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ORIGINAL PAPER

American options with liquidation penalties

Anna Battauz[1] · **Marzia De Donno**[2] · **Alessandro Sbuelz**[2]

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Abstract

This paper integrates liquidation costs into the pricing of American options in an arbitrage-free and otherwise frictionless market. The introduction of liquidation penalties changes the comparison between immediate payoff and continuation value for American option holders. Without these penalties, the continuation value is equal to the actual funds obtainable by selling the option. When the sale proceeds achievable upon liquidation are lower due to penalties, immediate exercise becomes more advantageous, leading to a wider optimal early exercise region. We start studying the impact of liquidation penalties in discrete time, and provide closed-form solutions for perpetual American call options in the binomial model. In the continuous-time lognormal model, we derive closed-form asymptotic solutions near maturity for the critical price that triggers optimal early exercise. We also provide explicit pricing formulas for perpetual American options with liquidation penalties. Our results are relevant for executive stock options (ESOs), which typically exhibit liquidation penalties, and for the American equity options for which there is evidence of liquidation costs.

Keywords Liquidation penalties · Executive stock options · Optimal stopping · American options · Early exercise

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✉ Anna Battauz
anna.battauz@unibocconi.it

Marzia De Donno
marzia.dedonno@unicatt.it

Alessandro Sbuelz
alessandro.sbuelz@unicatt.it

[1] Department of Finance, Baffi and IGIER, Bocconi University, Via Roentgen 1, 20136 Milan, Italy

[2] Department of Mathematics for Economics, Finance, and Actuarial Sciences, Catholic University of Milan, Via Necchi 9, 20123 Milan, Italy

1 Introduction

In an arbitrage-free market we study American options with liquidation penalties. American equity options incur regularly significant liquidation costs, as Battalio et al. (2020) document. In particular, these authors verify that, for in the money options within a couple of months of expiration, the best available bid across all option-trading venues is regularly well below the option's intrinsic value. Liquidation penalties are also ubiquitous in executive stock options (ESOs), in which liquidation can be completely forbidden or mildly deterred via liquidation penalties. In ESOs, liquidation penalties are designed to incentivize executives to focus on increasing the value of the underlying shares. This is because executives' call options become more profitable when the underlying share price rises. If they cannot easily sell their options for profit, they are supposed to prioritize working towards this goal in the long term. Together with ESOs' vesting periods, which allow the option to be exercised only after a given period of time or if a prespecified performance measure reaches the desired level, liquidation penalties aim at providing executives with the incentives to increase the company shares value over the long run.

In a no-arbitrage and frictionless market for the riskless asset and the underlying stock, we demonstrate that liquidation penalties in American options significantly alter the optimal exercise strategies for the holders. Without liquidation penalties, the holder compares immediate exercise with the standard continuation value, which is the market price of the American option that commences at the next period. Therefore, in the absence of friction, the continuation value represents the actual funds available to the option holder at the current date, which they can obtain by promptly selling the option after the potential current exercise date. However, if the liquidation price of the option is lower, then they must weigh the payoff achievable through immediate exercise against the funds they can realistically obtain by liquidating the option. This enhances the optimality of early exercise. Even mild liquidation penalties trigger the existence of optimal exercise opportunities for the American call options with liquidation constraints that are absent when the option can be liquidated without frictions. Thus, liquidation penalties make the holder exercise at underlying stock prices lower than the frictionless optimal critical price (that is the underlying price level that triggers optimal immediate exercise). The ESO liquidation penalties can go in the opposite direction of the desired incentive, given the possibility of selling the shares acquired via option exercise. This finding is consistent with the empirical literature that suggests the introduction of vesting periods in ESOs better aligns the interests of the firm with those of the executives, compared to traditional ESOs with liquidation penalties (e.g. Johnson and Tian 2000; Kuang and Qin 2009).

In the limiting case, our findings endorse the accounting practice of recording ESOs on balance sheets with an initial zero value when they are at-the-money or out-of-the-money at inception. We show that, when the limiting case of a full liquidation ban for American call options applies, their value to the holder and the writer's replication cost equal the immediate payoff of the call option, which is null by construction for at-the-money and out-of-the-money options.

Our paper contributes to the vast literature on American options (e.g. Broadie and Detemple 1996; Detemple and Tian 2002; Battauz et al. 2022), and more specifically to the literature on American options in markets with frictions. Lu et al. (2022) derive the price of American options in markets where the fundamental assets are subject to transaction costs following a utility-based approach. In contrast, we specifically examine the scenario of liquidation penalties solely for the American derivative within an otherwise frictionless market.

Figlewski (2022) observes that, in real-world markets, American stock options that are in the money and nearing expiration are seldom traded at their theoretical European option values or even at their intrinsic values. Even when stocks have negligible transaction costs, equity options display significant bid-ask spreads. Consequently, for American equity options there exists an incentive to exercise rather than accept the best available bid in the market for their liquidation before maturity. Figlewski (2022) approximates the continuation value of an American call option subject to liquidation penalties by estimating the optimal critical price through a portfolio of European options subjected to similar constraints in the presence of additive transaction costs. Jensen and Pedersen (2016) investigate the impact on the early exercise of American call options of short selling constraints on the underlying asset and of funding costs, determining an optimal exercise boundary for the call holder who exercises the option when the margins on short positions become prohibitively expensive. The characterization of the critical price with liquidation penalties provided in these papers is rather complex and can only be achieved numerically.

In contrast, we assume neither the value for the holder when exercising the option in relation to the portfolio of other investments nor specific risk preferences for the holder: we compare attainable cashflows (immediate payoff versus proceeds from sale) to determine the optimal exercise policy. In particular, we evaluate the actual amount of money available to the option holder through exercising the option or by liquidating it, while also accounting for the liquidation penalty. By concentrating solely on the liquidation cost for the American option and modelling it as an additional discount factor, our evaluation is carried out first in discrete time via an adaptation of the backward recursion approach in the presence of the liquidation penalty and then in the Black-Scholes model. In particular, we derive closed-form asymptotic solutions for the critical price near maturity and we obtain explicit pricing formulas for the perpetual case.

The paper is organized as follows: Sect. 2 addresses the optimal decision problem for the holder of an American option subject to liquidation penalties within a discrete-time framework. In Sect. 3, we examine the problem within the binomial market model. Sect. 3.1 focuses on the finite-maturity case, and Sect. 3.2 collects closed-form formulas for American call options in the perpetual case. Section 4 collects results in the Black-Scholes market model. Section 5 discusses the impact of liquidation penalties in various market cases. Finally, Sect. 6 concludes the paper.

2 American options with liquidation penalties in discrete-time

Consider an arbitrage-free and complete discrete-time market on $(\Omega, \mathcal{F}, \mathbb{Q})$ with $t = 0, \Delta t, \dots, T - \Delta t, T$, and an American option whose payoff at t is $X(t)$. Let $B(t) = e^{rt}$ denote the riskless asset value and denote with a $\widetilde{(\cdot)}$ the discounted quantities.

If there are no liquidation costs, the discounted minimal cost of the portfolio-consumption pair that delivers the payoff X whenever exercised is precisely equal to

$$\widetilde{V}(t) = \sup_{t \le \tau \le T} \mathbb{E}_t\left[\widetilde{X}(\tau)\right] \text{ for all } t = 0, \Delta t, \dots, T - \Delta t, T \tag{1}$$

where τ varies among all $\mathcal{F}$-stopping times with values from the current t to maturity T and the expectation is taken under the risk-neutral measure $\mathbb{Q}$. Thus, the discounted no-arbitrage price of the American option with payoff X is $\widetilde{V}$ in (1) , that is the Snell envelope of $\widetilde{X}$. In undiscounted terms

$$V(t) = \sup_{t \le \tau \le T} \mathbb{E}_t\left[X(\tau)e^{-r(\tau - t)}\right] \tag{2}$$

Thanks to the Bellman principle, V in (2) can be equivalently computed as

Lemma 1 (Backward recursion, frictionless case)

$$\begin{aligned} V(T) &= X(T) \\ V(t) &= \max\left\{X(t), CV(t)\right\} \text{ for } t = T - \Delta t, \dots, \Delta t, 0 \end{aligned} \tag{3}$$

where $CV(t)$ is the continuation value at t

$$CV(t) = \mathbb{E}_t\left[e^{-r\Delta t} V(t + \Delta t)\right] \tag{4}$$

$CV(t)$ is the cost at t of delivering the payoff at any future date after t, or, equivalently, the value at t of all future exercises.

When no liquidation costs are involved, $CV(t)$ defined in (4) is the amount of money the holder of an American option can obtain at t by selling the forward-starting option at $t + \Delta t$, whose price at t is $CV(t)$. Because of this, at any t the holder rightly compares the immediate payoff $X(t)$ to the $t-$value of selling the option forward-starting at $t + \Delta t$, that coincides with the continuation value $CV(t)$. Thus the value of the American call option is actually $V(t) = \max(X(t), CV(t))$. This value coincides with the proceedings of liquidating the option with no penalties at t as well.

When the option has liquidation penalties, the choice of selling the option at t at the full price $V(t)$ (if exercise date t has not yet passed) or $CV(t)$ (if exercise at t is no longer feasible) is no longer available to the holder. Instead, the holder can liquidate the option at a lower price. We define the liquidation penalty as follows:

Definition 2 The holder of the American option V can liquidate the option at the current t at the discounted price $\lambda V(t)$, where V is the frictionless American option value in (2) and $\lambda = e^{-\delta \Delta t} \in [0, 1]$ is the liquidation penalty with liquidation penalty rate $\delta > 0$. When $\delta = 0$ there is no liquidation penalty. When $\delta \to +\infty$ the liquidation of the American option is effectively forbidden (full liquidation ban), as $e^{-\delta \Delta t} V(t) \to 0$.

Over a period of time of 1 day $\Delta t = \frac{1}{250} = 0.004$, a liquidation penalty rate $\delta = 0.05$ corresponds to a liquidation penalty of $2bps$, as $\lambda = \exp(-0.004 \cdot 0.05) = 0.9998$. A liquidation penalty rate $\delta = 0.1$ corresponds to a liquidation penalty of $4bps$, as $\lambda = \exp(-0.004 \cdot 0.1) = 0.9996$.

In the presence of option liquidation penalties, the liquidation of the option is penalized at t and at all future dates. Therefore in the discrete-time framework the holder of the option faces a different monetary alternative to the usual Bellman recursion (3): she can either exercise at t getting the immediate payoff $X(t)$ or she can liquidate the option getting the penalized value $\lambda V(t) = e^{-\delta \Delta t} V(t)$. The whole continuation value $CV(t)$ defined in (4) is no longer an amount of money that is viable to the holder at t.

We thus have the modified backward recursion for the holder of the option with liquidation penalties:

Definition 3 (Backward recursion with liquidation penalties for option's holder) The cashflow-based value function for the holder of the American option with liquidation penalty rate δ is defined via the following recursion:

$$\begin{aligned} V_\delta(T) &= X(T) \\ V_\delta(t) &= \max\left\{X(t), e^{-\delta \Delta t} V(t)\right\} \text{ with } t = T - \Delta t, \dots, 0 \end{aligned} \tag{5}$$

When comparing actual monetary values as in (5), our approach remains neutral regarding the other investments in their portfolios, such as short positions on the underlying assets. Conveniently, we show in the next proposition that the cashflow-based value function defined in (5) is the Snell envelope with respect to the risk-neutral measure of the payoff X deflated at a liquidation-adjusted rate $r + \delta$. More precisely, we have the following result:

Proposition 4 *The value function* V_δ *defined in* (5) *can be rewritten as*

$$V_\delta(t) = \sup_{t \le \tau \le T} \mathbb{E}_t\left[e^{-(r+\delta)(\tau - t)} X(\tau)\right] \tag{6}$$

Proof At T the backward recursion (5) implies that $V_\delta(T) = X(T)$. At any other $t < T$, formula (5) implies

$$\begin{aligned} V_\delta(t) &= \max\left\{X(t), e^{-\delta\Delta t}V(t)\right\} \\ &= \max\left\{X(t), e^{-\delta\Delta t}\left(\max\left(X(t), \mathbb{E}_t\left[e^{-r\Delta t}V(t+\Delta t)\right]\right)\right)\right\} \\ &= \max\left\{X(t), e^{-\delta\Delta t}\mathbb{E}_t\left[e^{-r\Delta t}V(t+\Delta t)\right]\right\} \text{ as } 1 > e^{-\delta\Delta t} \\ &= \max\left\{X(t), \mathbb{E}_t\left[e^{-(r+\delta)\Delta t}V(t+\Delta t)\right]\right\} \end{aligned}$$

Thus, the recursion (5) can be rewritten as

$$\begin{cases} V_\delta(T) = X(T) & t = T \\ V_\delta(t) = \max\left\{X(t), \mathbb{E}_t\left[e^{-(r+\delta)\Delta t}V(t+\Delta t)\right]\right\} & t = T-\Delta t, \dots, 0 \end{cases}$$

which proves that V_δ is the Snell envelope of X discounted at the rate $r+\delta$ and with respect to the risk-neutral probability as in (6) . ■

As in the frictionless case, the cashflow-based value function V_δ defined in (5) dominates the payoff

$$V_\delta(t) \geq X(t)$$

for all t. The liquidation-adjusted rate $r+\delta$ used to discount the payoff in (6) results in a continuation value for the option holder that is lower than in the frictionless case (2) due to the liquidation penalties. This lower continuation value makes early exercise appear more advantageous to the holder facing liquidation penalties compared to a scenario without penalties.

