



Some new results about a conjecture by Brian Alspach

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Abstract. In this paper, we consider the following conjecture, proposed by Brian Alspach, concerning partial sums in finite cyclic groups: given a subset A of $\mathbb{Z}_n \setminus \{0\}$ of size k such that $\sum_{z \in A} z \neq 0$, it is possible to find an ordering (a_1, \dots, a_k) of the elements of A such that the partial sums $s_i = \sum_{j=1}^i a_j$, $i = 1, \dots, k$, are nonzero and pairwise distinct. This conjecture is known to be true for subsets of size $k \leq 11$ in cyclic groups of prime order. Here, we extend this result to any torsion-free abelian group and, as a consequence, we provide an asymptotic result in \mathbb{Z}_m . We also consider a related conjecture, originally proposed by Ronald Graham: given a subset A of $\mathbb{Z}_p \setminus \{0\}$, where p is a prime, there exists an ordering of the elements of A such that the partial sums are all distinct. Working with the methods developed by Hicks, Ollis, and Schmitt, based on Alon's combinatorial Nullstellensatz, we prove the validity of this conjecture for subsets A of size 12.

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1. Introduction. In this paper, we give some new results about a conjecture, due to Brian Alspach, concerning finite cyclic groups. First of all, we introduce some notations. Given an abelian group $(G, +)$, a finite subset $A = \{x_1, x_2, \dots, x_k\}$ of $G \setminus \{0_G\}$ and an ordering $\omega = (x_{j_1}, x_{j_2}, \dots, x_{j_k})$ of its elements, we denote by $s_i = s_i(\omega)$ the partial sum $x_{j_1} + x_{j_2} + \dots + x_{j_i}$. Clearly, the ordering ω induces the permutation $\sigma_\omega = (j_1, j_2, \dots, j_k) \in \text{Sym}(k)$. Alspach's conjecture was originally proposed only for finite cyclic groups, see [6, 7]. However, it can be extended to any abelian group, [11, 19].

Conjecture 1.1 (Alspach). *Given an abelian group $(G, +)$ and a subset A of $G \setminus \{0_G\}$ of size k such that $\sum_{z \in A} z \neq 0_G$, it is possible to find an ordering ω of the elements of A such that $s_i(\omega) \neq 0_G$ and $s_i(\omega) \neq s_j(\omega)$ for all $1 \leq i < j \leq k$.*

The validity of this conjecture has been proved in each of the following cases:

- (1) $k \leq 9$ or $k = |G| - 1$, [3, 6, 7, 15];
- (2) $k = 10$ or $|G| - 3$ with G cyclic of prime order, [17];
- (3) $k = 11$ with G cyclic of prime order, [20];
- (4) $|G| \leq 21$, [7, 11];
- (5) G is cyclic and either $k = |G| - 2$ or $|G| \leq 25$, [6, 7].

Clearly, when $k = |G| - 3, |G| - 2, |G| - 1$, G is assumed to be finite. Alspach's conjecture is worth to be studied also in connection with sequenceability and strong sequenceability of groups, see [2, 3, 18], and simplicity of Heffter arrays, see [4, 5, 12, 14].

In Section 2, we explain how the validity of Conjecture 1.1 for sets of size k in cyclic groups \mathbb{Z}_p , for infinitely many primes p , implies the validity for sets of size k in any torsion-free abelian group. As a consequence, in Section 3, we provide an asymptotic result for sets of size $k \leq 11$ in finite cyclic groups: this has been achieved without any direct or recursive construction (that we believe can hardly be obtained) but only with some theoretical nonconstructive arguments.

Another conjecture, very close to the Alspach one, was originally proposed by R.L. Graham in [16] for cyclic groups of prime order, and by D.S. Archdeacon, J.H. Dinitz, A. Mattern, and D.R. Stinson for any finite cyclic group, see [6].

Conjecture 1.2 (G-ADMS). *Let $A \subseteq \mathbb{Z}_n \setminus \{0\}$. Then there exists an ordering of the elements of A such that the partial sums are all distinct.*

In [6], the authors proved that Conjecture 1.1 in a group \mathbb{Z}_n for sets of size at most k implies Conjecture 1.2 in the same group \mathbb{Z}_n for sets of size at most k . As remarked in [11], the G-ADMS conjecture can be extended to any finite subset of an abelian group. This implies that our results on Alspach's conjecture can also be applied to the G-ADMS conjecture. In Section 4, we prove the validity of Conjecture 1.2 for subsets of size 12 of cyclic groups of prime order. This result is achieved using Alon's combinatorial Nullstellensatz and the techniques developed in [17]. As a consequence, we obtain a similar extension to torsion-free abelian groups and a similar asymptotic result.

