DE GRUYTER Adv. Nonlinear Anal. 2019; 8: 1043–1056

Research Article

Raffaela Capitanelli*, Salvatore Fragapane and Maria Agostina Vivaldi

Regularity results for p-Laplacians in pre-fractal domains

https://doi.org/10.1515/anona-2017-0248
Received October 31, 2017; revised December 21, 2017; accepted January 28, 2018

Abstract: We study obstacle problems involving p-Laplace-type operators in non-convex polygons. We establish regularity results in terms of weighted Sobolev spaces. As applications, we obtain estimates for the FEM approximation for obstacle problems in pre-fractal Koch Islands.

Keywords: Degenerate elliptic equations, smoothness and regularity of solutions, FEM, fractals

MSC 2010: 35J70, 35B65, 65N30, 28A80

1 Introduction

In this paper, we deal with obstacle problems involving p-Laplace-type operators in bad domains in $\mathbb{R}^2$. This kind of problems occurs in many mathematical models of physical processes: nonlinear diffusion and filtration, power-law materials and quasi-Newtonian flows (see, for example, [17] and references therein).

Let Ω_ω denote a conical domain (see Section 2 for definitions and properties) and let us consider the two obstacle problem:

$$\text{find } u \in \mathcal{K} \text{ such that} \quad a_p(u, v-u) - \int_{\Omega_\omega} f(v-u)\,dx\,dy \geq 0 \quad \text{for all } v \in \mathcal{K}, \tag{1.1}$$

where

$$a_p(u,v) = \int_{\Omega_\omega} (k^2 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \nabla v\,dx\,dy$$

and

$$\mathcal{K} = \{v \in W_0^{1,p}(\Omega_\omega) : \varphi_1 \leq v \leq \varphi_2 \text{ in } \Omega_\omega\}.$$

Then, under *natural* assumptions (see (2.2)), there exists a unique function u that solves problem (1.1). Properties of first-order derivatives have been established by Li and Martio in [25] and by Lieberman in [27] (see also the references quoted there). In this paper, we face the study of the regularity of the second-order derivatives. To our knowledge, for $p > 2$ there are no second-order L^2 regularity results concerning obstacle problems even if the differentiability of the data and the smoothness of the boundary are assumed; in particular, recent results by Brasco, Santambrogio [5] and by Mercuri, Riey, Sciunzi [29] do not seem to work for obstacle problems. Global regularity results in terms of Sobolev (or Besov) spaces with smoothness index greater than 1 are up to now only established for solutions of obstacle problems for $p = 2$ (see [11]).

***Corresponding author: Raffaela Capitanelli,** Dipartimento di Scienze di Base e Applicate per l'Ingegneria, "Sapienza" Università di Roma, Via A. Scarpa 16, 00161 Roma, Italy, e-mail: raffaela.capitanelli@uniroma1.it. https://orcid.org/0000-0003-1771-1522
Salvatore Fragapane, Maria Agostina Vivaldi, Dipartimento di Scienze di Base e Applicate per l'Ingegneria, "Sapienza" Università di Roma, Via A. Scarpa 16, 00161 Roma, Italy, e-mail: salvatore.fragapane@uniroma1.it, maria.vivaldi@sbai.uniroma1.it. https://orcid.org/0000-0003-1931-2402

Open Access. © 2019 Walter de Gruyter GmbH, Berlin/Boston.
This work is licensed under the Creative Commons Attribution 4.0 Public License.

In this paper, we establish a regularity result for the solution of obstacle problem (1.1) in terms of the weighted Sobolev spaces, where the weight is the distance from the conical point (see Theorem 3.1). In our approach, the Lewy–Stampacchia inequality (see Proposition 2.2) plays a crucial role. We note that this result is new not only for obstacle problems but also in the case of Dirichlet problems. In fact, there is a huge literature about the regularity in the Hölder classes for both the solution u and the gradient ∇u (see [22] and the references quoted there), while the smoothness of the second derivatives is little investigated in such type of irregular domains. Actually, on this topic we have only the contributions by Borsuk and Kondratiev [4] and by Cianchi and Maz'ya [16]. More precisely, Borsuk and Kondratiev (see [4, Theorems 8.43, 8.44, 8.46]) deal with Dirichlet problems in conical domains, but they require a stronger assumption of the datum and prove a weaker regularity. In particular, the exponent of the weight in [4] is greater than the one in our paper (see (3.2)). On the other side, Cianchi and Maz'ya (see [16, Theorem 2.4]) deal with Dirichlet problems in domains that either satisfy [16, condition (2.12)]) or are convex: here the domain Ω_ω is not convex and $\partial\Omega_\omega$ does not satisfy [16, condition (2.12)]. Actually, we use some ideas from [16] in order to obtain local estimates and estimates far away from the conical point. We note that in this part the boundedness of the data (f and $A_p(\varphi_i)$, $i = 1, 2$) is not required, but only the belonging to $L^2(\Omega_\omega)$ (see Theorems 3.6 and 3.7). To establish estimates near the conical point we follow the approach of Tolksdorf [34] and Dobrowolski [19].

In the present paper, we prove also the boundedness of the gradient far away from the conical point (see Theorem 4.3). Essential tools are some results by Tolksdorf [34], Cianchi and Maz'ya [15] and Barret and Liu [2] (for $k = 0$).

We think that the established results are interesting in themselves and also from the point of view of numerical analysis. In fact, as is well known, the regularity results are crucial tools to establish error estimates for the FEM approximation (see, for instance, [6, 8]). To face the numerical approach of the solutions of obstacle problems in fractal domains, it is natural to consider the solutions of obstacle problems in pre-fractal approximating domains and the corresponding FEM-solutions and to evaluate the approximation error. In this spirit, we apply Theorems 3.1 and 4.3 in the study of the obstacle problems in pre-fractal Koch Islands. More precisely, in Theorem 5.5 we prove a *sharp* error estimate for the FEM approximations using the sharp approach of Grisvard [20]. We remark that for $p = 2$ Theorem 5.5 gives the *sharp* result of Grisvard (see [20, Corollary 8.4.1.7]). Moreover, Theorem 5.5 improves the results of [12]: in particular, estimate (5.11) gives a faster convergence than the convergence in [12, estimate (5.63)].

The plan of the paper is the following. In Section 2, we describe the geometry of our domain, we introduce the obstacle problems and we state existence, uniqueness, energy estimates, the Lewy–Stampacchia inequality and a first regularity result for the solutions in terms of the Besov spaces. In Section 3, we establish our main result in terms of the weighted Sobolev spaces. In Section 4, we establish some further results concerning the boundedness of the gradient. In the last section, we show an application of these estimates.

2 Preliminary

Let Ω_ω denote a plane domain with a polygonal boundary $\partial\Omega_\omega$ union of a finite number N of linear segments Γ_j numbered according to the positive orientation. We denote by ω_j the angle between Γ_j and Γ_{j+1} and we assume that $\omega_j < \pi$ for any $j < N$ and $\omega_N = \omega > \pi$. For simplicity, we assume that the corner point between Γ_N and Γ_1 is the origin and that Γ_1 is included in the positive abscissa axis.

We consider the two obstacle problem:

$$\text{find } u \in \mathcal{K} \text{ such that} \quad a_p(u, v - u) - \int_{\Omega_\omega} f(v - u)\, dx\, dy \geq 0 \quad \text{for all } v \in \mathcal{K}, \tag{2.1}$$

where

$$a_p(u, v) = \int_{\Omega_\omega} (k^2 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \nabla v\, dx\, dy, \qquad k \in \mathbb{R},$$

and

$$\mathcal{K} = \{v \in W_0^{1,p}(\Omega_\omega) : \varphi_1 \leq v \leq \varphi_2 \text{ in } \Omega_\omega\}.$$

By using the Poincaré inequality (see, e.g., [28]), the monotonicity properties of the p-Laplacian and choosing $v = \varphi_2 \wedge (\varphi_1 \vee 0)$ as test function in (2.1), we can prove the following result.