It's important to note that V_δ, the cashflow-based value for the option holder who is subject to liquidation constraints as in (6), is not the fair price of the American option with liquidation penalties, as it does not represent the value of the portfolio that delivers the payoff when exercised optimally. To define the no-arbitrage price of the American option with liquidation penalties, we first determine the optimal exercise policy τ^*_δ for the option holder subject to the liquidation penalty rate δ. Then, we compute the no-arbitrage price (that is the replication cost) of the option exercised according to this optimal policy.

If the option has not yet been exercised at time t, the optimal exercise time for the holder of the option with liquidation penalty rate δ is

$$\tau^*_\delta(t) = \inf\left\{s \geq t \,:\, V_\delta(s) = X(s)\right\} \wedge T \tag{7}$$

At time 0 the optimal exercise time is

$$\tau^*_\delta = \inf\left\{t = 0, \dots, T \,:\, V_\delta(t) = X(t)\right\} \wedge T \tag{8}$$

Once the optimal exercise policy has been established, we can determine the replication cost.

Definition 5 For the writer of the option, the cost of replicating the option optimally exercised at (7) by the holder subject to liquidation penalty rate δ is

$$RC_\delta(t) = \mathbb{E}_t\left[e^{-r\left(\tau^*_\delta(t)-t\right)}X(\tau^*_\delta(t))\right] \tag{9}$$

for any time $t = 0, \Delta t, \dots, T$.

The payoff X, the cashflow-based value function V_δ defined in (5), the replication cost RC_δ defined in (9), and the frictionless value function V in (2) satisfy the following chain of inequalities for all $t = 0, \dots, T$

$$X(t) \le V_\delta(t) \le RC_\delta(t) \le V(t). \tag{10}$$

The first inequality $X(t) \le V_\delta(t)$ follows from (6) . The second inequality $V_\delta(t) \le RC_\delta(t)$ is due to the fact that, in (9) , the payoff is discounted at the rate r rather than at the higher adjusted rate $r + \delta$ in (6) . The third inequality $RC_\delta(t) \le V(t)$ is due to the fact that $V(t)$ is the replication cost of the payoff exercised at any date $\tau \ge t$, which includes $\tau^*_\delta(t)$ in (7) .

The ratio

$$\frac{V(t) - RC_\delta(t)}{V(t)} \ge 0$$

is the fraction of the option's value at time t that is lost by the holder, due to the liquidation frictions.

In the next section, we investigate the impact of liquidation penalties on American call options in the binomial setting.

3 American call options with liquidation penalties in the binomial market

Using the well-known binomial approximation of the continuous-time lognormal underlying asset price (as outlined by Battauz and Rotondi (2022); Mulinacci and Pratelli (1998)), the numerical results we obtain here in the binomial model are an effective approximation for the Black-Scholes case that we investigate in the next section. Given $B(0) = 1$ and $S(0) = S_0$, at time$t = n \cdot \Delta t$ for $n \in \mathbb{N}$ we have the riskless asset $B(t + \Delta t) = B(t)e^{r\Delta t}$ growing at the riskless interest rate r, and a risky stock S with risk-neutral binomial distribution:

$$S(t+\Delta t) = \begin{cases} S(t)e^{\sigma\sqrt{\Delta t}} & \text{with } \mathbf{q} = \frac{e^{(r-q)\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} \\ S(t)e^{-\sigma\sqrt{\Delta t}} & \qquad 1 - \mathbf{q} \end{cases}$$

where σ is the underlying volatility and q is the dividend yield of the risky security S. As $\Delta t \to 0$, the binomial process S converges to the continuous-time lognormal one.

3.1 Finite-maturity call option in the binomial model

Equation (6) implies that the cashflow-based value function V_δ is the Snell envelope of the payoff X discounted at the liquidation adjusted rate $r + \delta$. For an American call option on the underlying asset with immediate payoff

$$X(t) = (S(t) - K)^+,$$

convexity, value dominance and monotonicity imply that there exists a time-varying free boundary $s^*_\delta = s^*_\delta(t) > K$ such that V_δ in (6) coincides with the payoff $V_\delta(t) = (S(t) - K)^+$ for all $S(t) \geq s^*_\delta(t)$, while $V_\delta(t) > (S(t) - K)^+$ for all $S(t) < s^*_\delta(t)$. More precisely, once V_δ in (6) has been computed along the binomial tree, the critical price can be retrieved numerically as follows:

$$s^*_\delta(t) = \inf\left\{S(t) \,:\, V_\delta(t) = (S(t) - K)^+\right\} \tag{11}$$

When $T = 1$, $r = 0.05$, $q = 0$, $\sigma = 0.25$, $K = S(0) = 1$, the frictionless American ATM (at-the-money) call option does not have any optimal early exercise opportunity. On the contrary, the American call option subject to a liquidation penalty rate $\delta = 0.01$ has a non-empty early exercise region. This region is defined by the

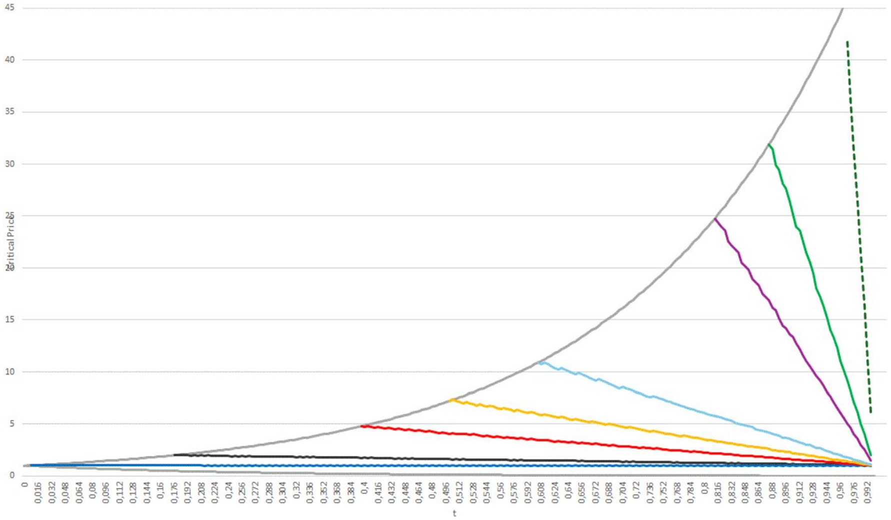

Fig. 1 Critical Price s^*_δ for an American call option with $K = S(0) = 1, T = 1$, in the binomial model with $\Delta t = 1/250$, $r = 0.05$, $q = 0$, $\sigma = 0.25$, and with liquidation penalty rate $\delta = 0.01$ (green, dashed), $\delta = 0.05$ (green, solid), $\delta = 0.1$ (purple), $\delta = 0.5$ (light blue), $\delta = 1$ (orange), $\delta = 2$ (red), $\delta = 10$ (black), $\delta = 500$ (blue). The upper and lower branch of the binomial tree are plotted in grey. (colour figure online)

values of $S(t)$ not below the critical price s^*_δ defined in (11) and plotted with a green dashed line in Fig. 1. For larger values of the liquidation penalty rate δ, the critical price becomes progressively lower and flatter, approaching the constant strike price $K = 1$. Figure 1 shows that, for the markedly high value $\delta = 500$, the critical price (plotted in blue) almost coincides with the constant strike price. This indicates that as δ approaches infinity, our cashflow-based value function converges to the immediate payoff function. This intuition will be confirmed in the next section.

The replicating cost RC_δ for the call option as in (9) is the no-arbitrage price of a barrier knock-in American call option that is activated when the underlying price S triggers the time-varying barrier s^*_δ defined in (11) . American knock-in options can be parsimoniously markovianized in the binomial setting by introducing an auxiliary state variable[1] that accounts for the barrier activation. In particular, we consider the pair $(S(t), \Upsilon(t))$ where

$$\Upsilon(t) = \begin{cases} -1 & \text{if barrier activated in the past, i.e. } S(u) \geq s^*_\delta(u) \text{ for some } u < t \\ 0 & \text{if barrier never activated before nor at } t, \text{ i.e. } S(u) < s^*_\delta(u) \text{ for all } u \leq t \\ 1 & \text{if barrier activated for the first time at } t \text{ with } S(t) \geq s^*_\delta(t) \end{cases}$$

The pair (S, Υ) is Markovian, as

$$\Upsilon(t + \Delta t) = \begin{cases} -1 & \text{if } \Upsilon(t) = -1 \text{ or } \Upsilon(t) = 1 \\ 0 & \text{if } \Upsilon(t) = 0 \text{ and } S(t + \Delta t) < s^*_\delta(t + \Delta t) \\ 1 & \text{if } \Upsilon(t) = 0 \text{ and } S(t + \Delta t) \geq s^*_\delta(t + \Delta t) \end{cases}$$

Thus depending on the current value of $\Upsilon(t)$, for each node of the tree $S(t)$ we have at the following period $t + \Delta t$

$$\begin{tikzcd}
 & {[S(t + \Delta t) = S(t) \cdot u, \Upsilon(t + \Delta t) = -1]} \\
{[S(t), \Upsilon(t) = -1, 1]} \arrow[ur, "\mathbf{q}"'] \arrow[dr, "1-\mathbf{q}"'] & \\
 & {[S(t + \Delta t) = S(t) \cdot d, \Upsilon(t + \Delta t) = -1]}
\end{tikzcd}$$

when $\Upsilon(t) = -1$ and when $\Upsilon(t) = 1$. When $\Upsilon(t) = 0$, in case of an upward/downward movement we may activate the barrier at $t + \Delta t$ or not, leading to

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[1] See Battauz and Staffolani (2024) and Battauz and Rotondi (2004) for more details.

$$\begin{tikzcd}
 & {\begin{array}{c}[S(t+\Delta t) = S(t)\cdot u, \Upsilon(t+\Delta t)] \text{ where} \\ \Upsilon(t+\Delta t) = 0 \text{ if } S(t+\Delta t) < s^*_\delta(t+\Delta t) \\ \text{or } \Upsilon(t+\Delta t) = 1 \text{ if } S(t+\Delta t) \geq s^*_\delta(t+\Delta t)\end{array}} \\
{[S(t), \Upsilon(t) = 0]} \arrow[ur, "\mathbf{q}"'] \arrow[dr, "\mathbf{1-q}"'] & \\
 & {\begin{array}{c}[S(t+\Delta t) = S(t)\cdot d, \Upsilon(t+\Delta t)] \text{ where} \\ \Upsilon(t+\Delta t) = 0 \text{ if } S(t+\Delta t) < s^*_\delta(t+\Delta t) \\ \text{or } \Upsilon(t+\Delta t) = 1 \text{ if } S(t+\Delta t) \geq s^*_\delta(t+\Delta t)\end{array}}
\end{tikzcd}$$

We notice that the last updating formula for the pair (S, Υ) differs from the constant barrier case. In the scenario of an up-and-in constant barrier, if the stock price is below the barrier at time t, it can only reach the barrier through an upward movement. However, in our current situation, the barrier exhibits a significant time-to-maturity effect, decreasing toward the strike price as it approaches maturity. Therefore, the barrier may be hit at time $t + \Delta t$ even with a downward movement[2] from the value $S(t)$ at time t.

On the tree (S, Υ) we compute the optimally exercised cashflows for $t = T - \Delta t, \dots, 0$

$$X_\delta(t) = \begin{cases} 0 & \text{if } \Upsilon(t) = -1 \text{ or } \Upsilon(t) = 0 \\ (S(t) - K)^+ & \text{if } \Upsilon(t) = 1 \end{cases}$$

Thus the backward recursion to compute RC_δ with respect to the pair (S, Υ) is

$$RC_\delta(T) = \begin{cases} 0 & \text{if } \Upsilon(T) = -1 \\ (S(T) - K)^+ & \text{if } \Upsilon(T) = 0 \text{ or } \Upsilon(T) = 1 \end{cases}$$

$$RC_\delta(T - \Delta t) = \begin{cases} 0 & \text{if } \Upsilon(T - \Delta t) = -1 \text{ or } \Upsilon(T - \Delta t) = 1 \\ \mathbb{E}^{\mathbb{Q}}_{T-\Delta t}\left[e^{-r\Delta t} RC_\delta(T))\right] & \text{if } \Upsilon(T - \Delta t) = 0 \end{cases}$$

and for $t = T - 2\Delta t, \dots, 0$

$$RC_\delta(t) = \begin{cases} 0 & \text{if } \Upsilon(t) = -1 \text{ or } \Upsilon(t) = 1 \\ \mathbb{E}^{\mathbb{Q}}_t\left[e^{-r\Delta t}\left(RC_\delta(t + \Delta t) + X_\delta(t + \Delta t)\right)\right] & \text{if } \Upsilon(t) = 0 \end{cases}$$

This recursion allows to compute the fraction of the initial American option premium that is lost due to the liquidation penalties:

$$\Lambda_\delta = \frac{V(0) - RC_\delta(0)}{V(0)} \geq 0 \tag{12}$$

Table 1 collects the values of $V(0)$, $RC_\delta(0)$, $V_\delta(0)$ and Λ_δ for different values of δ for the ATM call option with $K = S(0) = 1$, $T = 1$, $r = 0.05$, $q = 0$, and $\sigma = 0.25$. As δ increases, the cashflow-based value function $V_\delta(0)$ and the replication cost $RC_\delta(0)$

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[2] The authors are grateful to the research assistant Vlad Damaschin for highlighting this detail.

