UNIVERSITÀ CATTOLICA DEL SACRO CUORE

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Scuola di Dottorato in Politica Economica
Ciclo XXV
S.S.D. SECS-P/01, SECS-S/06

Two-Country OLG Models With Integrated Financial Market

Tesi di Dottorato di FRANCESCO RILLOSI

Matricola 3810395

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Introduction

The present work investigates some of the consequences on the economic growth of countries involved in the globalization process, which consists on the opening of domestic markets to the international exchanges.

The main references have been taken from the researcher Kiminori Matsuyama, in particular from his fundamental paper "*Financial Market Globalization, Symmetry-Breaking and Endogenous Inequality of Nations*" [10] and from the adaptation to the two-country case, operated by Tomoo Kikuchi and John Stachurski in "*Endogenous inequality and fluctuations in a two-country model*" [8] (see below for further details).

Therefore, our basic scheme consists in describing the effects from autarky to the case of two-country integrated markets, making different hypotheses about which markets are open: factor markets solely (Chapter 1), factor and financial markets (Chapter 1), financial markets solely (Chapter 2 and 3). In each case an Overlapping Generations model is considered.

The essay is composed by two parts and three chapters.

In both parts, an OLG model, from Diamond's family, is employed.[1]

A sequence of overlapping generations is considered, each one living through two different periods. Agents are respectively denoted as "young" and "old". The young have an endowment of labor; they supply it inelastically to the labor market and receive an income.

In the first chapter, the young have to decide how much to consume and how much to save. Then they have to decide whether to lend their savings or to become entrepreneurs, in order to maximize their second period consumption.

In the second and third chapter, the young do not consume in the first part of their life, but invest all their income, for the second period-second age. Eventually the old receive their output from the productive process of final goods, consume it and exit.

There is only one kind of good, suitable for every economic dealing.

There are two productive processes. The first produces consumer goods; it starts and ends during each period and utilizes two factors, labor and capital, according to a neoclassical productive function. The second concerns the production of physical capital and covers two periods. It takes in input savings (goods not consumed by the young agents) and, in the next period, converts them in capital factors; its technology is supposed to be linear.[2].

In the first part, perfect markets are considered, while in the second part an hypothesis of credit rationing is admitted.

[1] For a description of the Diamond model see Romer [14]

[2] If we interpret physical capital as "human", then this second production process may be intended as the educational system.

Part I - Chapter 1.

In the first part, consisting of a single chapter, we use a standard OLG model to analyze the effects on economic growth caused by the opening of perfect markets of factors and credit. We take under consideration different cases: at first, factor markets are integrated, and financial markets are closed; then, all markets are assumed to be open. We prove that no matter if the markets are open or closed, in the long run, economies always converge to an equilibrium of steady state, influenced by the efficiency of the countries in producing capital and consumer goods. We also prove that the map, whose repeated application represents the dynamic behavior of the economic system, is a continuous one, but with some non differentiable points, and we are able to characterize the conditions under which production of final goods occurs in the most efficient region, or in both.

Part II - Chapters 2 and 3.

In the second part, structured in two chapters (the second and the third of the thesis), we introduce a new axiomatic picture that allows us to obtain more articulated results in the long run dynamics, characterized by endogenous fluctuations.

In order to put specific attention on financial globalization phenomena, only the credit markets are assumed to be open, while the others are closed to the international exchanges.

Now agents are supposed to consume only during the second period of their life; in this way all their wages become savings. This particular hypothesis (commonly accredited in specific literature) induces, of course, a simplification of the model, but it allows us to concentrate on the main maximizing choice of the young agents: to lend or to borrow the resources available in the economy. Particularly, under some reasonable hypotheses, we can prove that agents need to borrow in order to become entrepreneurs.

The main hypothesis of this second part is that credit markets are assumed to be imperfect: borrowers are supposed to repay their debts only if the cost of the obligation is smaller than the cost of the default, which is taken to be a fraction $\lambda \in (0, 1)$ of the project output. According to the condition by which agents need to borrow in order to start their investment project, this hypothesis introduces a borrowing constraint in addition to the more common profitability constraint. Moreover, when the countries lie in autarky, credit market imperfection doesn't affect equilibrium levels; in this case it has an impact only on the interest rate, that adjusts in order to balance savings and investment. Differently, when financial markets are integrated, the credit market imperfections have significant consequences on the steady states of the long run. The map describing the two-country economic development becomes a piece-wise two-dimensional one, and the plane of significant states is divided into four regions, conforming to the fact that the j-country is borrowing-not borrowing constrained (for $j = home$ / *abroad,* respectively).

In Chapter 2 we focus on the case of perfect symmetry, assuming that the economies are identical in all their basic features.

Until their markets are closed, countries stay in their long run equilibrium of autarky, that is unique and asymptotically stable. Opening their financial markets, multiple steady states may occur, introducing, in this way, asymmetrical position between the countries. We also prove that the equilibrium of autarky still remain, but it might change its features of stability. In particular, even when it becomes a saddle, it preserves a stable set of positive Lebesgue measure acting, in this way, as a Milnor attractor.

Choosing a production function from the Cobb-Douglas family we are able to depict a global analysis of the economic environment.

In the third Chapter, we gradually introduce heterogeneity between the countries.

Firstly, as in Kikuchi and Stachurski [8], we consider only the differences in the population size. We prove that the autarky equilibrium preserves some features analogous to the ones of the symmetric case. Then we depict a periodic dynamic constituted by a Neimark-Sacker bifurcation.

Our thesis is that periodic phenomena derive not only by differences in population size, as in Kikuchi and Stachurski, but are also due to heterogeneity in technology and credit market imperfection.

In doing that we introduce the hypothesis of complete heterogeneity between the two countries and prove the existence of periodic dynamics, characterized by cyclical phenomena and a crater bifurcation. Particularly, we prove the possibility of long run behavior in which countries run continuously from constrained and not constrained credit situations.

Finally we introduce the hypothesis of "quasi heterogeneity" assuming differences in technologies and credit market imperfections, but equal populations. Even in this case we are able to prove the existence of a crater bifurcation and, in this way, we consider definitely proved our thesis.

In the following, before introducing the specific analysis of the models, we offer an overview of the main literature on the topic.

An overview on the main literature.

Over the past two decades, macroeconomic implications of credit market imperfections, in the globalization process, have been investigated by a number of researchers; their efforts have constituted a conspicuous body of results and offer a reach literature on the topic. Hence we will propose a brief summary of some of the basic papers on the subject.

