martingale. A simple inspection shows that M_t is of finite variation and by Revuz and Yor (1999), Proposition IV.1.2, it follows that it must be constant. In particular then $M_t=M_0=1$ which entails

$$\exp\left\{\frac{1}{2}\int_0^t\|\theta_u\|^2\beta^2\gamma\mathrm{d}u\right\}=\mathbb{E}_0\left[\exp\left\{\frac{1}{2}\int_0^t\|\theta_u\|^2\beta^2\gamma\mathrm{d}u\right\}\right],$$

for any $t\in[0,\infty)$. Observing that we can differentiate with respect of time t under the expectation operator, taking the derivative on both sides we find that

$$\mathbb{E}_0\left[\|\theta_t\|^2\right]=\|\theta_t\|^2,$$

and since the above holds for Lebesgue a.e. $t\in[0,\infty)$, we conclude that the variance of $\|\theta_t\|^2$ must be 0.
The converse implication is trivially satisfied as condition (2.18) clearly holds if $Var(\|\theta_t\|^2)=0$. $\square$

2.5 Stochastic risk aversion

Motivated by the previous discussion, we are now ready to illustrate the main contribution of this chapter. We consider an agent who plans to invest on the market which has a **stochastic market price of risk** $(\theta_t)_{t\geq 0}$ (see Assumption (A*) below). The agent faces uncertainty on their future risk aversion and may be willing to adjust the terminal date of the investment over time. We assume that the preferences of the agent are represented by a state-dependent utility function of the exponential type $\mathbb{u}_t : \Omega \times \mathbb{R} \to \mathbb{R}$ for $t \in [0, \infty)$, such that

$$\mathbb{u}_t(\omega, x) = -\frac{1}{\gamma_t(\omega)} e^{-\gamma_t(\omega) x}. \tag{2.19}$$

The uncertain risk aversion is modeled through a general adapted process $\big(1/\gamma_t\big)_{t\geq 0}$, satisfying the following SDE:

$$\begin{cases} \mathrm{d}\left(\frac{1}{\gamma_t}\right) = \frac{1}{\gamma_t}(\eta_t \mathrm{d}t + \beta_t' \, \mathrm{d}W_t^{\mathbb{Q}}), \\ \frac{1}{\gamma_0} > 0, \end{cases} \tag{2.20}$$

where $\mathrm{d}W_t^{\mathbb{Q}} = \mathrm{d}W_t - \theta_t \mathrm{d}t$ is a Brownian motion under $\mathbb{Q}$. The use of stochastic risk-aversion is widespread in the literature on forward utilities (see e.g. Musiela and Zariphopoulou, 2009, 2010; Žitković, 2009), especially for utilities of exponential type. Other similar attempts to model risk-aversion as a random variable can be found for instance in Desmettre and Steffensen (2023). Finally, an alternative approach to model stochastic utility through a multiplicative source of randomness is studied in Section 2.8.

We will be working with the following set of assumptions in addition to the Standing Assumptions 2.2.3:

Assumption (A). *The scalar process* $(\eta_t)_{t\geq 0}$ *and the d-dimensional process* $(\beta_t)_{t\geq 0}$ *are progressively measurable and for any* $t > 0$ *there exists a constant* $K_t > 0$ *such that*

$$\mathbb{P}\left(\sup_{0\leq u\leq t} |\eta_u| < K_t, \sup_{0\leq u\leq t} \|\beta_u\| < K_t \right) = 1.$$

Assumption (A*). *Assumption (A) is in force and for any* $t > 0$ *we have* $Var(\|\theta_t\|^2) \neq 0$.

Remark 2.5.1. Assumption (A*) restricts the financial market model to the case of a stochastic market price of risk. However, Proposition 2.5.6 below addresses the complementary scenario of a deterministic market price of risk with stochastic risk aversion, thereby providing a comprehensive perspective on the problem.
Assumption (A) is needed to formally justify a few technical steps in the proofs, although the reader will notice that the boundedness conditions imposed on $(\eta_t)_{t\geq 0}$ and $(\beta_t)_{t\geq 0}$ are not too restrictive in light of the results of the following theorem (i.e. $\eta_t = 0$ and $\beta_t = -\theta_t/2$).

Theorem 2.5.2. *Suppose Assumption (A*) holds and consider* $\mathbb{u}_t : \Omega \times \mathbb{R} \to \mathbb{R}$ *of the form in* (2.19)*, with* $(1/\gamma_t)_{t\geq 0}$ *a solution to* (2.20)*, then*

(i) Problem (2.8) *is consistent if and only if* $\eta_t = 0$ *and* $\beta_t = -\theta_t/2$.

Moreover, for the choice of η_t *and* β_t *in* (i)*, the following hold:*

(ii) The optimal wealth process is given by

$$\xi_t^* = \frac{1}{\gamma_t}(\gamma_0 x - \ln(Z_t)),$$

(iii) The optimal strategy α_t^* *is given by*

$$(\alpha_t^*)' = -\frac{1}{\gamma_t \xi_t^*}\left(\theta_t' + \frac{\gamma_t \xi_t^*}{2}\theta_t'\right)\sigma_t^{-1},$$

(iv) for all $t, s \in [0, \infty)$ *such that* $t > s$

$$\mathbb{E}_{\mathbb{P}}\left[\mathbb{u}_t(V_t^{\alpha^*})\middle|\mathcal{F}_s\right] = \mathbb{u}_s(V_s^{\alpha^*}).$$

Remark 2.5.3 (On the implications of Theorem 2.5.2). Under the reference measure $\mathbb{P}$ the dynamics of $(1/\gamma_t)_{t\geq 0}$ are given by

$$d\left(\frac{1}{\gamma_t}\right) = \frac{1}{\gamma_t}\left(\frac{\theta_t^2}{2}dt - \frac{\theta_t}{2}dW_t\right),$$

and consequently

$$d\gamma_t = \gamma_t\left(-\frac{\theta_t^2}{4}dt + \frac{\theta_t}{2}dW_t\right).$$

First, we observe that $(\gamma_t)_{t\geq 0}$ is a positive supermartingale and hence there exists a random variable $\gamma_\infty \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ such that $\gamma_t \to \gamma_\infty$ $\mathbb{P}$-a.s. as $t \to \infty$ (see Revuz and Yor, 1999, Corollary II.2.11). In particular, if $\mathbb{P}(\gamma_\infty \neq 0) = 1$ then one obtains $1/\gamma_t \to 1/\gamma_\infty$ $\mathbb{P}$-a.s. as well. One can therefore think of $1/\gamma_\infty$ as defining the agent's unknown true preferences $(\mathbb{u}, \mathbb{P})$ with $\mathbb{u}(\omega, x) = -[1/\gamma_\infty(\omega)]\, e^{-\gamma_\infty(\omega)x}$. Since $\left(1/\gamma_s\right)_{0\leq s\leq t}$ is a $\mathbb{Q}$-martingale for all t, for every pair $s < t$, the process $1/\gamma_s$ can be seen as the prediction at time s of the future risk tolerance $1/\gamma_t$. If, moreover, there existed an equivalent probability measure[3] $\tilde{\mathbb{Q}}$ on $\mathcal{F}$ we could interpret $1/\gamma_t$ as the "most rational" prediction of the real risk tolerance $1/\gamma_\infty \in L^1(\Omega, \mathcal{F}, \mathbb{Q})$, through the relation $1/\gamma_t = \mathbb{E}_{\mathbb{Q}}[1/\gamma_\infty|\mathcal{F}_t]$. Therefore, in devising their investment strategy, the agent is adopting a consistent estimate of their true risk aversion given the information available on the market. The novelty of this approach lies in the fact that the longer the agent engages with the market, the more they become aware of their true risk aversion This leads to a strong interplay between the time evolution of the agent's

[3]Although intuitively appealing, this assertion requires a stronger integrability assumption on $(\theta_t)_{t\geq 0}$ in order to ensure the existence of a suitable equivalent probability measure $\tilde{\mathbb{Q}}$ on $(\Omega, \mathcal{F})$ and $(1/\gamma_t)_{t\geq 0}$ uniformly integrable. Under this assumptions the conclusion follows from a straightforward application of martingale convergence Theorem (see Revuz and Yor, 1999, Theorem II.3.1).

strategy and the updating of their preference order. We present a few economic insights related to the previous theorem. As the process $1/\gamma_t$ evolves with positive drift, on average, the agent tends to become more risk-tolerant, or equivalently less risk-averse, with time. This is coherent with the previous idea that the agent gradually uncovers their true risk aversion by interacting with the market. However, trajectory-wise the risk-aversion is influenced by the movements of the market through the Brownian motion. The correction term in α^* further emphasizes the point: $\gamma_t \xi_t^*$ depends on the whole trajectory of the Brownian motion W_t up to time t through $\ln(Z_t)$. Therefore in order to be consistent the agent may need to invest contrary to the market movements.

Conversely, there is no systematic bias in the foreseen dynamics of risk tolerance under the pricing measure $\mathbb{Q}$. Indeed, Theorem 2.5.2 points out that a biased perception of risk tolerance dynamics would rule out any possible consistency.

Moreover, the result perfectly fits into the theory of forward performances, as the martingale property of $\left(\mathbb{u}_t(V_t^{\alpha^*})\right)_{t\geq 0}$ is recovered, and in Proposition 2.5.4 we shall also prove that for any strategy $\alpha \in \mathcal{A}$, $(\mathbb{u}_t(V_t^{\alpha}))_{t\geq 0}$ is a supermartingale. This result is extremely interesting in view of the existence of infinitely many forward performances with shape as in (2.19), as the following proposition will point out.

Proposition 2.5.4. *Suppose Assumption (A) holds and let $(1/\gamma_t)_{t\geq 0}$ evolve according to (2.20) with $u_0(x) = -\frac{1}{\gamma_0}e^{-\gamma_0 x}$. Consider the process $(\eta_t)_{t\geq 0}$ defined by*

$$\eta_t = \frac{\theta_t'(\theta_t + 2\beta_t)}{2(\gamma_t V_t^* + 1)}, \tag{2.21}$$

where V_t^* is the solution to the following stochastic differential equation

$$\mathrm{d}V_t^* = \frac{(\gamma_t V_t^* \beta_t' - \theta_t')}{\gamma_t} \sigma_t^{-1} \left((\mu_t - r\mathbf{1})\mathrm{d}t + \sigma_t \, \mathrm{d}W_t\right). \tag{2.22}$$

Then for every choice of $(\beta_t)_{t\geq 0}$ such that the stochastic differential equations (2.20) and (2.22) admit a solution, the stochastic utility function $\mathbb{u}_t(\omega, x) = -\frac{1}{\gamma_t(\omega)}e^{-\gamma_t(\omega)x}$ is a forward performance and for $(\alpha_t^*)' = \frac{(\gamma_t V_t^* \beta_t' - \theta_t')}{\gamma_t V_t^*}\sigma_t^{-1}$ the process $\mathbb{u}_t(V_t^{\alpha^*})$ is a $\mathbb{P}$-martingale.
In particular, for the choice of $(\eta_t)_{t\geq 0}, (\beta_t)_{t\geq 0}$ in Theorem 2.5.2 and setting $1/\gamma_0 = \mathbb{E}_{\mathbb{Q}}\left[1/\gamma_t\right]$, $(\mathbb{u}_t)_{t\geq 0}$ is a forward performance.