Table 1 $V(0), V_\delta(0), RC_\delta(0)$ and Λ_δ for different $\delta$with K $S(0) = 1,\ T = 1, r = 0.05, q = 0, \sigma = 0.25, \Delta t = 1/250$

	$\delta = 0.01$	$\delta = 0.05$	$\delta = 0.1$	$\delta = 0.5$
$V(0)$	0.1232612	0.1232612	0.1232612	0.1232612
$RC_\delta(0)$	0.1232612	0.1232605	0.1232494	0.1230601
$V_\delta(0)$	0.1232562	0.1232365	0.1232119	0.1230149
Λ_δ	3.37×10^{-16}	5.04×10^{-6}	9.54×10^{-5}	0.0016314
	$\delta = 1$	$\delta = 2$	$\delta = 10$	$\delta = 500$
$V(0)$	0.1232612	0.1232612	0.1232612	0.1232612
$RC_\delta(0)$	0.1228273	0.1223616	0.1187336	0.0273299
$V_\delta(0)$	0.1227691	0.1222790	0.1184280	0.0166816
Λ_δ	0.0035194	0.0072981	0.0367311	0.7782761

decrease. Since the frictionless American call option value $V(0)$ is constant, the fraction Λ_δ increases. This behavior is confirmed in the next sections within the perpetual binomial model and the continuous-time lognormal setting.

3.2 Perpetual call option in the binomial model

In the perpetual case, the cashflow-based value function for the call option V_δ in (6) is time-independent and given by

$$V_{\infty,\delta}(x) = \sup_{\tau \geq 0} \mathbb{E}\left[e^{-(r+\delta)\tau}(S(\tau) - K)^+\right] \text{ with } x = S(0), \tag{13}$$

where τ can take any value among $t_n = n\Delta t$, with $n \in \mathbb{N}$ and Δt fixed. In the perpetual case the critical price defined in (11) is independent of time t, and coincides with the lowest attainable value on the binomial tree that triggers immediate optimal exercise. The cashflow-based value function for the call option in the perpetual binomial model becomes a perpetual barrier option

$$\begin{aligned} V_{\infty,\delta}(x) &= \sup_{b > S(0)} \mathbb{E}\left[e^{-(r+\delta)\tau_b}\left(S\left(\tau_b\right) - K\right)^+\right] \\ &= \sup_{b > S(0)} \mathbb{E}\left[e^{-(r+\delta)\tau_b}\right](b - K)^+ \\ \text{with } \tau_b &= \inf\{t > 0 : S(t) = b\}. \end{aligned} \tag{14}$$

The barrier b has to be determined among the realizations of the binomial tree $S(0)u^k$ for $k \in \mathbb{N}$, by maximizing

$$\mathbb{E}\left[e^{-(r+\delta)\tau_b}\left(S\left(\tau_b\right) - K\right)^+\right] = \mathbb{E}\left[e^{-(r+\delta)\tau_b}\right](b - K)^+$$

By following this approach, in the next proposition we provide the closed-form solution for the value of the perpetual cashflow-based value function in the binomial model by using the probability generating function of the hitting time of a random walk (see Zitkovic 2024).

Proposition 6 *The cashflow-based perpetual value function* (13) *in the binomial model is*

$$V_{\infty,\delta}(x) = \begin{cases} x - K & x \geq b^* \\ \left(\frac{x}{b^*}\right)^{\frac{-C}{\sigma\sqrt{\Delta t}}}(b^* - K) & x < b^* \end{cases}$$

where

$$C = \ln\left(\frac{1 - \sqrt{1 - 4\mathbf{q}(1-\mathbf{q})a^2}}{2(1-\mathbf{q})a}\right) < 0 \text{ with } a = e^{-(r+\delta)\Delta t}$$

and

$$b^* \text{ is the ceiling node of } \frac{CK}{C + \sigma\sqrt{\Delta t}} > K \text{ on the binomial tree,} \tag{15}$$

i.e. the smallest value in the tree that is greater than or equal $\frac{CK}{C+\sigma\sqrt{\Delta t}}$.

Proof See the Appendix. □

The replication cost in the perpetual case as defined in (9) for $T \to +\infty$ is independent of time t, and is the no-arbitrage price of the American perpetual barrier call option that is activated as soon as the underlying price reaches the upper barrier b^*. We provide the closed form solution in the following

Proposition 7 *The replication cost of the perpetual American call option subject to liquidation penalty rate* δ *is*

$$RC_{\infty,\delta}(x) = (b^* - K)\left(\frac{x}{b^*}\right)^{\frac{-C_{RC}}{\sigma\sqrt{\Delta t}}}$$

where $x = S(0), b^*$ *is defined in* (15) *and*

$$C_{RC} = \ln\left(\frac{1 - \sqrt{1 - 4\mathbf{q}(1-\mathbf{q})e^{-2r\Delta t}}}{2(1-\mathbf{q})e^{-r\Delta t}}\right) \tag{16}$$

Moreover $C_{RC} > C$, *and therefore* $RC_{\infty,\delta}(x) > V_{\infty,\delta}(x)$.

Proof See the Appendix. □

In Table 2, we apply our closed-form solutions to the perpetual binomial model for various values of delta. As the dividend yield is zero, the price of the frictionless

Table 2 $V_\infty, V_{\infty,\delta}, RC_{\infty,\delta}$ and $\Lambda_{\infty,\delta}$ for different $\delta$with $K = S(0) = 1, r = 0.05, q = 0,$ $\sigma = 0.25$ and $\Delta t = 1/250$

	$\delta = 0.01$	$\delta = 0.05$	$\delta = 0.1$	$\delta = 0.5$
V_∞	1	1	1	1
$RC_{\infty,\delta}$	0.89465	0.66054	0.52315	0.108 35
$V_{\infty,\delta}$	0.686 37	0.379 11	0.266 36	0.255 92
$\Lambda_{\infty,\delta}$	0.105 35	0.339 46	0.476 85	0.744 08
	$\delta = 1$	$\delta = 2$	$\delta = 10$	$\delta = 500$
V_∞	1	1	1	1
$RC_{\infty,\delta}$	0.181 54	0.127 94	$5.631\,9 \times 10^{-2}$	$5.891\,0 \times 10^{-3}$
$V_{\infty,\delta}$	$7.357\,6 \times 10^{-2}$	$5.032\,4 \times 10^{-2}$	$2.132\,2 \times 10^{-2}$	$2.173\,6 \times 10^{-3}$
$\Lambda_{\infty,\delta}$	0.81846	0.87206	0.94368	0.99411

perpetual call option V_∞ is equivalent to the initial stock price (see Jeanblanc et al. 2009, Section 3.11.1). The cashflow-based value function $V_{\infty,\delta}$ quickly decreases to the immediate payoff value (which is zero, since the option is at-the-money at inception) as δ increases. Similarly, the replication cost $RC_{\infty,\delta}$ decreases as delta increases. The proportion of the initial premium lost to liquidation penalties defined in (12) can be explicitly computed in the perpetual setting. In particular, we see that $\Lambda_{\infty,\delta}$ as in (12) rapidly converges to 100% as delta approaches infinity. These findings confirm the results presented in Table 1 for the finite maturity case.

The results obtained in the perpetual binomial model closely align with those in the perpetual lognormal model, as will be demonstrated in the following section.

The closed-form solutions obtained for the perpetual binomial model in Proposition 6 and 7 depend on the monitoring period Δt. However, this dependence is very mild, as can be seen in Fig. 7 in the next section, where we portray the behavior of the perpetual critical price in the binomial model defined in Equation (15) as a function of Δt. Indeed Fig. 6 shows that the perpetual barrier in the binomial case very mildly decreases with Δt and, in the limiting case of Δt approaching zero, it perfectly coincides with the constant perpetual value in the lognormal model.

4 American options with liquidation penalties in the Black-Scholes market

We consider in this section a continuous-time Black-Scholes setting on $(\Omega, \mathcal{F}, \mathbb{Q})$ with $t \in [0, T]$. Under the risk-neutral probability $\mathbb{Q}$ the stock price S has the following dynamics

$$\frac{dS(t)}{S(t)} = (r - q)dt + \sigma dW(t) \tag{17}$$

where W is the risk-neutral Brownian motion and $\sigma \in \Re$ is the underlying stock volatility.

Inspired by the discrete-time setting (6), and given the payoff function $X(t) = f(S(t))$ with $f : \mathfrak{R}^+ \to \mathfrak{R}^+$, we define the cashflow-based value of the American option with liquidation penalty rate δ for the holder as

$$\begin{aligned} V_\delta(t) &= \sup_{t \le \tau \le T} \mathbb{E}_t\left[e^{-(r+\delta)(\tau-t)} X(\tau)\right] = F_\delta(t, S(t)) \\ \text{where } F_\delta(t,x) &= \sup_{0 \le \theta \le T-t} \mathbb{E}\left[e^{-(r+\delta)\theta} f(S(\theta))\right] \text{ with } x = S(t), \end{aligned} \tag{18}$$

thanks to the strong Markov property of the lognormal stock price.

We define for any δ the Early Exercise Region $ER_\delta(t)$ and the Continuation Region $CR_\delta(t)$ at t as follows:

$$\begin{aligned} ER_\delta(t) &= \left\{x \in \mathfrak{R}^+ : F_\delta(t,x) = f(x)\right\} \\ CR_\delta(t) &= \left\{x \in \mathfrak{R}^+ : F_\delta(t,x) > f(x)\right\} \end{aligned}$$

The Early Exercise Region $ER_\delta(t)$ is empty at $t < T$ if early exercise is never optimal at t.

As in our discrete-time setting, the optimal exercise policy at t when $S(t) = x$ for the holder with liquidation penalty rate δ is

$$\tau^*_\delta(t,x) = \inf\left\{s \in [t,T] : F_\delta(s, S(s)) = f(S(s))\right\} \wedge T \tag{19}$$

The no-arbitrage value (fair price) of the option subject to liquidation penalties and optimally exercised by the holder according to (19) coincides with its replicating cost, that we can compute as

$$RC_\delta(t,x) = \mathbb{E}\left[e^{-r\left(\tau^*_\delta(t,x)-t\right)} f\left(S(\tau^*_\delta(t,x))\right)\right], \tag{20}$$

following equation (9) in our discrete-time setting.

In the remaining part of the paper we focus on American call options with payoff

$$f(x) = (x-K)^+.$$

As in the binomial case, convexity, monotonicity and value dominance imply that early exercise for the penalized call option (18) occurs if the underlying stock price at t is not below $s^*_\delta(t)$, which is the critical price at t defined as

$$s^*_\delta(t) = \inf\left\{x > 0 : F_\delta(t,x) = (x-K)^+\right\} \tag{21}$$

We provide closed-form solutions in the perpetual case and an asymptotic characterization of the critical price of the American option near maturity. We also show that, given (t, x), for $\delta \to +\infty$ then $F_\delta(t,x) = f(x)$, i.e. it is always optimal to exercise immediately the option. We show that the holder's cashflow-based value function (18) converges to the immediate payoff as $\delta \to +\infty$, or, equivalently, that the continuation region $CR_\delta(t)$ progressively tends to the empty set as $\delta \to +\infty$. We also prove that the replicating cost of the call option exercised at (19) converges to the immediate call payoff when the liquidation ban is full, that is for $\delta \to +\infty$.

We start rewriting the underlying stock price dynamics as

$$\begin{aligned} \frac{dS(t)}{S(t)} &= \left(\hat{r} - \hat{q}\right)dt + \sigma dW(t), \\ \text{where } \hat{r} &= r + \delta, \hat{q} = q + \delta \end{aligned} \tag{22}$$

Thus, the value function $F_\delta(t, x)$ defined in equation (18) is the value of a frictionless American option with interest rate $\hat{r} = r + \delta$ and dividend yield[3] $\hat{q} = q + \delta$. The cashflow-based value function for a perpetual American call option with liquidation penalty rate δ has the following closed-form expression:

Proposition 8 *The cashflow-based value function of the perpetual American call option with liquidation penalty rate* $\delta > 0$ *is*

$$\begin{aligned} F_{\infty,\delta}(x) &= \begin{cases} \left(x^*_{\infty,\delta} - K\right)\left(\frac{x}{x^*_{\infty,\delta}}\right)^\alpha & \text{for } x < x^*_{\infty,\delta} \text{ CR} \\ x - K & \text{for } x \geq x^*_{\infty,\delta} \text{ ER} \end{cases} \\ \text{with } \alpha &= \frac{-\left(r - q - \frac{\sigma^2}{2}\right) + \sqrt{\left(r - q - \frac{\sigma^2}{2}\right)^2 + 2(r+\delta)\sigma^2}}{\sigma^2} > 1 \\ \text{and } x^*_{\infty,\delta} &= K\frac{\alpha}{\alpha - 1} > K. \end{aligned} \tag{23}$$

When liquidation is impossible, i.e. for $\delta \to +\infty$, *then*

$$F_{\infty,\delta}(x) \underset{\delta \to +\infty}{\to} (x - K)^+$$
$$\text{and } x^*_{\infty,\delta} \underset{\delta \to +\infty}{\to} K$$

Proof See the Appendix. ☐

Since the perpetual option dominates the finite-maturity one, we can prove that, if liquidation is forbidden, then the cashflow-based value function for the finite-maturity call option coincides with its immediate payoff. More rigorously

Corollary 9 *Consider the American call option with liquidation penalty rate* δ *defined in equation* (18) *with* $f(x) = (x - K)^+$. *When* $\delta \to +\infty$

$$\lim_{\delta \to +\infty} F_\delta(t, x) = (x - K)^+ \text{ for all } t \in [0, T], x > 0$$

and the critical price defined in (21)

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[3] Incorporating the modeling of liquidation penalties on American options as an additional discount factor that lowers the continuation value is essential for effectively integrating liquidation costs in a tractable manner. Results of Al-Hadad and Palmowksi (2024) may allow to include stock-dependent liquidation costs in the perpetual case, as future research.