In his paper "Financial market globalization, symmetry-breaking, and endogenous inequality of nations" [10] Matsuyama focuses on the effects of financial market globalization on the inequality of the nations. According to the mainstream, globalization would activate financial flows from the rich countries to the poor ones, constituting an impulse of developing for the poor economies. But as many researchers have highlighted, this mechanism fails in the case of imperfect markets. In that case, the more financial security that a rich and well-developed country can offers rather than a poor and not well-developed one, may inverts the natural flow of the wealth; when this happens, globalization magnifies the inequality, the poor countries become poorer and the rich richer. For these reasons, some researchers believe that governments of poor countries must act in order to prevent the outflows of domestic savings and that the rich countries must aid the poor ones to overtake their lack of development. Considering that the debut often experiences a strong polarity of the opposite factions, the author tries to conciliate these contrasting opinions. In doing that, he draws a Diamond overlapping generations model relative to a continuum of inherently identical countries, in which two generations, the old an the young, coexist in each period and respectively supply labor and physical capital, as input factors for the production of a unique final good. The goods produced in period t, may be consumed or invested in the production of physical capital that will become available in period $t + 1$. For hypothesis the marginal return of capital is decreasing relative to the capital itself and this makes the productivity of investment higher in the poor countries than in the rich ones. This is an equalizing force that tends to transfer the wealth from the rich to the poor economies. Another key hypothesis is the imperfection of the credit market, depicted in this way: lenders believe that borrowers, having to choose between the default and the repay of their debts, will run the first option rather than to pay more than a fixed percentage (the measure of imperfection) of their earnings. This makes an unequalizing force, that pushes wealth in the opposite direction relative to the one due to the hypothesis of decreasing productivity. When it is in autarky, in spite of credit imperfection, the world economy has a unique, symmetric and globally stable steady state. Within each country, the interest rate adjusts in order to balance savings and investments. So different countries face with different domestic interest rates, but reach the same steady state. When financial integration starts to operate, under certain conditions, symmetry breaking may occur: the symmetric steady state loses its stability and others stable asymmetric steady states come to exist. The fact is that, after the financial integration, there exists a unique interest rate in the world; this means that the interest rate continues to balance savings and investment but only globally; instead it stops to make equilibrium within the single country. Now savings can freely run across the frontiers and tend to escape from the more disadvantaged countries and move towards the more advantaged ones. The situation of stable asymmetric steady state pictures a world polarized into poor and rich countries; the first involved by the borrowing constraint, the second free. In this case the rich are richer, the poor are poorer and the total world output is smaller than in the symmetric case. These results are important for two reasons. First, they prove that in some cases, globalization may magnifies inequality between nations, and give a theoretical support to the popular belief that globalization sometimes makes the rich countries richer at the expense of the poor ones. Second, the model depicts a mechanism that depends on relative wealth and tends to reproduces itself continuously; this means that there is no advantage in cutting off the rich countries and connecting the poor: the division between rich and poor will replicate itself into the new group. This is contrary to the popular belief. However, symmetry-breaking is not unavoidable; it occurs under certain parameter values; particularly, it needs a sufficiently large credit market imperfection. The paper depicts sufficient and necessary conditions for the symmetry breaking equilibrium and, more in general, for all the possible steady states of the system. The fundamental policy implication of the theory is that situations of undeveloped economies can not be taken on as isolated problems, but need to be considered under a global approach, as part of an interrelated net of many countries. Depending on a set of initial conditions, globalization may bring a real progress for all the participating countries, but, under other conditions, may cause a declining for the poorer ones.

Kikuchi and Stachurski [8] have developed a two-country version of Matsuyama's model described above, in which the world economy is made by two large countries, with the same economic features. Agents live for two periods, work in the first and consume in the second; the young agents supply labor, while the old ones supply capital factors. After the production and the distribution of the output are completed, the old consume and exit the model, while the young invest all their wages. The credit market is imperfect, and borrowers face two constraints: the first marks the convenience to invest rather than to lend, while the second comes from the rationing of the credit due to the imperfection of the market. Like in Matsuyama's the financial imperfection is represented assuming that debtors are supposed to not credibly pay more than a fraction $\lambda \in [0, 1]$ of their expected earnings. When the countries are isolated, their interest rate balances their domestic savings and investment, and in this way the imperfection of credit markets never binds the accumulation of capital, that tends to a symmetric, asymptotically stable point of steady state. On the contrary, when financial markets are integrated, savings can freely flow from one country to another and equivalence between domestic savings and investment is no more guaranteed. In this case, under certain parametrization, the symmetric steady state becomes unstable and asymmetric stable steady states start existing. In addition, considering a diversification on population sizes of the two countries, the authors prove the existence of endogenous periodic phenomena like stable cycles. Summarizing the model depicts phenomena of symmetry-breaking like in the Matsuyama's one but, adopting the hypothesis of two countries, is able to extend the analysis outside of the steady states and to intercept the presence of periodic dynamics.

Continuing on the fundamental question, whether or not financial globalization is good for all the participating countries, Mendoza, Quadrini and Rios-Rull [12] highlight the thesis by which only a well-designed domestic financial system, as result of well-designed domestic political and social systems, allows to enjoy the positive effects of the financial globalization. Taking this suggestion, they investigate the possible consequences of joining credit markets, for countries whose domestic financial markets are not sufficiently and uniformly developed. In doing that they hypothesize the presence of markets limited in the amount of borrowing, assuming that limit as measure of the degree of financial undevelopment; then they consider a model of financial integration among countries heterogeneous in their degree of domestic financial development. They prove that, in this case, welfare consequences of liberalization may be different from one country to another. Particularly, the most financially developed country will experience an increase in its aggregate welfare, and the less developed one, a decrease. This occurs because the financial integration increases the cost of borrowing, relative to the autarky situation, to poor agents in the less developed country and decreases it in the more developed one. The authors conclude affirming that, this adverse phenomenon can be contrasted by a redistribution of initial wealth in the less developed countries; this can lead to a situation in which the aggregate welfare effects are positive for all the participants.

Angeletos and Panousi [3] deal with the theoretical problem to justify the empirical evidence that, in many cases, the direction of capital flows seems to be in contrast with the neoclassical paradigm: not from the rich to the poor countries, neither from slow-growing to the fast-growing, but actually in the opposite sense. Their paper analyzes the global macroeconomic effects of financial integration, in the presence of an uninsurable idiosyncratic entrepreneurial risk. This risk induces entrepreneurs to make a self-insure against it and generates a wedge between the interest rate and the marginal product of capital, because of the private assurance the entrepreneurs pretend in compensation for the risk they face in their activity. This wedge is likely to vary across countries, and in some of them is much higher than in others. In their model, the authors consider the case of two economies, the "North" and the "South", populated by a continuum of families, each family including a worker and an entrepreneur. The two countries differ for the level of uninsurable risk, that is low in the North and high in the South. Before the financial integration, the South faces with a lower interest rate than the North, due to its stronger demand for precautionary savings, and with a lower capital stock due to a higher wedge between the marginal product of capital and the interest rate. After the financial integration, the interest rate tends to rise up in the South, increasing the opportunity cost of the capital and, in this way, depressing the capital stock, the wages and the final product. The South experiences a depressed period, while the North knows a boom. This explains why financial globalization may initially increase cross-country inequality and capital may often fail to flow from the rich countries to the poor ones. However the long run effects may be different. During the time the agents of the South, saving abroad at higher return than domestically, will be able to accumulate more and more wealth, in such a way increasing their willingness to assume risk and diminishing the wedge between the interest rate and the return of capital. Now the situation is overturned, the economy of the North tends to slacken, while the one of the South to develop. As a result, the South can reach a better condition than the previous one of autarky. Anyway this happens only in the long run. It is to be noted that, in the short-run the South's poor tends to become poorer for two reasons: the increase in the cost of borrowing due to the increase in the interest rate, and the reduction of the wages as a consequence of the initial outflow of capital due to the greater convenience, for the South's rich, to invest in the North. On the contrary the South's rich gain because of the higher return of their savings and the lower cost of labor in their private activities.

Summarizing, the South has to survive to the first inequality period of globalization; later the new generation of agents will generally gain the benefits of the reform.