The intricate relationship between the processes η_t and β_t carries both economic and technical implications. From an economic perspective, the mutual dependence between $\mathbb{u}_t$ (through the risk aversion parameter) and the optimal portfolio $V_t^{\alpha^*}$ implies that the pair $(\mathbb{u}_t, \mathbb{P})_{t\geq 0}$ cannot be regarded as an *a priori* system of dynamic preferences. An appealing aspect of the result in Theorem 2.5.2 is that $1/\gamma_t$ is completely disentangled from the optimum, hence providing a regular dynamic preference structure for the agent.
Besides, because of the coupling of the processes γ_t and V_t^*, guaranteeing the existence of a solution to (2.20) and (2.22) for a given choice of β_t is not straightforward. To address this, the following example introduces a parameter family, indexed by a constant vector $\boldsymbol{\delta}$, that guarantees the existence of a unique strong solution to both stochastic differential equations and that satisfies the relationship in (2.21).

Example 2.5.5. Consider the auxiliary process

$$\begin{cases} \mathrm{d}H_t = \left(\frac{1}{2(1+H_t^2)}\theta_t'\boldsymbol{\delta} - \frac{\|\theta_t\|^2}{2}\right)\mathrm{d}t - \theta_t'\,\mathrm{d}W_t^{\mathbb{Q}}, \\ H_0 = \gamma_0 x \in \mathbb{R}, \end{cases} \tag{2.23}$$

where $\boldsymbol{\delta} \in \mathbb{R}^d$ is an arbitrary vector. Recalling that θ_t is bounded by Assumption 2.2.3, the process in (2.23) has a unique solution H_t. Set

$$\beta_t = \frac{1+H_t}{2(1+H_t^2)}\boldsymbol{\delta} - \frac{\theta_t}{2}, \quad \eta_t = \frac{1}{2(1+H_t^2)}\theta_t'\boldsymbol{\delta}.$$

and note that the dynamics of H_t can be rewritten as $\mathrm{d}H_t = (\theta_t'\beta_t - \eta_t H_t)\,\mathrm{d}t - \theta_t'\,\mathrm{d}W_t^{\mathbb{Q}}$. Observe that for this choice of parameters the SDE (2.20) for $1/\gamma_t$ has a unique strong solution as well. In particular, $1/\gamma_t$ is an exponential process and a straightforward application of Itō's Lemma yields the process $\gamma_t = (1/\gamma_t)^{-1}$. Finally, we define $\tilde{V}_t := H_t/\gamma_t$ and we show that $\tilde{V}_t$ satisfies the dynamics of the process V_t^* in (2.22). The stochastic differential of $\tilde{V}_t$ can be computed as

$$\begin{aligned} \mathrm{d}\tilde{V}_t = \mathrm{d}\left(\frac{1}{\gamma_t}H_t\right) &= \frac{1}{\gamma_t}\,\mathrm{d}H_t + H_t\,\mathrm{d}\frac{1}{\gamma_t} + \mathrm{d}\langle\frac{1}{\gamma_\cdot}, H_\cdot\rangle_t \\ &= \frac{1}{\gamma_t}\left((\theta_t'\beta_t - \eta_t H_t)\,\mathrm{d}t - \theta_t'\,\mathrm{d}W_t^{\mathbb{Q}}\right) + H_t\frac{1}{\gamma_t}\left(\eta_t\mathrm{d}t + \beta_t'\,\mathrm{d}W_t^{\mathbb{Q}}\right) - \frac{1}{\gamma_t}\theta_t'\beta_t\mathrm{d}t \\ &= \frac{1}{\gamma_t}\left(H_t\beta_t' - \theta_t'\right)\mathrm{d}W_t^{\mathbb{Q}} \\ &= \frac{1}{\gamma_t}\left(H_t\beta_t' - \theta_t'\right)\left[-\theta_t\mathrm{d}t + \mathrm{d}W_t\right]. \end{aligned} \tag{2.24}$$

Observing that by definition $H_t = \gamma_t\tilde{V}_t$ and recalling that $\theta_t = -\sigma_t^{-1}(\mu_t - r\mathbf{1})$, a simple algebraic rearrangement of (2.24) yields

$$\mathrm{d}\tilde{V}_t = \frac{\left(\gamma_t\tilde{V}_t\beta_t' - \theta_t'\right)}{\gamma_t}\sigma_t^{-1}\left[(\mu_t - r\mathbf{1})\mathrm{d}t + \sigma_t\,\mathrm{d}W_t\right].$$

It follows that $\tilde{V}_t$ satisfies the SDE in (2.22) and thus it is indistinguishable from V_t^*. Ultimately, we notice that the coefficients β_t and η_t satisfy condition (2.21) once H_t is substituted with $\gamma_t V_t^*$ and therefore Proposition 2.5.4 ensures that $(\mathbb{u}_t)_{t\geq 0}$ is a forward performance. We point out that for $\boldsymbol{\delta} = 0$ we recover the parametrization of Theorem 2.5.2.

In Proposition 2.5.6, we complete Theorem 2.5.2 by highlighting the following somewhat counterintuitive result: whenever an agent endowed with a utility function with stochastic risk aversion (analogous to that in Theorem 2.5.2) invests in a financial market with deterministic market price of risk, the problem is consistent regardless of the choice of $(\beta_t)_{t\geq 0}$, as long as it is a deterministic function. Nevertheless, the unique parametrization of $(\beta_t)_{t\geq 0}$ that makes $\mathbb{u}_t(\omega, x)$ a forward performance is the one resulting from Theorem 2.5.2.

Proposition 2.5.6. *Suppose Assumption (A) holds, let the map $t \mapsto \theta_t$ be deterministic and consider $\mathbb{u}_t : \Omega \times \mathbb{R} \to \mathbb{R}$ of the form in (2.19), with $(1/\gamma_t)_{t\geq 0}$ a solution to (2.20).Problem (2.8) is consistent if and only if $\eta_t = 0$ and $\beta_t = f(t)$ for some deterministic function $f : [0,\infty) \to \mathbb{R}^d$. Moreover $(\mathbb{u}_t)_{t\geq 0}$ is a forward performance if and only if $\beta_t = -\theta_t/2$.*

Remark 2.5.7. Consider an agent with utility function $\mathbb{u}_t : \Omega \times \mathbb{R} \to \mathbb{R}$ of the form (2.19) with $(\gamma_t)_{t\geq 0}$ parametrized as in Theorem 2.5.2.(i). Given consistency of the problem (2.8), the agent is able to extend/reduce arbitrarily the investment horizon determined at inception. A natural question would then be whether an optimal random time exists to exit the investment plan. Optimal stopping problems in a utility maximization framework have been studied, for example, in Karatzas and Wang (2000) and Henderson and Hobson (2007). We now present a heuristic argument to illustrate why consistency offers the advantage of ensuring no regret, regardless of the exit time chosen by the agent.
Consider the family $\mathcal{T}$ of $\mathbb{P}$ almost surely finite stopping times $\tau : \Omega \to (0, +\infty)$, then the process $\xi^*_{t\wedge\tau}$ is a $\mathbb{Q}$-martingale for every $\tau \in \mathcal{T}$ and it optimizes $\mathbb{E}_\mathbb{P}[\mathbb{u}_{t\wedge\tau}(\cdot)]$. Under some further technical assumptions it is possible to infer that the agent is indifferent among all the possible terminal dates, as the expected reward of optimizing over all possible stopping rules does not exceed the one obtained without stopping, i.e.

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}_\mathbb{P}[\mathbb{u}_\tau(\xi^*_\tau)] = \mathbb{E}_\mathbb{P}[\mathbb{u}_t(\xi^*_t)], \quad \forall\, t > 0.$$

Furthermore, one can find that ξ^*_τ maximizes $\mathbb{E}_\mathbb{P}[\mathbb{u}_\tau(\cdot)]$ over all the $\mathcal{F}_\tau$-measurable random variables ξ such that $\mathbb{E}_\mathbb{P}[|\mathbb{u}_\tau(\xi)|] < \infty$ and $\mathbb{E}_\mathbb{Q}[\xi] = x$.

2.6 Proof of Theorem 2.5.2

Similarly to previous sections, we shall recast the utility maximization problem as the infinite dimensional static optimization problem of computing the optimal terminal wealth profile ξ^*. In order to do so, in Appendix B we first provide an extension of the characterization of the maximizer for state-dependent utilities. Furthermore, before proving the main result we need the following auxiliary lemma.

Lemma 2.6.1. *Suppose that Assumption (A) holds. Consider $\mathbb{u}_t : \Omega \times \mathbb{R} \to \mathbb{R}$ as in (2.19), with $(1/\gamma_t)_{t\geq 0}$ being the solution of the SDE*

$$\mathrm{d}\left(\frac{1}{\gamma_t}\right) = \frac{1}{\gamma_t}\beta'_t\,\mathrm{d}W^\mathbb{Q}_t, \qquad \gamma_0 > 0, \tag{2.25}$$

then problem (2.8) is consistent if and only if one of the following is verified:

- *$\beta_t = f(t)$ and $\theta_t = g(t)$ with $f, g : [0,\infty) \to \mathbb{R}^d$ deterministic functions of time;*
- *$\beta_t = -\theta_t/2$.*

Proof of Lemma 2.6.1. Let $t > 0$ be fixed, and observe that since the market is assumed to be complete, Problem (2.8) is equivalent to the following infinite dimensional problem

$$\sup\left\{\mathbb{E}_\mathbb{P}[\mathbb{u}_t(\xi)] \mid \xi \in L^1(\Omega, \mathcal{F}_t, \mathbb{Q}) \text{ and } \mathbb{E}_\mathbb{Q}[\xi] = x\right\}.$$

Applying Lemma B.1 we can show that the optimal profile is

$$\xi_t^* = \frac{1}{\gamma_t}\left\{c\left(x + \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\gamma_t}\ln(Z_t)\right]\right) - \ln(Z_t)\right\}, \tag{2.26}$$

where $c = (\mathbb{E}_{\mathbb{Q}}[1/\gamma_t])^{-1}$ (which does not depend on t by assumption) and $Z_t = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$. Observe that, by virtue of Assumption 2.2.3, we have that $\ln(Z_t) \in L^2(\Omega, \mathcal{F}, \mathbb{Q})$, and $1/\gamma_t \in L^2(\Omega, \mathcal{F}, \mathbb{Q})$ follows from Assumption (A). Cauchy-Schwartz inequality then implies that $\ln(Z_t)/\gamma_t \in L^1(\Omega, \mathcal{F}, \mathbb{Q})$ and consequently also ξ_t^*.