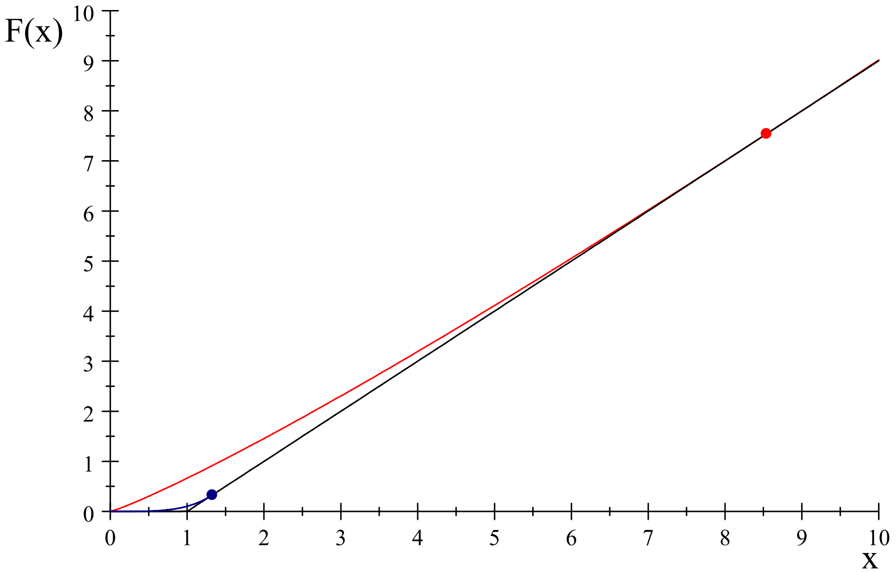

Fig. 2 $F_{\infty,\delta}(x)$ with $\delta = 0$ in red and $\delta = 0.5$ in blue. The dots denote the critical prices. Parameters' values: $r = 0.05$, $q = 0.01$, $\sigma = 0.25$, $K = 1$

$$s^*_\delta(t) = \inf\left\{S(t) \,:\, F_\delta(t, S(t)) = (S(t) - K)^+\right\} \downarrow K \text{ for all } t \in [0, T].$$

Proof See the Appendix. □

This corollary shows that liquidation penalties incentivize option holders to exercise an American call option early. If the liquidation ban is fully enforced, the cashflow-based value of the option for the holder is equal to the immediate payoff, making it optimal to exercise the option as soon as it is in the money. For Executive Stock Options (ESOs), the introduction of liquidation penalties thus contradicts the intended incentive.

As we already observed, the value of the American call option with liquidation penalty rate δ defined in equation (18) is the cashflow-based value for the option's holder. The cost of the replicating strategy of the optimally exercised American option with liquidation penalties is higher and will be computed for a perpetual American call option in Proposition 10.

The explicit expression for the perpetual call option (23) allows us to understand the impact of the liquidation penalty rate on the optimal exercise policy. In Fig. 2, for $r = 0.05$, $q = 0.01$, $\sigma = 0.25$, $K = 1$, we plot the value function of the perpetual call $F_{\infty,\delta}(x)$ with no liquidation penalties ($\delta = 0$ in red) and with liquidation penalty $\delta = 0.5$ (in blue). When the option is frictionless, in order to optimally exercise the holder has to wait until $S(t) \geq x^*_{\infty,\delta=0} = 8.5395$. On the contrary, if the holder has a liquidity penalty rate $\delta = 0.5$, she exercises optimally the option as soon as $S(t) \geq x^*_{\infty,\delta} = 1.3271$.

In Fig. 3, for the same parameter values as Fig. 2 ($r = 0.05$, $q = 0.01$, $\sigma = 0.25$, $K = 1$), we plot the perpetual call's critical price $x^*_{\infty,\delta}$ as a function of δ. We see that

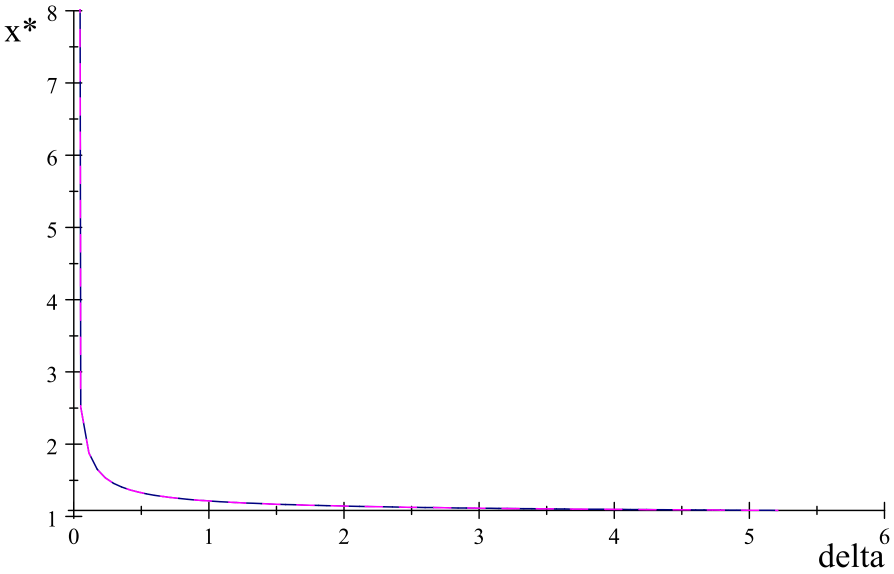

Fig. 3 The lognormal critical price $x^{\star}_{\infty,\delta}$ (blue) and the binomial critical price b^{*} (magenta, dashed with $\Delta t = 1/250$) as a function of δ: For large δ, $x^{\star}_{\infty,\delta} \to K = 1$. Parameters' values: $r = 0.05$, $q = 0.01$, $\sigma = 0.25$, $K = 1$. (colour figure online)

the critical price $x^{*}_{\infty,\delta}$ decreases very rapidly to the strike price $K = 1$ as δ increases. In the same figure, we also plot the critical price for the binomial perpetual case as given in Equation (15) in magenta. The graph for the binomial critical price with a fixed $\Delta t = 1/250$ is indistinguishable from the lognormal one. This overlap persists even for smaller and larger lengths of the time period of the binomial model Δt.

In the perpetual case, the cost of replicating the option optimally exercised in the presence of liquidation penalties as defined in (20) does not depend on time t, since the time to maturity is always infinite, and the optimal exercise policy becomes the hitting time of the constant perpetual critical price $x^{*}_{\infty,\delta}$ defined in (23) :

$$\tau^{*}_{\infty,\delta}(t,x) = \tau^{*}_{\infty,\delta}(x) = \inf\left\{ s \geq 0 \,:\, S(s) = x^{*}_{\infty,\delta} \right\} \text{ given } S(0) = x.$$

Thus the cost of the perpetual American call option with liquidation penalty rate δ optimally exercised at $\tau^{*}_{\infty,\delta}(x)$ is

$$\begin{aligned} RC_{\infty,\delta}(x) &= \mathbb{E}\left[e^{-r\tau^{*}_{\infty,\delta}(x)} f\Big(S(\tau^{*}_{\infty,\delta}(x)) \Big) \right] \text{ where } x = S(0) \\ &= \mathbb{E}\left[e^{-r\tau^{*}_{\infty,\delta}(x)} f\Big(x^{*}_{\infty,\delta} \Big) \right] \end{aligned} \tag{24}$$

We find the closed form solution for $RC_{\infty,\delta}(x)$ in the following

Proposition 10 *The replication cost of the perpetual American call option with liquidation penalties exercised at the optimal perpetual exercise time defined above is*

$$RC_{\infty,\delta}(x) = \mathbb{E}\left[e^{-r\tau^*_{\infty,\delta}(x)} f\left(S(\tau^*_{\infty,\delta}(x))\right)\right] = \mathbb{E}\left[e^{-r\tau^*_{\infty,\delta}(x)}\right]\left(x^*_{\infty,\delta} - K\right)$$
$$= \left(\frac{x}{x^*_{\infty,\delta}}\right)^{\beta}\left(x^*_{\infty,\delta} - K\right) \text{ for } x < x^*_{\infty,\delta},$$

where $x^*_{\infty,\delta}$ *is defined in* (23) *and*

$$\beta = \frac{-\left(r - q - \frac{\sigma^2}{2}\right) + \sqrt{\left(r - q - \frac{\sigma^2}{2}\right)^2 + 2r\sigma^2}}{\sigma^2}$$

Moreover,

$$\lim_{\delta\to+\infty} RC_{\infty,\delta}(x) = \lim_{y\to K}\left(\frac{x}{y}\right)^{\beta}(y - K) = 0.$$

Proof See the Appendix. ☐

In general, for non-negative payoffs *f*, the discrete-time chain of inequalities (10) translates for the different value functions in the Black-Scholes setting as follows:

$$F_{\infty,\delta}(x) = \mathbb{E}\left[e^{-(r+\delta)\tau^*_{\infty,\delta}(x)} f\left(S(\tau^*_{\infty,\delta}(x))\right)\right] < \mathbb{E}\left[e^{-r\tau^*_{\infty,\delta}(x)} f\left(S(\tau^*_{\infty,\delta}(x))\right)\right]$$
$$= RC_{\infty,\delta}(x) < F_{\infty,\delta=0}(x),$$

since $\delta > 0$.

For a perpetual call option, Fig. 4 shows that the value $F_{\infty,\delta}$ (blue) dominates the immediate payoff (black) and is dominated by $RC_{\infty,\delta}$ (green). The red dot describes the point $\bar{x}$, defined in the Appendix in Equation (26), where the distance between $RC_{\infty,\delta}$ and $F_{\infty,\delta}$ is maximized. The point $\bar{x}$ is pivotal in determining the rate of convergence of $RC_{\infty,\delta}$ to $F_{\infty,\delta}$ as δ diverges to $+\infty$ in Proposition 11.

For an ATM American call option we saw that, in the binomial setting, the fraction Λ_δ of the American option premium that is lost due to the liquidation penalties is increasing with respect to δ (Table 1). In the continuous-time setting, we can use Proposition 10 to compute the fraction of the American call option premium that is lost to the liquidation penalties as a function of $(x;\delta)$:

$$\Lambda_\infty(x;\delta) = \frac{F_{\infty,\delta=0}(x) - RC_{\infty,\delta}(x)}{F_{\infty,\delta=0}(x)} \geq 0.$$

Figure 5 shows that $\Lambda_\infty(x;\delta)$ increases as $\delta \to +\infty$.

In the next Proposition, we show that the replication cost $RC_{\infty,\delta}(x)$ converges to the cashflow-based value function $F_{\infty,\delta}(x)$ for $x < x^*_{\infty,\delta}$.

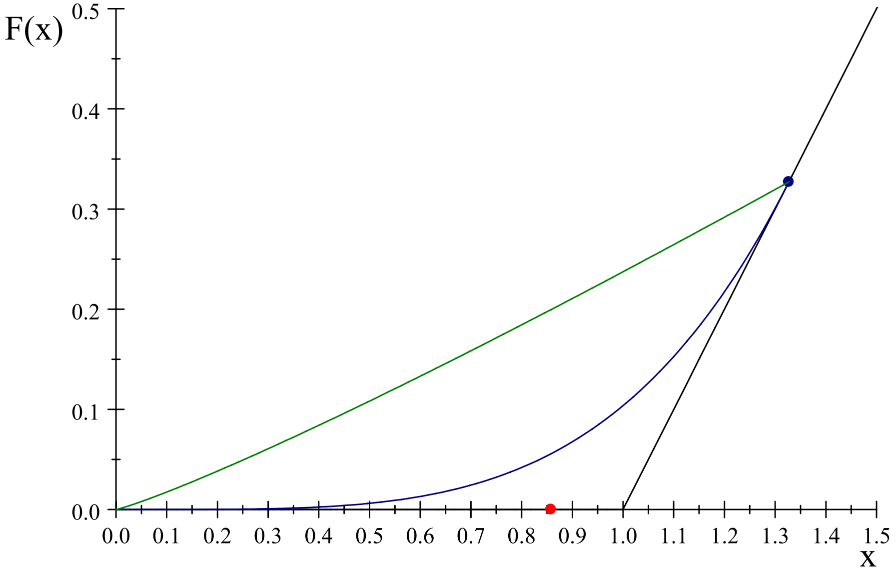

Fig. 4 Replicating cost $RC_{\infty,\delta}(x)$ in green and the cashflow -based value of the option $F_{\infty,\delta}$(x) in blue. Parameters' values: δ= 0.5, $r = 0.05$, $q = 0.01$, $\sigma = 0.25$, $K = 1$