Again, in order to justify the evidence that many less-developed countries receive little net credit or no credit at all, from the more developed ones, and that, in general terms, globalization seems not to work in the direction of an international equalization of capital stocks and productive levels, Boyd and Smith [5] develop a model based on the concept of *costly state verification* (CSV). It is known that CSV problem has been analyzed by Townsend [15] in the theory of contracts and consists in the optimization problem connected to a contract, in which lender has to decide whether to pay for monitoring borrower's performance or not. On one hand, verification has obviously a cost for the sponsor; on the other hand, no verification allows the borrower to hide the realized profits. In general terms, this problem introduces an external cost on the financing exchange. The authors prove the possibility of endogenous cyclical fluctuations due to the presence of credit rationing, because of the CSV phenomenon. They consider a two-country Diamond model in which economies open only their financial markets and in which allocation of funds are penalized by the CSV problem, so that credit is rationed. Countries are identical in every aspect, but differ in their initial capital stocks. When closed, each country lies in a unique, not trivial equilibrium, asymptotically stable. Economy converges to it without fluctuations and the only effect of the asymmetric information is the reduction of the capital stock of equilibrium, relative to the one of full information. When the two countries join their financial markets, the situation changes significantly. Under some particular hypotheses, the authors can prove the existence of at least three nontrivial steady states. One of these is the "old" steady state of autarky; it is symmetric and, in that case, the net-capital-flow from one country to the other is zero. The other two are asymmetric, with one country that has a permanently higher level of capital stock, income and final good. This situation is particularly unequal because the rich country is richer and the poor is poorer than in the autarky case. Also, the total amount of capital stock in the world is less than in autarky and the poor country is a net lender to the rich one. Differences in capital stocks across countries affect labor incomes and, in this way, the capacity of entrepreneurs to contrast the CSV obstacle.This is the basic mechanism. When in autarky economies identically converge to the unique point of steady state, but when the financial markets are integrated, the country with a higher capital stock has a bigger capacity in contrasting the CSV effect and it can attract more resources from the international loans. This last fact balances the difference in the marginal product of capital between the two countries (the country with the higher capital stock has the lower marginal product) and makes an asymmetrical equilibrium in which one country permanently has a higher capital stock than the other.

Before Boyd and Smith, Bernanke and Gertler [4] highlighted the weight of borrower solvency, introducing the concept of costly state verification (CSV). They consider the borrower net worth as inversely correlated to the CSV and derive two implications. First, considering that the net worth of borrowers is generally pro-cyclical, because entrepreneurs are more solvent when the economy is going well than when it is going bad, there will be a decline of agency costs during the periods of economic development and an increase during the recession ones. The authors prove that this may produce investment fluctuations and persistent cycles. Second, they also prove that shocks to borrower net worth, which are independent of aggregate output, may also cause occurrence of real fluctuations.

Carrying on the branch of research above, in his paper "Endogenous Inequality" [9] Matsuyama deals with the question of which are the determinant factors of the wealth distribution across households in the long run. His model considers a population made by infinitely-lived households that may differ only for the level of wealth. In each period, the distribution of wealth influences the supply and the demand for credit; the credit, in turn, influences the distribution of wealth of the consecutive period. For hypothesis the credit market is borrowing constrained, because the entrepreneurs are supposed to opt for the cost of the default, rather than pay more than a fraction of their project output. Because the more convenient project requires a minimum level of investment and credit is constrained, it follows that only the relatively rich households can borrow and become entrepreneurs, while the relatively poor families can only become lenders. The interest rate of equilibrium determines endogenously the threshold between the rich and the poor. The model identifies two possible scenarios with opposite consequences. In a "virtuous" scenario the strong demand for credit tends to produce an increase of the interest rate and of the wealth of the lenders. The gap between the rich and the poor tends to disappear, and these two categories move towards a unique steady state in which all the agents will be rich enough to borrow and to become entrepreneurs. In this case a net flow of wealth runs from the rich to the poor agents; these last are pulled out of their poverty by the rich. On the contrary, another scenario depicts a situation in which the population is polarized into the rich and the poor and changing is not possible. The poor agents are unable to borrow and can only lend their savings to the rich ones that, in turn, pay the poor an interest rate lower than the project return. So, the rich become richer and the poor poorer. Summarizing, the model offers a theoretical justification to the fact that, sometimes, the rich owe their wealth to the poor.

Also Matsuyama, in "Credit Traps and Credit Cycles" [11], tries to analyze the consequences of credit market imperfections on aggregate investment through the composition of the credit. The author observes that composition is normally considered as important as the volume of credit, since many government institutions have the explicit goal of redirecting the credit flow towards more social useful forms of investment. Hence he proposes a model of credit market imperfections with heterogeneous investment projects. Credit tends to go towards the project that generates the highest rate of return. Moreover because of the borrowing constraint, the most profitable project for the lenders is not necessarily the most productive one. In fact the borrower net worth is, among others thinks, responsible of the determination of the project with the highest return. Therefore, movements in the borrower net worth can endogenously affect the investment projects. In this way, the investment technologies are influenced by the credit composition. In turn, the changes in investment technologies affect the borrower net worth. This interaction leads to a number of nonlinear phenomena such as credit traps, credit cycles, etc.

Aghion, Bacchetta and Banerjee [1] analyze the role of financial factors as a source of instability for a small open economy. In their model they hypothesize the case of a small open economy, which produces a single tradeable good, employing two factors, capital and a country-specific skilled labor factor. For hypothesis, firms cannot borrow more than μ times the amount of their current level of investible funds; in this way, μ represents a measure of the degree of development (undevelopment) of the credit market. The authors intercept a basic mechanism by which economies that know an intermediate level of financial development - not very developed, neither very undeveloped - are the more exposed to destabilizing effects due to the financial globalization. This is the sequence: greater investment leads to greater output and profits, which permit to borrow more savings and realize more investment. The economy of the country starts to rise and new capital flows from the foreign countries. But, at the same time, the rise in investment increases the demand for the country-specific factor and, consequently, its price relative to the final good. This depresses the profits and now a new opposite chain of cause-and-effect begins: less profits, less credit, less investment and, finally, a fall in the output level. This mechanism does not significantly involve firms that enjoy a very high level of financial development, because they are not generally constrained by cash-flow levels, neither it involves firms that lie in a very low level of development, because these last are anyway constrained. On the contrary, "intermediate" economies may experience periods of instability due to cash-flow shocks that hit their capacity to obtain credit. This endogenous instability may cause persistent effects and, in extreme cases, limit cycles dynamics.

Daniel and Jones [7] face the problem that, as confirmed in many cases, financial liberalization often leads to financial crises. Even if they recognize the weight of explanations based on the weakness of banking systems, they develop a more dynamic model in which they prove how financial liberalization can produce, in and of itself, instability and banking crises. In their model, they consider a small open economy, just embarked on the international financial markets, therefore during a transition period that is the riskiest, even if its banking system is well-designed and its banks are safe, relatively to the long run equilibrium. It is reasonable to assume that, at the beginning of the financial liberalization, there are a small capital stock and a limited bank net worth. For hypothesis the financial sector of the model is composed by some foreign competitors and a single domestic bank that initially takes advantage from the inexperience of the foreign creditors and enjoys the high rate of internal marginal product, due to the small stock of capital of the country. Therefore the bank obtains high interest rates for its loans, it faces little default risk, even in presence of high leverage, it tends to retain most of its earnings and it experiences a period of net worth growing. But, through time, the capital stock grows and its marginal product falls; the foreign lenders become more expert and start to grant cheaper loans; the domestic bank's profit starts to fall and is expected to fall even further. This is the riskiest period, in which banking crises are most likely. If the bank will survive it probably will adopt a more conservative behavior and a lower leverage. The authors conclude with a policy recommendation, that is increasing bank capitalization, especially before the liberalization. Of course this has to be compared with the welfare loss due to diverted resources (the ones of capitalization) from alternative uses.

Part I.

A two-Country OLG model with a perfect, financial market

We consider an infinite-horizon overlapping generations model, in discrete time, belonging to the Diamond's family.

The economy is assumed to be "perfectly neoclassical", factors are paid at their marginal return, markets are competitive and there are no informative asymmetries between the agents.

First we will consider a closed economy, then the case of two countries, that interact with each other.

The aim of the model is essentially to explain the challenges from the closed situation, to the opening one, particularly referring to the process of capital accumulation.