2. Alspach's conjecture for torsion-free abelian groups. In this paper, we will say that a finite subset A of an abelian group $(G, +)$ is nice if $0_G \notin A$ and $\sum_{z \in A} z \neq 0_G$. Also, with an abuse of notation, we will say that Alspach's conjecture is true in G for any subset of size k if it is true for any nice subset of size k .

Given a nice subset A of an abelian group G , by $\Delta(A)$ we mean the set $\{x_1 - x_2 : x_1, x_2 \in A, x_1 \neq x_2\}$. This allows us to define the set

$$\Upsilon(A) = A \cup \Delta(A) \cup \left\{ \sum_{z \in A} z \right\}.$$

Given two integers a, b with $a \leq b$, the subset $\{a, a + 1, \dots, b\} \subset \mathbb{Z}$ will be denoted by $[a, b]$. Our aim is to extend the known results on Alspach’s conjecture in cyclic groups of prime order to any torsion-free abelian group.

Lemma 2.1. *Let G_1 and G_2 be abelian groups such that Alspach’s conjecture holds in G_2 for any subset of size k . Given a nice subset A of G_1 of size k , suppose there exists an homomorphism $\varphi : G_1 \rightarrow G_2$ such that $\ker(\varphi) \cap \Upsilon(A) = \emptyset$. Then Alspach’s conjecture is true for the subset A .*

Proof. Since no element of $\Upsilon(A)$ belongs to the kernel of φ , $\varphi(A)$ is a nice subset of G_2 of size k . Therefore, there exists an ordering $\omega_2 = (x_1, x_2, \dots, x_k)$ of the elements of $\varphi(A)$ such that $s_i(\omega_2) \neq 0_{G_2}$ and $s_i(\omega_2) \neq s_j(\omega_2)$ for all $1 \leq i < j \leq k$. However, considering the ordering $\omega_1 = (z_1, z_2, \dots, z_k)$ of the elements of A , where $\varphi(z_i) = x_i$ for all $i = 1, \dots, k$, we obtain that the partial sums $s_i(\omega_1)$ in G_1 are still pairwise distinct and nonzero. \square

Proposition 2.2. *Let k be a positive integer and suppose that, for infinitely many primes p , Alspach’s conjecture holds in \mathbb{Z}_p for any subset of size k . Then Alspach’s conjecture holds in \mathbb{Z} for any subset of size k .*

Proof. Consider a nice subset A of \mathbb{Z} of size k . Let $p > \max_{z \in \Upsilon(A)} |z|$ be a prime such that Alspach’s conjecture holds in \mathbb{Z}_p for subsets of size k . Then $\Upsilon(A)$ and the kernel of the canonical projection $\pi_p : \mathbb{Z} \rightarrow \mathbb{Z}_p$ are disjoint sets. Hence, the statement follows from Lemma 2.1. \square

We now consider the free abelian group \mathbb{Z}^n of rank n .

Proposition 2.3. *Suppose that Alspach’s conjecture holds in \mathbb{Z} for any subset of size k . Then it holds in \mathbb{Z}^n for any $n \geq 2$ and any subset of size k .*

Proof. Fix a nice subset $A = \{a^1, a^2, \dots, a^k\}$ of \mathbb{Z}^n of size k , and set $B = \Upsilon(A)$. Given an integer

$$M > \max_{(z_1, \dots, z_n) \in B} \max_{j \in [1, n]} n|z_j|,$$

we define the homomorphism $\varphi : \mathbb{Z}^n \rightarrow \mathbb{Z}$ as follows:

$$\varphi(x_1, \dots, x_n) = \sum_{i=1}^n x_i M^{i-1}.$$

Because of the choice of M , the subset B and the kernel of φ are disjoint. Namely, suppose that there exists $b = (y_1, y_2, \dots, y_n) \in B$ such that $\varphi(b) = 0$. We can assume that y_{s_1}, \dots, y_{s_c} are all nonnegative integers and that y_{t_1}, \dots, y_{t_d} are all negative integers. Then we can write

$$\sum_{j=1}^c y_{s_j} M^{s_j-1} = \sum_{j=1}^d (-y_{t_j}) M^{t_j-1}.$$

We can look at the two sides of this equality as two expansions in base M of the same nonnegative integer since the coefficients $y_{s_1}, \dots, y_{s_c}, y_{t_1}, \dots, y_{t_d}$ all belong to the set $[0, M - 1]$. The uniqueness of such an expansion implies that all these coefficients are zero, i.e., that $b = 0$. It follows from Lemma 2.1 that Alspach’s conjecture holds for the subset A . \square

From the previous proposition, we deduce this result.