Proposition 11 *For the perpetual American call option with liquidation penalty rate* δ*, for* $x < x^*_{\infty,\delta}$,

$$\left|RC_{\infty,\delta}(x) - F_{\infty,\delta}(x)\right| \leq K\left(\frac{1}{\alpha - 1}\right)\left(\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\beta-\alpha}} - \left(\frac{\alpha}{\beta}\right)^{\frac{\alpha}{\beta-\alpha}}\right)$$

$$\sim K\left(\frac{1}{\sqrt{\frac{2}{\sigma^2}\delta}}\right) \text{ as } \delta \to +\infty$$

and

$$\Lambda_\infty(x;\delta) = \frac{F_{\infty,\delta=0}(x) - RC_{\infty,\delta}(x)}{F_{\infty,\delta=0}(x)} \to \frac{F_{\infty,\delta=0}(x) - (x-K)^+}{F_{\infty,\delta=0}(x)} \text{ as } \delta \to +\infty$$

Proof See the Appendix. □

Thanks to the proposition, in the out-of-the-money region $x < K$, we have that the fraction of the American call option premium that is lost to the liquidation penalties as a function of $(x;\delta)$

$$\Lambda_\infty(x;\delta) = \frac{F_{\infty,\delta=0}(x) - RC_{\infty,\delta}(x)}{F_{\infty,\delta=0}(x)} \underset{\delta\to+\infty}{\to} \frac{F_{\infty,\delta=0}(x) - 0}{F_{\infty,\delta=0}(x)} = 1.$$

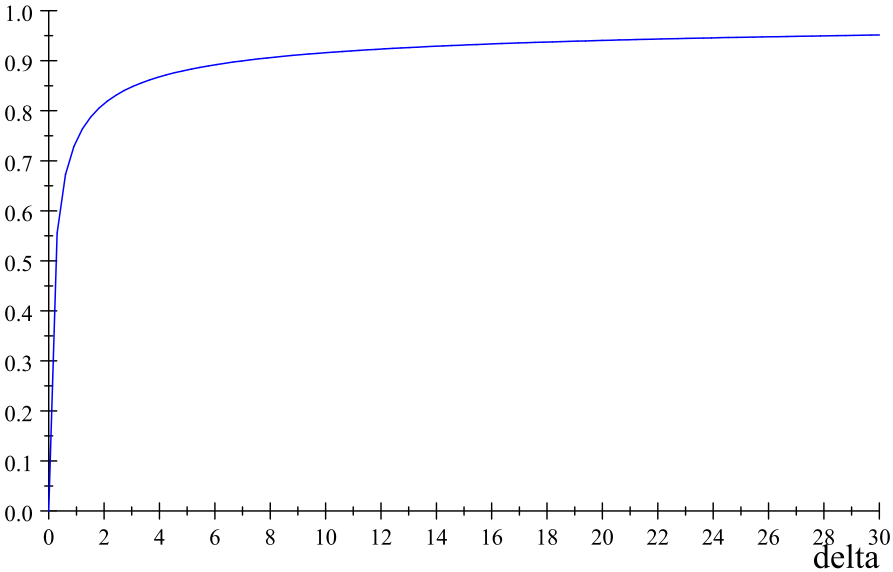

Fig. 5 fraction Λ_∞ (x; δ) of the premium lost to liquidation penalties in the OTM region for $x = 0.5$ as a function of δ Parameter values: $r = 0.05$, $q = 0.01$, $\sigma = 0.25$, $K = 1$

The result is consistent with Fig. 5, that shows that $\Lambda_\infty(x;\delta) \to 100\%$ as $\delta \to +\infty$.

Figure 6 plots the critical prices for a perpetual call option subject to various liquidation penalty rates. The lines describe the critical price in the binomial model described in the previous section as a function of the monitoring frequency Δt. We observe that when Δt is zero, the value of the binomial critical price coincides with the lognormal one, denoted by a dot. Moreover, as δ increases to infinity, i.e., when liquidation becomes increasingly penalized, the critical price converges to the strike price, confirming the result that was already highlighted in Fig. 3 for the lognormal model.

In the finite-maturity case, there are no closed form solutions for either the value function (18) nor for its replication cost, as the critical price is not known in closed form. By using Equation (22) for the dynamics of S, we are able to obtain a good approximation of $s^*_\delta(t)$ in (21) near maturity, that is for $t \to T$, in the following:

Proposition 12 *For $t \to T$, the critical price for the penalized American call option is*

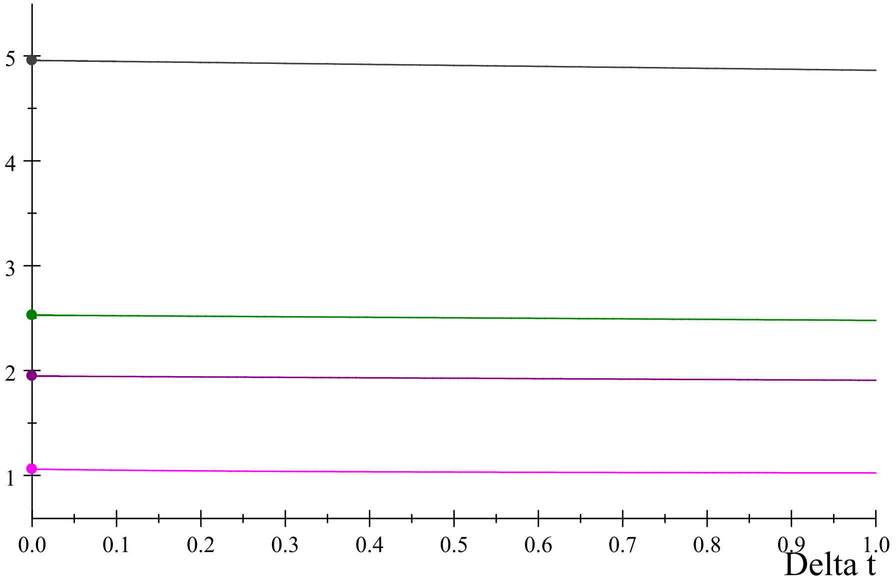

Fig. 6 Perpetual call option critical price with $q = 0.01$, $\sigma = 0.25$, $r = 0.05$, $K = 1, T = 1$. The binomial critical price as a function of Δt (solid line) and the lognormal critical price (dot) , for different values of δ ($\delta = 0.01$ gray, $\delta = 0.05$ green, $\delta = 0.1$ purple, $\delta = 10$ magenta). (colour figure online)

$$s_\delta^*(t) = \frac{K}{1 - \sigma\sqrt{(T-t)\ln\frac{1}{T-t}}} \text{ for } q > r$$

$$s_\delta^*(t) = \frac{K}{1 - \sqrt{2}\sigma\sqrt{(T-t)\ln\frac{1}{T-t}}} \text{ for } r = q$$

$$s_\delta^*(t) = \frac{K}{\frac{\hat{q}}{\hat{r}}\left(1 - 0.638\sigma\sqrt{(T-t)}\right)} \text{ for } q < r$$

Proof See the Appendix. □

With liquidation costs early exercise is profitable even if $q = 0$. In Fig. 7 we plot the analytical approximations for the free-boundary of the call option with $\sigma = 0.25$, $r = 0.05, q = 0, K = 1, T = 1$, for different values of δ. Since $q = 0$ in the frictionless case the option is optimally exercised at maturity only. Introducing the liquidation penalty rate $\delta = 0.05$ we see that immediate exercise opportunities appear whenever the stock price is above the green curve. If the penalty cost δ increases, the critical price lowers, triggering optimal early exercise at lower levels of the underlying price. For $\delta \geq 170$ all the critical prices overlap and are very close to $K = 1$ over the entire life of the option. This means that the option is optimally exercised as soon as it is in the money.

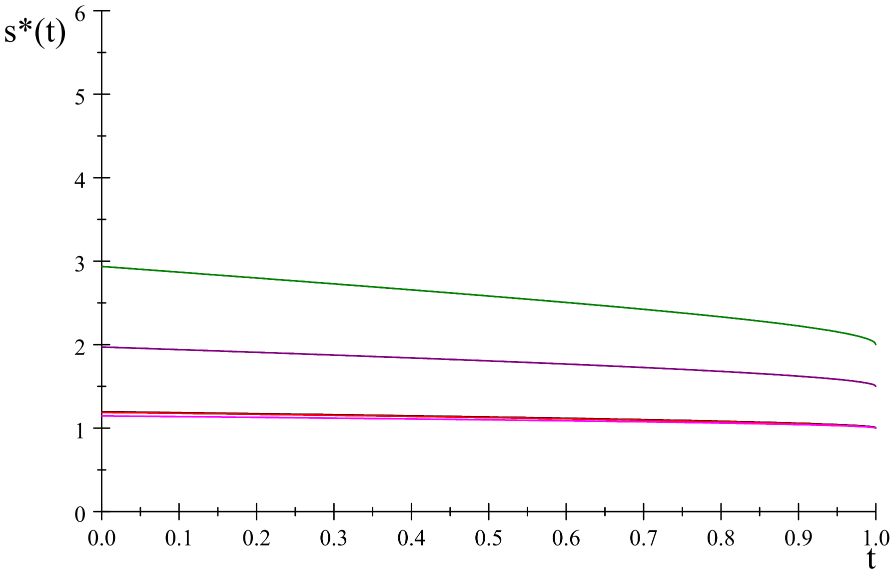

Fig. 7 The (analytical approximation for the) critical price for a call option $q = 0$, $\sigma = 0.25$, $r = 0.05$, $K = 1, T = 1$, for different values of δ ($\delta = 0.05$ green, $\delta = 0.1$ purple, $\delta = 10$ black, $\delta = 170$ magenta, $\delta = 500$ blue, $\delta = 1000$ red). (colour figure online)

In the following two propositions, we study the finite maturity early exercise premium (EEP) and the replication cost of the optimally exercised option subject to liquidation penalties when delta approaches infinity, i.e., when liquidation becomes impossible.

Proposition 13 (Cashflow-based early exercise premium with full liquidation ban) *Let* $C^E_\delta(t) = \mathbb{E}_t\left[e^{-(r+\delta)(T-t)}(S(T) - K)^+\right]$ *denote the price of the European call option at t. In the lognormal framework*

$$C^E_\delta(t) = S(t)e^{-(q+\delta)(T-t)}N\left(d_1\right) - Ke^{-(r+\delta)(T-t)}N\left(d_2\right),$$

where N(z) is the distribution function of a standard normal random variable, i.e.

$$N(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz,$$

while

$$d_{1,2} = \frac{1}{\sigma\sqrt{(T-t)}}\left(\ln\left(\frac{S(t)}{K}\right) + \left(r - q \pm \frac{1}{2}\sigma^2\right)(T-t)\right).$$

Denote with $C_\delta(t) = F_\delta(t, S(t))$ *defined in* (18) *the cashflow-based value function of the penalilzed American call option. The Early Exercise premium at t for the American option subject to liquidation penalties is*

$$EEP_\delta(t) = C_\delta(t) - C^E_\delta(t) \geq 0 \text{ for all } t \in (0, T)$$

and

$$\lim_{\delta \to +\infty} EEP_\delta(t) = (S(t) - K)^+$$

Proof See the Appendix. ☐

This result shows that when liquidation is fully prohibited, the premium for the cash flow-based optimal early exercise coincides with the immediate payoff of the option, i.e., its intrinsic value. This means that the extra premium compared to the European option is solely the possibility of exercising immediately rather than waiting until maturity.

Proposition 14 (Finite-maturity replication cost with full liqudation ban) *The replication cost* (20) *of the finite-maturity American call option with liquidation penalty rate* δ *exercised at* (19) *is*

$$RC_\delta(t, x) = \mathbb{E}\left[e^{-r\left(\tau^*_\delta(t,x)-t\right)}\left(S(\tau^*_\delta(t, x)) - K\right)^+\right] \to (x - K)^+$$

as $\delta \to +\infty$, *for all* $t \leq T$, $x > 0$.

Proof See the Appendix. ☐

The proposition above shows that, when liquidation is fully prohibited, the replication cost of the option optimally exercised according to (19) converges to its immediate payoff. Because the replication cost can be identified with the no-arbitrage price of the derivative, this proposition justifies the practice of computing the fair value of non-transferable American options at their intrinsic value, i.e., their immediate payoff. This is the case of Employee Stock Options (ESOs), which are traditionally recorded on firm balance sheets at their initial intrinsic value.

5 Sensitivity analysis with market data

In this section we develop a sensitivity analysis based on real-world cases. Data are taken from Figlewski (2022), that conducts a detailed analysis to understand the liquidity-related value of American options, focusing on the bid-ask spread as a measure of the liquidation penalty in the options market. The study utilizes real-world data from options trading on 24 major stocks, selected based on their options trading volumes during the period of 2016-2017, to provide insights into how market liquidity affects option pricing and trading strategies. The dataset includes options on prominent stocks such as Apple Inc., Bank of America Corp., and Facebook Inc., among others, categorized into three groups based on their options trading activity: most active, medium active, and less active. This categorization allows for a nuanced examination of liquidity dynamics across different segments of the options market. The data, sourced from the Options Clearing Corporation and OptionMetrics, focuses on regular third-Friday-of-the-month expirations, excluding deep in-the-money or out-of-the-money options, as well as those with low open interest or excessively wide spreads, ensuring the analysis reflects typical market conditions. The bid-ask spread is a critical measure of market liquidity: a wider spread signifies a higher cost of illiquidity, as investors must accept a larger discount to liquidate their option positions prematurely.