The case of the two countries will be investigated without any distinction about relative dimensions, but under the hypothesis of differentiated relative capacity in producing capital goods. We will see that, in presence of a neoclassical axiomatic context, even in case of open economies, the law of motion of capital accumulation has no jumping, and tends to a unique steady state point, in a continuous manner. Moreover, we will prove that the entire process is governed by the interest factor, supposing agents take it as given, without trying to influence it. Time will be assumed discrete.

1. A two-Country OLG model with a perfect, financial market

1.1. Basic assumptions.

We consider an infinite-horizon overlapping generations model in discrete time, with two sectors, consumption and production. There's only one kind of goods, useful for every economic dealing.

In each period there are two generations of agents, the "old" and the "young". So, people live for two periods. When they are young, they work and when they become old, they retire. The young divide their wages in two parts, one for consumption, one for savings, in order to assure consumption during the second part of their life, when they will become old and they will be out of the labor market.

The production sector. It is supposed to be an infinitely lived neoclassical firm, which combines two factors of production, capital and labor, entirely consumed at the end. The final goods may be consumed in the first period, or invested for the second to produce new capital. So there are two productive processes, respectively of consumer goods and of capital goods. The first starts and ends during each period, while the second starts in a period and ends in the later.

Final goods never survive their period of production, so the only way they have to be transferred to the next period, is to be invested. What is not consumed is saved and what is saved is invested.

The production of consumer goods. Per capita output in period t, is obtained by $y_t = f(k_t)$, where k_t denotes capital per worker and $f : R^+ \to R^+$ is the production function in intensive form.

For every $k_t \in R^+$ it is assumed that:

$i_0)$ $f(k_t) \in C^2(R^+)$

$i_1)$ $f(0) = 0$

$i_2)$ $f'(k_t) > 0$

$i_3)$ $f''(k_t) < 0$

First Inada's condition:

$i_4)$ $\lim_{k_t \to 0^+} f'(k_t) = +\infty$

Second Inada's condition:

$i_5)$ $\lim_{k_t \to +\infty} f'(k_t) = 0$

Non vanishing labor share property:

$i_6)$ $\lim_{k_t \to 0^+} \frac{k_t f'(k_t)}{f(k_t)} \in [0,1)$ [1]

Perfect competitiveness of factor markets, these last remunerated at their marginal productiveness:

$i_7)$ $w_t \equiv W(k_t) = f(k_t) - k_t f'(k_t)$ *(for the remuneration of labor)*[2]

$i_8)$ $\rho_t = f'(k_t)$ *(for the remuneration of capital)*

Minimum elasticity of substitution property:

$i_9)$ $\sigma(k_t) > 1 - \frac{k_t}{f(k_t)}$,

where $\sigma(k_t) = \frac{f'(k_t)W(k_t)}{f(k_t)W'(k_t)}$ is the elasticity of substitution between capital and labor.

The production of capital goods. The production of capital goods starts in the first period and ends in the second one. We assume that 1 unit of goods invested in t (saved in t) returns $R > 0$ units of capital in period $t+1$.

Consumption and investment. The consumption sector is represented by two periods living overlapping generations of consumers, the old and the young. Each generation consists of a continuum of homogeneous agents, with unit mass and, for hypothesis, there is no population growth.[3] Young consumers are endowed with one unit of labor, which they supply inelastically to the competitive labor market.

[1]If the productive factors are remunerated at their marginal productiveness, then output $f(k_t)$ has to be divided into two shares, $\frac{kf'(k)}{f(k)}$ and $1 - \frac{kf'(k)}{f(k)}$, in order to remunerate capital and labor respectively. Until $\frac{kf'(k)}{f(k)} \neq 1$, the labor share will not be zero.

[2]Being $W'(k_t) = -k_t f''(k_t)$, from i_3 it follows that $W(k_t)$ is an increasing function of k_t.

[3]The hypothesis of normalization of population consists on that agents must be thought like the set of real numbers covering a unit segment I.

Suppose a quantity $q(x)$ of a certain endowment (wage, savings, capital, etc.) belongs to each agent x, from a group covering a portion λ of population I. Then the total quantity of the endowment into the population is $\int^{\lambda} q(x)\,dx$. Particularly, if $q(x) = q = constant$, the total quantity is $\int^{\lambda} q dx = q\lambda$. For example, if half of young earn a wage w, then the total wage of the entire population is $\frac{w}{2}$. Most of all, if each agent has an endowment q, the quantity of the endowment that can be found into the population is exactly q.

Investment project requires a minimum of one unit of consumption goods for investment in period t and returns $R > 0$ units of physical capital in period $t+1$. Produced capital goods can be rented to final commodity producing firm.

For formal simplicity, it can be assumed that returns of the period will be disposable at the end of the same one. So, at the end of the first period (instant t), each worker will earn w_t; he will consume the share c_{1t} and invest $s_t = w_t - c_{1t}$, in order to assure the consumption c_{2t+1}, from the production of second period $(t+1)$; the latter obtained because of new capital goods derived from investment s_t.

After received his income w_t, agent has to decides how much to consume and how much to save. Again, he has to decide whether to become a depositor, or an entrepreneur, and run a discrete, indivisible investment project. So, at time t, agent has two possible alternatives (let r_{t+1} be the second period lending/borrowing return):

a) if he becomes a depositor, then his second period consumption will be

$c_{2t+1} = s_t r_{t+1}$;

b) if he becomes an entrepreneur, at the end, he will consume

$c_{2t+1} = Rf'(k_{t+1}) - (1 - s_t)\, r_{t+1}$

Options a) and b) involve a non-arbitrage condition:

$s_t r_{t+1} = Rf'(k_{t+1}) - (1 - s_t)\, r_{t+1} \Rightarrow$

$$\mathbf{r_{t+1} = Rf'(k_{t+1})} \qquad [\mathbf{1.1}]$$

It means: on offering one unit of goods (not consuming it) agent has contributed to produce $Rf'(k_{t+1})$ units of goods. This is exactly what is expected to gain on the exchange. [4]

The resource constraint. Considering that one unit of savings invested in period t produces R units of capital in period $t+1$, and that factors are entirely consumed during the production process, it follows that the total stock of capital in the second period is:

$$\mathbf{k_{t+1} = Rs_t} \quad \textit{\textbf{resource constraint}} \qquad [\mathbf{1.2}]$$

[4] Relation [1.1] can be seen as a result of the *profitability condition:*
$Rf'(k_{t+1}) - (1 - s_t)\, r_{t+1} \geq s_t r_{t+1}$; this pushes the agents to become entrepreneurs, instead of lenders.

Remark. Relation [1.2] may be interpreted as the traditional inter-temporal constraint over the two periods, for an overlapping generations model.

In fact, $s_t = w_t - c_{1t}$ and, from [1.1] , $R = \frac{r_{t+1}}{f'(k_{t+1})}$. Substituting into [1.2]:

$k_{t+1}f'(k_{t+1}) = r_{t+1}(w_t - c_{1t})$. The left hand side of the latter relation is the share of per capita product, due to capital factor owners: $c_{2t+1} = k_{t+1}f'(k_{t+1})$. Then the relation becomes $c_{1t} + \frac{c_{2t+1}}{r_{t+1}} = w_t$ that is the usual inter-temporal consumer constraint.

1.1.1. The optimization process

The optimum choice of the young agents.

