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Research article

A note on the Kuramoto-Sivashinsky equation with discontinuity†

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Abstract: In this work we consider differential equations of the type

$$\pm u^{(k)} = f(u),$$

and study the extinction profile of their solutions. Emphasis is placed on the special case $-u^{(4)} = \operatorname{sgn}(u)$, which is related to the Kuramoto-Sivashinsky equation. In this case we describe in more detail the extinction phenomenon and prove a conjecture by Galaktionov and Svirshchevskii.

Keywords: discontinuous differential equations; periodic solutions; finite time extinction; extinction profile; oscillations

To Alberto with esteem.

1. Introduction

Main subject of this work are the following two equations:

$$u^{(4)} + \operatorname{sgn}(u) = 0 \tag{1.1}$$

and

$$w^{(4)} + 10w''' + 35w'' + 50w' + 24w + \operatorname{sgn}(w) = 0 \tag{1.2}$$

where $\operatorname{sgn}(x) = x/|x|$ for $x \neq 0$, $\operatorname{sgn}(0) = 0$, and $u^{(4)}$ (resp. $w^{(4)}$) denotes the fourth derivative of u (resp. w).

The two equations are related by the transformation

$$u(t) = (1 + t)^4 w(\ln(1 + t)). \tag{1.3}$$

In fact, if w solves (1.2), then u given by (1.3) solves (1.1). Viceversa, if u solves (1.1), then w given by

$$w(s) = e^{-4s} u(e^s - 1) \tag{1.4}$$

solves (1.2).

Equations (1.1) and (1.2) are simple prototypes of differential equations with a discontinuous nonlinearity. This kind of differential equations are widely used as a model to describe phenomena typically occurring in several disparate fields such as control systems, friction mechanics, nonlinear oscillations, economics and biology. See [12–15] and references therein for a survey on these applications.

In [10], Galaktionov and Svirshchevskii consider Eqs (1.1) and (1.2) when studying extinction phenomena and maximal regularity at the interface for solutions of the Kuramoto–Sivashinsky equation with absorption. To do so, they present the PDE

$$v_t = -v_{xxxx} + \kappa(v_{xx})^2 - \operatorname{sgn}(v), \tag{1.5}$$

for which they look for traveling wave solutions $v(x, t) = u(x - \lambda t)$. Such a function u must solve the ODE

$$-\lambda u' = -u^{(4)} + \kappa(u'')^2 - \operatorname{sgn}(u). \tag{1.6}$$

Studying Eqs (1.1) and (1.2) serves as a tool to gain insight on the more complicated Eq (1.6). Eqs (1.1) and (1.2) also arise in the study of maximal regularity of oscillatory solutions having zero contact angle at the interface of some quasilinear wave equations (see [10, pp. 256–258]), and play a role in the study of oscillatory solutions of a fifth-order nonlinear dispersive PDE (see [10, pp. 185–189]). See also [9] for more problems to which these equations are associated.

Of key importance in the analysis made in [10] is the existence of periodic solutions of (1.2). Therein, the authors state that "existence and uniqueness of periodic solutions of (1.2) are OPEN", and formulate the following Conjecture (supporting it by numerical experiments):

Conjecture 1 ([10, Conjecture 3.2])**.** *The ODE* (1.2) *has a unique nontrivial periodic solution* $w = w(s)$ *which is asymptotically stable for* $s \to +\infty$*.*

Existence and stability of periodic orbits for discontinuous ODEs has received a lot of attention in the literature. Many of the classical tools in the qualitative theory of ODEs have been extended or adapted to the discontinuous case. See for instance [3] for a theoretical account and [1] for a review of numerical methods. These tools do not adapt well to the equations under investigation, that allow for a direct and constructive approach.

Below we unfold the main results of this work. The first one gives a partial (positive) answer to Conjecture 1.

Theorem 14. *There exists a nontrivial periodic solution of* (1.2) *which is asymptotically stable forwards in time.*

Uniqueness remains an open problem, see Remark 8. The following result follows from Theorem 14 through a wise use of transformations (1.3) and (1.4).

Theorem 15. *For any $T \in \mathbb{R}$:*

i) there exist solutions of (1.1) *that* disappear oscillating *at T and solutions of* (1.1) *that* appear oscillating *at T;*

ii) there exist solutions of (1.2) *that* disappear oscillating *at T and solutions of* (1.2) *that* appear oscillating *at T.*

By "solution that disappears oscillating at T" we mean, roughly speaking, a nontrivial solution that vanishes together with all of its continuous derivatives as $t \to T^-$ (in [10], the authors refer to this phenomenon as *extinction*) changing sign infinitely many times. The exact meaning of the terms *appear*, *disappear* and *oscillating* will be clarified in Definitions 5 and 7.

We point out that Eq (1.1) is a special case of

$$u^{(4)} + |u|^{q-1}u = 0, \tag{1.7}$$

with $q = 0$. If $q > 1$, it is known that any solution of (1.7) blows up in finite (forward and/or backward) time, see [6]. In [5], the authors study the blow-up profile of those solutions. To do so, they apply a change of variable similar to (1.4) and recast the problem as that of studying periodic solutions of an auxiliary equation akin to (1.2). For instance, they show that solutions of (1.7), for q in an open neighbourhood of 3, that blow up at $T = 1$, do so through progressively wider oscillations that can be described as the amplitude modulation of $w(-\ln(1-t))$ by a singular function, with w being a periodic function. In this work, via an analogous argument, we show that disappearance of solutions of (1.1) at $T = 1$ can be described as the amplitude modulation of $w(\ln(1-t))$ by a vanishing function, with w being a periodic solution of (1.2), see Figure 1.

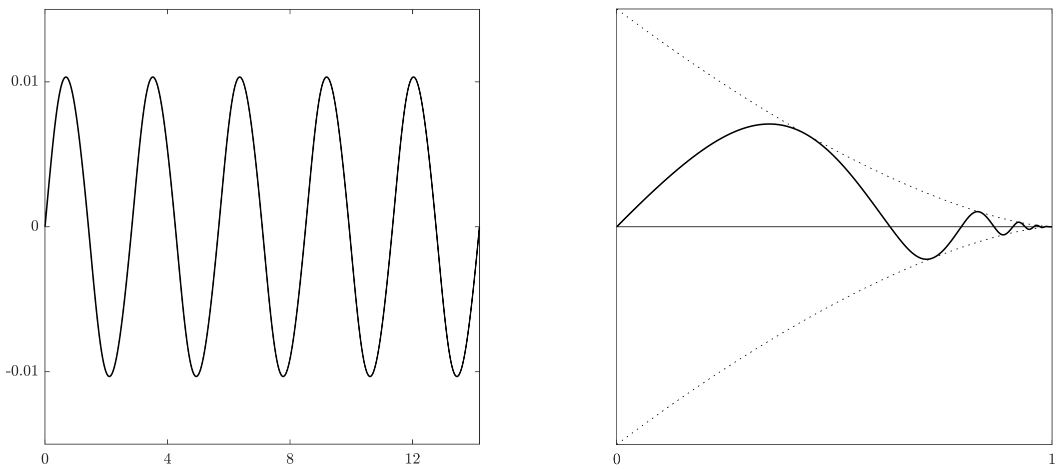

Figure 1. On the left a periodic solution of Eq (1.2); on the right a solution of (1.1) that disappears at 1.

Throughout the rest of the work, the previous results are generalized in several ways. For instance, in Theorem 12 we show that *all* solutions of (1.1) and (1.2) that disappear or appear do so through infinitely many oscillations. We also extend, by a lifting argument, the results of Theorem 15 to Eq (1.6).

Theorem 20. *For any κ, λ and $T \in \mathbb{R}$ there exist solutions of* (1.6) *that* disappear oscillating *at T and solutions of* (1.6) *that* appear oscillating *at T.*

Equation (1.1) is a particular case of

$$\pm\ u^{(k)} = f(u), \tag{1.8}$$

where $u^{(k)}$ denotes the k-th derivative of u.

It is of interest investigating the existence of appearing and disappearing solutions of equations of type (1.8), as well as the profile through which they appear or disappear. Such equations arise in several contexts, see for instance [4, 6, 7, 9, 10]. The technique that will be used to study (1.1) and (1.2) can be adapted to other cases of (1.8). For instance, in Section 5 we obtain analogous results on higher order instances of (1.1) and (1.2). In Section 4 we provide a partial picture of the general case (1.8) and study in more detail the cases $k = 1, 2, 3, 4$. We believe that our analysis for the case $f(u) = \operatorname{sgn}(u)$ could suggest what to expect for other kinds of nonlinearities, e.g., $f(u) = |u|^{q-1}u$ with $0 < q < 1$.

Part of this work has its origin in 2015 from a joint project that involved the authors and Prof. Jean-Philippe Lessard of the Department of Mathematics at McGill University. A proof of the existence of a periodic orbit for Eq (1.2) that follows ideas similar to those used in this work is present in Gallo's Undergraduate Thesis [11]. In [2], Alama and Lessard present a computer-assisted proof of existence and local stability of the same periodic orbit.

The work is organized as follows. In Section 2 we introduce definitions and notions that are required throughout the rest of the work. In Section 3 we present the main results of the work. In Section 4, we extend our analysis of extinction profiles to appearing/disappearing solutions of Eq (1.8). In Section 5, we state some of our results to higher order equations. We conclude with an Appendix in which we present some bounds on solutions of Eqs (1.1) and (1.2).

2. Preliminary notions and definitions

Some definitions and facts are required before we head to the main results of this work.

Definition 2. *We say that $u = u(t)$ is a solution of Eq* (1.1) *for t in some open interval I if:*

i) u is of class $C^3(I)$ and u''' is absolutely continuous on every compact subset of I;

ii) $u^{(4)}(t) + \operatorname{sgn}(u(t)) = 0$ *almost everywhere on I.*[1]

Analogous definition holds for solutions of (1.2), mutatis mutandis. Note that the function

$$u(t) = 0 \text{ for all } t \in I$$

is a solution according to definition 2, which we call *trivial* solution on I. Accordingly, a solution is called *nontrivial* in I if it is not constantly equal to zero on I. A solution is called *global* if $I = \mathbb{R}$.

[1] Equivalently, given the smoothness required for u, one could replace condition ii) with the following: $u(t)$ solves $u^{(4)}(t)+\operatorname{sgn}(u(t)) = 0$ in distributional sense.

Theorem 3. *For any $t_0 \in \mathbb{R}$ and any vector $\boldsymbol{u}_0 \in \mathbb{R}^4$, there exists a solution u of* (1.1) *defined on $\mathbb{R}$ such that*

$$\left[u(t_0),\ u'(t_0),\ u''(t_0),\ u'''(t_0)\right] = \boldsymbol{u}_0.$$

Furthermore:

i) if $\left[u(t_0),\ u'(t_0),\ u''(t_0),\ u'''(t_0)\right] \neq \left[0,\ 0,\ 0,\ 0\right]$, the solution is unique in a neighbourhood of t_0;

ii) if $u(t_0) \neq 0$, the solution is smooth (actually analytic) in a neighbourhood of t_0.

Proof. Existence of a global solution of (1.1) for any initial condition $\boldsymbol{u}_0 \in \mathbb{R}^4$ could be inferred by arguing just as in the classical Peano Theorem[2]. Local uniqueness and smoothness can be deduced from a straightforward analysis of the sign of a solution in a neighbourhood of t_0. □

Remark 4. We point out that Theorem 3 also applies to solutions of Eq (1.2). Moreover, uniqueness is only local. More precisely, solutions having nontrivial initial conditions enjoy right (resp., left) uniqueness up to the point where they possibly disappear (resp., appear).