Proposition 2.1. *Let*

$$\begin{cases} f \in W^{-1,p'}(\Omega_\omega), \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad \varphi_i \in W^{1,p}(\Omega_\omega),\ i = 1, 2, \\ \varphi_1 \leq \varphi_2 \text{ in } \Omega_\omega, \quad \varphi_1 \leq 0 \leq \varphi_2 \text{ in } \partial\Omega_\omega. \end{cases} \tag{2.2}$$

Then there exists a unique function u that solves problem (2.1). *Moreover,*

$$\|u\|_{W^{1,p}(\Omega_\omega)} \leq C\{\|f\|_{W^{-1,p'}(\Omega_\omega)} + \|\varphi_1\|_{W^{1,p}(\Omega_\omega)} + \|\varphi_2\|_{W^{1,p}(\Omega_\omega)} + |k|\}. \tag{2.3}$$

From now on, we denote by C possibly different constants.

We recall that the solution u to problem (2.1) realizes the minimum on the convex $\mathcal{K}$ of the functional

$$J_p(u) = \min_{v \in \mathcal{K}} J_p(v), \quad \text{where} \quad J_p(v) = \frac{1}{p} \int_{\Omega_\omega} (k^2 + |\nabla v|^2)^{\frac{p}{2}}\, dx\, dy - \int_{\Omega_\omega} fv\, dx\, dy.$$

Now we introduce the Lewy–Stampacchia inequality that plays an important role in our approach to the regularity of the solution. We set

$$A_p(u) = -\operatorname{div}((k^2 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u).$$

Proposition 2.2. *We assume hypothesis* (2.2) *and*

$$f, A_p(\varphi_i) \in L^{p'}(\Omega_\omega), \quad i = 1, 2, \ \frac{1}{p} + \frac{1}{p'} = 1. \tag{2.4}$$

Let u be the solution of (2.1). *Then*

$$A_p(\varphi_2) \wedge f \leq A_p(u) \leq A_p(\varphi_1) \vee f \quad \text{in } \Omega_\omega. \tag{2.5}$$

The Lewy–Stampacchia inequality was first proved in [24] for superharmonic functions which solve a minimum problem, the proof being deeply based on the properties of the Green function. This result has been extended to more general (linear) operators and more general obstacles by Mosco and Troianiello in [31], and for T-monotone operators like the p-Laplacian in [30]. Actually, inequalities (2.5) hold under assumptions weaker than (2.4) according to [36, Remark 1 in Chapter 4.5].

Proposition 2.3. *We assume hypotheses* (2.2) *and* (2.4). *Then the solution u of problem* (2.1) *is the solution of the Dirichlet problem*

$$\begin{cases} A_p(u) = f^* & \text{in } \Omega_\omega, \\ u = 0 & \text{in } \partial\Omega_\omega, \end{cases} \tag{2.6}$$

where f^* belongs to the space $L^{p'}(\Omega_\omega)$ and

$$\|f^*\|_{L^{p'}(\Omega_\omega)} \leq C\{\|f\|_{L^{p'}(\Omega_\omega)} + \|A_p(\varphi_1)\|_{L^{p'}(\Omega_\omega)} + \|A_p(\varphi_2)\|_{L^{p'}(\Omega_\omega)}\}.$$

By using the Lewy–Stampacchia inequality and [32, Theorem 2], we stated in [12] for $k = 0$ the following regularity result in terms of Besov spaces; the case $k \neq 0$ can be treated analogously. We recall a characterization of Besov spaces

$$\begin{aligned} B^{1-\lambda}_{p,q}(\Omega_\omega) &:= (W^{1,p}(\Omega_\omega), L^p(\Omega_\omega))_{\lambda,q}, \\ B^{2-\lambda}_{p,q}(\Omega_\omega) &:= (W^{2,p}(\Omega_\omega), W^{1,p}(\Omega_\omega))_{\lambda,q} = \{u \in W^{1,p}(\Omega_\omega) : \nabla u \in B^{1-\lambda}_{p,q}(\Omega_\omega : \mathbb{R}^2)\}, \end{aligned}$$

where $\lambda \in [0, 1]$, $p, q \in [1, +\infty]$ and $(\cdot, \cdot)_{\lambda,q}$ is the real interpolation functor (see [3]).

Theorem 2.4. *We assume hypotheses* (2.2) *and* (2.4). *Let u be the solution of* (2.1). *Then u belongs to the Besov space $B^{1+1/p}_{p,+\infty}(\Omega_\omega)$. Moreover,*

$$\|u\|_{B^{1+1/p}_{p,+\infty}(\Omega_\omega)} \leq C\{1 + \|f\|^{p'/p}_{L^{p'}(\Omega_\omega)} + \|A_p(\varphi_1)\|^{p'/p}_{L^{p'}(\Omega_\omega)} + \|A_p(\varphi_2)\|^{p'/p}_{L^{p'}(\Omega_\omega)}\}.$$

Note that, putting $p = 2$ in the previous theorem, we get $u \in H^{3/2-\epsilon}(\Omega_\omega)$ in the Sobolev scale. We point out that the previous result is, in some sense, *the best possible* as it holds for any value of $\omega \in (\pi, 2\pi)$, and as $\omega \to 2\pi$, the domain becomes very *bad*.

A *natural* question is then if we can expect sharper regularity results if we consider a fixed value of ω. Having in mind the by now classical results of Kondratiev (see [21]), we think that the natural spaces to study regularity properties in non-convex polygons are the weighted Sobolev spaces of which we now recall the definition.

Let $L_{2,\mu}(\Omega_\omega)$ be the completion of the space $C(\bar{\Omega}_\omega)$ with respect to the norm

$$\|v\|_{L_{2,\mu}(\Omega_\omega)} = \Big\{ \int_{\Omega_\omega} |v|^2 \rho^{2\mu}\, dx \Big\}^{1/2},$$

where ρ denotes the distance function from the origin.

The weighted Sobolev space

$$H^{2,\mu}(\Omega_\omega) = \{v \in W^{1,2}(\Omega_\omega) : D^\beta v \in L_{2,\mu}(\Omega_\omega) \text{ for all } |\beta| = 2\}, \quad \beta = (\beta_1, \beta_2),\ \beta_1, \beta_2 \in \mathbb{N} \cup \{0\},$$

is a Hilbert space with the norm

$$\|v\|_{H^{2,\mu}(\Omega_\omega)} = \Big\{ \sum_{|\beta|=2} \|D^\beta v\|^2_{L_{2,\mu}(\Omega_\omega)} + \|v\|^2_{W^{1,2}(\Omega_\omega)} \Big\}^{1/2}.$$

In the next section, we state our regularity result in terms of weighted Sobolev spaces.

3 Main result

In this section, we state our regularity result in terms of weighted Sobolev spaces.

Theorem 3.1. *Assume hypotheses* (2.2) *and*

$$\begin{cases} k \neq 0, \\ f, A_p(\varphi_i) \in L^\infty(\Omega_\omega), \quad i = 1, 2, \\ A_p(\varphi_2) \wedge f \geq 0. \end{cases} \tag{3.1}$$

Then the solution u of obstacle problem (2.1) *in Ω_ω belongs to the weighted Sobolev space*

$$H^{2,\mu}(\Omega_\omega), \quad \mu > 1 - \gamma, \tag{3.2}$$

where

$$\gamma = \gamma(p, \chi) = 1 + \frac{p(1-\chi)^2 + (1-\chi)\sqrt{p^2 - \chi(2-\chi)(p-2)^2}}{2\chi(2-\chi)(p-1)} \tag{3.3}$$

with $\chi = \frac{\omega}{\pi}$.