We choose an utility function of *constant relative risk aversion (CRRA)* type:

$$\underset{(\beta>\mathbf{0})}{\mathbf{U}(\mathbf{c_{1t}}, \mathbf{c_{2t+1}})} = \begin{cases} \frac{\mathbf{c_{1t}^{1-\gamma}-1}}{\mathbf{1-\gamma}} + \beta\frac{\mathbf{c_{2t+1}^{1-\gamma}-1}}{\mathbf{1-\gamma}} & if\ 0<\gamma<1 \\ ln(c_{1t}) + \beta ln(c_{2t+1}) & if\ \gamma = 1 \end{cases}$$

Proposition 1.1. [5]

Depending on γ, the optimum choices of young agents and the marginal propensities of savings are:

For $\gamma = 1$:

$$\begin{cases} \hat{\mathbf{c}}_{\mathbf{1t}} = \frac{\mathbf{w_t}}{\mathbf{1}+\beta} \\ \hat{\mathbf{c}}_{\mathbf{2t+1}} = \frac{\beta}{\mathbf{1}+\beta}\mathbf{w_t r_{t+1}} \end{cases} \qquad [\mathbf{1}.3]$$

and

$$\mathbf{S}(\beta) = \frac{\beta}{\mathbf{1}+\beta} \qquad [\mathbf{1}.4]$$

For $0 < \gamma < 1$:

$$\begin{cases} \hat{\mathbf{c}}_{\mathbf{1t}} = \frac{\mathbf{w_t}}{\mathbf{1}+\frac{(\beta\mathbf{r_{t+1}})^{\frac{1}{\gamma}}}{\mathbf{r_{t+1}}}} \\ \hat{\mathbf{c}}_{\mathbf{2t+1}} = \frac{\mathbf{w_t}(\beta\mathbf{r_{t+1}})^{\frac{1}{\gamma}}}{\mathbf{1}+\frac{(\beta\mathbf{r_{t+1}})^{\frac{1}{\gamma}}}{\mathbf{r_{t+1}}}} \end{cases} \qquad [\mathbf{1}.5]$$

[5] *For this and the following propositions, see the Appendix to Chapter 1, for a detailed proof.*

and

$$\mathbf{S}\left(\mathbf{r_{t+1}}\right) = \frac{\left(\beta\mathbf{r_{t+1}}\right)^{\frac{1}{\gamma}}}{\mathbf{r_{t+1}} + \left(\beta\mathbf{r_{t+1}}\right)^{\frac{1}{\gamma}}} \qquad [\mathbf{1.6}]$$

The law of motion. The previous results and the relations [1.1], [1.2] permit to derive the law of motion of the model:

$$\begin{cases} r_{t+1} = Rf'\left(k_{t+1}\right) \\ k_{t+1} = RS\left(r_{t+1}\right)W\left(k_t\right) \end{cases} \Rightarrow$$

$$\frac{\mathbf{k_{t+1}}}{\mathbf{S}\left(\mathbf{Rf'}\left(\mathbf{k_{t+1}}\right)\right)} = \mathbf{RW}\left(\mathbf{k_t}\right) \qquad [\mathbf{1.7}]$$

Proposition 1.2.

k_{t+1} is a monotonic increasing function of k_t and R:

$$\mathbf{k_{t+1}} = \psi\left(\mathbf{k_t}, \mathbf{R}\right) \qquad [\mathbf{1.8}]$$

Proposition 1.3.

$(0; R)$ is a forward invariant set for $k_{t+1} = \psi\left(k_t, R\right)$.

There is only one global, not trivial steady state $k^\star \neq 0$, asymptotically stable.

Remark. From the last proposition and the resource constraint [1.2], it follows that $Rs_t < R$; hence $s_t < 1$. Therefore it is proved that agents need to borrow, in order to become entrepreneurs.

Remark. Because $0 < s_t < 1$ it follows that some agents will be borrowers and others will be lenders.

Let m be the number of the borrowers. Then $n = 1 - m$ is the number of the lenders. The total borrowed goods will be $m\left(1 - s_t\right)$ and the total lent goods will be ns_t. These two quantities must be equal, so $m\left(1 - s_t\right) = ns_t$. This, with the previous $m + n = 1$, implies $m = s_t$ and $n = 1 - s_t$.[6]

[6] Alternatively, if m is the number of agents that run the investment project, because everyone brings one unit of savings, it must be $m = s_t$.

1.2. Two Countries

Now let us consider a world economy composed of two countries of the previous kind. We refer to these countries using h meaning *home* and a meaning *abroad*.

The heterogeneity between these two countries consists in their initial capital holding and in their efficiency in producing capital goods. In particular, it can be can assumed, without loss of generality, that $R_h > R_a$.

Two different cases will be considered separately.

At first it will be analyzed the situation in which there is free trade between countries, but the financial markets are closed, so borrowing and lending take place only domestically.

Later we will study the condition of complete opening, with free trade in consumption, capital and financial goods across countries.

Remark. In these hypotheses, there will be some differences between the case

$0 < \gamma < 1$ and the case $\gamma = 1$.

Particularly, when $\gamma = 1$, savings do not depend on interest rate but they are always a fixed ratio of income. Whether or not financial markets are open, savings are the same in the two countries and are exactly $s_t^h = s_t^a = s_t = \left(\frac{\beta}{1+\beta}\right) W\left(k_t\right)$.

On the contrary, when $0 < \gamma < 1$, profitability condition will tend to concentrate investments in the more productive country.

1.2.1. Free trade, but financial market closed

In the following, we will suppose there is free trade between the two countries concerning consumer goods and productive factors. On the contrary, the financial markets will be considered closed.

Proposition 1.4.

In the two countries the following relations occur:

$k_t^h = k_t^a = k_t$, $r_t^h > r_t^a$, $s_t^h \geq s_t^a$.

Proposition 1.5.

The law of motion that represents the formation of the new capital is:

$$\mathbf{k_{t+1}} = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{W}\left(\mathbf{k_t}\right)\left[\mathbf{R_h S}\left(\mathbf{R_h f'}\left(\mathbf{k_{t+1}}\right)\right) + \mathbf{R_a S}\left(\mathbf{R_a f'}\left(\mathbf{k_{t+1}}\right)\right)\right] \qquad [\mathbf{1.9}]$$

Proposition 1.6.

$(0; R_h) \subset \left(0; R^{(+)}\right)$ *is a forward invariant interval for* k_{t+1}.[7]

k_{t+1} *converges to a unique, non trivial and globally asymptotically stable fixed point* $k^* \in (0; R_h)$.

1.2.2. Free trade and integrated deposit markets

Let us consider now the situation in which capital goods are freely tradeable across countries and deposit markets operate internationally. This implies equalization of capital stocks, as well as deposit rates: $\forall t: \quad k_t^h = k_t^a = k_t$ and $r_t^h = r_t^a = r_t$.

Following the previous hypothesis, countries are different in efficiency, so $R_h > R_a$.

If k_{t+1} is the new value of capital, that comes from a first period capital stock k_t, then the maximum that agents running an investment project in h can pay, in return for each unit of saving borrowed, is $R_h f'(k_{t+1})$; on the contrary, the entrepreneurs of a can only offer $R_a f'(k_{t+1})$.

Therefore $R_a f'(k_{t+1}) \leq r_{t+1} \leq R_h f'(k_{t+1})$. Notice that agents keep the interest rate as given and they never try to influence it.

When there are little savings, we can suppose that the production of capital goods should be concentrated on the h-Country.

The lower are the savings, the higher is its cost; the maximum will be:

$r_{t+1} = R_h f'(k_{t+1})$; that is the return of capital h-agents can obtain from one unit of investment.

On the contrary, when there are a lot of savings, production of capital goods will occur either in h and in a Country; in this case, the interest factor will be:

$r_{t+1} = R_a f'(k_{t+1})$.

Then there's an intermediate situation, when production occurs only in h, but r_{t+1} is less than the return of investment in h, but more than in a.

Now we know that the agent starting an investment project needs to offer an amount of 1 unit of goods.

Therefore, supposing a unit mass population, the total amount of goods invested, in each country, must not be bigger than 1.

But the total amount of savings is $s_t^h + s_t^a$ and this, in conjunction with the interest factor, offers a criterion to establish if production will take place in one or two countries.