Definition 5. *We say that a solution $z = z(t)$ of* (1.1) *or* (1.2) disappears *at T if:*

i) z is nontrivial on any left neighbourhood of T;

ii) $\lim_{t\to T^-} z^{(j)}(t) = 0$ *for* $j = 0, 1, 2, 3$.

In this case, we call z a disappearing *solution. Analogously, we say that z* appears *at T if:*

i) z is nontrivial on any right neighbourhood of T;

ii) $\lim_{t\to T^+} z^{(j)}(t) = 0$ *for* $j = 0, 1, 2, 3$.

We call z an appearing *solution.*

Remark 6. Note that solutions of (1.1) or (1.2) that disappear at some T_0 can be "glued" to solutions that appear at any time $T_1 \geq T_0$, and that this operation yields again a solution according to Definition 2.

Equations (1.1) and (1.2), written as first-order vector equations, belong to the class of so called differential equations with discontinuous right-hand side, which has been extensively studied in the literature. The books [8] and [3] are a comprehensive account on this subject. Although several definitions of solutions of differential equations with piecewise smooth right-hand side can be given (again, see [8]), they are all substantially equivalent to each other, and to Definition (2), unless *sliding motion* occurs. For a first-order autonomous vector equation

$$x' = F(x),$$

the discontinuity set Σ is defined as the set of x's at which F is discontinuous, and sliding motion occurs when a solution $x = x(t)$ belongs to Σ for t in some open interval. For Eqs (1.1) and (1.2), we have $\Sigma = \left\{\left[x_1,\ x_2,\ x_3,\ x_4\right] \in \mathbb{R}^4 : x_1 = 0\right\}$, and solutions u that approach Σ "from one side" (say with $u > 0$) naturally "pass to the other side" (i.e., with $u < 0$). In fact, we will show (see Remarks 6 and 21) that, in the equations of interest for us, sliding motion is made possible only by gluing appearing/disappearing

[2] Appropriate existence results can also be found in Filippov's book [8].

solutions with the trivial solution. Therefore, there is no need to further specify the behaviour of our solutions on the discontinuity set.

We will show (Theorem 12) that disappearance and appearance of solutions is characterized by oscillations.

Definition 7. *We say that a solution of* (1.1) *or* (1.2) *disappears (resp. appears)* oscillating *at T if the solution disappears (resp. appears) changing sign infinitely many times in any left (resp. right) neighbourhood of T.*

3. Main results

We begin this Section with the following Remark, in which we present an identity that will be useful throughout the rest of the work.

Remark 8. First, we note that transformations (1.3) and (1.4) preserve the multiplicity of any zero of u and w. That is, if $t_0 > -1$ is a zero of u, then $s_0 = \ln(1 + t_0)$ is a zero of w of the same multiplicity, and vice versa.

Now, let u be a solution of (1.1). Multiplying the equation by u and integrating by parts twice on $[a, b]$, we get

$$u'''(b)u(b) - u'''(a)u(a) - u''(b)u'(b) + u''(a)u'(a) + \int_a^b (u'')^2 + \int_a^b |u| = 0. \tag{3.1}$$

This simple identity has several consequences:

i) if $u(a) = u(b) = 0$ and $u'(a) = u'(b) = 0$, then $u(t) = 0$ for all t in $[a, b]$. Similarly, u is trivial on $[a, b]$ if $u(a) = u(b) = 0$ and $u''(a) = u''(b) = 0$;

ii) Eq (1.1) does not have any nontrivial periodic solution;

iii) no solution of (1.1) can appear at some T_0 and disappear at $T_1 > T_0$ (therefore, there are no compactly supported solutions);

iv) any nontrivial periodic solution w of (1.2) must have only simple zeros.

Below we argue that –any– nontrivial global solution of (1.1) or (1.2) must change sign infinitely many times.

Theorem 9. *Let u be a solution of* (1.1).

i) If u is eventually nonnegative forwards in time (i.e., there exists t_0 such that $u(t) \geq 0$ for all $t \geq t_0$), then u is eventually trivial forwards in time (i.e., there exists t_1 such that $u(t) = 0$ for all $t \geq t_1$). The same holds if u is eventually nonpositive forwards in time;

ii) if u is eventually nonnegative backwards in time (i.e., there exists t_0 such that $u(t) \geq 0$ for all $t \leq t_0$), then u is eventually trivial backwards in time (i.e., there exists t_1 such that $u(t) = 0$ for all $t \leq t_1$). The same holds if u is eventually nonpositive backwards in time.

Proof. It is enough to show claim i), as claim ii) follows immediately upon observing that, if $u = u(t)$ solves (1.1), also $u(-t)$ does. Moreover, we can restrict ourselves to the case of u being eventually nonnegative forwards in time (if u is eventually nonpositive forwards in time, just consider $-u$). There are two possible cases:

a) *u is eventually strictly positive.* In this case, we have $\operatorname{sgn}(u(t)) = 1$ for all $t > t_1$, and therefore $u(t) = -\frac{1}{24}t^4 + p_3(t)$ for $t > t_1$, where p_3 is a polynomial with $\deg(p_3) \leq 3$. Consequently, we would have that $u(t) \to -\infty$ as $t \to +\infty$, which contradicts our assumption;

b) *there exists $(t_n)_{n\geq 1} \nearrow \infty$ such that $u(t_n) = 0$ for all $n \geq 1$, with $t_1 > t_0$.* In this case, we must have $u'(t_n) = 0$ for all $n \geq 1$. It follows from Remark 8 (choose $a = t_1$ and $b = t_n$) that $u(t) = 0$ for $t_1 \leq t \leq t_n$. Letting $n \to +\infty$, we have the claim.

□

Corollary 10. *Let w be a solution of* (1.2). *If w is eventually nonnegative forwards in time, then w is eventually trivial forwards in time. The same holds if w is eventually nonpositive forwards in time.*

Remark 11. Note that there is no analog of ii) of Theorem 9 for solutions of Eq (1.2). In fact, there exist solutions of (1.2) that are eventually positive backwards in time without being eventually trivial backwards in time. For instance, let u be a solution of (1.1) such that $u(-1) = 1$. Through the change of variables (1.4), we have a solution w of (1.2) such that $w(s)e^{4s} \to 1$ as $s \to -\infty$.

In Theorem 15 we will explicitly construct a family of solutions of (1.1) that disappear/appear oscillating. Here we show that "oscillation" is a property of –any– solution that disappears or appears.

Theorem 12. *Any solution of* (1.1) *or* (1.2) *that disappears or appears, does so oscillating in the sense of Definition* (7).

Proof. We begin by considering Eq (1.1). The equation is autonomous. Therefore, without loss of generality, we can restrict ourselves to solutions u that disappear at 0. We will argue by contradiction. Assume that, for some $t_0 < 0$, $u(t) \geq 0$ for all $t \in (t_0, 0)$. There are two possible cases:

a) *$u(t) > 0$ for all $t \in (t_0, 0)$.* In this case, we have $\operatorname{sgn}(u(t)) = 1$ for all t in $(t_0, 0)$, and therefore $u(t) = -\frac{1}{24}t^4 + a_3t^3 + a_2t^2 + a_1t + a_0$ for all t in $(t_0, 0)$, with $a_j \in \mathbb{R}$ for all $j = 0, 1, 2, 3$. By assumptions, $u^{(j)}(t) \to 0$ as $t \to 0^-$ for $j = 0, 1, 2, 3$, from which it follows that $a_0 = a_1 = a_2 = a_3 = 0$. This means that $u(t) = -\frac{1}{24}t^4$, which is strictly negative on $(t_0, 0)$, a contradiction.

b) *$u(t_1) = 0$ for some $t_1 \in (t_0, 0)$.* Since $u'(t_1) = 0$, we can use Eq (3.1) with $a = t_1$ and $b = 0$ to obtain that $u(t) = 0$ for all $t_1 \leq t \leq 0$. This contradicts our assumption that u disappears at 0 (see i) of Definition 5).

Now, consider a disappearing solution w of (1.2). Without loss of generality, we can assume that w disappears at 0. Let u be defined by the change of variable (1.3). The function u solves (1.1) and disappears at 0. It follows from the previous part of this proof that u disappears –oscillating– at 0, and so does w (because (1.3) is sign preserving). Analogously one can show that the same conclusions hold for appearing solutions. □

Remark 13. See Theorem 23 for another proof of Theorem 12 in a more general case.

We now present the main results of this work.

Theorem 14. *There exists a nontrivial periodic solution of* (1.2) *which is asymptotically stable forwards in time.*

Before proving Theorem 14, we state and prove its main consequence.

Theorem 15. *For any $T \in \mathbb{R}$:*

i) there exist solutions of (1.1) *that disappear oscillating at T and solutions of* (1.1) *that appear oscillating at T;*

ii) there exist solutions of (1.2) *that disappear oscillating at T and solutions of* (1.2) *that appear oscillating at T.*

Proof. Let $\gamma = \gamma(s)$ be a periodic solution of (1.2) given by Theorem 14. Then, the function $u(t) = (1+t)^4\gamma(\ln(1+t))$ solves (1.1) and appears at -1. Recall that Eq (1.1) is autonomous and invariant by time reversal. Therefore, there exist solutions of (1.1) that *appear* at any time T and solutions of (1.1) that *disappear* at any time T. This proves the first claim. Now, let $u = u(t)$ be a solution of (1.1) that appears (resp. disappears) at 0. Then, $w(s) = e^{-4s}u(e^s - 1)$ solves (1.2) and appears (resp. disappears) at 0. The second claim follows upon observing that also Eq (1.2) is autonomous. □

Remark 16. We point out that it is possible to explicitly construct sets of initial conditions that lead to disappearance or appearance of solutions of (1.1) and (1.2). In fact, let

$$\boldsymbol{u} := \begin{bmatrix} u \\ u' \\ u'' \\ u''' \end{bmatrix}, \ \boldsymbol{w} := \begin{bmatrix} w \\ w' \\ w'' \\ w''' \end{bmatrix}, \ L := \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 16 & -7 & 1 & 0 \\ -64 & 37 & -9 & 1 \end{bmatrix}, \ D(\alpha) := \begin{bmatrix} \alpha^4 & 0 & 0 & 0 \\ 0 & \alpha^3 & 0 & 0 \\ 0 & 0 & \alpha^2 & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix}, \tag{3.2}$$

where $u = u(t)$ is a solution of (1.1) and $w = w(s)$ is a solution of (1.2). There is no loss of generality in assuming all initial conditions to be given at $t = 0$ (or $s = 0$), and we do so. Simple computations yield the following facts:

i) if u solves (1.1) with initial conditions $\boldsymbol{u}(0)$, then w given by (1.4) solves (1.2) with initial conditions $\boldsymbol{w}(0) = L\boldsymbol{u}(0)$;

ii) if w solves (1.2) with initial conditions $\boldsymbol{w}(0)$, then u given by (1.3) solves (1.1) with initial conditions $\boldsymbol{u}(0) = L^{-1}\boldsymbol{w}(0)$.