We calculate the implied real-world values for δ, which represent the option liquidation penalty rate. These δ values are derived using the median bid-ask spreads and the option pricing setup reported in Table 2 (caption and spread characteristics in Panel A) of Figlewski (2022) together with the Black-Scholes European call option price $c(S, K, T, r, q, \sigma)$, which corresponds to $C^E_{\delta}(0)$ with $\delta = 0$ in Proposition 13. The assumptions for this calculation are all taken from Figlewski (2022) and include at-the-money call options with both the stock price S and the strike price K set to \$100, a time to maturity T of 45 days, and a risk-free rate r of 1%, which is based on the average 3-month LIBOR rate during 2016-2017. The volatility σ used in the calculation is derived from the median implied volatility for each stock option category. The additional assumption of a non-dividend-paying underlying stock ($q = 0$) is used to establish a theoretical common baseline for the valuation of European and American call options to isolate the liquidation penalty rate in real-world markets. The cost of illiquidity is quantified as half of the median bid-ask spread, so that the option liquidation price (sell price) is

$$c(S, K, T, r, q, \sigma) - \frac{\text{median bid-ask spread}}{2}.$$

The implied liquidation penalty rate δ is calculated as the rate that equates the option liquidation price with the Black-Scholes European call price discounted by $\lambda = e^{-\delta \Delta t}$ as in Definition 2 with $\Delta t = 1/250$:

$$c(S, K, T, r, q, \sigma) - \frac{\text{median bid-ask spread}}{2} = c(S, K, T, r, q, \sigma) e^{-\delta\left(\frac{1}{250}\right)}.$$

The resulting δ values provide a clear measure of the liquidation penalty rate across different stock option categories, highlighting the varying costs of illiquidity in the options market. The table below presents these δ values, reflecting the option liquidation penalty rate for each category.

In Table 4 we study the impact of liquidation penalties on the optimal exercise policy (7) of American options across the four stock option categories of Table 3.

The presence of liquidation penalties affects investor behavior across all four categories presented in Table 3. Both the cashflow-based value function $V_{\delta}(0)$ and the fraction of the premium Λ_{δ} illustrate the impact of such liquidation penalties on investors' optimal behavior. Indeed, the cashflow-based value function $V_{\delta}(0)$

Table 3 Implied δ values across different stock option categories

Category	Median Bid-Ask Spread	Median Implied Volatility	Delta (δ)
All firms	0.141	0.223	4.608
Most active	0.062	0.193	2.322
Medium active	0.112	0.253	3.227
Less active	0.508	0.352	10.746

Table 4 $V(0), V_\delta(0), RC_\delta(0)$ and Λ_δ for the Stock Option Categories in Table 3 with $K = S(0) = 100,\ T = 0.18, r = 0.01, q = 0, \Delta t = 1/250$

	Most active	Medium active	Less active	All firms
$V(0)$	3.371658	4.390581	6.036395	3.881185
$RC_\delta(0)$	3.349887	4.35129	5.868741	3.836686
$V_\delta(0)$	3.340487	4.334271	5.782424	3.810302
Λ_δ	0.006457	0.008949	0.027774	0.011465

is consistently lower than the frictionless value of the American option $V(0)$. For the all-firms category, the dollar difference between the frictionless $V(0)$ and the cashflow-based value function $V_\delta(0)$ is 0.07 . An option holder who exercises optimally according to the optimal time defined in Equation (7) loses a fraction Λ_δ of the premium that is evident across all categories, as it can be seen in Table 4. For the most active options, Λ_δ remains mild, ranging around 0.64%. It gradually increases to 0.89% for medium-active options and reaches 2.7% for less-active options, with an average penalty of 1.1% for the all-firms option category.

The replication cost $RC_\delta(0)$ can be thought of as the no-arbitrage price of American options subject to liquidation penalties, representing the replication cost of the option exercised at the optimal stopping time defined in Equation (7) . $RC_\delta(0)$ is computed using the backward recursion presented in Section 3.1. We observe that $RC_\delta(0)$ is consistently lower than the frictionless value $V(0)$. The differences range from 0.02 dollars in the most-active category to 0.167 dollars in the least-active category. In the all-firm case, the difference is 0.044 dollars. We can therefore claim that the impact on the option prices $RC_\delta(0)$ is systematically present in all four categories of Table 3. We explore how the impact tends to increase with the option's life by recalculating our option valuation metrics in Table 5 for a longer time to maturity (1 year).

Table 5 shows how liquidation penalties change as the option matures over a year. The percentage premium loss Λ_δ escalates from 1.1% to 1.6% across all firms. Less active options experience a conspicuous penalty increase from 2.7% to 3.67%, which underscores the critical role of market liquidity in option pricing. The consistent depression of valuation metrics relative to frictionless scenarios confirms the systematic impact of these market imperfections. By quantifying how option liquidity and duration interact, the results in Tables 4 and 5 reveal how real-world frictions can influence option exercise policies and valuations.

Table 5 $V(0), V_\delta(0), RC_\delta(0)$ and Λ_δ for the Stock Option Categories in Table 3 with $K = S(0) = 100,\ T = 1, r = 0.01, q = 0, \Delta t = 1/250$

	Most active	Medium active	Less active	All firms
$V(0)$	8.149492	10.511575	14.312195	9.331358
$RC_\delta(0)$	8.081311	10.392375	13.786252	9.181433
$V_\delta(0)$	8.074150	10.376763	13.710033	9.160938
Λ_δ	0.008366	0.0113398	0.0367478	0.016066

6 Conclusions

This paper incorporates liquidation costs into the valuation of American options. The introduction of liquidation penalties alters the comparison between immediate payoff and continuation value for holders of American options. Without such penalties, the continuation value is the actual funds available to the holder by selling the option after a potential exercise date. However, lower liquidation prices prompt a comparison between immediate exercise payoff and lower funds obtainable at liquidation, encouraging early exercise. With liquidation penalties, cashflow-based early exercise of American call options can be optimal even when the underlying asset provides no payout. Due to our efficient modeling approach, we first obtain a discrete-time characterization of the cashflow-based problem and we then achieve closed-form asymptotic solutions near maturity for the critical price triggering optimal early exercise in the continuous-time lognormal market. We also work out explicit pricing formulas for perpetual American options with liquidation penalties. When the liquidation ban is fully in place, we show that the cashflow-based value function for the holder and the option replication cost both collapse to the immediate payoff for finite-maturity and perpetual call options. Our findings reveal that executive stock options, when structured as American call options with liquidation penalties, actually encourage executives to exercise the options earlier than intended, prompting them to acquire the underlying stock sooner. Contrary to their intended purpose of aligning long-term interests, option liquidation penalties can create an incentive that recent empirical research has highlighted as problematic for motivating executives.

Appendix

Proofs of Results

Proof of Proposition 6 *To compute* $\mathbb{E}\left[e^{-(r+\delta)\tau_b}\right]$ *in* (14) *we rewrite the hitting time in terms of the standardized log-price*

$$Y(n) = \frac{1}{\sigma\sqrt{\Delta t}} \ln \frac{S(n\Delta t)}{S(0)}$$

that is a random walk, as $Y(0) = 0$, *and for* $n > 1$

$$Y(n) - Y(n-1) = \begin{cases} +1 & \text{with } \mathbf{q} = \frac{e^{(r-q)\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} \\ -1 & 1 - \mathbf{q} \end{cases}$$

Since

$$S(t) = S(0)e^{\sigma\sqrt{\Delta t}Y(n)} \text{ with } n = \frac{t}{\Delta t}$$

for any $t = n\,\Delta t, n \in \mathbb{N}$, *the hitting time can be rewritten as*

$$\begin{aligned} \tau_b &= \inf\{t > 0 : S(t) = b\} \\ &= \inf\left\{ t > 0 : S(0)e^{\sigma\sqrt{\Delta t}Y\left(\frac{t}{\Delta t}\right)} = b \right\} \\ &= \Delta t \inf\left\{n > 0 : Y(n) = b_Y\right\} \text{ with } b_Y = \frac{1}{\sigma\sqrt{\Delta t}} \ln \frac{b}{S(0)} \in \mathbb{N} \\ &= \Delta t \cdot \tau_{b_Y} \text{ where } \tau_{b_Y} = \inf\left\{n > 0 : Y(n) = b_Y\right\} \end{aligned}$$

We compute $\mathbb{E}\left[e^{-(r+\delta)\tau_b}\right]$ *using the probability generating function of the hitting time* τ_{b_Y} *of the random walk* Y *to the barrier* b_Y (*see Zitkovic* 2024):

$$\mathbb{E}\left[a^{\tau_{b_Y}}\right] = G_{b_Y}(a) = \left(\frac{1 - \sqrt{1 - 4\mathbf{q}(1-\mathbf{q})a^2}}{2(1-\mathbf{q})a} \right)^{b_Y} \tag{25}$$

for $a = e^{-(r+\delta)\Delta t}$. *In Lemma* 15 *we show that the function* G_{b_Y} *is well defined on* $a = e^{-(r+\delta)\Delta t}$, *as the argument of the square root is positive and the argument of the power function is positive as well, i.e.* $\frac{1-\sqrt{1-4\mathbf{q}(1-\mathbf{q})a^2}}{2(1-\mathbf{q})a} > 0$.

This implies that the expected value we want to compute can be expressed as

$$\mathbb{E}\left[e^{-(r+\delta)\tau_b}\right] = \mathbb{E}\left[e^{-(r+\delta)\Delta t \cdot \tau_{b_Y}}\right] = G_{b_Y}(a).$$

Defining

$$C = \ln\left(\frac{1 - \sqrt{1 - 4\mathbf{q}(1-\mathbf{q})a^2}}{2(1-\mathbf{q})a} \right)$$

we rewrite

$$\begin{aligned}\mathbb{E}\left[e^{-(r+\delta)\tau_b}\right] &= \left(e^C\right)^{b_Y}\\ &= \left(e^C\right)^{\frac{1}{\sigma\sqrt{\Delta t}}\ln\frac{b}{S(0)}}\\ &=e^{\ln\frac{b}{S(0)}\frac{C}{\sigma\sqrt{\Delta t}}} = \left(\frac{b}{S(0)}\right)^{\frac{C}{\sigma\sqrt{\Delta t}}}\end{aligned}$$

since $b_Y = \frac{1}{\sigma\sqrt{\Delta t}}\ln\frac{b}{S(0)}$. *The perpetual barrier option initial value is*

$$\begin{aligned}\mathbb{E}\left[e^{-(r+\delta)\tau_b}\left(S\left(\tau_b\right)-K\right)^+\right] &= \mathbb{E}\left[e^{-(r+\delta)\tau_b}\right](b-K)\\ &= \left(\frac{b}{S(0)}\right)^{\frac{C}{\sigma\sqrt{\Delta t}}}(b-K)\end{aligned}$$

to be maximized over $b > S(0)$. *In terms of the random walk* Y, *we have to determine* $n^* \in \mathbb{N}$ *such that* $b^* = S(0)u^{n^*}$ *maximizes*

$$\left(\frac{b}{S(0)}\right)^{\frac{C}{\sigma\sqrt{\Delta t}}}(b-K) = \left(\frac{S(0)u^n}{S(0)}\right)^{\frac{C}{\sigma\sqrt{\Delta t}}}(S(0)u^n - K).$$

Let

$$\begin{aligned}\varphi(n) &= (u^n)^{\frac{C}{\sigma\sqrt{\Delta t}}}(S(0)u^n - K)\\ &= e^{Cn}\left(S(0)e^{\sigma\sqrt{\Delta t}n} - K\right) \text{ as } u = e^{\sigma\sqrt{\Delta t}}\end{aligned}$$

leading to

$$\varphi(n) = S(0)e^{\left(C+\sigma\sqrt{\Delta t}\right)n} - Ke^{Cn}$$

To maximize $\varphi(n)$, *we extend the function on* $\mathfrak{R}$ *and compute for* $n \geq 0$

$$\varphi'(n) = S(0)e^{\left(C+\sigma\sqrt{\Delta t}\right)n}\left(C+\sigma\sqrt{\Delta t}\right) - CKe^{Cn}.$$

Setting $\varphi'(n) \geq 0$ *we get*

$$\begin{aligned}S(0)e^{\left(C+\sigma\sqrt{\Delta t}\right)n}\left(C+\sigma\sqrt{\Delta t}\right) - CKe^{Cn} &\geq 0\\ S(0)e^{\sigma\sqrt{\Delta t}n}\left(C+\sigma\sqrt{\Delta t}\right) - CK &\geq 0\end{aligned}$$

$$e^{\sigma\sqrt{\Delta t}n} \leq \frac{1}{S(0)}\frac{CK}{C+\sigma\sqrt{\Delta t}}$$

since $C + \sigma\sqrt{\Delta t} \leq 0$, *as shown in Lemma* 15. *Thus the the function* $\varphi(n)$ *is increasing for* $n = 0, \dots, n^* - 1$ *and decreasing for* $n \geq n^*$ *where*

$$n^* = \lceil Y^* \rceil$$

where

$$Y^* = \frac{1}{\sigma\sqrt{\Delta t}} \ln \frac{1}{S(0)} \frac{CK}{C + \sigma\sqrt{\Delta t}}$$

and $\lceil Y^* \rceil$ *is the smallest integer above* Y^*.