By definition $(1/\gamma_t)_t$ is a strictly positive martingale in the filtration generated by the $\mathbb{Q}$-Brownian motion $\mathrm{d}W_t^{\mathbb{Q}} = \mathrm{d}W_t - \theta_t \mathrm{d}t$. Recalling that $\mathrm{d}\ln(Z_t) = \frac{1}{2}\|\theta_t\|^2 \mathrm{d}t + \theta_t' \,\mathrm{d}W_t^{\mathbb{Q}}$ and given the dynamics of $1/\gamma_t$ in (2.25) we have

$$\mathrm{d}\left(\frac{1}{\gamma_t}\ln(Z_t)\right) = \frac{1}{\gamma_t}\theta_t'\left(\frac{\theta_t}{2} + \beta_t\right)\mathrm{d}t + \frac{1}{\gamma_t}\left(\theta_t' + \ln(Z_t)\beta_t'\right)\mathrm{d}W_t^{\mathbb{Q}},$$

and hence the conditional expectation with respect to $\mathcal{F}_s$ reads

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\gamma_t}\ln(Z_t)\middle|\mathcal{F}_s\right] = \frac{1}{\gamma_s}\ln(Z_s) + \mathbb{E}_{\mathbb{Q}}\left[\int_s^t \frac{1}{\gamma_u}\theta_u'\left(\frac{\theta_u}{2} + \beta_u\right)\mathrm{d}u\middle|\mathcal{F}_s\right].$$

We now move to the consistency condition $\mathbb{E}_{\mathbb{Q}}[\xi_t^*|\mathcal{F}_s] = \xi_s^*$. Collecting previous computations we obtain that it is in fact equivalent to

$$\begin{aligned} & c\left(x + E_{\mathbb{Q}}\left[\frac{1}{\gamma_t}\ln(Z_t)\right]\right)\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\gamma_t}\middle|\mathcal{F}_s\right] - \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\gamma_t}\ln(Z_t)\middle|\mathcal{F}_s\right] \\ = \; & c\left(x + E_{\mathbb{Q}}\left[\frac{1}{\gamma_s}\ln(Z_s)\right]\right)\frac{1}{\gamma_s} - \frac{1}{\gamma_s}\ln(Z_s), \end{aligned}$$

and a few calculations then yield

$$\frac{c}{\gamma_s}\left(\mathbb{E}_{\mathbb{Q}}\left[\int_s^t \frac{1}{\gamma_u}\theta_u'\left(\frac{\theta_u}{2} + \beta_u\right)\mathrm{d}u\right]\right) = \mathbb{E}_{\mathbb{Q}}\left[\int_s^t \frac{1}{\gamma_u}\theta_u'\left(\frac{\theta_u}{2} + \beta_u\right)\mathrm{d}u\middle|\mathcal{F}_s\right]. \tag{2.27}$$

We define $L_t := c/\gamma_t$ and observe it is a positive martingale with $\mathbb{E}_{\mathbb{Q}}[L_t] = 1$ and hence we can specify a new measure $\mathbb{Q}_L$ through $\mathrm{d}\mathbb{Q}_L = L_t \,\mathrm{d}\mathbb{Q}$. Let us denote $\kappa_u := \theta_u'(\theta_u/2 + \beta_u)$, then Eq. (2.27) can be rewritten as

$$\mathbb{E}_{\mathbb{Q}_L}\left[\frac{1}{L_t}\int_s^t L_u\kappa_u \mathrm{d}u\right] = \mathbb{E}_{\mathbb{Q}_L}\left[\frac{1}{L_t}\int_s^t L_u\kappa_u \mathrm{d}u\middle|\mathcal{F}_s\right]. \tag{2.28}$$

The linearity of the expectation allows to rearrange the above equation as follows

$$\mathbb{E}_{\mathbb{Q}_L}\left[\frac{1}{L_t}\int_0^t L_u\kappa_u \mathrm{d}u\right] - \mathbb{E}_{\mathbb{Q}_L}\left[\frac{1}{L_s}\int_0^s L_u\kappa_u \mathrm{d}u\right] = \mathbb{E}_{\mathbb{Q}_L}\left[\frac{1}{L_t}\int_0^t L_u\kappa_u \mathrm{d}u\middle|\mathcal{F}_s\right] - \frac{1}{L_s}\int_0^s L_u\kappa_u \mathrm{d}u. \tag{2.29}$$

We notice that

$$\mathrm{d}\left(\frac{1}{L_t}\int_0^t L_u\kappa_u \mathrm{d}u\right) = \underbrace{\int_0^t L_u\kappa_u \mathrm{d}u \cdot \mathrm{d}\left(\frac{1}{L_t}\right)}_{=:\,\mathrm{d}M_t} + \frac{1}{L_t}L_t\kappa_t \mathrm{d}t,$$

where M_t is a $\mathbb{Q}_L$-martingale. In integral notation the latter reads

$$\frac{1}{L_t}\int_0^t L_u\kappa_u \mathrm{d}u = M_t + \int_0^t \kappa_u \mathrm{d}u,$$

and substitution into (2.29) yields

$$\mathbb{E}_{\mathbb{Q}_L}\left[\int_s^t \kappa_u du\right] = \mathbb{E}_{\mathbb{Q}_L}\left[\int_s^t \kappa_u du \middle| \mathcal{F}_s\right]. \tag{2.30}$$

We point out that the above is trivially verified whenever κ_t is a deterministic function. In turn, this holds when either:

- $\beta_t = f(t)$ and $\theta_t = g(t)$ with f, g deterministic $\mathbb{R}^d$-valued functions of time, or
- $\beta_t = -\theta_t/2$.

We now prove the converse implication. Consider the process

$$Y_t := \int_0^t \kappa_u \mathrm{d}u - \mathbb{E}_{\mathbb{Q}_L}\left[\int_0^t \kappa_u \mathrm{d}u\right].$$

Computing its conditional expectation and using (2.30) we find that it is a $\mathbb{Q}_L$-martingale. Simple inspections show that Y_t is of finite variation and hence, by Revuz and Yor (1999), Proposition IV.1.2, we conclude Y_t is constant. Therefore $u \mapsto \theta_u' \left(\theta_u/2 + \beta_u\right)$ must be a deterministic function. This occurs when both θ_t, β_t are deterministic, or when $\beta_t = -\theta_t/2$. □

Proof of Theorem 2.5.2. We first show (i). Fix $t > 0$, and consider Problem (2.8) for the state-dependent utility function $\mathbb{u}_t : \Omega \times \mathbb{R} \to \mathbb{R}$ defined in (2.19). We reformulate the maximization problem as

$$\sup\left\{\mathbb{E}_{\mathbb{P}}\left[\mathbb{u}_t(\xi)\right] \mid \xi \in L^1(\Omega, \mathcal{F}_t, \mathbb{Q}) \text{ and } \mathbb{E}_{\mathbb{Q}}\left[\xi\right] = x\right\}.$$

Applying Lemma B.1 to $\mathbb{u}_t(\omega, x)$ yields the t-optimal discounted profile

$$\xi_t^* = -\frac{1}{\gamma_t}\ln\left(\lambda Z_t\right). \tag{2.31}$$

One can then recover λ by plugging (2.31) into the budget constraint $\mathbb{E}_{\mathbb{Q}}\left[\xi_t^*\right] = x$

$$\ln(\lambda) = \frac{1}{\mathbb{E}_{\mathbb{Q}}\left[1/\gamma_t\right]}\left(-x - \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\gamma_t}\ln(Z_t)\right]\right).$$

Substituting in (2.31) and setting $c_t = \left(\mathbb{E}_{\mathbb{Q}}\left[1/\gamma_t\right]\right)^{-1}$ we finally obtain

$$\xi_t^* = \frac{1}{\gamma_t}\left\{c_t\left(x + \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\gamma_t}\ln(Z_t)\right]\right) - \ln(Z_t)\right\}. \tag{2.32}$$

Notice that $\xi_t^* \in L^1(\Omega, \mathcal{F}, \mathbb{Q})$: boundedness of θ_t ensures in fact that $\ln(Z_t) \in L^2(\Omega, \mathcal{F}, \mathbb{Q})$ and

$$\frac{1}{\gamma_t} = \exp\left(\int_0^t (\eta_u - \frac{1}{2}\|\beta_u\|^2)\mathrm{d}u + \int_0^t \beta_u'\,\mathrm{d}W_u^{\mathbb{Q}}\right)$$

is an element of $L^2(\Omega, \mathcal{F}, \mathbb{Q})$ from Assumption (A). Therefore, by Cauchy-Schwarz inequality $\mathbb{E}_{\mathbb{Q}}[\ln(Z_t)/\gamma_t] < +\infty$. Consider now some $s < t$ and denote by ξ_s^* the optimal discounted profile for the investment horizon s, the problem is consistent whenever the condition $\xi_s^* = \mathbb{E}_{\mathbb{Q}}[\xi_t^* | \mathcal{F}_s]$ is verified.

By assumption

$$\mathrm{d}\left(\frac{1}{\gamma_t}\right) = \frac{1}{\gamma_t}(\eta_t \mathrm{d}t + \beta_t' \, \mathrm{d}W_t^{\mathbb{Q}}),$$

for η_t, β_t adapted (respectively scalar and d-dimensional) processes and with $W_t^{\mathbb{Q}}$ a $\mathbb{Q}$-Brownian motion defined through $\mathrm{d}W_t^{\mathbb{Q}} = \mathrm{d}W_t - \theta_t \mathrm{d}t$. Observing that $\mathrm{d}\ln(Z_t) = \frac{1}{2}\|\theta_t\|^2 \mathrm{d}t + \theta_t' \, \mathrm{d}W_t^{\mathbb{Q}}$, stochastic Leibniz rule yields

$$\mathrm{d}\left(\frac{1}{\gamma_t}\ln(Z_t)\right) = \frac{1}{\gamma_t}\left(\frac{\|\theta_t\|^2}{2} + \eta_t \ln(Z_t) + \theta_t'\beta_t\right)\mathrm{d}t + \frac{1}{\gamma_t}\left(\theta_t' + \ln(Z_t)\beta_t'\right)\mathrm{d}W_t^{\mathbb{Q}}. \tag{2.33}$$

Recall that θ, η, β satisfy the boundedness conditions in Assumption (A). Then in particular we can rewrite the latter in integral form and take the expectation conditional on $\mathcal{F}_s$ to obtain

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\gamma_t}\ln(Z_t)\middle|\mathcal{F}_s\right] = \frac{1}{\gamma_s}\ln(Z_s) + \mathbb{E}_{\mathbb{Q}}\left[\int_s^t \frac{1}{\gamma_u}\left(\frac{\|\theta_u\|^2}{2} + \eta_u \ln(Z_u) + \theta_u'\beta_u\right)\mathrm{d}u\middle|\mathcal{F}_s\right]. \tag{2.34}$$

Plugging (2.32) into the time consistency condition we obtain

$$\begin{aligned} & c_t\left(x + \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\gamma_t}\ln(Z_t)\right]\right)\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\gamma_t}\middle|\mathcal{F}_s\right] - \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\gamma_t}\ln(Z_t)\middle|\mathcal{F}_s\right] \\ = \; & c_s\left(x + \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\gamma_s}\ln(Z_s)\right]\right)\frac{1}{\gamma_s} - \frac{1}{\gamma_s}\ln(Z_s). \end{aligned} \tag{2.35}$$

Let us set $k_t = c_t\left(x + \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\gamma_t}\ln(Z_t)\right]\right)$ and observing that

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\gamma_t}\middle|\mathcal{F}_s\right] = \frac{1}{\gamma_s} + \mathbb{E}_{\mathbb{Q}}\left[\int_s^t \frac{1}{\gamma_u}\eta_u \mathrm{d}u\middle|\mathcal{F}_s\right],$$

Eq. (2.35) boils down to

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\gamma_t}\ln(Z_t)\middle|\mathcal{F}_s\right] - \frac{1}{\gamma_s}\ln(Z_s) = (k_t - k_s)\frac{1}{\gamma_s} + k_t\,\mathbb{E}_{\mathbb{Q}}\left[\int_s^t \frac{1}{\gamma_u}\eta_u \mathrm{d}u\middle|\mathcal{F}_s\right].$$

Substituting (2.34) in the latter we find

$$\mathbb{E}_{\mathbb{Q}}\left[\int_s^t \frac{1}{\gamma_u}\left(\frac{\|\theta_u\|^2}{2} + \eta_u \ln(Z_u) + \theta_u'\beta_u\right)\mathrm{d}u\middle|\mathcal{F}_s\right] = (k_t - k_s)\frac{1}{\gamma_s} + k_t\,\mathbb{E}_{\mathbb{Q}}\left[\int_s^t \frac{1}{\gamma_u}\eta_u \mathrm{d}u\middle|\mathcal{F}_s\right],$$

and a simple algebraic rearrangement then provides

$$\frac{k_t - k_s}{\gamma_s} = \mathbb{E}_{\mathbb{Q}}\left[\int_s^t \frac{1}{\gamma_u}\left(\frac{\|\theta_u\|^2}{2} + \eta_u \ln(Z_u) + \theta_u'\beta_u - k_t\eta_u\right)\mathrm{d}u\middle|\mathcal{F}_s\right], \tag{2.36}$$

which has to hold for any couple $s, t \in [0, \infty)$ such that $s < t$.