Let Γ be the orbit that corresponds to a periodic solution of (1.2) given in Theorem 14, and let $\boldsymbol{w}(0) \in \Gamma$. For all $\alpha \neq 0$,

$$u_\alpha(t) := \alpha^4(u(t/\alpha))$$

solves (1.1) with initial condition $\boldsymbol{u}_\alpha(0) = D(\alpha)L^{-1}\boldsymbol{w}(0)$ and appears (resp., disappears) at $-\alpha$ if $\alpha > 0$ (resp., $\alpha < 0$). Moreover, for $\alpha < 1$ and $\alpha \neq 0$,

$$w_\alpha(s) := e^{-4s}u_\alpha(e^s - 1)$$

solves (1.2) and appears (resp., disappears) at $\ln(1-\alpha)$ if $0 < \alpha < 1$ (resp., $\alpha < 0$).

Therefore, the following set $\mathcal{U}$ is made of initial conditions that lead to disappearance or appearance of solutions of (1.1):

$$\mathcal{U} := \bigcup_{\alpha\neq 0} D(\alpha)L^{-1}\Gamma.$$

This is a two-dimensional manifold in $\mathbb{R}^4$, which is analytic at all points except those on the $\{u = 0\}$ hyperplane. Analogously, the set

$$\mathcal{W} := \bigcup_{\alpha<1,\alpha\neq 0} LD(\alpha)L^{-1}\Gamma$$

is made of initial conditions that lead to disappearance or appearance of solutions of (1.2). Actually, if we let

$$\mathcal{W}^- := \bigcup_{0<\alpha<1} LD(\alpha)L^{-1}\Gamma \text{ and } \mathcal{W}^+ := \bigcup_{\alpha<0} LD(\alpha)L^{-1}\Gamma,$$

we have that solutions $\boldsymbol{w} = \left[w,\, w',\, w'',\, w'''\right]^T$ of (1.2) (written as a first-order vector equation, see (3.3) below) having initial conditions in $\mathcal{W}^-$ approach the origin backwards in time, while those with initial conditions in $\mathcal{W}^+$ approach the origin forwards in time. That is, $\mathcal{W}^+$ and $\mathcal{W}^-$ act, respectively, as stable and unstable manifold for the origin. Furthermore, a more careful analysis (similar to the one used in [5, Theorem 2.13]) shows that orbits through points in $\mathcal{W}^-$ approach Γ forwards in time, and therefore act as connecting orbits between the origin and the periodic orbit Γ. It is the existence of points in $\mathcal{W}^+$ that prevents the periodic solution in Theorem 14 from being –globally– asymptotically stable.

Remark 17. Note that the existence of solutions of (1.1) and (1.2) that appear or disappear is ultimately related to the existence of nontrivial solutions of (1.2) that are bounded backwards in time. One of those solutions is provided by Theorem 14, but we could not rule out the existence of different solutions that share the same behaviour at $-\infty$.

Throughout the rest of the work we denote by $(\boldsymbol{x})_j$ the j-th component of a vector $x \in \mathbb{R}^n$.

Proof of Theorem 14. We split the proof into two parts: first existence and then stability.

Existence. Rewrite (1.2) as a vector equation:

$$\frac{d}{ds}\begin{bmatrix} w \\ w' \\ w'' \\ w''' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -24 & -50 & -35 & -10 \end{bmatrix}\begin{bmatrix} w \\ w' \\ w'' \\ w''' \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mathrm{sgn}(w) \end{bmatrix}. \tag{3.3}$$

If we let

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -24 & -50 & -35 & -10 \end{bmatrix}, \tag{3.4}$$

denote by $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_4$ the standard basis vectors of $\mathbb{R}^4$, and make use of the notation introduced in (3.2), then (3.3) can be written as

$$\boldsymbol{w}' = A\boldsymbol{w} - \mathrm{sgn}(w)\,\boldsymbol{e}_4. \tag{3.5}$$

For any nonzero $\boldsymbol{w}_0 \in \mathbb{R}^4$, denote by $\boldsymbol{\psi}(s, \boldsymbol{w}_0)$ the solution $\boldsymbol{w} = \boldsymbol{w}(s)$ of (3.5) such that $\boldsymbol{w}(0) = \boldsymbol{w}_0$. We refer to Theorem 3 and Remark 4 for considerations on existence and uniqueness of solutions of (3.5). Note that $\boldsymbol{\psi}$ enjoys the following symmetry:

$$\boldsymbol{\psi}(s, -\boldsymbol{w}_0) = -\boldsymbol{\psi}(s, \boldsymbol{w}_0). \tag{3.6}$$

Therefore, to produce a periodic solution of (3.5) it is enough to find $\boldsymbol{p} \in \mathbb{R}^4$ and $\tau > 0$ such that

$$\boldsymbol{\psi}(\tau, \boldsymbol{p}) = -\boldsymbol{p},$$

from which it follows that

$$\boldsymbol{\gamma}(s) := \boldsymbol{\psi}(s, \boldsymbol{p}) \tag{3.7}$$

is 2τ-periodic, and its first component is a periodic solution of (1.2).

By virtue of Corollary 10, there is no loss of generality in restricting ourselves to searching for vectors $\boldsymbol{p}$ and scalars $\tau > 0$ with $(\boldsymbol{p})_1 = 0$ and $(\boldsymbol{\psi}(s, \boldsymbol{p}))_1 > 0$ for all $s \in (0, \tau)$; therefore, we consider vectors $\boldsymbol{p}$ of the form

$$\boldsymbol{p} = \begin{bmatrix} 0 \\ \boldsymbol{\eta} \end{bmatrix} = \begin{bmatrix} 0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}, \text{ where } \boldsymbol{\eta} \in \mathbb{R}^3 \text{ with } \eta_1 > 0.$$

Again by Corollary 10, for any nonzero $\boldsymbol{p} \in \mathbb{R}^4$ there exists $s_1 > 0$ such that $w(s) = (\boldsymbol{\psi}(s, \boldsymbol{p}))_1 > 0$ for all $s \in (0, s_1)$ and $w(s_1) = 0$. At s_1, the solution of (3.5) has the form

$$\boldsymbol{\psi}(s_1, \boldsymbol{p}) = \begin{bmatrix} 0 \\ \boldsymbol{\xi} \end{bmatrix} = \begin{bmatrix} 0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}, \text{ for some } \boldsymbol{\xi} \in \mathbb{R}^3.$$

Explicit integration of (3.5) for $s \in (0, s_1)$ yields:

$$\begin{aligned} w(s) = &\frac{1}{6}\left(-11e^{-4s} + 42e^{-3s} - 57e^{-2s} + 26e^{-s}\right)\eta_1 + \frac{1}{2}\left(-2e^{-4s} + 7e^{-3s} - 8e^{-2s} + 3e^{-s}\right)\eta_2 \\ &+ \frac{1}{6}\left(-e^{-4s} + 3e^{-3s} - 3e^{-2s} + e^{-s}\right)\eta_3 + \frac{1}{24}\left(1 + e^{-4s} - 4e^{-3s} + 6e^{-2s} - 4e^{-s}\right). \end{aligned} \tag{3.8}$$

Therefore we have to look for $\tau > 0$, $\eta_1 > 0$, $\eta_2, \eta_3 \in \mathbb{R}$ such that:

$$\begin{aligned} w(\tau) &= 0, \\ w'(\tau) &= -\eta_1, \\ w''(\tau) &= -\eta_2, \\ w'''(\tau) &= -\eta_3. \end{aligned}$$

Solving the last 3 (linear) equations with respect to η_1, η_2, η_3, and setting $x := e^{-\tau}$, we find:

$$\boldsymbol{p} = \begin{bmatrix} 0 \\ \boldsymbol{\eta} \end{bmatrix} = \begin{bmatrix} 0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \frac{1}{6d} \begin{bmatrix} 0 \\ -\left(x^4 - x^3 - x^2 - x + 1\right)(x-1)^3 x \\ \left(x^4 - 4x^3 - 4x^2 - 4x + 1\right)(x-1)^3 x \\ -(x-1)\left(x^6 - 12x^5 + 17x^4 + 17x^2 - 12x + 1\right) x \end{bmatrix},$$

where $d := 4x^8 - 10x^7 + 12x^6 - 5x^5 + 6x^4 - 11x^3 + 11x^2 - 4x + 1$. Next, we solve $w(\tau) = 0$ with respect to τ. Through the substitution $x = e^{-\tau}$, this is equivalent to solving

$$\frac{\left(x^4 - 3x^3 - 4x^2 - 3x + 1\right)(x-1)^5}{24d} = 0,$$

with $x \in (0, 1)$.

Direct computation shows that the root of

$$p_4(x) := x^4 - 3x^3 - 4x^2 - 3x + 1$$

we are looking for is given by

$$x_\tau := \frac{1}{4}\left(3 + \sqrt{33} - \sqrt{26 + 6\sqrt{33}}\right), \tag{3.9}$$

and therefore:

$$\tau := -\log\left(\frac{1}{4}\left(3 + \sqrt{33} - \sqrt{26 + 6\sqrt{33}}\right)\right). \tag{3.10}$$

Many of the quantities that will follow are polynomial expressions in x evaluated at x_τ, and we recall that $p_4(x_\tau)$=0. To simplify the notation, we will express[3] those quantities in the minimal algebraic extension of $\mathbb{Q}$ containing the roots of p_4. We do, for instance, express $\boldsymbol{p}$ as:

$$\boldsymbol{p} = \frac{1}{24} \begin{bmatrix} 0 \\ -7x_\tau^3 + 31x_\tau^2 - 9x_\tau + 1 \\ 37x_\tau^3 - 165x_\tau^2 + 51x_\tau - 3 \\ -163x_\tau^3 + 727x_\tau^2 - 225x_\tau + 13 \end{bmatrix} = \frac{\sqrt{26 + 6\sqrt{33}}}{96} \begin{bmatrix} 0 \\ 29 - 5\sqrt{33} \\ 27\sqrt{33} - 155 \\ 683 - 119\sqrt{33} \end{bmatrix}. \tag{3.11}$$

Given the expression of $\boldsymbol{p}$ above, the function w in (3.8) reads as:

$$\begin{aligned} w(s) = \frac{1}{24}\Big[&(-2 + 3x_\tau - 13x_\tau^2 + 3x_\tau^3)e^{-4s} + (7 + 3x_\tau + 3x_\tau^2 - x_\tau^3)e^{-3s} \\ &+(-10 - 6x_\tau + 2x_\tau^2)e^{-2s} + (6 + 8x_\tau^2 - 2x_\tau^3)e^{-s} - 1\Big]. \end{aligned} \tag{3.12}$$

To finally conclude that w is a solution of (1.2) on $(0, \tau)$, we need to verify that $w(s) > 0$ for all $s \in (0, \tau)$. Through the transformation $x = e^{-s}$, this is equivalent to showing that the following polynomial is positive for $x \in (x_\tau, 1)$:

$$\begin{aligned} q(x) := &(-2 + 3x_\tau - 13x_\tau^2 + 3x_\tau^3)x^4 + (7 + 3x_\tau + 3x_\tau^2 - x_\tau^3)x^3 + (-10 - 6x_\tau + 2x_\tau^2)x^2 \\ &+(6 + 8x_\tau^2 - 2x_\tau^3)x - 1. \end{aligned} \tag{3.13}$$

[3]For instance, since $d = 4x^8 - 10x^7 + 12x^6 - 5x^5 + 6x^4 - 11x^3 + 11x^2 - 4x + 1 = (4x^4 + 2x^3 + 34x^2 + 117x + 495)p_4 + 2042x^3 + 2308x^2 + 1364x - 494$, evaluating d at x_τ yields the following expression: $d = 2042x_\tau^3 + 2308x_\tau^2 + 1364x_\tau - 494$.