[7] $R^{(+)}$ is the smallest number for which the equation $W(x)\, S\left(x f'(x)\right) = 1$ is satisfied. If this equation doesn't have solution, then $R^{(+)} = +\infty$. See Appendix to Chapter 1 - proof of proposition 1.3.

i) Suppose it is $r_{t+1} = R_h f'(k_{t+1})$. In this case, production of capital goods will take place only in h-Country; if not there will be a minor interest factor. Then it will be $s_t^h + s_t^a \leq 1$.

ii) Suppose it is $r_{t+1} = R_a f'(k_{t+1})$. The production of capital goods will necessary take place in each country, especially in the less efficient a, otherwise the interest factor will not be so low. It means that the production in the h-Country is saturated. Therefore, it must be $s_t^h + s_t^a > 1$.

iii) Finally, suppose it is $R_a f'(k_{t+1}) < r_{t+1} < R_h f'(k_{t+1})$. Capital goods will be produced only in the home Country, otherwise r_{t+1} would be $R_a f'(k_{t+1})$. But now savings are payed less than their maximum productivity. The only possibility is that $s_t^h + s_t^a = 1$; all agents in the h-Country are entrepreneurs and their profitability constraint is valid, with the sign "$>$", while agents in the a-Country are all lenders. These last strictly prefer to borrow than to start an investment project, because in that way their return is greater.

We will analyze these three situations in the following part, beginning with the hypothesis $0 < \gamma < 1$.

Then we will consider the alternative hypothesis $\gamma = 1$.

1.2.2.1. The case $r_{t+1} = R_h f'(k_{t+1})$ and $0 < \gamma < 1$

Savings are paid at their maximum value, capital goods are produced solely in the h-Country and $s_t^h + s_t^a \leq 1$. This last relation involves $s_t \leq \frac{1}{2}$, because in each country there will be the same share of savings: $s_t^h = s_t^a = s_t = W(k_t)\, S\left(R_h f'(k_{t+1})\right)$.

In equilibrium, only agents in home Country will be indifferent between becoming lenders or entrepreneurs, while agents in foreign Country would strictly prefer to become lenders.

Proposition 1.7.

Under hypotheses $r_{t+1} = R_h f'(k_{t+1})$ and $0 < \gamma < 1$, the law of formation of new capital is:

$$\frac{\mathbf{k_{t+1}}}{\mathbf{S}\left(\mathbf{R_h f'(k_{t+1})}\right)} = \mathbf{R_h W(k_t)} \qquad [\mathbf{1.10}]$$

Indeed, from the hypotheses and the resource constraint, it derives $2k_{t+1} = R_h(2s_t) \Rightarrow k_{t+1} = R_h s_t$. The thesis follows immediately substituting the optimized values.

Proposition 1.8.

The relation [1.10] *has a unique solution* $\mathbf{k_{t+1}} = \psi(\mathbf{k_t}, \mathbf{R_h})$.

Effectively the relation [1.10] is formally identical to the [1.7].

Proposition 1.9.

The capital values k_{t+1} *and* k_t *of* [1.10] *satisfy the following relations:*

$k_{t+1} \in (0, \frac{R_h}{2}]$ *and* $k_t \in (0, k_A]$, *where* $k_A = W^{-1}\left(\frac{1}{2S\left(R_h f'\left(\frac{R_h}{2}\right)\right)}\right)$.[8]

Proposition 1.10. *The interest factor* $r_{t+1} = R_h f'(k_{t+1})$ *is a decreasing, continuous function of* k_t, *defined into the interval* $(0, k_A]$.

Its minimum value is $R_h f'\left(\frac{R_h}{2}\right)$.

1.2.2.2. The case $r_{t+1} = R_a f'(k_{t+1})$ and $0 < \gamma < 1$

In these hypotheses, each country will produce capital goods.

Agents in a-Country will be indifferent to become lenders or entrepreneurs, while agents in h-Country will strictly prefer to start an investment project and enjoy, at maximum degree, the differential between the return of capital $R_h f'(k_{t+1})$ and the cost of borrowing $r_{t+1} = R_a f'(k_{t+1})$.

The resource constraint is $2k_{t+1} = (1)\, R_h + \left(s_t^h + s_t^a - 1\right) R_a$ and the law of formation of second period capital stock will be obtained substituting the optimal values of savings $\hat{s}_t^h$ and $\hat{s}_t^a$.

Second period consumption for agents in the home and in the foreign Country respectively, will be:

$$c_{2t+1}^h = (1)\, R_h f'(k_{t+1}) - \left(1 - s_t^h\right) R_a f'(k_{t+1})$$

and

$$c_{2t+1}^a = s_t^a R_a f'(k_{t+1}).^{9}$$

[8] *See Appendix to Chapter 1.*

[9] Note that $\hat{c}_{2t+1}^h + \hat{c}_{2t+1}^a = \left[R_h + R_a\left(\hat{s}_t^h + \hat{s}_t^a - 1\right)\right] f'(k_{t+1})$ and, from resource constraint, it follows $\hat{c}_{2t+1}^h + \hat{c}_{2t+1}^a = 2k_{t+1} f'(k_{t+1})$. That's correct, because $\hat{c}_{2t+1}^h + \hat{c}_{2t+1}^a$ is also the share of second period production, that pays the capital factors.

At this point, young agents will choose the level of savings, in order to maximize their utility, over the two periods.

Thus:

$\hat{s}_t^h$: $max\, U\left(c_{1t}^h\,;\,c_{2t+1}^h\right)$ i.e. $max\, U\left(W\left(k_t\right)-s_t^h\,;\,R_h f'\left(k_{t+1}\right)-\left(1-s_t^h\right)R_a f'\left(k_{t+1}\right)\right)$.

$\hat{s}_t^a$: $max\; U\left(c_{1t}^a\,;\,c_{2t+1}^a\right)$ i.e. $max\; U\left(W\left(k_t\right)-s_t^a\,;\,s_t^a R_a f'\left(k_{t+1}\right)\right)$.

Considering again the utility function $U\left(c_1\,;\,c_2\right)=\frac{c_1^{1-\gamma}-1}{1-\gamma}+\beta\frac{c_2^{1-\gamma}-1}{1-\gamma}$, we can derive the optimal values for savings, in h and in a-Country.

Finally we will be able to obtain the law of formation of the new capital stock k_{t+1}.

Proposition 1.11.

When $r_{t+1}=R_a f'\left(k_{t+1}\right)$ and $0<\gamma<1$, *the optimal level of savings, respectively in home and foreign Country are:*

$$\mathbf{\hat{s}_t^h}=\frac{\mathbf{W(k_t)}\beta^{\frac{1}{\gamma}}\left[\mathbf{R_a f'(k_{t+1})}\right]^{\frac{1}{\gamma}-1}-\left(\frac{\mathbf{R_h-R_a}}{\mathbf{R_a}}\right)}{\mathbf{1}+\beta^{\frac{1}{\gamma}}\left[\mathbf{R_a f'(k_{t+1})}\right]^{\frac{1}{\gamma}-1}}=$$

$$=\mathbf{W}\left(\mathbf{k_t}\right)\mathbf{S}\left(\mathbf{R_a f'}\left(\mathbf{k_{t+1}}\right)\right)-\left(\mathbf{1}-\mathbf{S}\left(\mathbf{R_a f'}\left(\mathbf{k_{t+1}}\right)\right)\right)\left(\frac{\mathbf{R_h-R_a}}{\mathbf{R_a}}\right) \qquad [\mathbf{1.11}]$$

$$\mathbf{\hat{s}_t^a}=\frac{\beta^{\frac{1}{\gamma}}\mathbf{W(k_t)}\left(\mathbf{R_a f'(k_{t+1})}\right)^{\frac{1}{\gamma}-1}}{\mathbf{1}+\beta^{\frac{1}{\gamma}}\left(\mathbf{R_a f'(k_{t+1})}\right)^{\frac{1}{\gamma}-1}}=\mathbf{W}\left(\mathbf{k_t}\right)\mathbf{S}\left(\mathbf{R_a f'}\left(\mathbf{k_{t+1}}\right)\right) \qquad [\mathbf{1.12}]$$

The law of motion of the economy is:

$$\frac{\mathbf{k_{t+1}}-\left(\frac{\mathbf{R_h-R_a}}{\mathbf{2}}\right)\mathbf{S}\left(\mathbf{R_a f'(k_{t+1})}\right)}{\mathbf{S}\left(\mathbf{R_a f'(k_{t+1})}\right)}=\mathbf{R_a W(k_t)} \qquad [\mathbf{1.13}]$$

Proposition 1.12.