Observe that, setting $\eta_t = 0$ and $\beta_t = -\theta_t/2$ for all $t \geq 0$ we have $k_t = k_s = x \cdot \gamma_0$ with $1/\gamma_0 = \mathbb{E}_{\mathbb{Q}}[1/\gamma_t]$ and therefore Eq. (2.36) is trivially verified.

We now show the reverse implication. For $s < t$ we define the following $\mathbb{Q}$-martingale:

$$M_s := \mathbb{E}_{\mathbb{Q}}\left[\int_0^t \frac{1}{\gamma_u}\left(\frac{\|\theta_u\|^2}{2} + \eta_u \ln(Z_u) + \theta_u'\beta_u - k_t\eta_u\right) \mathrm{d}u \,\middle|\, \mathcal{F}_s\right].$$

In light of Eq. (2.36) we may write

$$M_s = \int_0^s \frac{1}{\gamma_u}\left(\frac{\|\theta_u\|^2}{2} + \eta_u \ln(Z_u) + \theta_u'\beta_u - k_t\eta_u\right) \mathrm{d}u + \frac{k_t - k_s}{\gamma_s}. \tag{2.37}$$

Notice that

$$\mathrm{d}\left(\frac{k_t - k_s}{\gamma_s}\right) = (k_t - k_s)\left(\frac{1}{\gamma_s}(\eta_s \mathrm{d}s + \beta_s' \,\mathrm{d}W_s^{\mathbb{Q}})\right) - \frac{1}{\gamma_s}\,\mathrm{d}k_s,$$

with

$$\mathrm{d}k_s = \left(x + \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\gamma_s}\ln(Z_s)\right]\right)\left(-c_s^2\,\mathbb{E}_{\mathbb{Q}}\left[\frac{\eta_s}{\gamma_s}\right]\right)\mathrm{d}s + c_s\,\mathrm{d}\,\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\gamma_s}\ln(Z_s)\right]$$

and

$$\mathrm{d}\,\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\gamma_s}\ln(Z_s)\right] = \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\gamma_s}\left(\frac{\|\theta_s\|^2}{2} + \eta_s \ln(Z_s) + \beta_s'\theta_s\right)\right]\mathrm{d}s.$$

Since M_s is a martingale it has null drift and diffusion coefficient ${\gamma_s}^{-1}(k_t - k_s)\beta_s'$. Therefore we can write M_s as

$$M_0 + \int_0^s \frac{(k_t - k_u)}{\gamma_u}\beta_u' \,\mathrm{d}W_u^{\mathbb{Q}},$$

with $M_0 = \mathbb{E}_{\mathbb{Q}}[M_s]$. Letting $s \to t$ one would get

$$M_0 + \int_0^t \frac{(k_t - k_u)}{\gamma_u}\beta_u' \,\mathrm{d}W_u^{\mathbb{Q}} = \int_0^t \frac{1}{\gamma_u}\left(\frac{\|\theta_u\|^2}{2} + \eta_u \ln(Z_u) + \theta_u'\beta_u - k_t\eta_u\right) \mathrm{d}u, \tag{2.38}$$

Eq. (2.38) holds true if and only if both the left- and right-hand sides are null. This occurs if and only if

$$\mathbb{P}\left((k_t - k_u)\beta_u = 0 \text{ and } \eta_u = \frac{-\frac{\|\theta_u\|^2}{2} - \theta_u'\beta_u}{\ln(Z_u) - k_t}\right) = 1 \quad \text{for Lebesgue a.e. } u \in [0, \infty).$$

Indeed, up to a modification of the processes, we can assume that the previous property holds for every $u \in [0, \infty)$. On the other hand the choice of $t \in [0, \infty)$ is arbitrary and the definition of the process η cannot depend on t. This implies that $k_t = k$ for every $t \in [0, \infty)$. Since $k = k_t = c_t\left(x + \mathbb{E}_{\mathbb{Q}}[1/\gamma_t \ln Z_t]\right)$, this implies

$$0 = (c_t - c_s)x + c_t\,\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\gamma_t}\ln Z_t\right] - c_s\,\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\gamma_s}\ln Z_s\right] \quad \forall x \in \mathbb{R}.$$

As the previous identity holds for every possible choice of $x \in \mathbb{R}$, we must have $c_t = c_s$. Recalling that $c_t = (\mathbb{E}_{\mathbb{Q}}[1/\gamma_t])^{-1}$ and given the dynamics $\mathrm{d}(1/\gamma_t) = (1/\gamma_t)(\eta_t \mathrm{d}t + \beta_t' \,\mathrm{d}W_t^{\mathbb{Q}})$, this is the case if and only if $\mathbb{P}(\eta_t = 0) = 1$ for every $t \in [0, \infty)$ (again up to a modification of the process). This implies $c_t = c_s = \gamma_0$ and $\mathbb{E}_{\mathbb{Q}}[1/\gamma_t \ln Z_t] = 0$ for all $t \in [0, \infty)$. By Lemma 2.6.1 jointly with the assumption that $\mathrm{Var}(\|\theta_t\|^2) \neq 0$ we have that $\beta_t = -\theta_t/2$.

We turn to item (ii). In order to compute the optimal wealth ξ_t^*, we first notice that for $\eta_t = 0$ and $\beta_t = -\theta_t/2$, Eq. (2.33) collapses to

$$\mathrm{d}\left(\frac{1}{\gamma_t}\ln(Z_t)\right) = \frac{1}{\gamma_t}\left(\theta_t' - \ln(Z_t)\frac{\theta_t'}{2}\right)\mathrm{d}W_t^{\mathbb{Q}},$$

so that $\mathbb{E}_{\mathbb{Q}}[(1/\gamma_t)\ln(Z_t)] = 0$. Furthermore, $1/\gamma_t$ is a $\mathbb{Q}$-martingale, with $\mathbb{E}_{\mathbb{Q}}[1/\gamma_t] = 1/\gamma_0$ and in particular $c_t = \gamma_0$. Substituting these latter results into (2.32) yields

$$\xi_t^* = \frac{1}{\gamma_t}\left(\gamma_0 x - \ln(Z_t)\right).$$

As for item (iii), since the market is complete by assumption, the optimal strategy α^* is the hedging strategy for the above t-claim. The stochastic differential of ξ_t^* reads

$$\mathrm{d}\xi_t^* = -\left(\frac{\xi_t^*}{2}\theta_t' + \frac{1}{\gamma_t}\theta_t'\right)\mathrm{d}W_t^{\mathbb{Q}}.$$

Equating the coefficient above with the diffusion coefficient in the portfolio dynamics (2.6) yields the optimal strategy

$$(\alpha_t^*)' = -\frac{1}{\gamma_t \xi_t^*}\left(\theta_t' + \frac{\gamma_t \xi_t^*}{2}\theta_t'\right)\sigma_t^{-1}.$$

We conclude the proof by showing (iv), that is, for this choice of (η_t, β_t) and the optimal strategy α^* the process $\mathbb{u}_t(V_t^{\alpha^*})$ is a $\mathbb{P}$-martingale. Firstly, as $V_t^{\alpha^*}$ is the replicating portfolio for ξ_t^* we have that $\mathbb{P}$-a.s. $V_t^{\alpha^*} = \xi_t^*$, therefore plugging ξ_t^* into $\mathbb{u}_t$ we obtain

$$\mathbb{u}_t(\xi_t^*) = -\frac{1}{\gamma_t} Z_t e^{-\gamma_0 x}. \tag{2.39}$$

An application of Girsanov Theorem shows that the process $1/\gamma_t$ has $\mathbb{P}$-dynamics

$$\mathrm{d}\left(\frac{1}{\gamma_t}\right) = \frac{1}{\gamma_t}\left(-\frac{\theta_t'}{2}\right)\mathrm{d}W_t^{\mathbb{Q}} = \frac{1}{\gamma_t}\left(\frac{\|\theta_t\|^2}{2}\mathrm{d}t - \frac{\theta_t'}{2}\mathrm{d}W_t\right).$$

Consequently the stochastic differential of (2.39) is

$$\mathrm{d}\left(\mathbb{u}_t(\xi_t^*)\right) = -e^{-\gamma_0 x} Z_t \frac{1}{\gamma_t}\frac{\theta_t'}{2}\mathrm{d}W_t,$$

which in turn implies that $\mathbb{u}_t(V_t^{\alpha^*})$ is a martingale under the reference probability measure $\mathbb{P}$. □

2.7 Proofs of Proposition 2.5.4 and Proposition 2.5.6

The first part of this section focuses on proving Proposition 2.5.4. A key aspect of the argument is that the drift of the stochastic utility function takes the form of d-dimensional paraboloid in the control variable α. As a result, ensuring that $\mathbb{u}_t$ qualifies as a *forward performance*, requires that the parameters η_t and β_t satisfy a specific relationship.
The latter part of the section is dedicated to the proof of Proposition 2.5.6, which is mostly an application of Lemma 2.6.1 together with the computations in Section 2.6.