We recall that, by construction, $x = 1$ and $x = x_\tau$ are roots of q. Therefore, q can be written as

$$q(x) = (1 - x)(x - x_\tau)q_2(x),$$

with

$$q_2(x) := (2 - 3x_\tau + 13x_\tau^2 - 3x_\tau^3)x^2 + (-2 - 13x_\tau - 5x_\tau^2 + 2x_\tau^3)x + 3 + 4x_\tau + 3x_\tau^2 - x_\tau^3.$$

We need to show that q_2 is positive on $(x_\tau, 1)$. This follows upon observing that q_2 is strictly positive for any $x \in \mathbb{R}$, since it has leading coefficient

$$2 - 3x_\tau + 13x_\tau^2 - 3x_\tau^3 = 1 + \left(3 - \frac{\sqrt{33}}{2}\right)\sqrt{26 + 6\sqrt{33}} > 0$$

and discriminant

$$\begin{aligned}&(-2 - 13x_\tau - 5x_\tau^2 + 2x_\tau^3)^2 - 4(2 - 3x_\tau + 13x_\tau^2 - 3x_\tau^3)(3 + 4x_\tau + 3x_\tau^2 - x_\tau^3) = \\ &= 27 - 129x_\tau + 9x_\tau^2 + 5x_\tau^3 = \frac{1}{2}\left(117 - 21\sqrt{33} - 8\sqrt{2\left(27\sqrt{33} - 155\right)}\right) < 0.\end{aligned}$$

This concludes the proof of existence.

Stability. Recall that, by virtue of Corollary 10, we have that the first component of any solution $\boldsymbol{w}(s) = \boldsymbol{\psi}(s, \boldsymbol{w}_0)$ of Eq (3.5) having initial condition $\boldsymbol{w}_0 = \begin{bmatrix} 0 \\ \boldsymbol{\eta} \end{bmatrix} = \begin{bmatrix} 0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}$, where $\boldsymbol{\eta} \in \mathbb{R}^3$ with $\eta_1 \neq 0$, will vanish at some $s_1 > 0$. In particular, for any $\eta_1 > 0$ there exists $s_1 = s_1(\boldsymbol{\eta}) > 0$ such that $w(s) = (\boldsymbol{w}(s))_1 > 0$ for all $s \in (0, s_1)$ and $w(s_1) = 0$. At such value of s_1, the solution has the following form: $\boldsymbol{\psi}(s_1, \boldsymbol{w}_0) = \begin{bmatrix} 0 \\ \boldsymbol{\xi} \end{bmatrix} = \begin{bmatrix} 0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}$, with $\boldsymbol{\xi} \in \mathbb{R}^3$. Therefore, we define the "first return map" as the map P that sends $\boldsymbol{\eta}$ to $\boldsymbol{\xi}$. More precisely:

$$\boldsymbol{\eta} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} \in \mathbb{R}^3, \eta_1 > 0 \mapsto P(\boldsymbol{\eta}) = \Pi\, \boldsymbol{\psi}\left(s_1(\boldsymbol{\eta}), \begin{bmatrix} 0 \\ \boldsymbol{\eta} \end{bmatrix}\right) \in \mathbb{R}^3, \tag{3.14}$$

where Π is the following projection matrix:

$$\Pi := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We will be interested in the behaviour of P in a neighbourhood of $\tilde{\boldsymbol{p}} := \Pi\, \boldsymbol{p}$. One can easily see by direct computation that the vector field (3.5) is transversal to the set $\Sigma = \{\boldsymbol{x} \in \mathbb{R}^4 : (\boldsymbol{x})_1 = 0\}$ at $\tilde{\boldsymbol{p}}$.

A straightforward application of the Implicit Function Theorem yields that the maps $\boldsymbol{\eta} \mapsto s_1(\boldsymbol{\eta})$ and (consequently) P are smooth in a neighbourhood of $\tilde{\boldsymbol{p}}$. Note that $P(\tilde{\boldsymbol{p}}) = -\tilde{\boldsymbol{p}}$, or equivalently $-P(\tilde{\boldsymbol{p}}) = \tilde{\boldsymbol{p}}$. Because of the symmetry (3.6), the map $-P \circ -P$, appropriately restricted to a neighbourhood of $\tilde{\boldsymbol{p}}$, acts as Poincaré map for the periodic orbit that corresponds to the periodic solution $\boldsymbol{\gamma}$. Therefore, to conclude the proof we need to show that the Jacobian matrix of $-P \circ -P$ evaluated at $\tilde{\boldsymbol{p}}$ has all its eigenvalues strictly inside the unit circle, from which local stability of the periodic solution follows. To do so, it is enough to show that all the eigenvalues of the Jacobian matrix of P have modulus strictly less than 1.

Recall that $\boldsymbol{\psi} = \boldsymbol{\psi}(s, \boldsymbol{w}_0)$ is the flow of (3.5). We have:

$$\begin{aligned} \frac{\partial}{\partial \boldsymbol{\eta}} P(\boldsymbol{\eta}) &= \frac{\partial}{\partial \boldsymbol{\eta}}\left(\Pi\, \boldsymbol{\psi}\left(s_1(\boldsymbol{\eta}), \begin{bmatrix} 0 \\ \boldsymbol{\eta} \end{bmatrix}\right)\right) = \Pi\, \frac{\partial}{\partial \boldsymbol{\eta}} \boldsymbol{\psi}\left(s_1(\boldsymbol{\eta}), \begin{bmatrix} 0 \\ \boldsymbol{\eta} \end{bmatrix}\right) = \\ &= \Pi\left(\frac{\partial}{\partial s}\boldsymbol{\psi}\left(s, \begin{bmatrix} 0 \\ \boldsymbol{\eta} \end{bmatrix}\right)\right)_{s=s_1(\boldsymbol{\eta})} \frac{\partial s_1}{\partial \boldsymbol{\eta}} + \Pi\left(\frac{\partial}{\partial \boldsymbol{\eta}}\boldsymbol{\psi}\left(s, \begin{bmatrix} 0 \\ \boldsymbol{\eta} \end{bmatrix}\right)\right)_{s=s_1(\boldsymbol{\eta})}. \end{aligned} \tag{3.15}$$

Taking into account that $\frac{\partial}{\partial s}\boldsymbol{\psi}\left(s, \begin{bmatrix} 0 \\ \boldsymbol{\eta} \end{bmatrix}\right) = A\boldsymbol{\psi}\left(s, \begin{bmatrix} 0 \\ \boldsymbol{\eta} \end{bmatrix}\right) - \boldsymbol{e}_4$ for all $s \in (0, s_1(\boldsymbol{\eta}))$, and that $\boldsymbol{\psi}(\tau, \boldsymbol{p}) = -\boldsymbol{p}$, we have that the Jacobian matrix of P at $\tilde{\boldsymbol{p}}$ is given by:

$$\left(\frac{\partial}{\partial \boldsymbol{\eta}} P(\boldsymbol{\eta})\right)_{\eta=\tilde{p}} = \Pi\,(-A\boldsymbol{p} - \boldsymbol{e}_4)\left(\frac{\partial s_1}{\partial \boldsymbol{\eta}}\right)_{\eta=\tilde{p}} + \Pi\left(\frac{\partial}{\partial \boldsymbol{\eta}}\boldsymbol{\psi}\left(s, \begin{bmatrix} 0 \\ \boldsymbol{\eta} \end{bmatrix}\right)\right)_{s=\tau,\, \eta=\tilde{p}}. \tag{3.16}$$

Through simple but tedious computations, we obtain explicit expressions for the Jacobian matrix in (3.16) and for its eigenvalues $\{\mu_1, \mu_2, \mu_3\}$.

In fact, to compute $\left(\frac{\partial s_1}{\partial \boldsymbol{\eta}}\right)_{\eta=\tilde{p}}$, we apply the Implicit Function Theorem to the equation $w(s_1(\boldsymbol{\eta})) = 0$, obtaining

$$\left(\frac{\partial s_1}{\partial \boldsymbol{\eta}}\right)_{\eta=\tilde{p}} = \frac{1}{m}\begin{bmatrix} x_\tau\,(x_\tau - 1)\left(11x_\tau^2 - 31x_\tau + 26\right) \\ x_\tau\,(6x_\tau - 9)\,(x_\tau - 1)^2 \\ x_\tau\,(x_\tau - 1)^3 \end{bmatrix}, \tag{3.17}$$

where

$$\begin{aligned} m = {} & (4(\boldsymbol{p})_4 + 24(\boldsymbol{p})_3 + 44(\boldsymbol{p})_2 + 1)\,x_\tau^4 - (9(\boldsymbol{p})_4 + 63(\boldsymbol{p})_3 + 126(\boldsymbol{p})_2 + 3)\,x_\tau^3 + \\ & (6(\boldsymbol{p})_4 + 48(\boldsymbol{p})_3 + 114(\boldsymbol{p})_2 + 3)\,x_\tau^2 - ((\boldsymbol{p})_4 + 9(\boldsymbol{p})_3 + 26(\boldsymbol{p})_2 + 1)\,x_\tau. \end{aligned} \tag{3.18}$$

Equivalently, we can write (3.17) as:

$$\left(\frac{\partial s_1}{\partial \boldsymbol{\eta}}\right)_{\eta=\tilde{p}} = \begin{bmatrix} 83x_\tau^3 + 117x_\tau^2 + 91x_\tau - 4 \\ \frac{99}{2}x_\tau^3 + 63x_\tau^2 + 45x_\tau - \frac{15}{2} \\ 10x_\tau^3 + 12x_\tau^2 + 8x_\tau - 2 \end{bmatrix}.$$

Therefore, we have the following expression for $J_1 := \Pi\,(-A\boldsymbol{p} - \boldsymbol{e}_4)\left(\frac{\partial s_1}{\partial \boldsymbol{\eta}}\right)_{\eta=\tilde{p}}$:

$$J_1 = \begin{bmatrix} \frac{121}{6}x_\tau^3 - \frac{475}{6}x_\tau^2 - \frac{3}{2}x_\tau + \frac{9}{2} & \frac{19}{2}x_\tau^3 - 36x_\tau^2 - \frac{9}{2}x_\tau + 3 & \frac{4}{3}x_\tau^3 - \frac{29}{6}x_\tau^2 - x_\tau + \frac{1}{2} \\ -90x_\tau^3 + \frac{2087}{6}x_\tau^2 + \frac{37}{6}x_\tau - \frac{58}{3} & -\frac{85}{2}x_\tau^3 + 158x_\tau^2 + \frac{39}{2}x_\tau - 13 & -6x_\tau^3 + \frac{127}{6}x_\tau^2 + \frac{13}{3}x_\tau - \frac{13}{6} \\ \frac{1117}{6}x_\tau^3 - \frac{9997}{6}x_\tau^2 - \frac{317}{2}x_\tau + \frac{263}{2} & 68x_\tau^3 - 783x_\tau^2 - \frac{315}{2}x_\tau + \frac{165}{2} & \frac{10}{3}x_\tau^3 - \frac{677}{6}x_\tau^2 - 33x_\tau + \frac{29}{2} \end{bmatrix}.$$