The solution of [1.13] *is a continuous, increasing function of* k_t*:* $\mathbf{k_{t+1}}=\phi\left(\mathbf{k_t},\mathbf{R_h},\mathbf{R_a}\right)$.

Proposition 1.13.

Let $k_B = W^{-1}\left(\frac{\frac{R_h}{2}-\left(\frac{R_h-R_a}{2}\right)S\left(R_a f'\left(\frac{R_h}{2}\right)\right)}{R_a S\left(R_a f'\left(\frac{R_h}{2}\right)\right)}\right)$ *and* $k_A = W^{-1}\left(\frac{1}{2S\left(R_h f'\left(\frac{R_h}{2}\right)\right)}\right)$ *from proposition 1.9. Then, as always supposing* $R_h > R_a$*, it is* $k_B > k_A$*.*

Proposition 1.14.

The interest factor r_{t+1} *is a continuous, decreasing function of* k_t*; its maximum value, over the interval* $[k_B; R)$ *is* $R_a f'\left(\frac{R_h}{2}\right)$*.*

1.2.2.3. The case $R_a f'\left(\frac{R_h}{2}\right) < r_{t+1} < R_h f'\left(\frac{R_h}{2}\right)$ and $0 < \gamma < 1$

Under these hypotheses, capital goods are produced only in the home Country.

But now savings are payed less than their maximum productivity.

All agents in h-Country are entrepreneurs and their profitability constraint is valid, with the sign "$>$", while agents in the a-Country are all lenders. These last, strictly prefer to borrow than to start an investment project, therefore their return is bigger.

Proposition 1.15.

Under the hypotheses, $R_a f'\left(\frac{R_h}{2}\right) < r_{t+1} < R_h f'\left(\frac{R_h}{2}\right)$ and $0 < \gamma < 1$ *the law of formation of new capital is:*

$$\mathbf{k_{t+1}} = \frac{\mathbf{R_h}}{\mathbf{2}} \qquad [\mathbf{1.14}]$$

The thesis comes immediately reminding that $s_t^h + s_t^a = 1$ and that the production of capital goods takes place only in the home Country, characterized by the parameter of efficiency R_h.

Proposition 1.16.

Under the hypotheses, $R_a f'\left(\frac{R_h}{2}\right) < r_{t+1} < R_h f'\left(\frac{R_h}{2}\right)$ and $0 < \gamma < 1$*, it follows:*

$$\hat{\mathbf{s}}_{\mathbf{t}}^{\mathbf{h}} = \frac{\mathbf{W(k_t)}\beta^{\frac{1}{\gamma}}\mathbf{r}_{\mathbf{t+1}}^{\frac{1}{\gamma}} + \mathbf{r_{t+1}} - \mathbf{R_h f'}\left(\frac{\mathbf{R_h}}{\mathbf{2}}\right)}{\beta^{\frac{1}{\gamma}}\mathbf{r}_{\mathbf{t+1}}^{\frac{1}{\gamma}} + \mathbf{r_{t+1}}} \qquad [\mathbf{1.15}]$$

$$\mathbf{\hat{s}_t^a} = \frac{\mathbf{W(k_t)}\beta^{\frac{1}{\gamma}}\mathbf{r_{t+1}^{\frac{1}{\gamma}}}}{\beta^{\frac{1}{\gamma}}\mathbf{r_{t+1}^{\frac{1}{\gamma}}}+\mathbf{r_{t+1}}} \qquad [\mathbf{1.16}]$$

$$\mathbf{r_{t+1}}\left(\mathbf{k_t}\right) = \left[\frac{\mathbf{R_h f'}\left(\frac{\mathbf{R_h}}{\mathbf{2}}\right)}{\beta^{\frac{1}{\gamma}}\left(\mathbf{2W(k_t)-1}\right)}\right]^{\gamma} \qquad [\mathbf{1.17}]$$

1.2.2.4. Unifying the results, under the hypothesis $0 < \gamma < 1$

Proposition 1.17.

The interest factor r_{t+1} is a continuous, decreasing function of k_t, over the interval $(0; R_h)$.

Precisely, it is:

$$\mathbf{r_{t+1}}\left(\mathbf{k_t}\right) = \begin{cases} R_h f'\left(\psi\left(k_t, R_h\right)\right) & if \quad k_t \leq k_A \\ \left[\frac{R_h f'\left(\frac{R_h}{2}\right)}{\beta^{\frac{1}{\gamma}}\left(2W(k_t)-1\right)}\right]^{\gamma} & if \quad k_A < k_t < k_B \\ R_a f'\left(\phi\left(k_t, R_h, R_a\right)\right) & if \quad k_t \geq k_B \end{cases} \qquad [\mathbf{1.18}]$$

Proposition 1.18.

The law of motion of capital is a continuous, not decreasing function, whose expression is:

$$\mathbf{G}\left(\mathbf{k_t}, \mathbf{R_h}, \mathbf{R_a}\right) = \begin{cases} \psi\left(k_t, R_h\right) & if \quad k_t \leq k_A \\ \frac{R_h}{2} & if \quad k_A < k_t < k_B \\ \phi\left(k_t, R_h, R_a\right) & if \quad k_t \geq k_B \end{cases} \qquad [\mathbf{1.19}]$$

This result comes from the previous relation [1.14] and propositions 1.8, 1.12.

$G\left(k_t, R_h, R_a\right)$ is a function with “three branches”, the first and the third are increasing, the medium is flat.

About this last, it must be noted that new capital remains constant, while the cost of investments decreases from higher value $R_h f'\left(\frac{R_h}{2}\right)$, to the lower $R_a f'\left(\frac{R_h}{2}\right)$.

The first is the minimum return of investments, when production of capital goods is concentrated only in h-Country. The second is the maximum value when production is made in each country.

During the time in which r_{t+1} stays from these two values, aggregate savings remain equal to one, but the cost of the investments decreases continuously, until it reaches the lower value. At this point, production of capital goods starts also in a-Country.

1.2.3. Results under the hypothesis $\gamma = 1$

In this case interest factor doesn't bind savings decision, because savings are always the same, in each country: $s_t^h = s_t^a = s_t = \left(\frac{\beta}{1+\beta}\right) W\left(k_t\right)$.

Proposition 1.19.

If $k_t \leq k_A = W^{-1}\left(\frac{1+\beta}{2\beta}\right)$ then the aggregate savings are equal, or less than 1; production of capital goods occurs only in home Country and interest factor is $r_{t+1} = R_h f'\left(k_{t+1}\right)$.

The law of formation of new capital is:

$$\mathbf{k_{t+1}} = \psi\left(\mathbf{k_t}, \mathbf{R_h}\right) = \left(\frac{\beta}{\mathbf{1}+\beta}\right) \mathbf{R_h} \mathbf{W}\left(\mathbf{k_t}\right) \qquad [\mathbf{1.20}]$$

Proposition 1.20.

If $k_t > k_A = W^{-1}\left(\frac{1+\beta}{2\beta}\right)$ then the aggregate savings are greater than 1; production of capital goods takes place in each country and interest factor is $r_{t+1} = R_a f'\left(k_{t+1}\right)$.