Proof of Proposition 2.5.4. Observe that, given the dynamics of $1/\gamma_t$ under the martingale measure $\mathbb{Q}$ and recalling that $\mathrm{d}W_t^{\mathbb{Q}} = \mathrm{d}W_t - \theta_t \mathrm{d}t$, we can write

$$\begin{cases} \mathrm{d}\left(\frac{1}{\gamma_t}\right) = \frac{1}{\gamma_t}\left[(\eta_t - \theta_t'\beta_t)\mathrm{d}t + \beta_t'\,\mathrm{d}W_t\right], \\ \gamma_0 > 0 \end{cases}$$

and consequently applying Itō's formula to $f(1/\gamma_t)$ with $f : x \mapsto x^{-1}$ we have that $\mathrm{d}\gamma_t = -\gamma_t\left[(\eta_t - \theta_t'\beta_t - \|\beta_t\|^2)dt + \beta_t' dW_t\right]$. Let the portfolio process V_t^α be as in Eq. (2.6) and observe that

$$\begin{aligned} \mathrm{d}\left(-\gamma_t V_t^\alpha\right) &= -\left(V_t^\alpha\,\mathrm{d}\gamma_t + \gamma_t\,\mathrm{d}V_t^\alpha + \mathrm{d}\langle\gamma_\cdot, V_\cdot^\alpha\rangle_t\right) \\ &= -\Big(-\gamma_t V_t^\alpha\left[(\eta_t - \theta_t'\beta_t - \|\beta_t\|^2)\mathrm{d}t + \beta_t'\,\mathrm{d}W_t\right] + \\ &\qquad + \gamma_t V_t^\alpha\left[\alpha_t'(\mu_t - r\mathbf{1})\mathrm{d}t + \alpha_t'\sigma_t\,\mathrm{d}W_t\right] - \gamma_t V_t^\alpha \alpha_t'\sigma_t\beta\mathrm{d}t\Big) \\ &= \gamma_t V_t^\alpha\left[\left(\eta_t - \theta_t'\beta_t - \|\beta_t\|^2 + \alpha_t'\sigma_t\theta_t + \alpha_t'\sigma_t\beta\right)\mathrm{d}t + \left(\beta_t' - \alpha_t'\sigma_t\right)\mathrm{d}W_t\right]. \end{aligned}$$

Another application of Itō's formula to $\exp(-\gamma_t V_t^\alpha)$ then yields

$$\mathrm{d}\left(e^{-\gamma_t V_t^\alpha}\right) = e^{-\gamma_t V_t^\alpha}\gamma_t V_t^\alpha\left\{\left[(\alpha_t'\sigma_t - \beta_t')\left(\theta_t + \beta_t + \frac{1}{2}\gamma_t V_t^\alpha(\sigma_t'\alpha_t - \beta_t)\right)\right]\mathrm{d}t - (\alpha_t'\sigma_t - \beta_t')\,\mathrm{d}W_t\right\}$$

Collecting the results above, the stochastic differential of $\mathbb{u}_t(V_t^\alpha) = -(1/\gamma_t)\exp\{-\gamma_t V_t^\alpha\}$ reads

$$\begin{aligned} \mathrm{d}\left(\mathbb{u}_t(V_t^\alpha)\right) = -\frac{1}{\gamma_t}e^{-\gamma_t V_t^\alpha}\Big\{&\Big[\gamma_t V_t^\alpha(\alpha_t'\sigma_t - \beta_t')\left(\theta_t + \frac{1}{2}\gamma_t V_t^\alpha(\sigma_t'\alpha_t - \beta_t)\right) + \\ &+ \eta_t(\gamma_t V_t^\alpha + 1) - \theta_t'\beta_t\Big]\mathrm{d}t + \left[\beta_t' - \gamma_t V_t^\alpha\left(\alpha_t'\sigma_t - \beta_t'\right)\right]\mathrm{d}W_t\Big\}. \end{aligned} \tag{2.40}$$

To show that $\mathbb{u}_t(\omega, x)$ is a forward performance, we need to impose conditions on (η_t, β_t) to ensure that $\mathbb{u}_t(\omega, V_t^\alpha)$ is a supermartingale for all $\alpha \in \mathcal{A}$. Considering that $-(1/\gamma_t)\exp\{-\gamma_t V_t^\alpha\} < 0$ for any α, the latter condition is equivalent to requiring

$$\alpha_t'\frac{1}{2}\left(\gamma_t V_t^\alpha\right)^2\sigma_t\sigma_t'\alpha_t + \alpha_t'\sigma_t\gamma_t V_t^\alpha\left(\theta_t - \gamma_t V_t^\alpha\beta_t\right) + \frac{1}{2}(\gamma_t V_t^\alpha)^2\|\beta_t\|^2 + (\gamma_t V_t^\alpha + 1)(\eta_t - \theta_t'\beta_t) \geq 0. \tag{2.41}$$

Consider the quadratic function $\mathbf{x} \mapsto f(\mathbf{x}, y, \eta_t)$, with $\mathbf{x} \in \mathbb{R}^d$, defined as

$$f(\mathbf{x}, y, \eta_t) := \mathbf{x}'\frac{1}{2}(\gamma_t y)^2 \sigma_t\sigma_t'\mathbf{x} + \gamma_t y \left(\sigma_t \left[\theta_t - \gamma_t y \beta_t\right]\right)' \mathbf{x} + \\ + \frac{1}{2}(\gamma_t y)^2 \|\beta_t\|^2 + (\gamma_t y + 1)(\eta_t - \theta_t'\beta_t).$$

which is ascribable to

$$f(\mathbf{x}) = \mathbf{x}'A\mathbf{x} + b'\mathbf{x} + c,$$

where

$$\begin{aligned} A &= \frac{1}{2}(\gamma_t y)^2 \sigma_t'\sigma_t, \\ b &= \gamma_t y \sigma_t \left(\theta_t - \gamma_t y \beta_t\right), \\ c &= \frac{1}{2}(\gamma_t y)^2 \|\beta_t\|^2 + (\gamma_t y + 1)(\eta_t - \theta_t'\beta_t). \end{aligned}$$

Therefore, provided $A \in \mathbf{M}^{d\times d}(\mathbb{R})$ is invertible and $b \in \mathbb{R}^d$, $c \in \mathbb{R}$, it holds that

$$\mathbf{x}'A\mathbf{x} + b'\mathbf{x} + c \geq 0 \quad \forall \mathbf{x} \in \mathbb{R}^d$$

if and only if A is positive semi-definite and $c - \frac{1}{4}b'A^{-1}b \geq 0$. In fact, let us define $\mathbf{h} := -\frac{1}{2}A^{-1}b$ and $k := c - \frac{1}{4}b'A^{-1}b$. We can then write $b = -2A\mathbf{h}$ and after a suitable rearrangement we obtain:

$$f(\mathbf{x}) = \mathbf{x}'A\mathbf{x} + b'\mathbf{x} + c = (\mathbf{x} - \mathbf{h})'A(\mathbf{x} - \mathbf{h}) + k.$$

Clearly, $(\mathbf{x}-\mathbf{h})'A(\mathbf{x}-\mathbf{h})$ is a quadratic form and hence for A positive semi-definite it is non-negative. We conclude that $f(\mathbf{x}) \geq 0\ \forall \mathbf{x} \in \mathbb{R}^d$ if and only if $k \geq 0$. In particular, in order to ensure that our stochastic utility function is a forward performance, we want that $f(\mathbf{x}^*) = 0$ for a unique $\mathbf{x}^* \in \mathbb{R}^d$. This reduces to verifying that the matrix A is positive semi-definite and that $k = 0$. First observe that

$$\begin{aligned} \frac{1}{4}b'A^{-1}b &= \frac{1}{4}\left[\gamma_t y \sigma_t \left(\theta_t - \gamma_t y \beta_t\right)\right]' \frac{2}{(\gamma_t y)^2}(\sigma_t')^{-1}\sigma_t^{-1}\left[\gamma_t y \sigma_t \left(\theta_t - \gamma_t y \beta_t\right)\right] \\ &= \frac{1}{2}\|\theta_t - \gamma_t y \beta_t\|^2 \end{aligned}$$

and hence

$$\begin{aligned} 0 = c - \frac{1}{4}b'A^{-1}b &= \frac{1}{2}(\gamma_t y)^2\|\beta_t\|^2 + (\gamma_t y + 1)(\eta_t - \theta_t'\beta_t) - \frac{1}{2}\|\theta_t - \gamma_t y \beta_t\|^2 \\ &= (\gamma_t y + 1)\eta_t - \theta_t'\left(\beta_t + \frac{\theta_t}{2}\right). \end{aligned}$$

Therefore $k = 0$ is verified whenever

$$\eta_t(y) = \frac{\theta_t'\left(\theta_t + 2\beta_t\right)}{2(\gamma_t y + 1)},$$

provided that $\gamma_t y + 1 \neq 0$. Conversely, if $\gamma_t y = 0$, the condition is satisfied for $\theta_t = 0$, which is in contradiction with Assumption 2.2.3, or $\beta_t = -\theta_t/2$. Finally,

$A = \frac{1}{2}(\gamma_t y)^2 \sigma_t' \sigma_t$ is positive semi-definite by construction.
Thus for any fixed $y \in \mathbb{R}$ we have $f(\mathbf{x}, y, \eta_t(y)) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^d$ and $f(\alpha_t^y, y, \eta_t(y)) = 0$, which in particular implies $f(\alpha_t^*, V_t^*, \eta_t^*) = 0$ for

$$\alpha_t^* = (\sigma_t^{-1})' \frac{(\gamma_t V_t^* \beta_t - \theta_t)}{\gamma_t V_t^*}, \quad \eta_t^* = \frac{\theta_t'(\theta_t + 2\beta_t)}{2(\gamma_t V_t^* + 1)},$$

and V_t^* satisfying

$$\mathrm{d}V_t^* = \frac{(\gamma_t V_t^* \beta_t' - \theta_t')}{\gamma_t} \sigma_t^{-1} \left((\mu_t - r\mathbf{1})\mathrm{d}t + \sigma_t \, \mathrm{d}W_t \right).$$

Moreover, for every other strategy $\alpha \in \mathcal{A}$ we have the inequality $f(\alpha_t, V_t^*, \eta_t^*) \geq 0$. By setting $\mathbb{u}_t(\omega, x) = -[1/\gamma_t(\omega)] \exp\{-\gamma_t(\omega)x\}$ with β_t arbitrary and $\eta_t = \eta_t^*$ defined as above, then necessarily $\mathbb{u}_t(\omega, V_t^{\alpha^*})$ is a $\mathbb{P}$-martingale and for any other strategy (α_t) we have $\mathbb{u}_t(\omega, V_t^\alpha)$ is a $\mathbb{P}$-supermartingale. In fact once chosen η_t as η_t^* the quantity in (2.41) is necessarily positive for every α and therefore the drift in (2.40) is negative, and annihilates only for the choice $\alpha = \alpha^*$.

□

Proof of Proposition 2.5.6. First we observe that if $\theta_t = g(t)$ for some function $g : [0, \infty) \to \mathbb{R}^d$, $\eta_t = 0$ and $\beta_t = f(t)$ for some deterministic function $f : [0, \infty) \to \mathbb{R}^d$ then Problem 2.8 is consistent by Lemma 2.6.1.

Viceversa, repeating the same argument as that of Theorem 2.5.2, we can obtain the relation in Eq. (2.36), which we report below for simplicity

$$\frac{k_t - k_s}{\gamma_s} = \mathbb{E}_{\mathbb{Q}} \left[\int_s^t \frac{1}{\gamma_u} \left(\frac{\|\theta_u\|^2}{2} + \eta_u \ln(Z_u) + \theta_u' \beta_u - k_t \eta_u \right) \mathrm{d}u \middle| \mathcal{F}_s \right]. \tag{2.42}$$

An identical argument to that in the proof of Theorem 2.5.2 shows that Eq. (2.42) implies $\eta_t = 0$ for all $t \geq 0$. Subsequently, by Lemma 2.6.1 together with the assumption that $t \mapsto \theta_t$ is deterministic we conclude that $\beta_t = f(t)$ for some deterministic function $f : [0, \infty) \to \mathbb{R}^d$. □

2.8 Multiplicative Noise

We conclude by mentioning an alternative class of stochastic utilities, where a deterministic utility is perturbed by a multiplicative martingale noise. Similar approaches can be traced back to Duffie and Epstein (1992) and Schroder and Skiadas (1999), who show that, for diffusion models, recursive utility generalization introduces a variance multiplier that may be state-dependent. A comparable multiplicative term appears also in the definition of forward exponential performance in Musiela and Zariphopoulou (2008). While the expected utility of the agent on deterministic quantities is unaffected (and so are their preferences), the randomness of the market interacts with the martingale noise, distorting the agent's perceived utility from a portfolio. We find, however, that the problem is consistent only if the parameter of the noise process suitably counterbalances the randomness of the market. A further unappealing feature is that the consistency condition by itself is not enough to pin down a unique choice of parametrization of the noise, and additionally requiring that $\mathbb{u}_t(\omega, x)$ be a forward performance results in the agent not investing at all in the risky asset.