Differentiating (3.8) and making the appropriate substitutions, we obtain the following explicit expression for $J_2 := \Pi\left(\frac{\partial}{\partial \boldsymbol{\eta}}\boldsymbol{\psi}\left(s, \begin{bmatrix} 0 \\ \boldsymbol{\eta} \end{bmatrix}\right)\right)_{s=\tau, \boldsymbol{\eta}=\tilde{\boldsymbol{p}}}$:

$$J_2 = x_\tau \begin{bmatrix} \frac{22}{3}x_\tau^3 - 21x_\tau^2 + 19x_\tau - \frac{13}{3} & 4x_\tau^3 - \frac{21}{2}x_\tau^2 + 8x_\tau - \frac{3}{2} & \frac{2}{3}x_\tau^3 - \frac{3}{2}x_\tau^2 + x_\tau - \frac{1}{6} \\ -\frac{1}{3}(x_\tau - 1)^2 (88\, x_\tau - 13) & -16x_\tau^3 + \frac{63}{2}x_\tau^2 - 16x_\tau + \frac{3}{2} & -\frac{8}{3}x_\tau^3 + \frac{9}{2}x_\tau^2 - 2x_\tau + \frac{1}{6} \\ \frac{352}{3}x_\tau^3 - 189x_\tau^2 + 76x_\tau - \frac{13}{3} & 64x_\tau^3 - \frac{189}{2}x_\tau^2 + 32x_\tau - \frac{3}{2} & \frac{32}{3}x_\tau^3 - \frac{27}{2}x_\tau^2 + 4x_\tau - \frac{1}{6} \end{bmatrix}. \tag{3.19}$$

Finally, we have:

$$J_1 + J_2 = \begin{bmatrix} \frac{127}{6}x_\tau^3 - \frac{185}{6}x_\tau^2 + \frac{97}{6}x_\tau - \frac{17}{6} & 11x_\tau^3 - 12x_\tau^2 + 6x_\tau - 1 & \frac{11}{6}x_\tau^3 - \frac{7}{6}x_\tau^2 + \frac{5}{6}x_\tau - \frac{1}{6} \\ -115x_\tau^3 + \frac{385}{2}x_\tau^2 - \frac{155}{2}x_\tau + 10 & -59x_\tau^3 + 78x_\tau^2 - 27x_\tau + 3 & -\frac{19}{2}x_\tau^3 + \frac{17}{2}x_\tau^2 - \frac{7}{2}x_\tau + \frac{1}{2} \\ \frac{2095}{6}x_\tau^3 - \frac{6725}{6}x_\tau^2 + \frac{1135}{6}x_\tau + \frac{85}{6} & \frac{331}{2}x_\tau^3 - 495x_\tau^2 + 33x_\tau + \frac{37}{2} & \frac{131}{6}x_\tau^3 - \frac{397}{6}x_\tau^2 - \frac{7}{6}x_\tau + \frac{23}{6} \end{bmatrix},$$

whose eigenvalues can be explicitly computed and are given by:

$$\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = \begin{bmatrix} -x_\tau \\ -x_\tau^4 \\ -x_\tau^5 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} - \frac{1}{4}\sqrt{33} + \frac{1}{4}\sqrt{26 + 6\sqrt{33}} \\ -\frac{289}{4} - \frac{51}{4}\sqrt{33} + \frac{75}{8}\sqrt{26 + 6\sqrt{33}} + \frac{13}{8}\sqrt{33}\sqrt{26 + 6\sqrt{33}} \\ \frac{27}{4}\sqrt{33}\sqrt{26 + 6\sqrt{33}} + \frac{77}{2}\sqrt{26 + 6\sqrt{33}} - \frac{209}{4}\sqrt{33} - \frac{1203}{4} \end{bmatrix} \approx \begin{bmatrix} -2.4 \times 10^{-1} \\ -3.4 \times 10^{-3} \\ -8.3 \times 10^{-4} \end{bmatrix}. \tag{3.20}$$

This shows that $|\mu_j| < 1$ for $j = 1, 2, 3$, and concludes the proof. □

Remark 18. In [5, Conjecture 1], the authors conjecture the existence of a periodic solution for all differential equations of the form

$$w^{(4)} + 10w''' + 35w'' + 50w' + 24w + w|w|^{q-1} = 0, \text{ with } q > 1,$$

and also speculate on the explicit expression for its Floquet exponents. Theorem 14 and the expressions for the Floquet multipliers (3.20) provide a proof of this conjecture for the case $q = 0$.

3.1. Fourth order equation with lower order terms

We now consider Eq (1.6), that we rewrite below for convenience:

$$u^{(4)} - \kappa(u'')^2 - \lambda u' + \operatorname{sgn}(u) = 0. \tag{1.6}$$

Through transformation (1.3), the equation becomes

$$L(w) - \kappa e^{4s} K(w) - \lambda e^{3s} \Lambda(w) + \operatorname{sgn}(w) = 0, \tag{3.21}$$

where:

$$\begin{aligned} L(w) &:= w^{(4)} + 10w''' + 35w'' + 50w' + 24w, \\ \Lambda(w) &:= w' + 4w, \\ K(w) &:= (w'' + 7w' + 12w)^2 . \end{aligned} \tag{3.22}$$

The notion of solution of (1.6) and (3.21) is analogous to the one given in Definition 2. For $\lambda = 0$ and $\kappa = 0$, equation (3.21) corresponds to (1.2) and, therefore, admits the periodic solution stated in Theorem 14.

Theorem 19. *Let $\boldsymbol{\gamma} = \boldsymbol{\gamma}(s)$ be the periodic solution of* (1.2) *stated in Theorem 14. There exist $\kappa_0 > 0$ and $\lambda_0 > 0$ such that for any $\kappa \in (-\kappa_0, \kappa_0)$ and $\lambda \in (-\lambda_0, \lambda_0)$ Eq* (3.21) *admits solutions w with the following property: $w^{(j)}(s) - \gamma^{(j)}(s) \to 0$ as $s \to -\infty$, for $j = 0, 1, 2, 3$.*

Proof. Let $\boldsymbol{\gamma} = \boldsymbol{\gamma}(s)$ be the periodic solution of (1.2) as in (3.7), and let $\gamma := (\boldsymbol{\gamma})_1$. To prove the Theorem, it is convenient to introduce two further unknowns and equations, as follows:

$$\begin{cases} L(w) - yK(w) - z\Lambda(w) + \operatorname{sgn}(w) = 0, \\ y' = 4y, \text{ with } y(0) = \kappa, \\ z' = 3z, \text{ with } z(0) = \lambda. \end{cases} \tag{3.23}$$

This turns (3.21) back into an autonomous problem, that can be rewritten as a first order equation of dimension 6:

$$\frac{d}{ds}\begin{bmatrix} w \\ w' \\ w'' \\ w''' \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -24 & -50 & -35 & -10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} w \\ w' \\ w'' \\ w''' \\ y \\ z \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ \operatorname{sgn}(w) - y(w'' + 7w' + 12w)^2 - z(w' + 4w) \\ 0 \\ 0 \end{bmatrix}. \tag{3.24}$$

Clearly the orbit $\widehat{\boldsymbol{\gamma}} := \begin{bmatrix} \boldsymbol{\gamma} \\ 0 \\ 0 \end{bmatrix}$ is a periodic solution of (3.24), corresponding to the initial condition $\widehat{\boldsymbol{p}} := \begin{bmatrix} \boldsymbol{p} \\ 0 \\ 0 \end{bmatrix}$, where $\boldsymbol{p}$ is given (3.11). In order to study the stability of $\widehat{\boldsymbol{\gamma}}$, we need to appropriately extend the first return map P defined in (3.14) to a neighbourhood of $\widehat{\boldsymbol{p}}$ in $\{0\} \times \mathbb{R}^5$. This leads to the map

$$\widehat{\boldsymbol{\eta}} = \begin{bmatrix} \boldsymbol{\eta} \\ \zeta_1 \\ \zeta_2 \end{bmatrix} \in \mathbb{R}^5, \boldsymbol{\eta} \in \mathbb{R}^3 \text{ with } (\boldsymbol{\eta})_1 > 0 \mapsto \widehat{P}(\widehat{\boldsymbol{\eta}}) = \widehat{\Pi}\,\widehat{\boldsymbol{\psi}}\left(\widehat{s_1}(\widehat{\boldsymbol{\eta}}), \begin{bmatrix} 0 \\ \widehat{\boldsymbol{\eta}} \end{bmatrix}\right) \in \mathbb{R}^5, \tag{3.25}$$

where $\widehat{\Pi}$, $\widehat{\boldsymbol{\psi}}$ and $\widehat{s_1}$ are straightforward extensions of, respectively, Π, $\boldsymbol{\psi}$ and s_1 defined on page 12.

Direct computation yields:

$$\left(\frac{\partial}{\partial\widehat{\boldsymbol{\eta}}}\widehat{P}(\widehat{\boldsymbol{\eta}})\right)_{\widehat{\eta}=\widehat{p}} = \left[\begin{array}{ccc|cc} & & & \times & \times \\ & \left(\frac{\partial}{\partial\eta}P(\boldsymbol{\eta})\right)_{\eta=\tilde{p}} & & \times & \times \\ & & & \times & \times \\ \hline 0 & 0 & 0 & 4\tau & 0 \\ 0 & 0 & 0 & 0 & 3\tau \end{array}\right], \tag{3.26}$$

where $\tau \approx 1.4$ is given in (3.10) and $\widetilde{\boldsymbol{p}}$ is defined on page 12. The stability properties of $\widehat{\boldsymbol{\gamma}}$ are related to the eigenvalues of the Jacobian matrix of the Poincaré map $S \circ \widehat{P} \circ S \circ \widehat{P}$, where

$$S := \left[\begin{array}{c|c} -I_3 & \\ \hline & I_2 \end{array}\right],$$

and I_n is the $n \times n$ identity matrix. From expression (3.26), it follows that three of these eigenvalues are those given in (3.20) and have modulus smaller than 1, while the other two are given by 4τ and 3τ, and have modulus greater than 1. Therefore, the solution of (3.24) through $\widehat{\boldsymbol{p}}$ is a hyperbolic solution of period 2τ and possesses (locally) a three-dimensional unstable manifold and a four-dimensional stable manifold in $\mathbb{R}^6$. Solutions of (3.24) that have initial conditions on the local unstable manifold approach the periodic solution γ backwards in time and, through (3.23), provide the solutions of (3.21) whose existence we claim in the Theorem. Because of the local nature of the unstable manifold, initial conditions have to lie in a sufficiently small neighbourhood of the periodic orbit $\widehat{\boldsymbol{\gamma}}$, and this translates into the requirement that $|\lambda|$ and $|\kappa|$ need to be sufficiently small. □

As a corollary, we have the following Theorem.

Theorem 20. *For any κ, λ and $T \in \mathbb{R}$ there exist solutions of* (1.6) *that disappear oscillating at T and solutions of* (1.6) *that appear oscillating at T.*

Proof. By virtue of Theorem 19, if $\lambda \in (-\lambda_0, \lambda_0)$ and $\kappa \in (-\kappa_0, \kappa_0)$, there exists $w = w(s)$ solution of (3.21) that approaches asymptotically backwards in time the non trivial periodic solution γ of Theorem 14. Then, using (1.3), we have that $u(t) = (1+t)^4 w(\ln(1+t))$ solves Eq (1.6) and appears at -1. Since (1.6) is autonomous, we can easily construct a solution that appears at any arbitrary time $T \in \mathbb{R}$. To remove the restriction on the size of $|\lambda|$ and $|\kappa|$, we first observe that u solves (1.6) if and only if

$$u_\alpha(t) = \alpha^4(u(t/\alpha))$$

solves

$$-\frac{\lambda}{\alpha^3}u'_\alpha = -u_\alpha^{(4)} + \frac{\kappa}{\alpha^4}(u''_\alpha)^2 - \operatorname{sgn}(u_\alpha).$$

Now, fix arbitrarily λ, κ and T in $\mathbb{R}$, choose $\alpha > 0$ such that $\alpha^3\lambda \in (-\lambda_0, \lambda_0)$ and $\alpha^4\kappa \in (-\kappa_0, \kappa_0)$, and consider u that solves (1.6) for parameters' values $(\alpha^3\lambda, \alpha^4\kappa)$ and appears at αT, which exists by virtue of the considerations above. Then, $u_\alpha(t)$ solves (1.6) for parameters' values (λ, κ) and appears at T.