Moreover,

$$\|u\|_{H^{2,\mu}(\Omega_\omega)} \leq C\{1 + \|f\|_{L^\infty(\Omega_\omega)} + \|A_p(\varphi_1)\|_{L^\infty(\Omega_\omega)} + \|A_p(\varphi_2)\|_{L^\infty(\Omega_\omega)}\}. \tag{3.4}$$

We note that γ is the least positive eigenvalue and $\phi(\theta)$ is the corresponding eigenfunction of the problem (see [34] and [4, Theorem 8.12 and Remark 8.13])

$$\begin{cases} \partial_\theta\{(\lambda^2\phi^2 + |\partial_\theta\phi|^2)^{\frac{p-2}{2}}\partial_\theta\phi\} + \lambda(\lambda(p-1) + 2 - p)(\lambda^2\phi^2 + |\partial_\theta\phi|^2)^{\frac{p-2}{2}}\phi = 0 \quad \text{in } 0 < \theta < \omega, \\ \phi(0) = \phi(\omega) = 0. \end{cases} \tag{3.5}$$

Remark 3.2. To our knowledge, for $p > 2$ there are no second-order L^2 regularity results concerning obstacle problems even if the differentiability of the data and the smoothness of the boundary are assumed; in particular, recent results of Brasco, Santambrogio [5] and Mercuri, Riey, Sciunzi [29] do not seem to work for

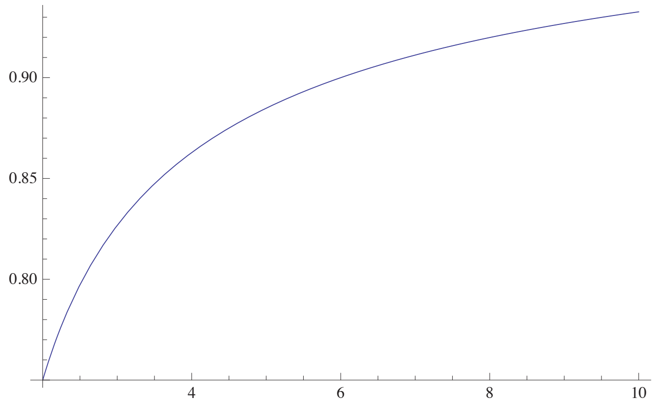

Figure 1: The function γ for $2 < p < 10$.

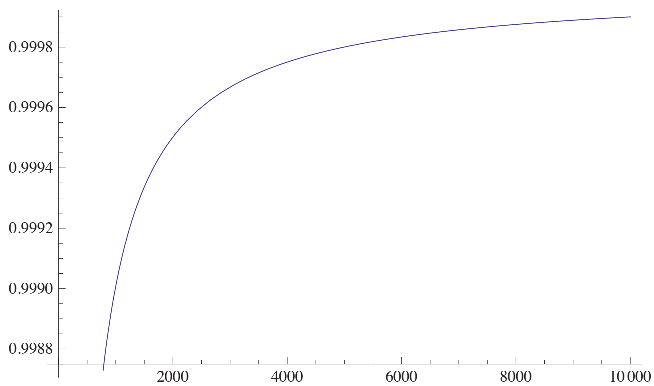

Figure 2: The function γ for $2 < p < 10.000$.

obstacle problems. For properties of first-order derivatives we refer to [25, 27] and to the references quoted there. Global regularity results in terms of Sobolev (or Besov) spaces with smoothness index greater than 1 for solutions of obstacle problems are up to now only established for $p = 2$ (see [11]).

Remark 3.3. We note that for any fixed value of $p > 2$ the function $\gamma(p, \cdot)$ decreases as the variable χ increases, and it tends to the value $\frac{p-1}{p}$ as $\chi \to 2$. Similarly, for any fixed value of $\chi < 2$ the function $\gamma(\cdot, \chi)$ increases as the variable p increases, and it tends to the value 1 as $p \to +\infty$. If we choose $\omega = \frac{4\pi}{3}$, then the expression for γ becomes

$$\gamma\Big(p, \frac{4}{3}\Big) = 1 + \frac{p - \sqrt{p^2 + 32p - 32}}{16(p-1)}.$$

Note that, putting $p = 2$ in the previous formula, we get $\gamma = \frac{3}{4}$ according to the by now classical results of Kondratiev for equations (see, e.g., [4]).

The behavior of $\gamma(p, \frac{4}{3})$ is shown in Figure 1 for $2 < p < 10$, and in Figure 2 for $2 < p < 10.000$.

Remark 3.4. We point out that the regularity result of Theorem 3.1, also in the case of Dirichlet problems with datum $F \in L^\infty$, cannot be deduced from [4, Theorems 8.43, 8.44, 8.46] since we do not assume the differentiability of F, and, for any $p > 2$, the exponent of the weight in [4] is greater than the one in (3.2). In fact, the exponent of the weight in formula [4, (8.4.35)] is required to be greater than $\frac{p}{2}(1-\gamma)$ (in our notation), it is increasing in p and its limit is equal to $\frac{1}{2}$ as $p \to \infty$, while μ in (3.2) is required to be greater than $(1-\gamma)$, it is decreasing in p and tends to 0 as $p \to \infty$.

Remark 3.5. We point out that this regularity result cannot be deduced from [16, Theorem 2.4] as our boundaries do not satisfy [16, condition (2.12)]. Actually, we use some ideas from [16] in order to obtain local estimates and estimates far away from the origin. We note that in this part the boundedness of the data (f and $A_p(\varphi_i)$, $i = 1, 2$) is not required, but only the belonging to $L^2(\Omega_\omega)$ (see Theorems 3.6 and 3.7).

The proof is obtained by combining some preliminary results that actually require weaker conditions than (3.1).

Theorem 3.6. *We assume hypothesis* (2.2) *and*

$$\begin{cases} k \neq 0, \\ f, A_p(\varphi_i) \in L^2_{\mathrm{loc}}(\Omega_\omega) \quad i = 1, 2. \end{cases} \tag{3.6}$$

Then the solution u of obstacle problem (2.1) *in Ω_ω belongs to $H^2_{\mathrm{loc}}(\Omega_\omega)$.*

Proof. From the Lewy–Stampacchia inequality (2.5) and assumption (3.6), we derive that the solution u of problem (2.1) is the solution of the equation $A_p(u) = f^*$, where f^* belongs to the space $L^2_{\mathrm{loc}}(\Omega_\omega)$ and

$$\|f^*\|_{L^2_{\mathrm{loc}}(\Omega_\omega)} \leq C\{\|f\|_{L^2_{\mathrm{loc}}(\Omega_\omega)} + \|A_p(\varphi_1)\|_{L^2_{\mathrm{loc}}(\Omega_\omega)} + \|A_p(\varphi_2)\|_{L^2_{\mathrm{loc}}(\Omega_\omega)}\}.$$

Moreover,

$$\sup_{t>0} \frac{(p-2)t^2(k^2+t^2)^{\frac{p-4}{2}}}{(k^2+t^2)^{\frac{p-2}{2}}} = p-2$$

and, as $k \neq 0$,

$$\inf_{t>0} \frac{(p-2)t^2(k^2+t^2)^{\frac{p-4}{2}}}{(k^2+t^2)^{\frac{p-2}{2}}} = 0.$$

Then we use [16, (5.11) in the proof of Theorem 2.1] and we obtain

$$\begin{aligned} &\int_{B_R} (k^2 + |\nabla u|^2)^{p-2} \sum_{|\beta|=2} |D^\beta u|^2 \, dx \\ &\quad \leq C\Big(\|f\|^2_{L^2(B_{2R})} + \|A_p(\varphi_1)\|^2_{L^2(B_{2R})} + \|A_p(\varphi_2)\|^2_{L^2(B_{2R})} + \frac{1}{R^4}\Big(\int_{B_{2R}} (k^2 + |\nabla u|^2)^{\frac{p-2}{2}} |\nabla u| \, dx\Big)^2\Big) \end{aligned} \tag{3.7}$$

for any ball $B_{2R} \Subset \Omega_\omega$ with C independent of k. Then we repeat [16, steps 2 and 3 of the proof of Theorem 2.1], and by (2.3) we obtain that $u \in H^2_{\mathrm{loc}}(\Omega_\omega)$. □

Now we derive estimates far away from the origin. Let $x \in \partial\Omega_\omega \setminus O$ and $\Omega_s(x) := B_s(x) \cap \bar{\Omega}_\omega$ for $s > 0$. Let $0 < R < \frac{\mathrm{dist}(x,O)}{4}$ be such that $\Omega_{2R}(x) := B_{2R}(x) \cap \bar{\Omega}_\omega$ is convex.