This suggests

$$b^* = \frac{K\frac{C}{\sigma\sqrt{\Delta t}}}{\frac{C}{\sigma\sqrt{\Delta t}} + 1} = \frac{KC}{C + \sigma\sqrt{\Delta t}} > K$$

as $C < 0$.■

Proof of Proposition 7 *The cashflow-based optimal strategy for the holder is to exercise as soon as the underlying equals* b^*. *The no-arbitrage price of the barrier call option exercised at* b^* *is*

$$\begin{aligned} RC_{\infty,\delta}(x) &= \mathbb{E}\left[e^{-r\tau_{b^*}}(b^* - K)\right] \\ &= (b^* - K)\mathbb{E}\left[e^{-r\tau_{b^*}}\right]. \end{aligned}$$

As in the previous proposition, we use the probability generating function of the hitting time τ_{b_Y} *of the random walk* Y *to the barrier* b_Y (*see Zitkovic* 2024) *to compute*

$$\mathbb{E}\left[e^{-r\tau_{b^*}}\right] = G_{b_Y}\left(e^{-r\Delta t}\right),$$

where G_{b_Y} *is defined in* (25) . *The definition of* C_{RC} *in* (16) *is well-posed, as the arguments of the square root and the logarithm are strictly positive* (*see Lemma* 15 *in the next subsection*). *We can rewrite*

$$\begin{aligned} \mathbb{E}\left[e^{-r\tau_b}\right] &= \left(e^{C_{RC}}\right)^{b_Y} \\ &= \left(e^{C_{RC}}\right)^{\frac{1}{\sigma\sqrt{\Delta t}} \ln \frac{b}{S(0)}} \\ &= e^{\ln \frac{b}{S(0)} \frac{C}{\sigma\sqrt{\Delta t}}} = \left(\frac{b}{S(0)}\right)^{\frac{C_{RC}}{\sigma\sqrt{\Delta t}}} \end{aligned}$$

Thus

$$\begin{aligned} RC_{\infty,\delta}(x) &= (b^* - K)\mathbb{E}\left[e^{-r\tau_{b^*}}\right] \\ &= (b^* - K)\left(\frac{b^*}{S(0)}\right)^{\frac{C_{RC}}{\sigma\sqrt{\Delta t}}} \end{aligned}$$

To show that $C_{RC} > C$, *we observe that*

$$f(\delta) = \frac{1 - \sqrt{1 - 4\mathbf{q}(1-\mathbf{q})e^{-2(r+\delta)\Delta t}}}{2(1-\mathbf{q})e^{-(r+\delta)\Delta t}}$$

is decreasing for $\delta \geq 0$ (*see Lemma* 16 *in the next subsection*). *This implies that* $C_{RC} = f(0) > f(\delta) = C$, *and then* $RC_{\infty,\delta}(x) > V_{\infty,\delta}(x)$, *as* $x < b^*$ *yields*

$$\frac{RC_{\infty,\delta}(x)}{V_{\infty,\delta}(x)} = \left(\frac{b^*}{x}\right)^{\frac{C_{RC}-C}{\sigma\sqrt{\Delta t}}} > 1$$

■

Proof of Proposition 8 *In the perpetual case, the value function defined in* (18) *depends only on* x, *as the time to maturity is constantly infinite. For* $f(x) = (x-K)^+$, *the perpetual value function in* (18) *is the price of a perpetual option on a lognormal stock with interest rate* $\widehat{r} = r + \delta$, *and dividend yield* $\widehat{q} = \delta + q$. *Thus we get*

$$F_{\infty,\delta}(x) = \begin{cases} Ax^{\alpha} & \text{for } x < x^*_{\infty,\delta} \\ x - K & \text{for } x \geq x^*_{\infty,\delta} \end{cases}$$

where $\alpha = \frac{-\left(r-q-\frac{\sigma^2}{2}\right)+\sqrt{\left(r-q-\frac{\sigma^2}{2}\right)^2+2(r+\delta)\sigma^2}}{\sigma^2} > 0$. *The exponent* α *is also larger than* 1, *as* $\alpha > 1$ *is equivalent to*

$$\sqrt{\left(r-q-\frac{\sigma^2}{2}\right)^2 + 2(r+\delta)\sigma^2} \overset{?}{>} \sigma^2 + \left(r-q-\frac{\sigma^2}{2}\right)$$

$$\left(r-q-\frac{\sigma^2}{2}\right)^2 + 2(r+\delta)\sigma^2 \overset{?}{>} \sigma^4 + \left(r-q-\frac{\sigma^2}{2}\right)^2 + 2\sigma^2\left(r-q-\frac{\sigma^2}{2}\right)$$

$$2(r+\delta) \overset{?}{>} \sigma^2 + 2\left(r-q-\frac{\sigma^2}{2}\right)$$

$$2\delta \overset{?}{>} 2(-q),$$

that is obviously satisfied, as $\delta > 0$ *and* $q \geq 0$.

The other constant quantities $x^*_{\infty,\delta}$ *and* A *are retrieved via value matching and smooth-pasting.*

When $\delta \to +\infty$, *we have that*

$$\alpha = \frac{-\left(r-q-\frac{\sigma^2}{2}\right)+\sqrt{\left(r-q-\frac{\sigma^2}{2}\right)^2+2(r+\delta)\sigma^2}}{\sigma^2} \underset{\delta\to+\infty}{\to} +\infty$$

and thus

$$x^*_{\infty,\delta} = K\frac{\alpha}{\alpha-1} \underset{\delta\to+\infty}{\to} K.$$

This implies that the CR and the ER for $\delta \to +\infty$ *converge to*

$$CR = [0, x^*_{\infty,\delta}[\underset{\delta\to+\infty}{\to} [0, K[, \text{ and } ER = [x^*_{\infty,\delta}, +\infty[\underset{\delta\to+\infty}{\to} [K, +\infty[$$

Since for $0 \le x < x^*_{\infty,\delta}$

$$0 \le \left(\frac{x}{x^*_{\infty,\delta}}\right)^{\alpha} < 1 \Rightarrow 0 \le \lim_{\delta\to+\infty}\left(\frac{x}{x^*_{\infty,\delta}}\right)^{\alpha} \le 1$$

When $\delta \to +\infty$, *for* $0 \le x < x^*_{\infty,\delta} \underset{\delta\to+\infty}{\to} K$

$$F_{\infty,\delta}(x) = \left(x^*_{\infty,\delta} - K\right)\left(\frac{x}{x^*_{\infty,\delta}}\right)^{\alpha} \underset{\delta\to+\infty}{\to} 0$$

and

$$\lim_{\delta\to+\infty} F_{\infty,\delta}(x) = \begin{cases} 0 & \text{for } x < K \\ x - K & \text{for } x \ge K \end{cases} = (x-K)^+$$

■

Proof of Corollary 9 *Since*

$$(x-K)^+ \le F_\delta(t,x) \le F_{\infty,\delta}(x) \text{ for all } (t,x)$$

Then, for all (t, x)

$$(x-K)^+ \le \lim_{\delta\to+\infty} F_\delta(t,x) \le \lim_{\delta\to+\infty} F_{\infty,\delta}(x) = (x-K)^+$$

Implying that $\lim_{\delta\to+\infty} F_\delta(t,x) = (x-K)^+$. *As for the critical price, we first observe that* $\delta_1 < \delta_2$ *implies* $F_{\delta_1}(t,x) \ge F_{\delta_2}(t,x)$, *and consequently* $s^*_{\delta_1}(t) \ge s^*_{\delta_2}(t)$. *Since for all* $t \in [0,T]$

$$K \le s^*_\delta(t) = \inf\left\{S(t) \,:\, F_\delta(t,S(t)) = (S(t)-K)^+\right\} \le x^*_{\infty,\delta}$$

and $x^*_{\infty,\delta} = \underset{\delta\to+\infty}{\to} K$, *we obtain that* $\lim_{\delta\to+\infty} s^*_\delta(t) \downarrow K$. ■

Proof of Proposition 10 *The replicating cost of the perpetual American call option with liquidation penalties optimally exercised at* $\tau^*_{\infty,\delta}$ *is the no-arbitrage price of an American perpetual barrier call option that is activated as soon as the underlying price reaches the upper barrier*

$$U = x^*_{\infty,\delta} = K\frac{\alpha}{\alpha-1} > x,$$

with $x^*_{\infty,\delta}$ *as in* (23). *Its replicating cost coincides with its no-arbitrage price given by*

$$\left(\frac{x}{U}\right)^{\beta}(U-K), \text{ where } U > K,$$

which delivers our statement. ■

Proof of Proposition 11 *For* $x < x^*_{\infty,\delta}$ *we have that*

$$\begin{aligned}\left|RC_{\infty,\delta}(x) - F_{\infty,\delta}(x)\right| &= \left(\frac{x}{x^*_{\infty,\delta}}\right)^{\beta}\left(x^*_{\infty,\delta}-K\right) - \left(x^*_{\infty,\delta}-K\right)\left(\frac{x}{x^*_{\infty,\delta}}\right)^{\alpha}\\ &= K\left(\frac{1}{\alpha-1}\right)\left[\left(\frac{x}{x^*_{\infty,\delta}}\right)^{\beta} - \left(\frac{x}{x^*_{\infty,\delta}}\right)^{\alpha}\right]\end{aligned}$$

The difference $RC_{\infty,\delta}(x) - F_{\infty,\delta}(x)$ *is maximal for* $\overline{x}$ *that solves*

$$\begin{aligned}&RC'_{\infty,\delta}(x) - F'_{\infty,\delta}(x) = 0\\ &\beta\left(\frac{1}{x^*_{\infty,\delta}}\right)^{\beta} x^{\beta-1} = \alpha\left(\frac{1}{x^*_{\infty,\delta}}\right)^{\alpha} x^{\alpha-1}\\ &x^{\beta-\alpha} = \frac{\alpha}{\beta}\left(x^*_{\infty,\delta}\right)^{\beta-\alpha}\\ &\overline{x} = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{\beta-\alpha}} x^*_{\infty,\delta}\end{aligned} \tag{26}$$

(*The red point in Fig.* 4 *denotes* $\overline{x}$). *It follows that*

$$\begin{aligned}RC_{\infty,\delta}(x) - F_{\infty,\delta}(x) \le& K\left(\frac{1}{\alpha-1}\right)\left[\left(\frac{\overline{x}}{x^*_{\infty,\delta}}\right)^{\beta} - \left(\frac{\overline{x}}{x^*_{\infty,\delta}}\right)^{\alpha}\right]\\ &= K\left(\frac{1}{\alpha-1}\right)\left[\left(\left(\frac{\alpha}{\beta}\right)^{\frac{1}{\beta-\alpha}}\right)^{\beta} - \left(\left(\frac{\alpha}{\beta}\right)^{\frac{1}{\beta-\alpha}}\right)^{\alpha}\right]\\ &= K\left(\frac{1}{\alpha-1}\right)\left(\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\beta-\alpha}} - \left(\frac{\alpha}{\beta}\right)^{\frac{\alpha}{\beta-\alpha}}\right)\\ &= K\left(\frac{1}{\alpha-1}\right)\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\beta-\alpha}}\frac{\alpha-\beta}{\alpha}\end{aligned}$$

Given our definition of α, β *we have that, as* $\delta \to +\infty$,

$$\alpha - \beta = \frac{\sqrt{\left(r - q - \frac{\sigma^2}{2}\right)^2 + 2(r+\delta)\sigma^2} - \sqrt{\left(r - q - \frac{\sigma^2}{2}\right)^2 + 2r\sigma^2}}{\sigma^2} \sim \sqrt{\frac{2}{\sigma^2}\delta}$$

$$\alpha = \frac{-\left(r - q - \frac{\sigma^2}{2}\right) + \sqrt{\left(r - q - \frac{\sigma^2}{2}\right)^2 + 2(r+\delta)\sigma^2}}{\sigma^2} \sim \sqrt{\frac{2}{\sigma^2}\delta}$$

$$\alpha - 1 = \frac{\left(-r + q - \frac{\sigma^2}{2}\right) + \sqrt{\left(r - q - \frac{\sigma^2}{2}\right)^2 + 2(r+\delta)\sigma^2}}{\sigma^2} \sim \sqrt{\frac{2}{\sigma^2}\delta}$$

Hence

$$RC_{\infty,\delta}(x) - F_{\infty,\delta}(x) \leq K\left(\frac{1}{\alpha - 1}\right)\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\beta-\alpha}} \frac{\alpha - \beta}{\alpha}$$

$$= K\left(\frac{1}{\sqrt{\frac{2}{\sigma^2}\delta}}\right)\left(\frac{\beta}{\sqrt{\frac{2}{\sigma^2}\delta}}\right)^{\frac{\beta}{\sqrt{\frac{2}{\sigma^2}\delta}}} \sim K\left(\frac{1}{\sqrt{\frac{2}{\sigma^2}\delta}}\right) \blacksquare$$

Proof of Proposition 12 *We use the analytical approximations of the critical price of American put options* (*see Lamberton and Villeneuve* 2003) *adapted to call options via the American put-call symmetry. In particular, the American put-call symmetry states that*

$$s^*_\delta(t) = \frac{K}{s^*_p(t)}$$

where $s^*_p(t)$ *is the critical price of the symmetric put option with interest rate* $\widehat{q}$, *dividend yield* $\widehat{r}$ *and strike price equal to* 1. *The critical price* $s^*_p(t)$ *when* $t \to T$ (*see Lamberton and Villeneuve* 2003) *is*

$$s^*_p(t) = 1 - \sigma\sqrt{(T-t)\ln\frac{1}{T-t}} \text{ for } r > q$$

$$s^*_p(t) = 1 - \sqrt{2}\sigma\sqrt{(T-t)\ln\frac{1}{T-t}} \text{ for } r = q$$

$$s^*_p(t) = \frac{r}{q}\left(1 - 0.638\sigma\sqrt{(T-t)}\right) \text{ for } r < q,$$

and this delivers our result. ■

Proof of Proposition 13 *The value of the European call option subject to liquidation penalties can be computed using the Black-Scholes call option formula with* $\widehat{q} = q + \delta$ *and* $\widehat{r} = r + \delta$. *Since* $d_{1,2}$ *do not depend on* δ, *we have that*

$$\lim_{\delta\to+\infty} C_\delta^E(t) = \lim_{\delta\to+\infty} \left(S(t)e^{-(q+\delta)(T-t)}N\left(d_1\right) - Ke^{-(r+\delta)(T-t)}N\left(d_2\right)\right) = 0.$$