Theorem 2.4. *Let k be a positive integer and suppose that, for infinitely many primes p , Alspach’s conjecture holds in \mathbb{Z}_p for any subset of size k . Then Alspach’s conjecture holds for any subset of size k in any torsion-free abelian group G .*

Proof. Let A be a nice subset of G of size k . Denote by H the subgroup of G generated by A . We can apply to H the structure theorem for finitely generated abelian groups, obtaining that H is isomorphic to a subgroup of \mathbb{Z}^k . So, we can view A as a nice subset of \mathbb{Z}^k . Since, by hypothesis, we are assuming the validity of Alspach’s conjecture in \mathbb{Z}_p for infinitely many primes p and for any subset of size k , by Propositions 2.2 and 2.3, Alspach’s conjecture holds in \mathbb{Z}^k for any subset of size k . In particular, it holds for A . □

Now, from Theorem 2.4 and the results cited in the introduction, we obtain:

Corollary 2.5. *Alspach’s conjecture holds for any subset of size $k \leq 11$ of any torsion-free abelian group.*

3. An asymptotic result. Given an element g of an abelian group G , we denote by $o(g)$ the cardinality of the cyclic subgroup $\langle g \rangle$ generated by g . Furthermore, we set

$$\vartheta(G) = \min_{0 \neq g \in G} o(g)$$

Now we are ready to prove that, if $\vartheta(G)$ is large enough, Alspach’s conjecture is true for $k \leq 11$. This result can be deduced from the compactness theorem of the first order logic but, here, we give a more direct proof.

Theorem 3.1. *Under the hypotheses of Theorem 2.4, there exists a positive integer $N(k)$ such that Alspach’s conjecture holds for any subset of size k of any abelian group G such that $\vartheta(G) > N(k)$.*

Proof. Let us suppose, for sake of contradiction, that such $N(k)$ does not exist. It means that, for any positive integer M , there exists an abelian group G_M such that $\vartheta(G_M) > M$ and there exists a nice subset $A_M = \{a_{M,1}, a_{M,2}, \dots, a_{M,k}\}$ of G_M of size k that contradicts Alspach’s conjecture. Therefore, for any ordering $\omega = (a_{M,j_1}, a_{M,j_2}, \dots, a_{M,j_k})$ of A_M , there exists a pair (i, j) , with $i, j \in [1, k]$, such that $s_i(\omega) = s_j(\omega)$. Choosing, for each ω , one of these pairs, for any positive integer M , we can define the function $f_M : \text{Sym}(k) \rightarrow [1, k] \times [1, k]$ that maps $\sigma_\omega = (j_1, j_2, \dots, j_k)$ into this chosen pair. Since there are only finitely many maps from $\text{Sym}(k)$ to $[1, k] \times [1, k]$, there exists an infinite sequence M_1, \dots, M_n, \dots such that $f_{M_1} = f_{M_i}$ for all $i \geq 1$.

Let us consider the group $G = \times_{i=1}^\infty G_{M_i}$ and the following equivalence relation on G . Given $x = (x_i), y = (y_i) \in G$, we set $x \approx y$ whenever $x_i \neq y_i$ only on a finite number of indices i . Since the equivalence class $[0]$ consists of the elements (x_i) of G that are nonzero on a finite number of coordinates x_i , and so it is a subgroup of G , the quotient set $H = G / \approx$ is still an abelian group.

Now we want to prove that H is torsion-free. Let us suppose, for sake of contradiction, that there exists an element $[0] \neq [x] \in H$ of finite order, say n . Let $\pi_j : G \rightarrow G_{M_j}$ be the canonical projection on G_{M_j} . For any i such that $M_i > n$, either $\pi_i(x) = 0_{G_{M_i}}$ or we have $n \cdot \pi_i(x) \neq 0_{G_{M_i}}$. However, since $n \cdot [x] = [0]$ in H and due to the definition of \approx , we should have $n \cdot \pi_i(x) = 0_{G_{M_i}}$ for i large enough. It follows that $\pi_i(x)$ is eventually zero but this is a contradiction since $[x]$ is nonzero. Therefore, H is torsion-free.