Theorem 3.7. *We assume hypothesis* (2.2) *and*

$$f, A_p(\varphi_i) \in L^2(\Omega_\omega) \quad i = 1, 2. \tag{3.8}$$

Then the solution u of obstacle problem (2.1) *in Ω_ω satisfies*

$$\begin{aligned} &\int_{\Omega_R(x)} (k^2 + |\nabla u|^2)^{p-2} \sum_{|\beta|=2} |D^\beta u|^2 \, dx \\ &\quad \leq C\Big(\|f\|^2_{L^2(\Omega_{2R}(x))} + \|A_p(\varphi_1)\|^2_{L^2(\Omega_{2R}(x))} + \|A_p(\varphi_2)\|^2_{L^2(\Omega_{2R}(x))} + \frac{1}{R^2}\int_{\Omega_{2R}(x)} (k^2 + |\nabla u|^2)^{p-2} |\nabla u|^2 \, dx\Big) \end{aligned} \tag{3.9}$$

for any $x \in \partial\Omega_\omega \setminus O$ and $R \in (0, \frac{\mathrm{dist}(x,O)}{4})$ such that $\Omega_{2R}(x) = B_{2R}(x) \cap \bar{\Omega}_\omega$ is convex.

Proof. From the Lewy–Stampacchia inequality (2.5) and assumption (3.8) we derive that the solution u of the problem is the solution of the Dirichlet problem (2.6), where f^* belongs to the space $L^2(\Omega_\omega)$ and

$$\|f^*\|_{L^2(\Omega_\omega)} \leq C\{\|f\|_{L^2(\Omega_\omega)} + \|A_p(\varphi_1)\|_{L^2(\Omega_\omega)} + \|A_p(\varphi_2)\|_{L^2(\Omega_\omega)}\}. \tag{3.10}$$

We point out that far away from the origin, according to the terminology of [16], the weak second fundamental form on $\partial\Omega_\omega$ is non-positive. We choose the cut function $\xi \in C_0^\infty(B_{2R}(x))$ with $\xi = 1$ in $B_R(x)$.

We proceed as in [16, step 1 of the proof of Theorem 2.4]. We observe that on $\Omega_\omega \cap \partial B_{2R}(x)$ we have $\xi = 0$ and on $\partial\Omega_\omega \cap B_{2R}(x)$ the Dirichlet condition holds, so the boundary integrals (see [16, (4.18)]) can be neglected. By using estimate (3.10), we obtain (see [16, (4.74)])

$$\int_{\Omega_\omega} \xi^2(k^2+|\nabla u|^2)^{p-2}\sum_{|\beta|=2}|D^\beta u|^2\,dx$$
$$\leq C\Big(\|\xi^2 f\|^2_{L^2(\Omega_\omega)}+\|\xi^2 A_p(\varphi_1)\|^2_{L^2(\Omega_\omega)}+\|\xi^2 A_p(\varphi_2)\|^2_{L^2(\Omega_\omega)}+\int_{\Omega_\omega}|\nabla\xi|^2(k^2+|\nabla u|^2)^{p-2}|\nabla u|^2\,dx\Big).$$

Then we repeat [16, steps 2, 3 and 4 of the proof of Theorem 2.3] and we achieve estimate (3.9), where the constant C is independent of k. □

The next theorem concerns estimates near the origin and it holds true for any $k\in\mathbb{R}$.

Theorem 3.8. *Assume hypotheses* (2.2), (2.4) *and*

$$A_p(\varphi_2)\wedge f\geq 0,\quad A_p(\varphi_1)\vee f\leq C_1 r^{\lambda_0}\quad \text{with } \lambda_0>\gamma(p-1)-p \text{ in } \Omega_\omega, \tag{3.11}$$

where γ is defined in (3.3). *Then the following estimates hold for the solution u of obstacle problem* (2.1)*:*

$$|u(x)|\leq Cr^\gamma,\quad |\nabla u(x)|\leq Cr^{\gamma-1},\quad |D^\beta u|\leq Cr^{\gamma-2},\quad |\beta|=2. \tag{3.12}$$

Proof. From the Lewy–Stampacchia inequality (2.5) and assumption (3.11), we derive that the solution u of problem (2.1) is the solution of the Dirichlet problem (2.6) with a datum f^* having the property

$$0\leq f^*\leq C_1 r^{\lambda_0}\quad \text{with } \lambda_0>\gamma(p-1)-p.$$

Moreover, we can suppose that $f^*\neq 0$. In fact, if $f^*=0$, then the unique solution u of problem (2.1) is identically zero and estimates (3.12) are trivial.

If $f^*\neq 0$, we use [19, Theorem 3 and the subsequent remarks] and we deduce that u admits the singular expansion

$$u(r,\theta)=C_2 r^\gamma\phi(\theta)+v(x) \tag{3.13}$$

with $C_2>0$, and

$$|v(x)|\leq C_3 r^{\gamma+\delta},\quad |\nabla v(x)|\leq C_3 r^{\gamma+\delta-1},\quad |D^\beta u|\leq C_3 r^{\gamma+\delta-2},\quad |\beta|=2. \tag{3.14}$$

Here γ is defined in (3.3), $\phi(\theta)$ is the corresponding eigenfunction in problem (3.5) and the maximum $\delta>0$ depends on γ and λ_0. We deduce estimates (3.12) from (3.13) and (3.14). □

We are now in a position to prove our main result.

Proof of Theorem 3.1. Since assumptions (2.2) and (3.1) imply the assumptions of Theorem 3.6, Theorem 3.7 and Theorem 3.8 (with $\lambda_0=0$), we combine all the results and we deduce that the solution u of problem (2.1) belongs to the weighted Sobolev space $H^{2,\mu}(\Omega_\omega)$ for any $\mu>1-\gamma$ as

$$r^\mu|D^\beta u|\in L^2(\Omega_\omega),\quad |\beta|=2.$$

Finally, estimate (3.4) follows from (2.3), (3.7), (3.9) and (3.12). □

4 Boundedness of the gradient far away from the origin

We now investigate boundedness of the gradient in L^∞ far away from the origin. We stress the fact that the results of Theorems 4.1 and 4.2 hold for any $k\in\mathbb{R}$.

Theorem 4.1. *We assume hypotheses* (2.2) *and*

$$f, A_p(\varphi_i)\in L^\infty(\Omega_\omega),\quad i=1,2. \tag{4.1}$$

Then the solution u of obstacle problem (2.1) *belongs to the Sobolev space* $W^{1,\infty}_{\mathrm{loc}}(\Omega_\omega)$.

Proof. From the Lewy–Stampacchia inequality (2.5) and assumption (4.1) we derive that the solution u of problem (2.1) is the solution of the Dirichlet problem (2.6) with datum $f^*\in L^\infty(\Omega_\omega)$. Then the thesis follows from [35, Theorem 1] (see also [18, 26, 37]). □

Theorem 4.2. *We assume hypotheses* (2.2) *and* (4.1). *Then the solution u of obstacle problem* (2.1) *belongs to the Sobolev space $W^{1,\infty}(\Omega_R(x))$ for any $x \in \partial\Omega_\omega \setminus O$ and $R \in (0, \frac{\operatorname{dist}(x,O)}{4})$ such that $\Omega_{2R}(x) = B_{2R}(x) \cap \bar{\Omega}_\omega$ is convex.*

Proof. From the Lewy–Stampacchia inequality (2.5) and assumption (4.1) we derive that the solution u of problem (2.1) is the solution of the Dirichlet problem (2.6) with datum $f^* \in L^\infty(\Omega_\omega)$.