The conclusion follows immediately as $\lim_{\delta\to+\infty} C_\delta(t) = (S(t)-K)^+$.■

Proof of Proposition 14 *To compute the replication cost defined in* (20) *of the call option exercised optimally at* (19) *we distinguish between the two cases* (*i*) $x > K$ *and* (*ii*) $x \le K$. *In Corollary* 9 *we have shown that* $s_\delta^*(t) \searrow K$ *as* $\delta \to +\infty$. *Thus in case* (*i*), *as* $S(t) = x > K$, *there exists* $\tilde{\delta}$ *such that* $x \ge s_\delta(t)$ *for all* $\delta \ge \tilde{\delta}$, *that is* $x \in ER_\delta(t)$ *for all* $\delta \ge \tilde{\delta}$. *This implies* $\tau_\delta^*(t,x) = t$ *and*

$$RC_\delta(t,x) = \mathbb{E}\left[e^{-r(t-t)}(S(t)-K)^+\right] = x - K > 0$$

for $\delta \ge \tilde{\delta}$. *In case* (*ii*) $S(t) = x \le K$, *the option is OTM and* $\tau_\delta^*(t,x) \ge t$. *We can therefore distinguish three possible final events*

$$\begin{aligned} A_1 &= (\tau_\delta^*(t,x) < T) \cup (\tau_\delta^*(t,x) = T, S(T) = s_\delta^*(t)),\\ A_2 &= (\tau_\delta^*(t,x) = T, S(T) \le K),\\ A_3 &= (\tau_\delta^*(t,x) = T, K < S(T) < s_\delta^*(t)) \end{aligned}$$

The replication cost becomes

$$RC_\delta(t,x) = \sum_{i=1}^{3} \mathbb{E}\left[e^{-r(\tau_\delta^*(t,x)-t)}\left(S(\tau_\delta^*(t,x)) - K\right)^+ \mathbf{1}_{A_i}\right].$$

On the event A_1, *we have* $S(\tau_\delta^*(t,x)) = s_\delta^*(t)$, *therefore*:

$$0 \le \mathbb{E}\left[e^{-r(\tau_\delta^*(t,x)-t)}\left(S(\tau_\delta^*(t,x)) - K\right)^+ \mathbf{1}_{A_1}\right] = \mathbb{E}\left[e^{-r(\tau_\delta^*(t,x)-t)}\,\mathbf{1}_{A_1}\right]\left(s_\delta^*(t) - K\right) \to 0$$

as $\delta \to +\infty$. *On the event* A_2, *we have* $S(\tau_\delta^*(t,x)) = S(T) \le K$, *therefore the option closes OTM at T and we get*

$$0 \le \mathbb{E}\left[e^{-r(\tau_\delta^*(t,x)-t)}\left(S(\tau_\delta^*(t,x)) - K\right)^+ \mathbf{1}_{A_2}\right] = 0$$

On the event A_3, *we have* $S(\tau_\delta^*(t,x)) = S(T) < s_\delta^*(t)$, *therefore*:

$$\begin{aligned} 0 \le& \mathbb{E}\left[e^{-r(\tau_\delta^*(t,x)-t)}\left(S(\tau_\delta^*(t,x)) - K\right)^+ \mathbf{1}_{A_3}\right] = e^{-r(T-t)}\,\mathbb{E}\left[(S(T)-K)^+ \mathbf{1}_{A_3}\right]\\ \le& \left(s_\delta^*(t) - K\right) \to 0 \quad \text{as } \delta \to +\infty \end{aligned}$$

Thus in case (*ii*) $S(t) = x \le K$ *we have* $f(x) = (x-K)^+ = 0$ *and*

$$RC_\delta(t,x) = \sum_{i=1}^{3} \mathbb{E}\left[e^{-r(\tau_\delta^*(t,x)-t)}\left(S(\tau_\delta^*(t,x)) - K\right)^+ \mathbf{1}_{A_i}\right] \to 0 \quad \text{as } \delta \to +\infty$$

and this concludes our proof. ■

Auxiliary lemmas

Lemma 15 *Details for the proof of Proposition 6*

1. *The argument of the square root* $\sqrt{1-4\mathbf{q}(1-\mathbf{q})a^2}$ *in the definition of* C

   $$C = \ln\left(\frac{1-\sqrt{1-4\mathbf{q}(1-\mathbf{q})a^2}}{2(1-\mathbf{q})a}\right)$$

   is positive for any $a \in (0,1)$.
2. *For any* $a \in (0,1)$*.we have* $1-\sqrt{1-4\mathbf{q}(1-\mathbf{q})a^2} > 0$ (*the log is well defined*), *and* $C \leq 0$
3. *For* $a = e^{-(r+\delta)\Delta t}$, *with* $r+\delta > 0$

   $$C + \sigma\sqrt{\Delta t} \leq 0$$

Proof

1. The argument of the square root is always positive:

   $$\begin{aligned}1-4\mathbf{q}(1-\mathbf{q})a^2 &>0\\ 4\mathbf{q}^2a^2-4\mathbf{q}a^2+1 &>0\\ a^2\mathbf{q}^2-a^2\mathbf{q}+\frac{1}{4} &>0\end{aligned}$$

   as the discriminant of the quadratic equation in $\mathbf{q}$ is negative

   $$\begin{aligned}a^4-a^2 &<0\\ a^2\left(a^2-1\right) &<0\\ \left(a^2-1\right) &<0,\end{aligned}$$

   as $a < 1$.
2. The first inequality $1-\sqrt{1-4\mathbf{q}(1-\mathbf{q})a^2} > 0$, is immediate as under the argument of the square root $1-4\,\mathbf{q}(1-\mathbf{q})a^2$ is positive and smaller than 1. We now prove that $C < 0$. This happens if the log argument is smaller than 1

$$\begin{aligned}
\frac{1-\sqrt{1-4\mathbf{q}(1-\mathbf{q})a^2}}{2(1-\mathbf{q})a} &< 1\\
1-\sqrt{1-4\mathbf{q}(1-\mathbf{q})a^2} &< 2(1-\mathbf{q}\ )a\\
1-2(1-\mathbf{q})a &< \sqrt{1-4\mathbf{q}(1-\mathbf{q})a^2}\\
1-4(1-\mathbf{q})a+4(1-\mathbf{q})^2a^2 &< 1-4\mathbf{q}(1-\mathbf{q})a^2\\
-4(1-\mathbf{q})a+4(1-\mathbf{q})^2a^2 &< -4\mathbf{q}(1-\mathbf{q})a^2\\
-1+(1-\mathbf{q})a &< -\mathbf{q}a\\
a &< 1
\end{aligned}$$

3. For $a = e^{-(r+\delta)\Delta t}$, and $r + \delta \geq 0$ we verify that

$$\begin{aligned}
&C + \sigma\sqrt{\Delta t} \leq 0\\
&C \leq -\sigma\sqrt{\Delta t}\\
&\ln\left(\frac{1-\sqrt{1-4\mathbf{q}(1-\mathbf{q})a^2}}{2(1-\mathbf{q})a}\right) \leq -\sigma\sqrt{\Delta t}\\
&\frac{1-\sqrt{1-4\mathbf{q}(1-\mathbf{q})a^2}}{2(1-\mathbf{q})a} \leq e^{-\sigma\sqrt{\Delta t}}\\
&1-\sqrt{1-4\mathbf{q}(1-\mathbf{q})a^2} \leq e^{-\sigma\sqrt{\Delta t}}2(1-\mathbf{q})a\\
&1-e^{-\sigma\sqrt{\Delta t}}2(1-\mathbf{q})a \leq \sqrt{1-4\mathbf{q}(1-\mathbf{q})a^2}\\
&1-e^{-\sigma\sqrt{\Delta t}}4(1-\mathbf{q})a+e^{-2\sigma\sqrt{\Delta t}}4(1-\mathbf{q})^2a^2 \leq 1-4\mathbf{q}(1-\mathbf{q})a^2\\
&-e^{-\sigma\sqrt{\Delta t}}+e^{-2\sigma\sqrt{\Delta t}}(1-\mathbf{q})a \leq -\mathbf{q}a
\end{aligned}$$

As $\mathbf{q} = \frac{e^{(r-q)\Delta t}-e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}}-e^{-\sigma\sqrt{\Delta t}}}$, we get that the previous equality is satisfied if

$$\begin{aligned}
&-e^{-\sigma\sqrt{\Delta t}} + e^{-2\sigma\sqrt{\Delta t}}\left(\frac{e^{\sigma\sqrt{\Delta t}} - e^{(r-q)\Delta t}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}\right)a \le -\frac{e^{(r-q)\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}a \\
&-e^{-\sigma\sqrt{\Delta t}}\left(e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}\right) + e^{-2\sigma\sqrt{\Delta t}}\left(e^{\sigma\sqrt{\Delta t}} - e^{(r-q)\Delta t}\right)a \\
&\quad \le -\left(e^{(r-q)\Delta t} - e^{-\sigma\sqrt{\Delta t}}\right)a \\
&-1 + e^{-2\sigma\sqrt{\Delta t}} + e^{-2\sigma\sqrt{\Delta t}}\left(e^{\sigma\sqrt{\Delta t}} - e^{(r-q)\Delta t}\right)a \le -\left(e^{(r-q)\Delta t} - e^{-\sigma\sqrt{\Delta t}}\right)a \\
&-1 + e^{-2\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}}a - e^{-2\sigma\sqrt{\Delta t}}e^{(r-q)\Delta t}a \le -ae^{(r-q)\Delta t} + e^{-\sigma\sqrt{\Delta t}}a \\
&-1 + e^{-2\sigma\sqrt{\Delta t}} - e^{-2\sigma\sqrt{\Delta t}}e^{(r-q)\Delta t}a \le -ae^{(r-q)\Delta t} \\
&-1 + e^{-2\sigma\sqrt{\Delta t}} - e^{-2\sigma\sqrt{\Delta t}}e^{(r-q)\Delta t}e^{-(r+\delta)\Delta t} \le -e^{-(r+\delta)\Delta t}e^{(r-q)\Delta t} \\
&-1 + e^{-2\sigma\sqrt{\Delta t}} - e^{-2\sigma\sqrt{\Delta t}}e^{-(\delta+q)\Delta t} \le -e^{-(\delta+q)\Delta t} \\
&-1 + e^{-2\sigma\sqrt{\Delta t}} \le e^{-2\sigma\sqrt{\Delta t}}e^{-(\delta+q)\Delta t} - e^{-(\delta+q)\Delta t} \\
&e^{-2\sigma\sqrt{\Delta t}} - 1 \le \left(e^{-2\sigma\sqrt{\Delta t}} - 1\right)e^{-(\delta+q)\Delta t} \\
&1 - e^{-2\sigma\sqrt{\Delta t}} \ge \left(1 - e^{-2\sigma\sqrt{\Delta t}}\right)e^{-(\delta+q)\Delta t} \\
&1 \ge e^{-(\delta+q)\Delta t}
\end{aligned}$$

which is true, as $\delta + q \ge 0$.

Lemma 16 *The function*

$$f(\delta) = \frac{1 - \sqrt{1 - 4\mathbf{q}(1-\mathbf{q})e^{-2(r+\delta)\Delta t}}}{2(1-\mathbf{q})e^{-(r+\delta)\Delta t}}$$

is decreasing for $\delta \ge 0$.

Proof We rewrite $f(\delta)$ as

$$f(\delta) = \frac{1}{2(1-\mathbf{q})e^{-r\Delta t}}\left(e^{\delta\Delta t} - \sqrt{e^{2\delta\Delta t} - 4\mathbf{q}(1-\mathbf{q})e^{-2r\Delta t}}\right)$$

So that

$$\begin{aligned}
f'(\delta) &= \frac{1}{2(1-\mathbf{q})e^{-r\Delta t}}\left(\Delta t e^{\delta\Delta t} - \frac{1}{2}\frac{2\Delta t e^{2\delta\Delta t}}{\sqrt{e^{2\delta\Delta t} - 4\mathbf{q}(1-\mathbf{q})e^{-2r\Delta t}}}\right) \\
&= \frac{\Delta t e^{\delta\Delta t}}{2(1-\mathbf{q})e^{-r\Delta t}}\left(1 - \frac{e^{\delta\Delta t}}{\sqrt{e^{2\delta\Delta t} - 4\mathbf{q}(1-\mathbf{q})e^{-2r\Delta t}}}\right) \\
&= \frac{\Delta t e^{\delta\Delta t}}{2(1-\mathbf{q})e^{-r\Delta t}}\left(1 - \frac{1}{\sqrt{1 - 4\mathbf{q}(1-\mathbf{q})e^{-2(r+\delta)\Delta t}}}\right) < 0
\end{aligned}$$

as

$$\sqrt{1-4\mathbf{q}(1-\mathbf{q})e^{-2(r+\delta)\Delta t}} < 1.\blacksquare$$

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