Now we consider the following subset A of H :

$$A = \{[z_1], [z_2], \dots, [z_k]\} \text{ where } (z_j)_\ell = a_{M_\ell, j}.$$

Clearly, A is a nice subset. Given an ordering $\omega = ([z_{j_1}], [z_{j_2}], \dots, [z_{j_k}])$ of A , we define the ordering $\omega_\ell = (a_{M_\ell, j_1}, a_{M_\ell, j_2}, \dots, a_{M_\ell, j_k})$ of A_{M_ℓ} that corresponds to the same permutation σ_ω . As $f_{M_1} = f_{M_\ell}$ for all $\ell \geq 1$, there exists a pair (i, j) such that $s_i(\omega_\ell) = s_j(\omega_\ell)$ for all ℓ . Since $(s_i(\omega_1), s_i(\omega_2), \dots, s_i(\omega_\ell), \dots)$ belongs to the equivalence class $s_i(\omega)$, it easily follows that $s_i(\omega) = s_j(\omega)$ also for the set A . It means that A is a counterexample to Alspach’s conjecture. Since this is a contradiction to Theorem 2.4, we have proved the statement. □

Corollary 3.2. *Let $k \leq 11$ be a positive integer. Then there exists a positive integer $N(k)$ such that Alspach’s conjecture holds in \mathbb{Z}_n for any subset of size k whenever the prime factors of n are all greater than $N(k)$.*

4. Implications on other conjectures. We consider here two conjectures related to the Alspach one. These conjectures have recently been studied mainly in relation to (relative) Heffter arrays and their application for constructing cyclic cycle decompositions of (multipartite) complete graphs, see [4, 5, 9, 12–14].

4.1. The G-ADMS conjecture. In [17], the validity of Conjecture 1.1 was proved for any cyclic group \mathbb{Z}_p , where p is a prime, whenever $k = |A| \leq 10$. This result was achieved using a polynomial method based on Alon’s combinatorial Nullstellensatz.

Theorem 4.1 ([1, Theorem 1.2]). *Let \mathbb{F} be a field and let $f = f(x_1, \dots, x_k)$ be a polynomial in $\mathbb{F}[x_1, \dots, x_k]$. Suppose the degree of f is $\sum_{i=1}^k t_i$, where each t_i is a nonnegative integer, and suppose the coefficient of $\prod_{i=1}^k x_i^{t_i}$ in f is nonzero. Then, if A_1, \dots, A_k are subsets of \mathbb{F} with $|A_i| > t_i$, there are $a_1 \in A_1, \dots, a_k \in A_k$ so that $f(a_1, \dots, a_k) \neq 0$.*

In order to apply this theorem for proving Alspach’s conjecture for subsets of \mathbb{Z}_p of size k , J. Hicks, M.A. Ollis, and J.R. Schmitt constructed a suitable homogeneous polynomial F_k of degree $k(k-1)-1$, identifying a monomial with nonzero coefficient such that the degree of each of its terms x_i is less than $|A| = k$ (clearly, we may assume $p > k$). The existence of values $a_1, \dots, a_k \in A$ such that $F_k(a_1, \dots, a_k) \neq 0$, given by Theorem 4.1, implies that $\omega = (a_1, \dots, a_k)$ is an ordering of the elements of A satisfying the requirements of Conjecture 1.1.

We recall here the polynomials given in [17]. For every $k \geq 2$, let

$$g_k(x_1, \dots, x_k) = \prod_{1 \leq i < j \leq k} (x_j - x_i)(x_i + \dots + x_j) \in \mathbb{Z}[x_1, \dots, x_k]$$

and

$$F_k(x_1, \dots, x_k) = \frac{g_k(x_1, \dots, x_k)}{x_1 + \dots + x_k} \in \mathbb{Z}[x_1, \dots, x_k]. \tag{4.1}$$

Observe that we can also define F_k recursively. To this purpose, define $G_\ell \in \mathbb{Z}[x_1, x_2, \dots, x_{\ell+1}]$, where $2 \leq \ell \leq k - 1$, as follows:

$$G_\ell = (x_1 + \dots + x_\ell)(x_{\ell+1} - x_1) \cdot \prod_{i=2}^{\ell} (x_{\ell+1} - x_i)(x_i + \dots + x_{\ell+1}).$$