Proposition 2.8.1. *Let $u : \mathbb{R} \to \mathbb{R}$ be a utility function of the type of Eq. (2.9) and $(X_t)_{t\geq 0}$ a solution to $dX_t = X_t\beta_t' dW_t$, $X_0 = 1$, with $(\beta_t)_{t\geq 0}$ satisfying Assumption (A)[4]. Consider $\mathbb{u}_t(\omega, x) := u(x) \cdot X_t(\omega)$. Problem (2.8) is consistent if and only if $\|\theta_t - \beta_t\|^2 = k(t)$ with $k : [0, \infty) \to \mathbb{R}$ an arbitrary deterministic function. The optimal strategy is given by*

$$(\alpha_t^*)' = \frac{(\theta_t' - \beta_t')}{\gamma \xi_t^*} \sigma_t^{-1}.$$

Furthermore, $(\mathbb{u}_t)_{t\geq 0}$ is a forward performance if and only if $k(t) = 0\ \forall t \geq 0$, in which case $\alpha_t^* = 0$.

Remark 2.8.2. Similarly to what we pointed out in Remark 2.3.2, it is possible to prove an analogous result assuming a power utility $u(x) = x^\gamma/\gamma$, with $\gamma \in (0, 1)$.

2.9 Proof of Proposition 2.8.1

First, we observe that the process X_t, defined as a solution to $dX_t = X_t\beta_t' dW_t$, $X_0 = 1$, defines a new probability measure $\mathbb{P}^*$ such that $X_t = \mathrm{d}\mathbb{P}^*/\,\mathrm{d}\mathbb{P}$. Consequently, fixing some investment horizon $t > 0$ and considering $\mathbb{u}_t(\omega, v) = u(v) \cdot X_t(\omega)$ we have that

$$\sup_{\substack{\alpha\in\mathcal{A}\\ V_0^\alpha = x}} \mathbb{E}_{\mathbb{P}}\left[\mathbb{u}_t(V_t^\alpha)\right] = \sup_{\substack{\alpha\in\mathcal{A}\\ V_0^\alpha = x}} \mathbb{E}_{\mathbb{P}}\left[u(V_t^\alpha) \cdot X_t\right] \sup_{\substack{\alpha\in\mathcal{A}\\ V_0^\alpha = x}} \mathbb{E}^*\left[u(V_t^\alpha)\right], \tag{2.43}$$

where the expectation on the right-hand side is taken with respect to the new probability measure $\mathbb{P}^*$. Similarly to the previous proofs, the problem above can be dealt with from the perspective of an infinite dimensional maximization problem. We then rewrite (2.43) as

$$\sup\left\{\mathbb{E}^*\left[u(\xi)\right] \mid \xi \in L^1(\Omega, \mathcal{F}_t, \mathbb{P}^*) \text{ and } \mathbb{E}_{\mathbb{Q}}\left[\xi\right] = x\right\}.$$

Observe that, whenever the function $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is of exponential form, as in Eq. (2.9), the above problem is analogous to the one of Proposition 2.3.1. Consider the process $\varphi_t := Z_t/X_t$. By a straightforward application of Itō's formula to $f(X_t)$ where $f : x \mapsto x^{-1}$ we find that $\mathrm{d}(1/X_t) = (1/X_t)\left(\|\beta_t\|^2 \mathrm{d}t - \beta_t' \,\mathrm{d}W_t\right)$. In particular, observing that $\mathrm{d}W_t^* = \mathrm{d}W_t - \beta_t \mathrm{d}t$ is a Brownian motion under the new measure $\mathbb{P}^*$, we can write $\mathrm{d}(1/X_t) = -(1/X_t)\beta_t' \,\mathrm{d}W_t^*$ and $\mathrm{d}Z_t = Z_t\left(\theta_t'\beta_t \mathrm{d}t + \theta_t' \,\mathrm{d}W_t^*\right)$. An application of stochastic Leibniz rule yields the dynamics of φ_t as follows:

$$\mathrm{d}\varphi_t = \mathrm{d}\left(\frac{Z_t}{X_t}\right) = \varphi_t\left(\theta_t' - \beta_t'\right) \mathrm{d}W_t^*.$$

In light of the boundedness assumptions on θ and β, φ_t is a $\mathbb{P}^*$ martingale and in particular $\mathbb{Q}$ has density φ_t with respect to $\mathbb{P}^*$. We can then directly apply the results

[4]That is: $(\beta_t)_{t\geq 0}$ is a progressively measurable d-dimensional process such that for any $t > 0$ there exists a constant $K_t > 0$ such that $\mathbb{P}\left(\sup_{0\leq u\leq t}\|\beta_u\| < K_t\right) = 1$.

of Proposition 2.3.1 to obtain that the optimization problem is consistent if and only if $\|\theta_t - \beta_t\|^2 = k(t)$, where $k : \mathbb{R} \to \mathbb{R}$ is some deterministic function.

Recalling that now φ_t plays the role of the measure change, i.e. $\mathrm{d}\mathbb{Q} = \varphi_t \mathrm{d}\mathbb{P}^*$, we identify the optimal t-claim as

$$\xi_t^* = x + \frac{1}{\gamma}\,\mathbb{E}_{\mathbb{Q}}\left[\ln(\varphi_t)\right] - \frac{1}{\gamma}\ln(\varphi_t).$$

Simple computations yield

$$\mathrm{d}\,\mathbb{E}_{\mathbb{Q}}\left[\ln(\varphi_t)\right] = \mathrm{d}\,\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{2}\int_0^t \|\theta_s - \beta_s\|^2 \mathrm{d}s + \int_0^t (\theta_s' - \beta_s')\,\mathrm{d}W_s^{\mathbb{Q}}\right] = \frac{1}{2}\|\theta_t - \beta_t\|^2 \mathrm{d}t,$$

where the last identity follows imposing consistency of the maximization problem. The stochastic differential of ξ_t^* then reads

$$\mathrm{d}\xi_t^* = (\theta_t' - \beta_t')\,\mathrm{d}W_t^{\mathbb{Q}}.$$

Equating the latter to the diffusion coefficient in equation (2.1) and solving for α_t yields the optimal strategy

$$(\alpha_t^*)' = \frac{(\theta_t' - \beta_t')}{\gamma_t \xi_t^*}\sigma_t^{-1}.$$

Lastly, assuming $u(v) = -e^{-\gamma v}/\gamma$, we show that $\mathbb{u}_t(\omega, v) := u(v)\cdot X_t(\omega)$ with $\mathbb{u}_0(v) = u(v)$, is a forward performance if and only if $k(t) \equiv 0$. Observe that, given the dynamics of V_t^α in (2.6) we have that

$$\mathrm{d}\left(u(V_t^\alpha)\right) = V_t^\alpha e^{-\gamma V_t^\alpha}\left[\left(\alpha_t'(\mu_t - r\mathbf{1}) - \frac{1}{2}\gamma V_t^\alpha \|\alpha_t'\sigma_t\|^2\right)\mathrm{d}t + \alpha_t'\sigma_t\,\mathrm{d}W_t\right]$$

and hence the stochastic differential of $\mathbb{u}_t(V_t^\alpha) = u(V_t^\alpha)\cdot X_t$ is given by

$$\mathrm{d}\mathbb{u}_t(V_t^\alpha) = X_t e^{-\gamma V_t^\alpha}\left[\left(-\frac{1}{2}\gamma (V_t^\alpha)^2\|\alpha_t'\sigma_t\|^2 - V_t^\alpha \alpha_t'\sigma_t(\theta_t - \beta_t)\right)\mathrm{d}t + \left(V_t^\alpha\alpha_t'\sigma_t - \frac{1}{\gamma}\beta_t'\right)\mathrm{d}W_t\right]. \tag{2.44}$$

We examine the quadratic function $\mathbf{x} \mapsto f(\mathbf{x}, y)$ defined as $f(\mathbf{x}, y) := -\frac{1}{2}\gamma y^2\|\mathbf{x}'\sigma_t\|^2 - y\mathbf{x}'\sigma_t(\theta_t - \beta_t)$ for $\mathbf{x} \in \mathbb{R}^d, y \in \mathbb{R}$. We omit the second argument for simplicity and we write $f(\mathbf{x}) = \mathbf{x}'A\mathbf{x} + b'\mathbf{x}$ with $A = -\frac{1}{2}\gamma y^2 \sigma_t'\sigma_t$ and $b = -y\sigma_t(\theta_t - \beta_t)$, for $y \in \mathbb{R}$. Using a completion of the square argument analogous to that in the proof of Proposition 2.5.4 we can write $f(\mathbf{x}) = (\mathbf{x} - \mathbf{h})'A(\mathbf{x} - \mathbf{h}) + c$, where $\mathbf{h} = -\frac{1}{2}A^{-1}b$ and $c = -\frac{1}{4}b'A^{-1}b$. Now, given that A is negative semi-definite by construction, requiring that $f(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in \mathbb{R}^d$ is equivalent to imposing that $c \leq 0$. Simple calculations show

$$c = -\frac{1}{4}b'A^{-1}b = \frac{1}{2}\|\theta_t - \beta_t\|^2 \geq 0,$$

which in turn requires $\|\theta_t - \beta_t\|^2 = k(t) = 0$. Hence, for $k(t) = 0$ we have, $f(\mathbf{x}, y) \leq 0$ for all $\mathbf{x} \in \mathbb{R}^d$ and $y \in \mathbb{R}$, so that in particular $f(\mathbf{x}, V_t^\alpha) \leq 0$ and thereby also $f(\alpha_t, V_t^\alpha) \leq 0$.

Finally, for this choice of $k(t)$ a simple inspection of (2.44) shows that the unique strategy $\alpha^* \in \mathcal{A}$ such that $\left(\mathbb{u}_t(V_t^{\alpha^*})\right)_{t\geq 0}$ is a $\mathbb{P}$-martingale, that is the quadratic function described above is null, is $\alpha_t^* = 0$ for all $t \geq 0$.

Conclusions

Chapter 1 examines issues pertaining to the representation of state-dependent preferences and establishes a representation theorem for consistent families of nonlinear conditional expectations in terms of conditional certainty equivalents. The first contribution of this chapter consists in a refinement of the preference representation result originally developed by Wakker and Zank (1999). In particular, we provide additional regularity properties on the associated state-dependent utility functions, thereby ensuring enough tractability for further applications.
A second contribution clarifies the role of the “Sure-Thing Principle” in the consistent conditioning of preferences and shows how conditional Chisini means arise as conditional certainty equivalents with respect to suitable state-dependent utilities.
Notably, extending the literature on conditional nonlinear expectations, our third result characterizes the families of consistent nonlinear expectations as those that admit a representation in terms of a conditional certainty equivalent with respect to the aforementioned state-dependent utilities.