Reversing the direction of time (i.e., considering $u(-t)$), we obtain solutions of (1.6) corresponding to arbitrary pairs $(-\lambda, \kappa)$ that disappear at arbitrary $-T$. □

Remark 21. Just as we pointed out in Remark 6, disappearing solutions of (1.6) can be appropriately glued to appearing solutions. Therefore, as a corollary of Theorem 20, we have that for any κ, λ and $T_0 \leq T_1 \in \mathbb{R}$ there exist solutions of (1.6) (and therefore also of (1.1)) that *disappear* at T_0 and *appear* at T_1.

4. Further results

We begin this Section with some considerations on (dis)appearance and oscillations in more general equations.

4.1. (Dis)appearance and oscillations in more general equations

In Section 3 we showed that all solutions of (1.1) that disappear or appear must do so oscillating according to Definition 7. A question that naturally arises is to which extent oscillations characterize appearance and disappearance of solutions in equations of the form

$$\pm\ u^{(k)} = \mathrm{sgn}(u), \tag{4.1}$$

for an arbitrary integer $k \geq 1$; or even, more generally, in equations of the form

$$\pm\ u^{(k)} = f(u). \tag{4.2}$$

This kind of equations arises in several contexts, see [9, 10] for a detailed account, and [4, 7] for related results.

From here on, we require

$$\begin{aligned} &\text{i) } f = f(x) \text{ locally bounded and measurable;} \\ &\text{ii) } f(0) = 0 \text{ and } f(x)x > 0 \text{ for all } x \neq 0. \end{aligned} \tag{4.3}$$

Note that the class of functions f we are considering here includes the nonlinearity in Eq (1.1) and more generally all functions of the form $f(x) = x|x|^{q-1}$ with $q \geq 0$. In what follows, by solution of (4.1) or (4.2) we mean a function $u : I \to \mathbb{R}$ of class C^{k-1} such that $u^{(k-1)}$ is locally absolutely continuous and satisfies $u^{(k)}(t) = f(u(t))$ almost everywhere.[4] Generalizing Definitions 5 and 7, we say that a solution u of Eqs (4.1) or (4.2) *disappears* at T if u is nontrivial on any left neighbourhood of T and $\lim\limits_{t\to T^-} u^{(j)}(t) = 0$ for $j = 0, \ldots, k-1$, and we say that u disappears *oscillating* at T if u disappears at T and changes sign infinitely many times on any left neighbourhood of T. In addition, we say that u disappears *monotonically* at T if it disappears at T and is monotone in a left neighbourhood of T. Analogous definitions hold for solutions that appear (oscillating or monotonically) at some T.

Remark 22. We note that, for all integers $k \geq 1$, equations

$$u^{(k)} = \mathrm{sgn}(u) \tag{4.4}$$

admit the following solutions that appear monotonically at $T = 0$:

$$u_k(t) := \frac{|t|^k + |t|^{k-1}t}{2k!}. \tag{4.5}$$

[4] Or in distributional sense on I, see also footnote 1.

For k even, time reversal invariance yields monotonically disappearing solutions $u_k(-t)$ of (4.4). Functions of the form (4.5) are not useful in the "monotone disappearance" case when k is odd. In fact, interestingly, we will see (it is a straightforward consequence of Theorem 23) that, for k odd, equations of the form

$$u^{(k)} = f(u)$$

admit no solutions that disappear monotonically.

We now turn our attention to equations of the following form:

$$-u^{(k)} = f(u). \tag{4.6}$$

The case $k = 4$ and $f(x) = \text{sgn}(x)$ has been one of the main objects of investigation of this work, and we know from Theorem 12 that solutions of (4.6) that disappear and/or appear must do so oscillating. On the other hand, for k odd, $u_k(-t)$, where u_k has been defined in (4.5), is a monotonically disappearing solution of (4.6) with $f(x) = \text{sgn}(x)$. It is natural to investigate in more detail the role played by k in the "monotone vs. oscillating" dichotomy of disappearing and appearing solutions of (4.6). It turns out that monotonicity only pertains to disappearing solutions with k odd. More precisely, we have the following

Theorem 23. *For all integers $k \geq 1$, solutions of* (4.6) *that appear, do so oscillating. Consequently, if k is even solutions that disappear do so oscillating.*

Proof. Since the equations are autonomous, there is no loss of generality in considering solutions that appear at 0. Suppose by contradiction that u is a solution of (4.6) that appears at 0 and is such that

$$u(t) \geq 0 \text{ for all } t \text{ in } (0, \delta), \text{ for some } \delta > 0.$$

Integrating (4.6) on $[0, t]$ and using $u^{(k-1)}(t) \to 0$ as $t \to 0^+$, we have

$$-u^{(k-1)}(t) = \int_0^t f(u(x))\, dx > 0, \text{ for all } t \text{ in } (0, \delta),$$

since $f(u(x))$ is not trivial in any neighborhood of 0. If $k \geq 2$, integrating the previous inequality $k - 1$ more times, and using $u^{(j)}(t) \to 0$ as $t \to 0^+$ for $j = k - 2, \dots, 0$, we get $u(t) < 0$ on $(0, \delta)$, a contradiction. For k even we obtain the sought result by time reversal. □

Remark 24. We want to point out that, if u changes sign infinitely many times, then also u' must change sign infinitely many times. Therefore, a more general definition for solutions that *appear oscillating* at T could have been that u appears at T and u' changes sign in any right neighbourhood of T. However, arguing as in the proof of Theorem 23 above, one can see that, if u is a solution of (4.2) that appears at T and is such that u' changes sign infinitely many times in any right neighbourhood of T, then also u changes sign infinitely many times in any right neighbourhood of T. The same argument allows to conclude that appearing solutions only have the following two alternatives: to appear oscillating or to appear monotonically.

We now present an *ad hoc* analysis of the cases $k = 1, 2, 3, 4$ for (4.1), whose notable consequence is the following

Theorem 25. *Among all equations of the form*

$$\pm u^{(k)} = f(u),$$

$k = 4$ is the smallest order for which there exists a function f such that the equation admits both solutions that appear oscillating and *solutions that disappear oscillating.*

In what follows we will often only consider appearance and disappearance at 0. We recall that there is no loss of generality in doing so, since all equations are autonomous. We will also make frequent use of the antiderivative of f, that we denote by F:

$$F(t) := \int_0^t f(x)\,dx. \tag{4.7}$$

Note that we have $F(t) > 0$ for all $t \neq 0$ because of our hypotheses on f.

First order. A straightforward qualitative analysis yields the following

Theorem 26. *Let $u : I \to \mathbb{R}$ be a solution of*

$$u' = f(u), \tag{4.8}$$

then the only alternatives are:

i) u is strictly positive and monotone increasing,

ii) u is strictly negative and monotone decreasing,

iii) u is a monotonically appearing solution.

In particular no solution can appear or disappear oscillating, and no solution has compact support.

Analogous conclusions easily follow for $-u' = f(u)$. We omit the details.

Remark 27. In particular, it is easy to see that all global solutions of $u' = \operatorname{sgn}(u)$ are a translation of $\pm u_1$, where u_1 is defined in (4.5), and therefore all are appearing solutions. Similarly, all global solutions of $-u' = \operatorname{sgn}(u)$ are a translation of $\pm u_1(-t)$, and therefore all are disappearing solutions.

Second order. We split the analysis of this case into two Theorems.

Theorem 28. *Solutions of*

$$u'' = f(u) \tag{4.9}$$

that appear or disappear, do so monotonically.

Proof. Let u be a solution of (4.9) that appears at $T = 0$. Multiplying (4.9) by u and integrating by parts we have:

$$\int_0^t f(u(s))u(s)ds = \int_0^t u''(s)u(s)ds = u'(t)u(t) - u'(0)u(0) - \int_0^t (u'(s))^2\,ds.$$

Since u is nontrivial in a right neighbourhood of 0, for $t > 0$ we have

$$u'(t)u(t) = \int_0^t f(u(s))u(s)ds + \int_0^t (u'(s))^2\,ds > 0,$$

which implies that u' does not change sign for $t > 0$. By time reversal we get the same conclusion for disappearing solutions. □

Theorem 29. *Equation*

$$-u'' = f(u) \tag{4.10}$$

has no solutions that appear or disappear.

Proof. Suppose by contradiction that u is a solution of (4.10) that appears at 0. Multiplying (4.10) by u' and integrating by parts we get

$$F(u(t)) = F(u(t)) - F(u(0)) = \int_0^t f(u(s))u'(s)\,ds = -\int_0^t u''(s)u'(s)\,ds = \\ = -\frac{1}{2}\left((u'(t))^2 - (u'(0))^2\right) = -\frac{1}{2}(u'(t))^2 .$$

But $F(u(t)) \geq 0$ for all $t > 0$, therefore $u'(t) = 0$. Since t is arbitrary, we get that u is trivial, a contradiction. □

Third order. Also for this case, we split the analysis into two Theorems.

Theorem 30. *Solutions of*

$$u''' = f(u)\,. \tag{4.11}$$

that appear, do so monotonically.

Proof. Multiply the equation by u' and integrate on $[0, t]$ to get

$$u''(t)u'(t) - u''(0)u'(0) - \int_0^t (u''(s))^2\,ds = \int_0^t u'''(s)u'(s)\,ds = \\ = \int_0^t f(u(s))u'(s)\,ds = F(u(t)) - F(u(0)).$$

This means that

$$u''(t)u'(t) = \int_0^t (u''(s))^2\,ds + F(u(t)) > 0,$$

and implies $u'(t) \neq 0$. Since $t > 0$ is arbitrary, we get the claim. □

As a consequence of Theorems 23 and 30, we have that solutions of

$$-u''' = f(u)\,, \tag{4.12}$$

that appear (if any exist) do so oscillating, while solutions that disappear (if any exist) do so monotonically.

A concrete case of equation of the form (4.12) in which solutions that appear/disappear do in fact exist is the following:

$$-u''' = \mathrm{sgn}(u)\,. \tag{4.13}$$

Indeed, the function $u_3(-t)$ (see (4.5)) and its translations provide solutions that disappear. Instead, solutions that appear oscillating can be explicitly constructed applying to (4.13) suitable modifications of transformations (1.3) and (1.4), namely

$$u(t) = (1+t)^3 w(\ln(1+t)) \text{ and } w(s) = e^{-3s}u(e^s - 1). \tag{4.14}$$

If u solves (4.13), then w given by (4.14) solves

$$w''' + 6w'' + 11w' + 6w + \mathrm{sgn}(w) = 0\,. \tag{4.15}$$

The very same arguments used in Section 3 for the fourth order Eq (1.2) allow to prove the following result. We omit the details of the proof.

Theorem 31. *Equation* (4.15) *has an asymptotically stable periodic solution with initial conditions*

$$w(0) = 0,\ w'(0) = w''(0) = \frac{\sqrt{5}}{30}\,,$$

period $\tau = 2\,\mathrm{arcosh}(3/2)$ *and nontrivial Floquet exponents* -1 *and* -5.

This implies that both (4.13) and (4.15) have solutions that appear oscillating.