Then $F_\ell \in \mathbb{Z}[x_1, x_2, \dots, x_\ell]$ can be defined as

$$F_2 = x_2 - x_1 \quad \text{and} \quad F_\ell = F_{\ell-1} \cdot G_{\ell-1} \quad \text{for } 3 \leq \ell \leq k.$$

Now, consider the monomial

$$\tilde{c}_{k,j} = c_{k,j} x_1^{k-1} \dots x_{j-1}^{k-1} \cdot x_j^{k-2} \cdot x_{j+1}^{k-1} \dots x_k^{k-1}$$

of F_k , where $1 \leq j \leq k$ and $c_{k,j} \in \mathbb{Z}$. In [17, Table 1], the authors described the coefficients $c_{k,j}$ for $k \leq 10$, showing that either $\gcd(c_{k,1}, \dots, c_{k,k})$ equals 1 or its prime factors are all less than k . This means that, for any prime $p > k$, there exists a coefficient $c_{k,j}$ which is nonzero modulo p , proving the validity of Conjecture 1.1 for subsets of \mathbb{Z}_p of size k .

We can use Alon’s combinatorial Nullstellensatz to prove the G-ADMS conjecture. Also in this case, we use the polynomial provided by [17]. For any $k \geq 2$, let

$$f_{k+1}(x_1, \dots, x_{k+1}) = g_k(x_2, x_3, \dots, x_{k+1}) \cdot \prod_{j=2}^{k+1} (x_j - x_1). \tag{4.2}$$

Note that $f_{k+1} \in \mathbb{Z}[x_1, \dots, x_{k+1}]$ is a homogeneous polynomial of degree k^2 . By Theorem 4.1, to prove the validity of Conjecture 1.2 for subsets of \mathbb{Z}_p of size $k + 1$, it suffices to find a monomial of f_{k+1} with nonzero coefficient such that the degree of each of its terms x_i is less than $k + 1$.

Instead of working directly with the polynomial f_{k+1} , we show how to use the polynomial F_k also for proving the validity of the G-ADMS conjecture via Alon’s combinatorial Nullstellensatz. The main advantage is computational: instead of working with a polynomial of degree k^2 in $k + 1$ indeterminates, we work with a polynomial of degree $k^2 - k - 1$ in k indeterminates. Hence, consider the monomial

$$\tilde{d}_{k+1,j} = d_{k+1,j} \cdot x_1^k \dots x_{j-1}^k \cdot x_{j+1}^k \dots x_k^k$$

of $f_{k+1}(x_1, \dots, x_{k+1})$, where $j \in [1, k + 1]$ and $d_{k+1,j} \in \mathbb{Z}$.

In order to apply Theorem 4.1, we would like to determine the values of the coefficients $d_{k+1,j}$. To this purpose, take the monomial

$$\tilde{e}_{k,j} = e_{k,j} \cdot x_1^k \dots x_{j-1}^k \cdot x_{j+1}^k \dots x_k^k$$

of $g_k(x_1, \dots, x_k)$, where $j \in [1, k]$ and $e_{k,j} \in \mathbb{Z}$. If $j \geq 2$, then x_1^k divides $\tilde{d}_{k+1,j}$ and so, by (4.2), the value of $(-1)^k \cdot d_{k+1,j}$ coincides with the coefficient of

$x_2^k \cdots x_{j-1}^k \cdot x_{j+1}^k \cdots x_{k+1}^k$ in $g_k(x_2, \dots, x_{k+1})$. In other words, we obtain that

$$d_{k+1,j+1} = (-1)^k e_{k,j} \quad \text{for all } j \in [1, k].$$

Hence, the problem of computing the values of the coefficients $d_{k+1,j+1}$ is equivalent to the problem of computing the coefficients $e_{k,j}$ of $g_k = (x_1 + \cdots + x_k) \cdot F_k$, see (4.1). So, for any $i, j \in [1, k]$ such that $i \neq j$, take the monomial

$$\tilde{a}_{i,j}^{(k)} = a_{i,j}^{(k)} \cdot x_i^{k-1} \cdot \prod_{\substack{1 \leq r \leq k \\ r \neq i,j}} x_r$$

of F_k , where $a_{i,j}^{(k)} \in \mathbb{Z}$. Then, for all $j \in [1, k]$, we have

$$(-1)^k d_{k+1,j+1} = e_{k,j} = a_{1,j}^{(k)} + \cdots + a_{j-1,j}^{(k)} + a_{j+1,j}^{(k)} + \cdots + a_{k,j}^{(k)}.$$