Then we can proceed as in [15, Theorem 2.2 and Remark 2.7]: more precisely, we replace [15, Lemma 5.4] by a localized version involving a cut-off function $\xi \in C_0^\infty(B_{2R}(x))$ with $\xi = 1$ in $B_R(x)$ and we obtain, for a smooth function v such that $v = 0$ on $\partial\Omega_\omega$,

$$\begin{aligned} C(k^2 + t^2)^{\frac{p-2}{2}} t \int_{\{|\nabla v| = t\}} \xi^2 |\nabla|\nabla v|| \, d\mathcal{H}^1(x) &\le t \int_{\{|\nabla v| = t\}} \xi^2 \big|\operatorname{div}((k^2 + |\nabla v|^2)^{\frac{p-2}{2}} \nabla v)\big| \, d\mathcal{H}^1(x) + \int_{\{|\nabla v| > t\}} \xi^2 \frac{1}{(k^2 + |\nabla v|^2)^{\frac{p-2}{2}}} \big|\operatorname{div}((k^2 + |\nabla v|^2)^{\frac{p-2}{2}} \nabla v)\big|^2 \, dx \\ &\quad + C \int_{\{|\nabla v| > t\}} \xi^2 |\nabla v|^p \, dx. \end{aligned}$$

We have exploited the fact that the weak second fundamental form on $\partial\Omega_\omega \cap B_{2R}(x)$ is non-positive. □

We now state a further property for the gradient, useful for the application we have in mind when $k = 0$ (see [2, Lemma 4.2]). Here, as before, for any $x \in \partial\Omega_\omega \setminus O$ we set $\Omega_{2R}(x) = B_{2R}(x) \cap \bar{\Omega}_\omega$ and $R \in (0, \frac{\operatorname{dist}(x,O)}{4})$ is chosen in such a way that $\Omega_{2R}(x) = B_{2R}(x) \cap \bar{\Omega}_\omega$ is convex.

Theorem 4.3. *We assume* (2.2), (4.1) *and*

$$k = 0, \quad A_p(\varphi_2) \wedge f \ge c^* > 0. \tag{4.2}$$

We suppose that the solution u of obstacle problem (2.1) *belongs to the space $W^{2,s}_{\text{loc}}(\Omega_\omega)$, and for any $x \in \partial\Omega_\omega \setminus O$ the restriction of u to the set $\Omega_{2R}(x)$ belongs to $W^{2,s}(\Omega_R(x))$, $s \in [1, 2]$. Then, for any $q \ge 1$, $p > 2$, we obtain*

$$|\nabla u|^{-\frac{(p-t)q}{t-q}} \in L^1(\Omega_\omega)$$

with

$$t \ge \frac{q(p + (p-2)s)}{q + (p-2)s}.$$

Proof. From the Lewy–Stampacchia inequality (2.5) and assumption (4.1) we derive that the solution u of problem (2.1) is the solution of the Dirichlet problem (2.6) with datum $f^* \in L^\infty(\Omega_\omega)$, and by (4.2) also $f^* \ge c^* > 0$. In particular, assumption (3.11) of Theorem 3.8 is satisfied with $\lambda_0 = 0$. We deduce from (3.13) that $|\nabla u|$ behaves like $r^{\gamma-1}$ in a neighborhood of O, and hence $|\nabla u|^{-1} \in L^\infty$ near O. Far away from the origin, we apply Theorem 4.1 to obtain that $u \in W^{1,\infty}_{\text{loc}}(\Omega_\omega)$.

Let G be a domain with $G \Subset \Omega_\omega$. Then $(v_1, v_2) \equiv \nabla u \in (W^{1,s}(G))^2$ and $v \equiv |\nabla u| \in L^\infty(G)$. It follows that $v \in W^{1,s}(G)$ and $\nabla v = (v_1 \nabla v_1 + v_2 \nabla v_2)/v$. Moreover, we have that

$$f^* = -\operatorname{div}(v^{p-2}\nabla u) = -\Big\{ v^{p-2}(v_{1x_1} + v_{2x_2}) + (p-2)v^{p-2} \frac{v_1 v_{x_1} + v_2 v_{x_2}}{v} \Big\}.$$

Then

$$c^* \le f^* \le M(x)|\nabla u|^{p-2} \quad \text{a.e. in } G,$$

where $M(x) \in L^s(G)$.

We obtain

$$\int_G |\nabla u|^{-\frac{(p-t)q}{t-q}} \, dx \le C \int_G (M(x))^{\frac{(p-t)q}{(p-2)(t-q)}} \, dx,$$

and if $t \ge \frac{q(p+(p-2)s)}{q+(p-2)s}$, then $\frac{(p-t)q}{(p-2)(t-q)} \le s$.

We repeat the previous proof by replacing G by $\Omega_R(x)$ and Theorem 4.1 by 4.2 to complete the proof. □

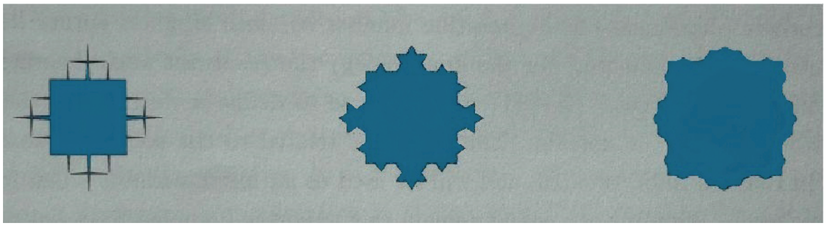

Figure 3: Pre-fractal Koch Islands Ω_α^2 with $\alpha = 2.1$, $\alpha = 3$ and $\alpha = 3.75$, respectively.

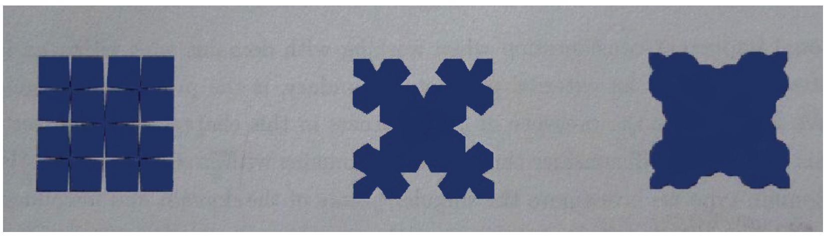

Figure 4: Pre-fractal Koch Islands Ω_α^2 with $\alpha = 2.1$, $\alpha = 3$ and $\alpha = 3.75$, respectively.

5 Error estimates

Obstacle problems in fractal domains have been studied in [10] in the framework of reinforcement problems. To face the numerical approach to the solutions of obstacle problems in fractal domains, it is natural to consider the solutions of obstacle problems in pre-fractal approximating domains and the corresponding FEM-solutions and to evaluate the approximation error. We consider the pre-fractal Koch Islands Ω_α^n that are polygonal domains having as sides pre-fractal Koch curves. We start by a regular polygon and we replace each side by a pre-fractal Koch curve (see Figures 3 and 4); we refer to [12, Section 2] for the definition and details.

In [12, Section 3], we showed that, assuming some natural conditions, the solutions u_n of the obstacle problem in Ω_α^n converge to the *fractal* solution of the obstacle problem in the Koch Island Ω_α.

For any (fixed) n, the number of reentrant angles is fixed and hence we can prove, for the solution u_n of the obstacle problem in Ω_α^n, all the results of previous sections with $\chi = \frac{\omega}{\pi}$, where

$$\omega = \begin{cases} \pi + \theta(\alpha) & \text{if the sides of the polygons are obtained by outward curves,} \\ \pi + 2\theta(\alpha) & \text{if the sides of the polygons are obtained by inward curves.} \end{cases} \tag{5.1}$$

We recall that by $\theta(\alpha)$ we denote the opening of the rotation angle of the similarities involved in the construction of the Koch curve, that is,

$$\theta(\alpha) = \arcsin\Big(\frac{\sqrt{\alpha(4-\alpha)}}{2}\Big).$$

Then $\chi \in (1, \frac{3}{2})$ in the case of outward curves or $\chi \in (1, 2)$ in the case of inward curves.