Remark 32. In [10, Conjecture 3.1] the authors formulate a Conjecture about a higher order ODE, of which Eq (4.15) is a special case. The Conjecture is supported by numerical experiments and proved in a particular case. Theorem 31 provides a proof of that Conjecture for Eq (4.15), a case that was left open.

Fourth order. Equation $-u^{(4)} = \mathrm{sgn}(u)$ has been extensively studied throughout the work. To complete the analysis, we state the following

Theorem 33. *Solutions of*

$$u^{(4)} = f(u) \tag{4.16}$$

that appear or disappear, do so monotonically.

Proof. We shall show that, if a solution u appears at 0, then we have that $u'(t) \neq 0$ for any $t > 0$. The argument goes as follows.
Step 1. Let $a < b$ be such that $u(a) = u'(a) = u''(a) = 0$ and $u'(b) = 0$, then $u(b) = u''(b) = 0$. In fact, multiplying the equation by u' and integrating by parts we have

$$F(u(b)) = u'''(b)u'(b) - \frac{1}{2}\,(u(b)'')^2 = -\frac{1}{2}\,(u(b)'')^2 \geq 0.$$

We deduce that $u''(b) = 0$ and $F(u(b)) = 0$, which implies $u(b) = 0$.
Step 2. Let $a < b$ be such that $u(a) = u'(a) = u''(a) = 0$ and $u'(b) = 0$. Then there exists a decreasing sequence $(b_n)_{n\geq 1}$ such that $b_n \to a^+$ and $u(b_n) = u'(b_n) = u''(b_n) = 0$ for all $n \geq 1$. In fact, from step 1 we have that $u(a) = u(b) = 0$. Applying Rolle's Theorem we have that $u'(b_1) = 0$ for some $b_1 \in (a, b)$. Applying again step 1 to a and b_1 we get that b_1 is the first element of the sequence. Iterating the argument above to the pair a, b_{n-1} for $n = 2, 3, \ldots$ we obtain the sequence recursively.
Step 3. Let $u(0) = u'(0) = u''(0) = 0$ and $u'(b) = 0$. Then, $u(t) = 0$ for $0 \leq t \leq b$. In fact, let us define

$$A := \{t \in [0, b] : u'(t) = 0\}.$$

By step 1 we have

$$A = \{t \in [0, b] : u(t) = u'(t) = u''(t) = 0\}.$$

Note that $0, b \in A$. We claim that the set A is dense in $[0, b]$. If not, then there would exist an open maximal interval (c, d) such that $(c, d) \cap A = \emptyset$. It follows by the continuity of u' and the maximality of (c, d) that $u'(c) = 0$ (otherwise, $u'(c) \neq 0$ would imply $u'(t) \neq 0$ for t in a left neighbourhood of c, contradicting the maximality of (c, d)). Applying step 2 to the pair c, b we have that there exists a sequence $(b_n)_{n\geq 1}$ such that $b_n \to c^+$ and $u'(b_n) = 0$ for all $n \geq 1$. That is, $b_n \in (c, d) \cap A$ for n large enough. This is a contradiction, and concludes the argument. □

Summing up the previous results, we can state that for the cases $k = 1, 2, 3, 4$ no solution of (4.2) can have compact support. This is a property we have already observed for Eq (1.1), see iii) of Remark 8. In the general case, we have the following

Theorem 34. *i) Let u be a solution of*

$$\pm u^{(k)} = f(u)$$

that appears at 0. If u appears monotonically, then $u'(t)$ does not change sign for $t > 0$ and, hence, u cannot disappear at any $T > 0$.

ii) Let u be a solution of

$$-u^{(k)} = f(u)$$

that appears at 0, with k odd or $k = 4m$ with m a positive integer. Then u cannot disappear at any $T > 0$.

iii) Let u be a solution of

$$u^{(4m+2)} = f(u)$$

that appears at 0, with m a positive integer. Then u cannot disappear at any $T > 0$.

Proof. Ideas and tools needed to prove these results have been extensively used throughout this section. Therefore, we only provide a sketch of the proofs and leave the details to the reader.

For case i), we note that if $u : I \to \mathbb{R}$ appears monotonically at 0, then integrating the equation k times we get that either $u' > 0$ or $u' < 0$ on $I \cap (0, +\infty)$. It follows that u cannot disappear.

For the remaining cases we distinguish the cases k odd and k even. Let $m \in \mathbb{N}$ and $k = 2m + 1$. Multiplying the equation by u' and integrating m times by parts we have

$$0 = F(u(T)) - F(u(0)) = (-1)^m \int_0^T \left((u^{(m)}\right)^2,$$

which contradicts the assumption.

Now let $k = 4m$. Multiplying the equation by u and integrating $2m$ times by parts we have

$$0 < \int_0^T f(u)u = -\int_0^T \left(u^{(2m)}\right)^2.$$

The case $k = 4m + 2$ is handled in a similar way. □

4.2. Open questions

The results of this section provide a picture for the behaviour of solutions of Eq (4.2), as far as "appearance and/or disappearance" and "oscillating vs monotonically" is concerned. We wish to point out that this picture is far from being complete. Below we list some open questions.

The third order Eq (4.13) admits solutions that appear oscillating and solutions that disappear monotonically, and all appearing (resp. disappearing) solutions share the same behaviour. A question that arises naturally is whether there exist an integer k and a function f satisfying (4.3) such that one of the Eq (4.2) admits –both– solutions that appear monotonically and solutions that appear oscillating.

Another question that arises naturally upon reading Theorem 34 is whether there exist an integer k and a function f satisfying (4.3) such that one of the Eq (4.1) admits a solution with compact support.

Furthermore, let k be an integer and f be a function satisfying (4.3). Is it true that all the appearing solutions of

$$u^{(k)} = f(u)$$

appear monotonically? A positive answer for $k = 1, 2, 3, 4$ is given by Theorems 26, 28, 30 and 33 respectively, for any f. These results provide evidence that the answer may be positive for all k, but the cases $k \geq 5$ are open. It would be interesting to solve the question at least for some special classes of functions f, e.g., f non decreasing with $f(0) = 0$ and $f(x)x > 0$ for all $x \neq 0$, or $f(x) = x|x|^{q-1}$ with $q \geq 0$.

The phenomenon of appearance of solutions for equations

$$-u''' = \operatorname{sgn}(u) \quad \text{and} \quad -u^{(4)} = \operatorname{sgn}(u)$$

relies on fact that certain "auxiliary" equations possess nontrivial periodic solutions. See, respectively, Theorems 31 and 14. We believe that the picture could be much the same for all equations

$$-u^{(k)} = \operatorname{sgn}(u)$$

with $k \geq 5$. In fact, making use of the very same ideas and tools used to prove Theorem 14, also for orders $k = 5, 6, 7$ we obtain existence of a periodic solution for the corresponding auxiliary equations, see (5.3) below. Moreover, our approach allows us to reveal two distinct periodic solutions for the case $k = 7$, see Theorem 37. Since the proofs are very similar to those given in Section 3, we present the respective statements in Section 5.

We have performed a substantial number of numerical experiments on the "uniqueness" part of Conjecture 3.2 in [10], and have worked out some bounds (see (A.3) in the Appendix) that have been useful in restricting the domain where to search for periodic solutions of (1.2). All we have found is numerical evidence that the periodic orbit explicitly constructed in Theorem 14 is the unique nontrivial periodic orbit for (1.2), but the problem remains open.

Note that Theorem 37 shows that, in general, uniqueness of periodic solutions of equations of the form (5.3) cannot be expected.

5. Appearance and periodic solutions for higher order equations

To exemplify how the ideas in Section 3 can be successfully applied to other cases, here we present results that are the analogue of Theorems 14 and 31 for equations

$$-u^{(k)} = \operatorname{sgn}(u) \tag{5.1}$$

with order $k = 5, 6, 7$.

We recall that a key tool needed to obtain the results in Section 3 are the transformations (1.3) and (1.4). It is straightforward to generalize those transformations to Eq (5.1), as follows:

$$u(t) = (1+t)^k w(\ln(1+t)) \text{ and } w(s) = e^{-ks}u(e^s - 1). \tag{5.2}$$

We have that, if u solves (5.1), then w given in (5.2) solves

$$w^{(k)} + c_{k-1}w^{(k-1)} + \cdots + c_1 w' + c_0 w + \operatorname{sgn}(w) = 0\,, \tag{5.3}$$

where the coefficients c_j are given by the polynomial identity

$$x^k + c_{k-1}x^{k-1} + \cdots + c_1 x + c_0 = (x+1)(x+2)\cdots(x+k).$$

With $k = 5$, if u solves

$$-u^{(5)} = \operatorname{sgn}(u),$$

then w given in (5.2) solves

$$w^{(5)} + 15w^{(4)} + 85w''' + 225w'' + 274w' + 120w + \operatorname{sgn}(w) = 0\,. \tag{5.4}$$

The very same arguments used in Section 3 for the fourth order Eq (1.2) allow to prove the following result. We omit the details of the proof.

Theorem 35. *Equation* (5.4) *has a periodic solution with period* $\tau = -2\log(x_\tau)$*, where* x_τ *is the smallest real root in the interval* $(0, 1)$ *of the polynomial*

$$x^8 - 6x^7 + 2x^6 - x^5 + 12x^4 - x^3 + 2x^2 - 6x + 1,$$

and initial conditions

$$\begin{aligned}
w(0) &= 0,\\
w'(0) &= \frac{1}{120}(8 - 56x_\tau + 85x_\tau^2 - 25x_\tau^3 + 13x_\tau^4 - 50x_\tau^5 + 31x_\tau^6 - 4x_\tau^7),\\
w''(0) &= \frac{1}{240}(-87 + 673x_\tau - 1093x_\tau^2 + 344x_\tau^3 - 132x_\tau^4 + 685x_\tau^5 - 461x_\tau^6 + 61x_\tau^7),\\
w'''(0) &= \frac{1}{240}(421 - 3331x_\tau + 5735x_\tau^2 - 1988x_\tau^3 + 536x_\tau^4 - 3799x_\tau^5 + 2747x_\tau^6 - 371x_\tau^7),\\
w^{(4)}(0) &= \frac{1}{120}(-978 + 7862x_\tau - 14261x_\tau^2 + 5389x_\tau^3 - 993x_\tau^4 + 9908x_\tau^5 - 7585x_\tau^6 + 1040x_\tau^7).
\end{aligned}$$

Now consider $k = 6$. If u solves

$$-u^{(6)} = \operatorname{sgn}(u),$$

then w given in (5.2) solves

$$w^{(6)} + 21w^{(5)} + 175w^{(4)} + 735w''' + 1624w'' + 1764w' + 720w + \operatorname{sgn}(w) = 0\,. \tag{5.5}$$