Despite its generality, the approach we present in Chapter 1 could be further developed along few different directions which we briefly remark. The notion of conditionability in Definition 1.3.4 is the natural extension of conditional expectation to nonlinear functionals. However, such a definition is inadequate for some conditional functionals that are frequent in modern mathematical finance. Alternative formulations for the conditional object have been proposed (see for example Amarante, 2022), highlighting the fundamental differences arising as a result. As for nonlinear conditional expectations, we stressed that the consistency property in Definition 1.27 is quite strong and *de facto* leads to the collapse of the conditional nonlinear expectation to a conditional certainty equivalent. It is often the case, in the framework of dynamic risk measures, that one assumes a weaker form of consistency (see for example *acceptance/rejection consistency* in Acciaio and Penner, 2011). A further development of this chapter could be to assume a weaker type of consistency and investigate the structural connections, if any, with the family of conditional certainty equivalents.

In Chapter 2 of this Thesis we study consistency in optimal portfolio choice problems, in the framework of a d-dimensional complete market. A first surprising result is that if the market is stochastic enough, the classical Merton problem with deterministic exponential utility function is never consistent (in the sense of Definition 2.2.4). To overcome this issue, we consider an agent whose preferences are represented by a state-dependent exponential utility function. Specifically, we allow the risk-aversion parameter γ to be the solution of a generic exponential SDE. It turns out that im-

posing that the optimization problem be consistent is equivalent to requiring that the risk-aversion process is a martingale under the (unique) pricing measure with a specific diffusion coefficient.
This result fits also in the forward performance literature as, for this parametrization, the resulting state-dependent utility is also a forward performance. Nevertheless, we show that it is only an element of an infinite family. We also propose an explanation for this apparent inconsistency. Indeed, we suggest that the consistency condition serves as a criterion to pin down the state-dependent utility representing the a priori dynamic preference structure of the optimizing agent. Conversely, the mutual dependence between forward performances and optimal strategies prevents them from being considered as an a priori preference structure.

This chapter relies fundamentally on the assumption of market completeness. The main arguments are in fact carried out by way of the "martingale method" to compute explicitly the maximizer for the expected utility functional. A more realistic, and far more challenging, scenario would be to investigate consistency issues in incomplete markets. We note that previous literature on forward performances (see for example Musiela and Zariphopoulou, 2009, 2010; Žitković, 2009) studies consistency in the framework of incomplete markets, which offers greater generality with respect to the sources of risk. Extending our analysis to incomplete markets, would foster the understanding of the relationship between the notion of consistency we propose and that arising from forward performances. Moreover, we could expand the market to consider price processes with jumps and stochastic interest rates, which are more representative of a real world scenario. A complementary research extension is provided by the theory of stochastic filtering. In this chapter we assume the agent's risk aversion is driven by a known (fully observable) SDE. Although this assumption makes the optimization problem tractable, it is reasonable to posit that the risk aversion may depend also on some unobservable process representing, for instance, the state of the economy. As a result the agent would have to suitably filter their risk aversion in order to devise the optimal strategy, which may in turn have implications on the consistency of the problem.

Appendix A

Auxiliary results for Chapter 1

Lemma A.1. *Suppose $\succeq$ satisfies (SM) and (PC), then $\succeq$ is monotone on $\mathcal{L}^\infty(\Omega, \mathcal{F})$, that is, for $f, g \in \mathcal{L}^\infty(\Omega, \mathcal{F})$, $f(\omega) \geq g(\omega)$ for all $\omega \in \Omega$ implies $f \succeq g$.*

Proof. We begin by showing the result for simple functions. Consider two simple functions $f, g \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ such that $f(\omega) \geq g(\omega)$ for all $\omega \in \Omega$. It is possible to find a common partition $\pi := \{A_1, A_2, ..., A_N\} \subseteq \mathcal{F}$ for some $N \in \mathbb{N}$ such that we can write $f = \sum_{i=1}^N x_i \mathbb{1}_{A_i}$, $g = \sum_{i=1}^N y_i \mathbb{1}_{A_i}$, with $x_i, y_i \in \mathbb{R}$ for $i = 1, ..., N$. We have

$$f = \sum_{i=1}^N x_i \mathbb{1}_{A_i} \succeq y_1 \mathbb{1}_{A_1} + \sum_{i=2}^N x_i \mathbb{1}_{A_i} \succeq y_1 \mathbb{1}_{A_1} + y_2 \mathbb{1}_{A_2} + \sum_{i=3}^N x_i \mathbb{1}_{A_i} \succeq \ldots \succeq \sum_{i=1}^N y_i \mathbb{1}_{A_i} = g,$$

where $\succeq$ will be $\sim$ if $x_i = y_i$ and $\succ$ if $x_i > y_i$ by (SM). We conclude $f \succeq g$ for any pair of simple functions.
Now consider $f, g \in \mathcal{L}^\infty(\Omega, \mathcal{F})$ such that $f(\omega) \geq g(\omega)$ for all $\omega \in \Omega$. Let $(f_n)_n, (g_n)_n \subseteq \mathcal{L}^\infty(\Omega, \mathcal{F})$ be sequences of simple functions such that $f_n \downarrow_n f$ and $g_n \uparrow_n g$ so that $f_n \geq g_n$ for every $n \in \mathbb{N}$. Therefore $f_n \succeq g_n$ from the previous point, for every $n \in \mathbb{N}$. Now assume by contradiction that $g \succ f$, then applying (PC) we find n_1 such that $g_{n_1} \succ f$ and applying again (PC) we find n_2 such that $g_{n_1} \succ f_{n_2}$. By monotonicity on simple functions we have $g_n \succeq g_{n_1} \succ f_{n_2} \succeq f_n$ for every $n \geq n_1 \vee n_2$ and hence a contradiction. □

Lemma A.2. *Let $u : \Omega \times \mathbb{R} \to \mathbb{R}$ and assume that for every $\omega \in \Omega$ the function $x \mapsto u(\omega, x)$ is right continuous with left limit (RCLL), nondecreasing and $u(\omega, 0) = 0$. Suppose additionally that $u(\omega, x) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ for every $x \in \mathbb{R}$. Then:*

(i) u is $\mathcal{F} \otimes \mathscr{B}_\mathbb{R} - \mathscr{B}_\mathbb{R}$ measurable;

(ii) for every $f \in \mathcal{L}^\infty(\Omega, \mathcal{F})$, $\omega \mapsto u(\omega, f(\omega)) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$

Proof. We prove Item (i) in the spirit of Karatzas and Shreve (1991) (Proposition 1.1.13 and Remark 1.1.14) as follows. Let $m > 0$. For $n \geq 1$, $k = 0, 1, ..., 2 \cdot 2^n - 1$, we define

$$u^{(n)}(\omega, s) := u\left(\omega, \frac{(k+1)\,m}{2^n} - m\right) \qquad \text{for} \quad k\frac{m}{2^n} - m \leq s < \frac{(k+1)\,m}{2^n} - m,$$

and fix

$$\begin{cases} u^{(n)}(\omega, s) := u\left(\omega, -m\right), & \text{for } s < -m, \\ u^{(n)}(\omega, s) := u\left(\omega, m\right), & \text{for } s \geq m. \end{cases}$$

The newly constructed process is piecewise constant over intervals of length $\frac{m}{2^n}$ and right continuous. The map $(\omega, s) \mapsto u^{(n)}(\omega, s)$ is $\mathcal{F} \otimes \mathscr{B}_{\mathbb{R}}$-measurable. By right continuity of u we have that $u^{(n)}(\omega, s) \to u(\omega, s)$ for every pair $(\omega, s) \in \Omega \times [-m, m]$ as $n \to \infty$. Therefore we obtain that the map $(\omega, s) \mapsto u(\omega, s)$ is itself $\mathcal{F} \otimes \mathscr{B}_{[-m,m]}$-measurable as pointwise limit of a sequence of measurable functions. Finally, letting $m \to \infty$ we have that $(\omega, s) \mapsto u(\omega, s)$ is $\mathcal{F} \otimes \mathscr{B}_{\mathbb{R}}$-measurable. By the latter argument we have that $\omega \mapsto u(\omega, x)$ is $\mathcal{F} \otimes \mathscr{B}_{\mathbb{R}} - \mathscr{B}_{\mathbb{R}}$ measurable. Moreover, for any $\mathcal{F} - \mathscr{B}_{\mathbb{R}}$ measurable and bounded f, the map $\varphi : \omega \mapsto (\omega, f(\omega))$ is $\mathcal{F} - \mathcal{F} \otimes \mathscr{B}_{\mathbb{R}}$ measurable (see Aliprantis and Border, 2006, Lemma 4.49). Hence, the composition $(u \circ \varphi) : \omega \mapsto u(\omega, f(\omega))$ is $\mathcal{F} - \mathscr{B}_{\mathbb{R}}$ measurable. To conclude, since u is nondecreasing, $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}) \ni u(\omega, -\|f\|_\infty) \leq u(\omega, f(\omega)) \leq u(\omega, \|f\|_\infty) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ thus concluding the proof of Item (ii). ☐

Proposition A.3. *Let $u : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying*

(i) for every $\omega \in \Omega$ $u(\omega, \cdot)$ is RCLL, nondecreasing and $u(\omega, 0) = 0$;

(ii) $u(\cdot, x) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ for every $x \in \mathbb{R}$;

(iii) the functional $T_u : \mathcal{L}^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$ defined by $T_u(f) := \int_\Omega u(\omega, f(\omega)) \mathrm{d}\mathbb{P}$ is continuous from below [1].

Then u is continuous in the following sense:

$$A_{cont} := \{\omega \in \Omega \mid x \mapsto u(\omega, x) \text{ is continuous}\} \in \mathcal{F}$$

and $\mathbb{P}(A_{cont}) = 1$.

Proof. In the present proof we shall rely on few concepts related to the theory of stochastic processes and in particular on stopping times. We refer the reader to Karatzas and Shreve (1991), page 6, for an exhaustive treatment of the notions involved.

Lemma A.2 guarantees that $T_u : \mathcal{L}^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$ is well defined and finite valued. Fix $M > 0$ and define $X_t(\omega) := u(\omega, t - M)$ for $t \geq 0$, omitting the dependence on M for simplicity. Take also the filtration $\mathcal{F}_t := \mathcal{F}, t \geq 0$, which is trivially right continuous by definition: $\bigcap_{s>t} \mathcal{F}_s = \mathcal{F} = \mathcal{F}_t$. Then $(X_t)_t$ is an RCLL process, adapted to $(\mathcal{F}_t)_t$. Set $\Delta X_0 = 0$ and for $t > 0$ set $\Delta X_t := X_t - \sup_{0 \leq s < t} X_s = X_t - \lim_{s \uparrow t} X_s$.