In this framework, the involved weighted Sobolev space is

$$H^{2,\mu}(\Omega_\alpha^n) = \{v \in W^{1,2}(\Omega_\alpha^n) : D^\beta v \in L_{2,\mu}(\Omega_\alpha^n) \text{ for all } |\beta| = 2\}, \quad \beta = (\beta_1, \beta_2),\ \beta_1, \beta_2, \in \mathbb{N},$$

which is a Hilbert space with the norm

$$\|v\|_{H^{2,\mu}(\Omega_\alpha^n)} = \Big\{ \sum_{|\beta|=2} \|D^\beta v\|^2_{L_{2,\mu}(\Omega_\alpha^n)} + \|v\|^2_{W^{1,2}(\Omega_\alpha^n)} \Big\}^{1/2}.$$

Here $L_{2,\mu}(\Omega_\alpha^n)$ is the completion of the space $C(\bar{\Omega}_\alpha^n)$ with respect to the norm

$$\|v\|_{L_{2,\mu}(\Omega_\alpha^n)} = \Big\{ \int_{\Omega_\alpha^n} |v|^2 \rho^{2\mu}\, dx \Big\}^{1/2}$$

and $\rho = \rho_n(x)$ denotes the distance function from the set of vertices of the *reentrant* corners of Ω_α^n. In this setting, we state the following theorems.

Theorem 5.1. *We assume*

$$\begin{cases} \varphi_i \in W^{1,p}(\Omega_\alpha^n), & i = 1, 2, \\ \varphi_1 \leq \varphi_2 \text{ in } \Omega_\alpha^n, & \varphi_1 \leq 0 \leq \varphi_2 \text{ in } \partial\Omega_\alpha^n, \end{cases} \tag{5.2}$$

and

$$\begin{cases} k \neq 0 \\ f, A_p(\varphi_i) \in L^\infty(\Omega_\alpha^n), \quad i = 1, 2, \\ A_p(\varphi_2) \wedge f \geq 0. \end{cases} \tag{5.3}$$

Then the solution u_n of obstacle problem (2.1) *in Ω_α^n belongs to the weighted Sobolev space*

$$H^{2,\mu}(\Omega_\alpha^n), \quad \mu > 1 - \gamma, \tag{5.4}$$

where

$$\gamma = \gamma(p, \chi) = 1 + \frac{p(1-\chi)^2 + (1-\chi)\sqrt{p^2 - \chi(2-\chi)(p-2)^2}}{2\chi(2-\chi)(p-1)} \tag{5.5}$$

with $\chi = \frac{\omega}{\pi}$ and ω in (5.1).

Moreover,

$$\|u_n\|_{H^{2,\mu}(\Omega_\alpha^n)} \leq C\{1 + \|f\|_{L^\infty(\Omega_\alpha^n)} + \|A_p(\varphi_1)\|_{L^\infty(\Omega_\alpha^n)} + \|A_p(\varphi_2)\|_{L^\infty(\Omega_\alpha^n)}\}. \tag{5.6}$$

If $k = 0$, then an analog of Theorem 4.3 holds.

Theorem 5.2. *We assume* (5.2) *and*

$$\begin{cases} k = 0 \\ f, A_p(\varphi_i) \in L^\infty(\Omega_\alpha^n), \quad i = 1, 2, \\ A_p(\varphi_2) \wedge f \geq c^* > 0. \end{cases} \tag{5.7}$$

If the solution u_n of obstacle problem (2.1) *in Ω_α^n belongs to the space $H^{2,\mu}(\Omega_\alpha^n)$, then for any $q \geq 1$ and $p > 2$ we obtain*

$$|\nabla u_n|^{-\frac{(p-t)q}{t-q}} \in L^1(\Omega_\alpha^n), \tag{5.8}$$

with

$$t \geq \frac{q(p + (p-2)2)}{q + (p-2)2}.$$

We introduce the triangulation of the domain Ω_α^n in order to define the approximate solutions u_h according to the Galerkin method. Let T_h be a partitioning of the domain Ω_α^n into disjoint, open regular triangles τ, each side being bounded by h so that $\bar{\Omega}_\alpha^n = \bigcup_{\tau \in T_h} \bar{\tau}$. Associated with T_h, we consider the finite-dimensional spaces

$$S_h = \{v \in C(\bar{\Omega}_\alpha^n) : v|_\tau \text{ is affine for all } \tau \in T_h\} \quad \text{and} \quad S_{h,0} = \{v \in S_h : v = 0 \text{ on } \partial\Omega_\alpha^n\}.$$

By π_h we denote the interpolation operator $\pi_h : C(\bar{\Omega}_\alpha^n) \to S_h$ such that $\pi_h v(P_i) = v(P_i)$ for any vertex P_i of the partitioning T_h.

Definition 5.3. The family of triangulations T_h is *adapted* to the $H^{2,\mu}(\Omega_\alpha^n)$-regularity if the following conditions hold:

- The vertices of the polygonal curves $\partial\Omega_\alpha^n$ are nodes of the triangulations.
- The meshes are conformal and regular.
- There exists $\sigma^* > 0$ such that, as $h \to 0$,

$$\begin{aligned} h_\tau &\leq \sigma^* h^{\frac{1}{1-\mu}} && \text{for all } \tau \in T_h \text{ such that one of the vertices of } \tau \text{ belongs to } \mathcal{R}^n, \\ h_\tau &\leq \sigma^* h \cdot \inf_\tau \rho^\mu && \text{for all } \tau \in T_h \text{ with no vertix in } \mathcal{R}^n. \end{aligned}$$

Here $h = \sup\{h_\tau = \operatorname{diam}(\tau) : \tau \in T_h\}$ is the size of the triangulation and $\rho = \rho_n(x)$ denotes the distance of the point x from the set $\mathcal{R}^n$ of the vertices of the *reentrant* corners of Ω_α^n.

The construction of triangulations T_h *adapted* to the $H^{2,\mu}$-regularity was introduced by Grisvard in [20]. This tool has been fruitfully used for the FEM approximation of linear problems in pre-fractal domains by [1, 13, 14, 23, 38, 39].

Consider the two obstacle problem in the finite-dimensional space $S_{h,0}$:

$$\text{find } u \in \mathcal{K}_h \text{ such that} \quad a_p(u, v-u) - \int_{\Omega_\alpha^n} f(v-u)\,dx\,dy \geqslant 0 \quad \text{for all } v \in \mathcal{K}_h, \tag{5.9}$$

where

$$a_p(u,v) = \int_{\Omega_\alpha^n} (k^2 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \nabla v\,dx\,dy \quad \text{and} \quad \mathcal{K}_h = \{v \in S_{h,0} : \varphi_{1,h} \leqslant v \leqslant \varphi_{2,h} \text{ in } \Omega_\alpha^n\},$$

with $\varphi_{1,h} = \pi_h \varphi_1$ and $\varphi_{2,h} = \pi_h \varphi_2$.

Proposition 5.4. *Let us assume hypothesis* (5.2). *Then, for any* $f \in L^{p'}(\Omega_\alpha^n)$, *there exists a unique function* u_h *that solves problem* (5.9). *Moreover,*

$$\|u_h\|_{W^{1,p}(\Omega_\alpha^n)} \le C\{|k| + \|f\|_{L^{p'}(\Omega_\alpha^n)}^{p'/p} + \|\varphi_1\|_{W^{1,p}(\Omega_\alpha^n)} + \|\varphi_2\|_{W^{1,p}(\Omega_\alpha^n)}\}.$$

As previously, the solution u_h to problem (5.9) realizes the minimum on the convex $\mathcal{K}_h$ of the functional $J_p(\cdot)$, i.e.,

$$J_p(u) = \min_{v \in \mathcal{K}_h} J_p(v), \quad \text{where} \quad J_p(v) = \frac{1}{p} \int_{\Omega_\alpha^n} (k^2 + |\nabla v|^2)^{\frac{p}{2}}\,dx\,dy - \int_{\Omega_\alpha^n} fv\,dx\,dy.$$

Theorem 5.5. *Let us denote by* u_n *and* u_h *the solutions of problems* (2.1) *in* Ω_α^n *and* (5.9), *respectively. Let us assume hypotheses* (5.2), (5.3) *and*

$$\varphi_i \in H^{2,\mu}(\Omega_\alpha^n), \quad i = 1, 2. \tag{5.10}$$

Let T_h *be a triangulation of* Ω_α^n *adapted to the* $H^{2,\mu}(\Omega_\alpha^n)$*-regularity of the solution* u_n. *Then*

$$\|u_n - u_h\|_{W^{1,t}(\Omega_\alpha^n)} \le C h^{\frac{r}{t}} \|u_n\|_{H^{2,\mu}(\Omega_\alpha^n)} \tag{5.11}$$

for any

$$r \in \left[1, \frac{2\sqrt{p^2 - \chi(2-\chi)(p-2)^2}}{\sqrt{p^2 - \chi(2-\chi)(p-2)^2} + (\chi - 1)(p-2)}\right), \quad t \in [2, p].$$