For instance, using the routines for Magma that we give in [10]

we determine the values of $a_{i,j}^{(6)}$, see Table 1. More generally, for $k \in [3, 10]$, we obtain that

$$e_{k,j} = (-1)^{\lfloor \frac{k-1}{2} \rfloor} c_{k,j} \quad \text{for all } j \in [1, k].$$

It would be very interesting to prove this equality for every value of k .

Considering the case $k = 11$, with the same routines, we obtain the values of the coefficients $e_{11,1}$ and $e_{11,2}$:

$$e_{11,1} = 18128730243333160, \quad e_{11,2} = 46383022877233608.$$

Since $\gcd(e_{11,1}, e_{11,2}) = 2^3$, the following result follows.

Proposition 4.2. *The G-ADMS conjecture holds for subsets of size $k \leq 12$ of cyclic groups of prime order.*

One can easily adjust to the G-ADMS conjecture the arguments of Sections 2 and 3. Clearly, some small modifications are required. In particular, given an abelian group $(G, +)$, a finite subset A of G is nice for the G-ADMS conjecture if $0_G \notin A$; also, given a nice subset A of G , let

$$\Upsilon(A) = A \cup \Delta(A).$$

Hence, from Proposition 4.2, we can deduce the following results.

$j \setminus i$	1	2	3	4	5	6	$e_{6,j}$
1		-28	-40	-20	20	40	-28
2	28		-28	-40	-20	20	-40
3	40	28		-28	-40	-20	-20
4	20	40	28		-28	-40	20
5	-20	20	40	28		-28	40
6	-40	-20	20	40	28		28

TABLE 1. Values of the coefficients $a_{i,j}^{(6)}$

Corollary 4.3. *Given a torsion-free abelian group G , the G-ADMS conjecture is true for any subset $A \subset G \setminus \{0_G\}$ such that $|A| \leq 12$.*

Corollary 4.4. *There exists a positive integer N such that the G-ADMS conjecture is true for any subset A of $\mathbb{Z}_n \setminus \{0\}$ whenever $|A| \leq 12$ and the prime factors of n are all greater than N .*

4.2. The CMPP-conjecture. In [11], the following conjecture was proposed by the authors of the present paper, in collaboration with F. Morini and A. Pasotti.

Conjecture 4.5 (CMPP). *Let G be an abelian group. Let A be a finite subset of $G \setminus \{0_G\}$ such that no 2-subset $\{x, -x\}$ is contained in A and with the property that $\sum_{a \in A} a = 0_G$. Then there exists an ordering of the elements of A such that the partial sums are all distinct.*

Suppose that Conjecture 1.1 holds for subsets of size k of a given abelian group G . Let A be a $(k+1)$ -subset of $G \setminus \{0_G\}$ such that $\sum_{a \in A} a = 0_G$. Clearly, for any $a \in A$, the set $A \setminus \{a\}$ is a nice subset of G of size k : we can find an ordering $\omega = (a_1, a_2, \dots, a_k)$ such that all the partial sums $s_i(\omega)$ are nonzero and pairwise distinct ($1 \leq i \leq k$). Now, taking $\omega' = (a_1, a_2, \dots, a_k, a)$, we obtain an ordering of the elements of A such that $s_i(\omega') \neq s_j(\omega')$ for all $1 \leq i < j \leq k+1$. Therefore, Conjecture 1.1 for sets of size at most k in an abelian group implies Conjecture 4.5 for sets of size at most $k+1$ in the same group. It follows that:

Corollary 4.6. *Given a torsion-free abelian group G , the CMPP conjecture is true for any subset $A \subset G \setminus \{0_G\}$ such that $|A| \leq 12$, $\sum_{z \in A} z = 0_G$, and A does not contain pairs of type $\{x, -x\}$.*

Corollary 4.7. *There exists a positive integer N such that the CMPP conjecture is true for any subset A of $\mathbb{Z}_n \setminus \{0\}$ whenever $|A| \leq 12$, $\sum_{z \in A} z = 0$, A does not contain pairs of type $\{x, -x\}$, and the prime factors of n are all greater than N .*

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