Following the idea in Sokol (2013), we define τ^ε as the first jump time for the size $\varepsilon > 0$, namely

$$\tau^\varepsilon := \inf\{t \geq 0 \mid \Delta X_t > \varepsilon\} \overset{(\star)}{=} \inf\{t > 0 \mid X_t - \sup_{0 \leq s < t} X_s > \varepsilon\},$$

where equality $(\star)$ follows from posing $\Delta X_0 = 0$. Looking back at $u(\omega, \cdot)$, $\tau^\varepsilon(\omega)$ is the first time $u(\omega, \cdot)$ jumps, in the interval $(-M, +\infty)$, with a jump size strictly greater

[1] For any $(f_n)_n \subset \mathcal{L}^\infty(\Omega, \mathcal{F})$ such that $f_n(\omega) \uparrow_n f(\omega)$ for any $\omega \in \Omega$ we have $T_u(f_n) \to T_u(f)$

than ε, with the usual convention that such time is set to be $+\infty$ if no such jump actually occurs. By Sokol (2013), $\tau^{\varepsilon} : \Omega \to [0, +\infty]$ is a $(\mathcal{F}_t)_t$-stopping time, hence in particular it is $\mathcal{F}$-measurable by definition of such a filtration. Setting

$$A^M_{cont} := \{\omega \in \Omega \mid [0, +\infty) \ni t \mapsto X_t(\omega) \text{ is continuous}\},$$

we observe that

$$\Omega \setminus A^M_{cont} = \bigcup_{n\in\mathbb{N}} \{\tau^{\frac{1}{n}} < +\infty\},$$

hence the measurability of A^M_{cont} follows. Suppose now that $\mathbb{P}(A^M_{cont}) < 1$, then for some n we would have $\mathbb{P}(\tau^{\frac{1}{n}} < +\infty) > 0$ and by $\{\tau^{\frac{1}{n}} < +\infty\} = \bigcup_{N\in\mathbb{N}}\{\tau^{\frac{1}{n}} < N\}$ we could find N big enough such that $\mathbb{P}(\tau^{\frac{1}{n}} \leq N) > 0$.
Set, for a fixed pair n, N satisfying $\mathbb{P}(\tau^{\frac{1}{n}} \leq N) > 0$, $B := \{\tau^{\frac{1}{n}} \leq N\}$ and $Z := \tau^{\frac{1}{n}}\mathbb{1}_B \in \mathcal{L}^\infty(\Omega, \mathcal{F})$. For any fixed $\omega \in B$ we know that $\tau^{\frac{1}{n}}(\omega) = \inf\{t \geq 0 \mid \Delta X_t(\omega) > 1/n\} = \lim_{k\to\infty} t_k(\omega)$ for some ω-dependent, non-increasing sequence $t_k(\omega) \downarrow_k \tau^{\frac{1}{n}}(\omega)$ which satisfies $\Delta X_{t_k(\omega)}(\omega) > 1/n$ for every k. This means that, for any k,

$$\frac{1}{n} \leq X_{t_k(\omega)}(\omega) - \sup_{0\leq s<t_k(\omega)} X_s(\omega) \leq X_{t_k(\omega)} - \sup_{0\leq s<\tau^{\frac{1}{n}}(\omega)} X_s(\omega)$$

as $\sup_{0\leq s<t_k(\omega)} X_s(\omega) \geq \sup_{0\leq s<\tau^{\frac{1}{n}}(\omega)} X_s(\omega)$. Letting $k \to \infty$ this implies by right continuity of $(X_t)_t$

$$\frac{1}{n} \leq X_{\tau^{\frac{1}{n}}(\omega)}(\omega) - \sup_{0\leq s<\tau^{\frac{1}{n}}(\omega)} X_s(\omega). \tag{A.1}$$

Take now $s_k := (Z - \frac{1}{k}\mathbb{1}_B)^+$ for $k \in \mathbb{N}$. s_k is $\mathcal{F} - \mathscr{B}_{[0,+\infty)}$ measurable and bounded. Observe that since $(X_t)_t$ is RCLL, by Karatzas and Shreve (1991) Remark 1.1.14 we have $(t, \omega) \mapsto X_t(\omega)$ is $\mathscr{B}_{[0,+\infty)} \otimes \mathcal{F} - \mathscr{B}_{[0,+\infty)}$ measurable, and $\omega \mapsto (\omega, Y(\omega))$ is $\mathcal{F} - \mathcal{F} \otimes \mathscr{B}_{[0,+\infty)}$ measurable for every $\mathcal{F} - \mathscr{B}_{[0,+\infty)}$ measurable bounded $Y : \Omega \to [0, +\infty)$. Hence $\omega \mapsto X_{s_k(\omega)}(\omega)$ is $\mathcal{F} - \mathscr{B}_{[0,+\infty)}$ measurable. By continuity from below, argued at the beginning of this proof,

$$T_u\big((s_k - M)\mathbb{1}_B\big) \to_k T_u\big((Z - M)\mathbb{1}_B\big). \tag{A.2}$$

At the same time, $s_k(\omega) = Z(\omega) = 0$ for every $\omega \in \Omega \setminus B$ and using (A.1) $X_{s_k(\omega)}(\omega) \leq \sup_{0\leq s<\tau^{\frac{1}{n}}(\omega)} X_s(\omega) \leq -\frac{1}{n} + X_{\tau^{\frac{1}{n}}(\omega)}(\omega)$, for any $\omega \in B$. Recalling that $u(\omega, 0) = 0$ for every $\omega \in \Omega$, we can then write

$$\begin{aligned} T_u\big((s_k - M)\mathbb{1}_B\big) &= \int_B u(\omega, s_k(\omega) - M)\mathrm{d}\mathbb{P} = \int_B X_{s_k(\omega)}(\omega)\mathrm{d}\mathbb{P} \\ &\overset{(A.1)}{\leq} -\frac{1}{n}\mathbb{P}(B) + \int_B X_{\tau^{\frac{1}{n}}(\omega)}(\omega)\mathrm{d}\mathbb{P} \\ &= -\frac{1}{n}\mathbb{P}(B) + \int_\Omega u(\omega, Z(\omega) - M)\mathrm{d}\mathbb{P} \end{aligned}$$

so that $T_u((s_k - M)\mathbb{1}_B) + \frac{1}{n}\mathbb{P}(B) \leq T_u((Z - M)\mathbb{1}_B)$ reaching a contradiction with (A.2), as $\frac{1}{n}\mathbb{P}(B) > 0$ does not depend on k. Now, we conclude that $\mathbb{P}(A^M_{cont}) = 1$ for every arbitrarily fixed $M > 0$. Since $A_{cont} = \bigcup_{M\in\mathbb{N}} A^M_{cont}$ we conclude that both $A_{cont} \in \mathcal{F}$ and $\mathbb{P}(A_{cont}) = 1$. $\square$

Corollary A.4. *Under the same assumptions of Proposition A.3, there exists a function* $\widehat{u}$ *such that* $\widehat{u}(\omega,\cdot)$ *is nondecreasing and continuous for every* $\omega \in \Omega$, $E = \{\omega \in \Omega \mid u(\omega,x) = \widehat{u}(\omega,x)\, \forall x \in \mathbb{R}\} \in \mathcal{F}$ *and* $\mathbb{P}(E) = 1$. *In particular,* $T_u(f) = T_{\widehat{u}}(f)$, $\forall f \in \mathcal{L}^\infty(\Omega,\mathcal{F})$. *Finally, if* $u(\omega,\cdot)$ *is strictly increasing on* $\mathbb{R}$ *for every* $\omega \in \Omega$, $\widehat{u}(\omega,\cdot)$ *can be taken strictly increasing on* $\mathbb{R}$ *for every* $\omega \in \Omega$.

Proof. As a consequence of Proposition A.3, $A_{cont} \in \mathcal{F}$. Set $A \coloneqq A_{cont}$ and define now $\widehat{u}(\omega,x) \coloneqq u(\omega,x)\mathbb{1}_A(\omega) + x\mathbb{1}_{\Omega\setminus A}$. Then clearly $\widehat{u}(\omega,\cdot)$ is continuous and nondecreasing either $\omega \in A$ or $\omega \in \Omega \setminus A$. Further for any fixed $x \in \mathbb{R}$ the map $\omega \mapsto \widehat{u}(\omega,x)$ is $\mathcal{F}$ measurable. As $u(\omega,x) \in \mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P})$ implies $\widehat{u}(\omega,x) \in \mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P})$ for every $x \in \mathbb{R}$ we can invoke Lemma A.2 to obtain that $\widehat{u}$ is $\mathcal{F} \otimes \mathscr{B}_{\mathbb{R}} - \mathscr{B}_{\mathbb{R}}$ measurable, $T_{\widehat{u}}$ is well defined in $\mathcal{L}^\infty(\Omega,\mathcal{F})$ and $T_u \equiv T_{\widehat{u}}$.

□

Appendix B

Auxiliary results for Chapter 2

Lemma B.1. *Let* $\mathbb{u} : \Omega \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ *be* $\mathcal{F} \times \mathscr{B}_{\mathbb{R}}$*-measurable, with* $\mathbb{u}(\omega, \cdot)$ *strictly increasing and strictly concave for any* $\omega \in \Omega$*. If for any* $\omega \in \Omega$*,* $\mathbb{u}(\omega, \cdot)$ *is differentiable and its derivative is invertible, then for* $\mathbb{Q} \sim \mathbb{P}$ *the problem*

$$\sup \left\{ \int_\Omega \mathbb{u}(\omega, \xi(\omega)) \, \mathrm{d}\mathbb{P} \mid \xi \in L^1(\Omega, \mathcal{F}_t, \mathbb{Q}) \text{ and } \mathbb{E}_\mathbb{Q}[\xi] = x_0 \right\}.$$

has a unique solution $\xi^*(\omega) = F(\omega, \lambda^* Y(\omega))$ *where* $Y = \frac{d\mathbb{Q}}{d\mathbb{P}}$*,* $F(\omega, y) = (\mathbb{u}')^{-1}(\omega, y)$ *and* λ^* *solves the equation* $\mathbb{E}_\mathbb{Q}[\xi^*] = x_0$*.*

Proof. The Lagrangian function associated to the optimization problem reads

$$\int_\Omega \mathbb{u}(\xi) \, \mathrm{d}\mathbb{P} - \lambda \left(\int_\Omega \xi \, \mathrm{d}\mathbb{Q} - x_0 \right).$$

And by a change of probability measure we obtain

$$\int_\Omega \left(\mathbb{u}(\xi) - \lambda \cdot Y(\xi - x_0) \right) \, \mathrm{d}\mathbb{P}.$$

Indeed the pointwise first order condition with respect to ξ becomes

$$\mathbb{u}'(\omega, \xi) - \lambda Y(\omega) = 0,$$

where $\mathbb{u}' : \Omega \times \mathbb{R} \to \mathbb{R}$ denotes the ω-wise first order derivative of $\mathbb{u}$ with respect to the second argument. Hence, the candidate for the optimum is $F(\omega, \lambda Y(\omega))$. Simple inspections show that for any $\xi \in L^1(\mathbb{Q})$ with $\mathbb{E}_\mathbb{Q}[\xi] = x_0$ we have

$$\int_\Omega \mathbb{u}(\xi) d\mathbb{P} \le \int_\Omega \mathbb{u}(F(\cdot, \lambda Y)) d\mathbb{P} - \lambda \left(\int_\Omega F(\cdot, \lambda Y) d\mathbb{Q} - x_0 \right)$$

and therefore if λ^* solves the equation $\mathbb{E}_\mathbb{Q}[F(\cdot, \lambda^* Y)] = x_0$ then $F(\omega, \lambda^* Y(\omega))$ is optimal. Uniqueness follows from strict monotonicity and concavity. $\square$

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