Proof. For any $\sigma \in [0, p]$ we put

$$|v|_{(p,\sigma)} = \left(\int_{\Omega_\alpha^n} (|k| + |\nabla u_n| + |\nabla v|)^{p-\sigma} |\nabla v|^\sigma\,dx\,dy \right)^{\frac{1}{p}}. \tag{5.12}$$

Repeating the proof of [12, Lemma 5.2] (given for $k = 0$), we prove for any $v_h \in \mathcal{K}_h$ and

$$v \in \mathcal{K}_n := \{v \in W_0^{1,p}(\Omega_\alpha^n) : \varphi_1 \le v \le \varphi_2 \text{ in } \Omega_\alpha^n\}$$

that

$$|u_n - u_h|_{(p,t)}^p \le C\{|u_n - v_h|_{(p,r)}^p + \|f + A_p(u_n)\|_{L^2(\Omega_\alpha^n)} (\|u_n - v_h\|_{L^2(\Omega_\alpha^n)} + \|v - u_h\|_{L^2(\Omega_\alpha^n)})\}, \tag{5.13}$$

where $r \in [1, 2]$, $t \in [2, p]$ and the constant C does not depend on h. Now we evaluate the terms on the right-hand side in estimate (5.13) by choosing the test functions $v_h \in \mathcal{K}_h$ and $v \in \mathcal{K}_n$ in an appropriate way. According to Theorem 5.1, the function u_n belongs to the weighted Sobolev space $H^{2,\mu}(\Omega_\alpha^n)$ for any $\mu > 1 - \gamma$ (see (5.4) and (5.5)).

We choose $v_h = \pi_h u_n$, and by using approximation estimates of Grisvard (see [20, Section 8.4.1]), we derive

$$\|u_n - \pi_h u_n\|_{L^2(\Omega_\alpha^n)} \le C h^2 \|u_n\|_{H^{2,\mu}(\Omega_\alpha^n)}. \tag{5.14}$$

Then we choose $v = \varphi_2 \wedge (u_h \vee \varphi_1)$ and, as in [12, Lemma 4.4], we have

$$\|v - u_h\|^2_{L^2(\Omega^n_\alpha)} \leq \|\pi_h \varphi_2 - \varphi_2\|^2_{L^2(\Omega^n_\alpha)} + \|\pi_h \varphi_1 - \varphi_1\|^2_{L^2(\Omega^n_\alpha)}.$$

Again using Grisvard estimates and assumption (5.10), we derive

$$\|v - u_h\|_{L^2(\Omega^n_\alpha)} \leq Ch^2. \tag{5.15}$$

We compare the seminorm $|u_n - u_h|_{W^{1,t}(\Omega^n_\alpha)}$ with $|u_n - u_h|^p_{(p,t)}$ (defined in (5.12)) and we obtain

$$|u_n - u_h|^t_{W^{1,t}(\Omega^n_\alpha)} \leq \frac{C}{|k|^{p-t}} |u_n - u_h|^p_{(p,t)}. \tag{5.16}$$

We now evaluate the term $|u_n - v_h|^p_{(p,r)}$, where $v_h = \pi_h u_n$. By the embedding of weighted Sobolev spaces in the fractional Sobolev spaces (see, for instance, [33]), u_n belongs to the space $W^{\sigma_2,2}(\Omega^n_\alpha)$ for any $\sigma_2 < 1 + \gamma$. Taking into account the Sobolev embedding (see, for instance, [7]), we have

$$|\nabla u_n| \in L^{r^*}(\Omega^n_\alpha) \quad \text{with } r^* = \frac{2}{2 - \sigma_2}. \tag{5.17}$$

By the Hölder inequality, we obtain

$$|u_n - v_h|^p_{(p,r)} \leq C(r) |u_n - \pi_h u_n|^r_{W^{1,2}(\Omega^n_\alpha)}, \tag{5.18}$$

where we have used estimate (5.6) with $r = \frac{2(r^* - p)}{r^* - 2}$. Hence, as $\sigma_2 < 1 + \gamma$, r^* is given in (5.17) and γ in (5.5), we have to choose $r < p + \frac{2-p}{\gamma}$ and we obtain by calculations that

$$r < \frac{2\sqrt{p^2 - \chi(2-\chi)(p-2)^2}}{\sqrt{p^2 - \chi(2-\chi)(p-2)^2} + (\chi - 1)(p-2)}.$$

Now we use [20, Theorem 8.4.1.6] and we obtain

$$|u_n - \pi_h u_n|_{W^{1,2}(\Omega^n_\alpha)} \leq Ch. \tag{5.19}$$

By taking into account estimates (5.13)–(5.16), (5.18) and (5.19), we conclude the proof using once again the Poincaré inequality. □

We note that in Theorem 5.5 we assume $k \neq 0$; if $k = 0$ the following result holds.

Theorem 5.6. *Let us denote by u_n and u_h the solutions of problems* (2.1) *in Ω^n_α and* (5.9), *respectively. Let us assume hypotheses* (5.2), (5.7), (5.10) *and that the solution u_n belongs to the space $H^{2,\mu}(\Omega^n_\alpha)$. Let T_h be a triangulation of Ω^n_α adapted to the $H^{2,\mu}(\Omega^n_\alpha)$-regularity of the solution u_n. Then*

$$\|u_n - u_h\|_{W^{1,q}(\Omega^n_\alpha)} \leq Ch^{\frac{r}{t}} \|u_n\|_{H^{2,\mu}(\Omega^n_\alpha)}$$

for any

$$r \in \left[1, \frac{2\sqrt{p^2 - \chi(2-\chi)(p-2)^2}}{\sqrt{p^2 - \chi(2-\chi)(p-2)^2} + (\chi - 1)(p-2)}\right)$$

$t \in [2, p]$, $q \in [1, t]$, and for $q < p$ we require $t \geq \frac{q(p+(p-2)2)}{q+(p-2)2}$.

Proof. We proceed as in the proof of Theorem 5.5: we replace estimate (5.16) by

$$|u_n - u_h|^t_{W^{1,q}(\Omega^n_\alpha)} \leq \big\| |\nabla u_n|^{-\frac{(p-t)q}{t-q}} \big\|^{(t-q)/q}_{L^1(\Omega^n_\alpha)} \cdot \int\limits_{\Omega^n_\alpha} |\nabla(u_n - u_h)|^t |\nabla u_n|^{p-t} \, dx \, dy \leq C |u_n - u_h|^p_{(p,t)}.$$

Here we have used the Hölder inequality and estimate (5.8). □

Remark 5.7. From the previous proofs we deduce that, for the linear case $p = 2$, Theorem 5.5 gives the sharp result of Grisvard (see [20, Corollary 8.4.1.7]): in fact, we have $p = t = 2$ and, in particular, formula (5.18) holds true for $r = 2 = p$.

Remark 5.8. We note that Theorem 5.5 improves the results of [12]: in particular, estimate (5.11) gives a faster convergence than the convergence in [12, estimate (5.63)]. In fact, the solution u_n belongs to the weighted Sobolev space $H^{2,\mu}(\Omega_\alpha^n)$ for any $\mu = \mu(p) > 1 - \gamma$. This space is continuously embedded in the fractional Sobolev space $W^{\sigma_2,2}(\Omega_\alpha^n)$ for any $\sigma_2 < 2 - \mu$ (see, e.g., [33]). Hence, by the Sobolev embedding (see, e.g., [7]), for any $\sigma < \gamma + \frac{2}{p}$, $p \geq 2$, the fractional Sobolev space $W^{\sigma,p}(\Omega_\alpha^n)$ properly contains the weighted Sobolev space $H^{2,\mu}(\Omega_\alpha^n)$ for some $\mu = \mu(p) > 1 - \gamma$. Actually, for every $p \geq 2$ the exponent r in (5.11) is strictly greater than $\gamma + \frac{2}{p}$. Namely by writing the expression of γ in (5.5) in terms of the parameters $p \in [2, +\infty)$ and $\chi \in (1, 2)$, we obtain that $\gamma + \frac{2}{p} < r$ if and only if

$$\chi(2-\chi)(p-1)(p-2) + (\chi-1)\sqrt{p^2(\chi-1)^2 + 4\chi(2-\chi)(p-1)} > 0. \tag{5.20}$$

Of course, inequality (5.20) holds for any choice of the parameters.