Theorem 36. *Equation* (5.5) *has a periodic solution with period* $\tau = -2\log(x_\tau)$, *where* x_τ *is the smallest real root in the interval* $(0,1)$ *of the polynomial*

$$x^8 - 7x^7 - 2x^6 + 8x^5 + 17x^4 + 8x^3 - 2x^2 - 7x + 1,$$

and initial conditions

$$\begin{aligned}
w(0) &= 0,\\
w'(0) &= \frac{1217}{107640} - \frac{7067}{71760}\,x_\tau + \frac{19717}{107640}\,x_\tau^2 - \frac{8057}{107640}\,x_\tau^3 + \frac{3617}{107640}\,x_\tau^4 - \frac{4783}{43056}\,x_\tau^5\\
&\quad + \frac{14003}{215280}\,x_\tau^6 + \frac{11}{1560}\,x_\tau^7,\\
w''(0) &= -\frac{623}{7176} + \frac{173219}{215280}\,x_\tau - \frac{1270}{897}\,x_\tau^2 + \frac{9677}{17940}\,x_\tau^3 - \frac{419}{897}\,x_\tau^4 + \frac{189733}{215280}\,x_\tau^5\\
&\quad - \frac{10119}{23920}\,x_\tau^6 + + \frac{203}{4680}\,x_\tau^7,\\
w'''(0) &= \frac{15959}{26910} - \frac{25951}{4784}\,x_\tau + \frac{973879}{107640}\,x_\tau^2 - \frac{71911}{21528}\,x_\tau^3 + \frac{448421}{107640}\,x_\tau^4 - \frac{1245029}{215280}\,x_\tau^5\\
&\quad + \frac{494177}{215280}\,x_\tau^6 - \frac{343}{1560}\,x_\tau^7,\\
w^{(4)}(0) &= -\frac{137089}{35880} + \frac{7426871}{215280}\,x_\tau - \frac{493993}{8970}\,x_\tau^2 + \frac{356651}{17940}\,x_\tau^3 - \frac{140518}{4485}\,x_\tau^4 + \frac{1549397}{43056}\,x_\tau^5\\
&\quad - \frac{285827}{23920}\,x_\tau^6 + \frac{4931}{4680}\,x_\tau^7,\\
w^{(5)}(0) &= \frac{321424}{13455} - \frac{5114219}{23920}\,x_\tau + \frac{35530447}{107640}\,x_\tau^2 - \frac{12579947}{107640}\,x_\tau^3 + \frac{23296517}{107640}\,x_\tau^4 - \frac{9447361}{43056}\,x_\tau^5\\
&\quad + \frac{13318433}{215280}\,x_\tau^6 - \frac{7711}{1560}\,x_\tau^7.
\end{aligned}$$

The case $k = 7$ is slightly different. In this case the auxiliary equation possesses two distinct periodic solutions. If u solves

$$-u^{(7)} = \operatorname{sgn}(u),$$

then w given in (5.2) solves

$$w^{(7)} + 28w^{(6)} + 322w^{(5)} + 1960w^{(4)} + 6769w''' + 13132w'' + 13068w' + 5040w + \operatorname{sgn}(w) = 0. \tag{5.6}$$

Theorem 37. *Equation* (5.6) *has two distinct periodic solutions. More precisely, let* x_1 *and* x_2 *be, respectively, the smallest and largest root in* $(0,1)$ *of the polynomial*

$$1-9\,x+3\,x^2+4\,x^3+39\,x^4+10\,x^5+5\,x^6-47\,x^7-29\,x^8-47\,x^9+5\,x^{10} \\ +10\,x^{11}+39\,x^{12}+4\,x^{13}+3\,x^{14}-9\,x^{15}+x^{16}. \quad (5.7)$$

Consider the following initial conditions:

$$\begin{aligned}
w(0) &= 0,\\
w'(0) &= -\frac{7327}{7610400}+\frac{18031}{2536800}x_\tau+\frac{32621}{1268400}x_\tau^2-\frac{217613}{3805200}x_\tau^3-\frac{355169}{7610400}x_\tau^4-\frac{5597}{237825}x_\tau^5\\
&\quad+\frac{35053}{422800}x_\tau^6+\frac{146099}{3805200}x_\tau^7+\frac{43831}{1902600}x_\tau{}^8+\frac{109513}{3805200}x_\tau^9+\frac{41063}{3805200}x_\tau^{10}\\
&\quad-\frac{370571}{7610400}x_\tau^{11}-\frac{4963}{271800}x_\tau^{12}-\frac{85451}{3805200}x_\tau^{13}+\frac{20607}{845600}x_\tau^{14}-\frac{3821}{1522080}x_\tau^{15},\\
w''(0) &= \frac{558979}{3805200}-\frac{360503}{2536800}x_\tau-x_\tau^2+\frac{2106829}{3805200}x_\tau^3+\frac{63379}{120800}x_\tau^4+\frac{212899}{951300}x_\tau^5\\
&\quad-\frac{453623}{543600}x_\tau^6-\frac{1901377}{3805200}x_\tau^7+\frac{134591}{7610400}-\frac{69091}{237825}x_\tau^8-\frac{410563}{1268400}x_\tau^9-\frac{70733}{1268400}x_\tau^{10}\\
&\quad+\frac{197083}{362400}x_\tau^{11}+\frac{371423}{1902600}x_\tau^{12}+\frac{881863}{3805200}x_\tau^{13}-\frac{94859}{362400}x_\tau^{14}+\frac{41233}{1522080}x_\tau^{15},\\
w'''(0) &= -\frac{1295629}{7610400}+\frac{1217299}{845600}x_\tau+\frac{280229}{422800}x_\tau^2-\frac{16440131}{3805200}x_\tau^3-\frac{33494303}{7610400}x_\tau^4-\frac{1738721}{951300}x_\tau^5\\
&\quad+\frac{2849211}{422800}x_\tau^6+\frac{16927493}{3805200}x_\tau^7+\frac{5138047}{1902600}x_\tau^8+\frac{10158151}{3805200}x_\tau^9+\frac{1189301}{3805200}x_\tau^{10}\\
&\quad-\frac{35631557}{7610400}x_\tau^{11}-\frac{2994937}{1902600}x_\tau^{12}-\frac{7359077}{3805200}x_\tau^{13}+\frac{5571107}{2536800}x_\tau^{14}-\frac{49469}{217440}x_\tau^{15},\\
w^{(4)}(0) &= \frac{420419}{304416}-\frac{6060767}{507360}x_\tau-\frac{414343}{152208}x_\tau^2+\frac{3428267}{108720}x_\tau^3+\frac{16917367}{507360}x_\tau^4+\frac{387989}{27180}x_\tau^5\\
&\quad-\frac{38605201}{761040}x_\tau^6-\frac{5243389}{152208}x_\tau^7-\frac{2120779}{95130}x_\tau^8-\frac{994663}{50736}x_\tau^9-\frac{42723}{16912}x_\tau^{10}\\
&\quad+\frac{18618533}{507360}x_\tau^{11}+\frac{872135}{76104}x_\tau^{12}+\frac{11452751}{761040}x_\tau^{13}-\frac{8574917}{507360}x_\tau^{14}+\frac{2662909}{1522080}x_\tau^{15},\\
w^{(5)}(0) &= -\frac{79376797}{7610400}+\frac{231424841}{2536800}x_\tau+\frac{13107631}{1268400}x_\tau^2-\frac{853862843}{3805200}x_\tau^3-\frac{1840586159}{7610400}x_\tau^4\\
&\quad-\frac{103508843}{951300}x_\tau^5+\frac{470378249}{1268400}x_\tau^6+\frac{953785589}{3805200}x_\tau^7+\frac{330841291}{1902600}x_\tau^8+\frac{519388543}{3805200}x_\tau^9\\
&\quad+\frac{91122293}{3805200}x_\tau^{10}-\frac{2096700581}{7610400}x_\tau^{11}-\frac{151734601}{1902600}x_\tau^{12}-\frac{431110661}{3805200}x_\tau^{13}\\
&\quad+\frac{317312131}{2536800}x_\tau^{14}-\frac{19679291}{1522080}x_\tau^{15},\\
w^{(6)}(0) &= \frac{579183971}{7610400}-\frac{566243681}{845600}x_\tau-\frac{130830199}{3805200}x_\tau^2+\frac{6016843249}{3805200}x_\tau^3+\frac{4362175579}{2536800}x_\tau^4\\
&\quad+\frac{775393369}{951300}x_\tau^5-\frac{10173557141}{3805200}x_\tau^6-\frac{6737821837}{3805200}x_\tau^7-\frac{312931321}{237825}x_\tau^8\\
&\quad-\frac{1174896703}{1268400}x_\tau^9-\frac{283141273}{1268400}x_\tau^{10}+\frac{733162423}{362400}x_\tau^{11}+\frac{147827609}{271800}x_\tau^{12}\\
&\quad+\frac{3184360003}{3805200}x_\tau^{13}-\frac{767386251}{845600}x_\tau^{14}+\frac{142545493}{1522080}x_\tau^{15}.
\end{aligned}$$

Then, choosing respectively $x_\tau = x_1$ *and* $x_\tau = x_2$*, one obtains two distinct periodic solutions of period* $-2\log(x_\tau)$.

Transforming through (5.2) the periodic solutions established in Theorems 35, 36 and 37, we obtain solutions of (5.1) that appear at -1. Since all differential equations are autonomous, we have the following:

Corollary 38. *For any* $T \in \mathbb{R}$ *and* $k = 5, 6, 7$ *there exists a solution of* (5.1) *that appears oscillating at* T.

Remark 39. Numerical simulations strongly suggest that the periodic solution of Eq (5.6) corresponding to x_1 is stable, and that the one corresponding to x_2 is of "saddle" type, with nontrivial stable and unstable manifolds. We also point out that seven is the smallest order for equations of type (5.3) for which we were able to prove the existence of two distinct periodic solutions.

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Conflict of interest

The authors declare no conflict of interest.

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A. Appendix

Here we derive some a priori bounds on solutions of Eqs (1.1) and (1.2).

Theorem 40. *Let u be a solution of* (1.1). *Then, there exists a polynomial q with* $\deg(q) \le 3$ *such that, for all $t \in \mathbb{R}$, we have:*

$$\begin{aligned} -\frac{1}{24}t^4 + q(t) \le u(t) \le \frac{1}{24}t^4 + q(t), \quad & -\frac{1}{6}t^3 + q'(t) \le u'(t) \le \frac{1}{6}t^3 + q'(t), \\ -\frac{1}{2}t^2 + q''(t) \le u''(t) \le \frac{1}{2}t^2 + q''(t), \quad & -t + q'''(t) \le u'''(t) \le t + q'''(t). \end{aligned} \tag{A.1}$$

Proof. The claim follows by integrating by parts four times the inequalities $-1 \le u^{(4)} \le 1$. □

The following result states that all solutions of (1.2) are uniformly bounded forwards in time.

Theorem 41. *Let w be a solution of* (1.2). *Then, for any $\varepsilon > 0$ there exists s_0 such that for all $s \ge s_0$ we have:*

$$|w(s)| \le \frac{1}{24} + \varepsilon, \quad |w'(s)| \le \frac{1}{3} + \varepsilon, \quad |w''(s)| \le \frac{10}{3} + \varepsilon, \quad |w'''(s)| \le 32 + \varepsilon. \tag{A.2}$$

In particular, if w is a periodic solution of (1.2)*, then for any $s \in \mathbb{R}$ we have:*

$$|w(s)| \le \frac{1}{24}, \quad |w'(s)| \le \frac{1}{3}, \quad |w''(s)| \le \frac{10}{3}, \quad |w'''(s)| \le 32. \tag{A.3}$$

Proof. Let w be a solution (1.2) and let be u be defined by transformation (1.3). Then u solves (1.1) and the bounds (A.1) hold true. Hence, again by (1.3), we have:

$$|w(\ln(1+t))| = \left|\frac{u(t)}{(1+t)^4}\right| \le \frac{1}{24}\frac{t^4}{(1+t)^4} + \frac{|q(t)|}{(1+t)^4}.$$

We obtain the first bound by taking the limit for $t \to +\infty$. All other bounds follow by analogous arguments after having expressed the derivatives of w in terms of u through (1.4). □

Remark 42. Note that one could easily extend the results of Theorems 40 and 41 above to all equations of the form (5.1) and (5.3).