Remark 5.9. We note that the constant C in estimate (5.11) does not depend on n. However, to deduce from (5.11) error estimates for the *fractal* solution we have to bound the norms $\|u_n\|_{H^{2,\mu}(\Omega_\alpha^n)}$ uniformly in n. Up to now, this type of results is only established for $p = 2$ (see [9, 11]).

Funding: The authors are members of GNAMPA (INdAM) and are partially supported by Grant Ateneo “Sapienza” 2017. Capitanelli and Vivaldi are partially supported by INdAM GNAMPA Project 2017.

References

[1] P. Bagnerini, A. Buffa and E. Vacca, Mesh generation and numerical analysis of a Galerkin method for highly conductive prefractal layers, *Appl. Numer. Math.* **65** (2013), 63–78.
[2] J. W. Barrett and W. B. Liu, Finite element approximation of the p-Laplacian, *Math. Comp.* **61** (1993), no. 204, 523–537.
[3] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Grundlehren Math. Wiss. 223, Springer, Berlin-New York, 1976.
[4] M. Borsuk and V. Kondratiev, *Elliptic Boundary Value Problems of Second Order in Piecewise Smooth Domains*, North-Holland Math. Libr. 69, Elsevier Science, Amsterdam, 2006.
[5] L. Brasco and F. Santambrogio, A sharp estimate à la Calderón–Zygmund for the p-Laplacian, *Commun. Contemp. Math.* **20** (2018), no. 3, Article ID 1750030.
[6] F. E. Brezzi, Finite element methods for nonlinear problems, *Confer. Sem. Mat. Univ. Bari* (1979), no. 158–162, 47–68.
[7] F. Brezzi and G. Gilardi, Fundamentals of P.D.E. for numerical analysis, in: *Finite Element Handbook*, McGraw-Hill, New York (1987), 1–121.
[8] F. Brezzi, W. W. Hager and P.-A. Raviart, Error estimates for the finite element solution of variational inequalities, *Numer. Math.* **28** (1977), no. 4, 431–443.
[9] R. Capitanelli and M. A. Vivaldi, Uniform weighted estimates on pre-fractal domains, *Discrete Contin. Dyn. Syst. Ser. B* **19** (2014), no. 7, 1969–1985.
[10] R. Capitanelli and M. A. Vivaldi, Reinforcement problems for variational inequalities on fractal sets, *Calc. Var. Partial Differential Equations* **54** (2015), no. 3, 2751–2783.
[11] R. Capitanelli and M. A. Vivaldi, Weighted estimates on fractal domains, *Mathematika* **61** (2015), no. 2, 370–384.
[12] R. Capitanelli and M. A. Vivaldi, FEM for quasilinear obstacle problems in bad domains, *ESAIM Math. Model. Numer. Anal.* **51** (2017), no. 6, 2465–2485.
[13] M. Cefalo and M. R. Lancia, An optimal mesh generation algorithm for domains with Koch type boundaries, *Math. Comput. Simulation* **106** (2014), 133–162.
[14] M. Cefalo, M. R. Lancia and H. Liang, Heat-flow problems across fractal mixtures: regularity results of the solutions and numerical approximation, *Differential Integral Equations* **26** (2013), no. 9–10, 1027–1054.
[15] A. Cianchi and V. G. Maz’ya, Global boundedness of the gradient for a class of nonlinear elliptic systems, *Arch. Ration. Mech. Anal.* **212** (2014), no. 1, 129–177.
[16] A. Cianchi and V. G. Maz’ya, Second-order two-sided estimates in nonlinear elliptic problems, *Arch. Ration. Mech. Anal.* **229**, (2018), 569–599.

[17] J. I. Díaz, *Nonlinear Partial Differential Equations and Free Boundaries. Vol. I: Elliptic Equations*, Res. Notes Math. 106, Pitman, Boston, 1985.
[18] E. DiBenedetto, $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, *Nonlinear Anal.* **7** (1983), no. 8, 827–850.
[19] M. Dobrowolski, On quasilinear elliptic equations in domains with conical boundary points, *J. Reine Angew. Math.* **394** (1989), 186–195.
[20] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Monogr. Stud. Math. 24, Pitman, Boston, 1985.
[21] V. A. Kondrat'ev, Boundary value problems for elliptic equations in domains with conical or angular points, *Trudy Moskov. Mat. Obšč.* **16** (1967), 209–292.
[22] T. Kuusi and G. Mingione, Guide to nonlinear potential estimates, *Bull. Math. Sci.* **4** (2014), no. 1, 1–82.
[23] M. R. Lancia, M. Cefalo and G. Dell'Acqua, Numerical approximation of transmission problems across Koch-type highly conductive layers, *Appl. Math. Comput.* **218** (2012), no. 9, 5453–5473.
[24] H. Lewy and G. Stampacchia, On the smoothness of superharmonics which solve a minimum problem, *J. Anal. Math.* **23** (1970), 227–236.
[25] G. Li and O. Martio, Stability and higher integrability of derivatives of solutions in double obstacle problems, *J. Math. Anal. Appl.* **272** (2002), no. 1, 19–29.
[26] G. M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, *Nonlinear Anal.* **12** (1988), no. 11, 1203–1219.
[27] G. M. Lieberman, Regularity of solutions to some degenerate double obstacle problems, *Indiana Univ. Math. J.* **40** (1991), no. 3, 1009–1028.
[28] V. G. Maz'ya and S. V. Poborchi, *Differentiable Functions on Bad Domains*, World Scientific, River Edge, 1997.
[29] C. Mercuri, G. Riey and B. Sciunzi, A regularity result for the p-Laplacian near uniform ellipticity, *SIAM J. Math. Anal.* **48** (2016), no. 3, 2059–2075.
[30] U. Mosco, Implicit variational problems and quasi variational inequalities, in: *Nonlinear Operators and the Calculus of Variations*, Lecture Notes in Math. 543, Springer, Berlin (1976), 83–156.
[31] U. Mosco and G. M. Troianiello, On the smoothness of solutions of unilateral Dirichlet problems, *Boll. Un. Mat. Ital. (4)* **8** (1973), 57–67.
[32] G. Savaré, Regularity results for elliptic equations in Lipschitz domains, *J. Funct. Anal.* **152** (1998), no. 1, 176–201.
[33] V. A. Solonnikov and M. A. Vivaldi, Mixed type, nonlinear systems in polygonal domains, *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **24** (2013), no. 1, 39–81.
[34] P. Tolksdorf, On the Dirichlet problem for quasilinear equations in domains with conical boundary points, *Comm. Partial Differential Equations* **8** (1983), no. 7, 773–817.
[35] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, *J. Differential Equations* **51** (1984), no. 1, 126–150.
[36] G. M. Troianiello, *Elliptic Differential Equations and Obstacle Problems*, Univ. Ser. Math., Plenum Press, New York, 1987.
[37] K. Uhlenbeck, Regularity for a class of non-linear elliptic systems, *Acta Math.* **138** (1977), no. 3–4, 219–240.
[38] E. Vacca, *Galerkin approximation for highly conductive layers*, Ph.D. thesis, Dipartimento di Metodi e Modelli Matematici per l'Ingegneria, Università degli Studi di Roma "La Sapienza", 2005.
[39] R. Wasyk, *Numerical solution of a transmission problem with a prefractal interface*, Ph.D. thesis, Department of Mathematical Sciences, Worcester Polytechinique Institute, 2007.