The law of formation of new capital is:

$$\mathbf{k_{t+1}} = \phi\left(\mathbf{k_t}, \mathbf{R_h}, \mathbf{R_a}\right) = \frac{\mathbf{R_h} - \mathbf{R_a}}{\mathbf{2}} + \left(\frac{\beta}{\mathbf{1}+\beta}\right) \mathbf{R_a} \mathbf{W}\left(\mathbf{k_t}\right) \qquad [\mathbf{1.21}]$$

The proof of this proposition comes naturally from the opposite of the previous proposition 1.19.

Particularly, $k_t > k_A$ involves $s_t > \frac{1}{2}$ then, the home Country cannot run out all the supply of savings. Consequently, production of capital goods takes place in each country.

Interest factor must be $r_{t+1} = R_a f'\left(k_{t+1}\right)$, because of the minor efficiency of the foreign Country. Agents in the home Country will strictly prefer to borrow, while agents in the foreign Country will be indifferently lenders, or borrowers.

In this case, $2k_{t+1} = (1) R_h + (2s_t - 1) R_a$. Substituting $s_t = \left(\frac{\beta}{1+\beta}\right) W\left(k_t\right)$ we obtain the relation [1.21].

Remark. It's immediate to observe that:
$\lim_{k_t \to k_A^-} \psi(k_t, R_h) = \lim_{k_t \to k_A^+} \phi(k_t, R_h, R_a) = \frac{R_h}{2}$
This result assures the continuity of the function of new capital formation, as we will see in the following proposition.

Proposition 1.21.
When the parameter γ is equal to 1, the following consequences occur:

– *The interest factor is a continuous, decreasing function of k_t:*

$$\mathbf{r_{t+1}}(\mathbf{k_t}) = \begin{cases} R_h f'(\psi(k_t, R_h)) & if \quad k_t \leq k_A \\ R_a f'(\phi(k_t, R_h, R_a)) & if \quad k_t > k_A \end{cases} \qquad [\mathbf{1.22}]$$

– *The law of the new capital formation is a continuous, increasing function of k_t:*

$$\mathbf{G}(\mathbf{k_t}, \mathbf{R_h}, \mathbf{R_a}) = \begin{cases} \psi(k_t, R_h) = \left(\frac{\beta}{1+\beta}\right) R_h W(k_t) & if \quad k_t \leq k_A \\ \phi(k_t, R_h, R_a) = \frac{R_h - R_a}{2} + \left(\frac{\beta}{1+\beta}\right) R_a W(k_t) & if \quad k_t > k_A \end{cases} \qquad [\mathbf{1.23}]$$

These assertions directly follow from propositions 1.19 and 1.20.

Particularly, comparing the relation [1.22] with the [1.18] and [1.23] with [1.19], we can see the case $\gamma = 1$ as a limit one of $0 < \gamma < 1$.

In this case the law of motion has not "flat branches" and, consequently, there is no longer an interval of values for k_t, by which the interest factor is decreasing, but the total amount of savings remains equal to 1.

Only a particular value of k_t may cause a total unitary supply of savings. A greater value of k_t immediately expands the production of capital goods also in the foreign Country.

1.3. Numerical examples

Now it will be considered the last case concerning two countries with no economic barriers, with same specific hypotheses about the production function and parameters. At first, we derive some useful outcomes, when the production function is of Cobb-Douglas type.

1.3.1. The case of Cobb-Douglas production function

Let $\mathbf{f}(\mathbf{k_t}) = \mathbf{A}\mathbf{k_t^{\alpha}}$, with $0 < \alpha < 1$.

Proposition 1.22.

When $f(k_t) = Ak_t^{\alpha}$, all properties from i_0 to i_9 are satisfied.

Proposition 1.23.

The relations [1.19] *and* [1.23] *respectively become:*

$$\underset{(0<\gamma<1)}{\mathbf{G}(\mathbf{k_t},\mathbf{R_h},\mathbf{R_a})}:$$

$$\begin{cases} k_{t+1}^{\frac{\gamma+(1-\alpha)(1-\gamma)}{\gamma}} + (\alpha A)^{\frac{1-\gamma}{\gamma}} \beta^{\frac{1}{\gamma}} R_h^{\frac{1-\gamma}{\gamma}} k_{t+1} = (1-\alpha)(\alpha)^{\frac{1-\gamma}{\gamma}} R_h^{\frac{1}{\gamma}} A^{\frac{1}{\gamma}} \beta^{\frac{1}{\gamma}} k_t^{\alpha} & if \quad k_t \leq k_A \\ \frac{R_h}{2} \quad if \quad k_A < k_t < k_B & \\ k_{t+1}^{\frac{\gamma+(1-\alpha)(1-\gamma)}{\gamma}} + (\alpha A)^{\frac{1-\gamma}{\gamma}} \beta^{\frac{1}{\gamma}} R_a^{\frac{1-\gamma}{\gamma}} k_{t+1} = (\alpha A)^{\frac{1-\gamma}{\gamma}} \beta^{\frac{1}{\gamma}} R_a^{\frac{1-\gamma}{\gamma}} \left(\frac{R_h - R_a}{2} + (1-\alpha) A R_a k_t^{\alpha}\right) & if \quad k_t \geq k_B \end{cases} \qquad [\mathbf{1.24}]$$

$$\underset{(\gamma=1)}{\mathbf{G}(\mathbf{k_t},\mathbf{R_h},\mathbf{R_a})}:$$

$$\begin{cases} k_{t+1} = \left(\frac{\beta}{1+\beta}\right)(1-\alpha) A R_h k_t^{\alpha} & if \quad k_t \leq k_A \\ k_{t+1} = \frac{R_h - R_a}{2} + \left(\frac{\beta}{1+\beta}\right)(1-\alpha) A R_a k_t^{\alpha} & if \quad k_t > k_A \end{cases} \qquad [\mathbf{1.25}]$$

Later on, it will be considered a particular choice of parameters, in order to solve analytically the law of motion.

At first we are going to set $\alpha = \frac{1}{3}$ and $\gamma = 1$, then we will see what happens when $\alpha = \frac{1}{3}$ and $\gamma = \frac{2}{5}$.

1.3.2. A particular choice of parameters: $\alpha = \frac{1}{3}$, $\gamma = 1$

From [1.25], being $k_A = \left[\frac{3(1+\beta)}{4A\beta}\right]^3$, we have:

$$\mathbf{G}\left(\mathbf{k_t},\mathbf{R_h},\mathbf{R_a}\right) : \begin{cases} k_{t+1} = \frac{2}{3}AR_h\left(\frac{\beta}{1+\beta}\right)k_t^{\frac{1}{3}} & if \quad k_t \leq \left(\frac{3(1+\beta)}{4A\beta}\right)^3 \\ k_{t+1} = \frac{R_h-R_a}{2} + \frac{2}{3}AR_a\left(\frac{\beta}{1+\beta}\right)k_t^{\frac{1}{3}} & if \quad k_t > \left(\frac{3(1+\beta)}{4A\beta}\right)^3 \end{cases} \qquad [\mathbf{1.26}]$$

Now we are going to study the process of convergence for $G\left(k_{t,}R_h, R_a\right)$.

Proposition 1.24.

If $\frac{R_h}{2} \leq k_A$, then the first branch of $G\left(k_{t,}R_h, R_a\right)$ converges to a stable, fixed point $0 < k_\psi^* \leq k_A$ and economy reaches its steady state, when production of capital goods is concentrated in the home Country.

On the contrary, if $\frac{R_h}{2} > k_A$, the second branch converges to a stable point $k_\phi^* > k_A$ and the production of capital goods expands over the two countries.

Fig. 1.1

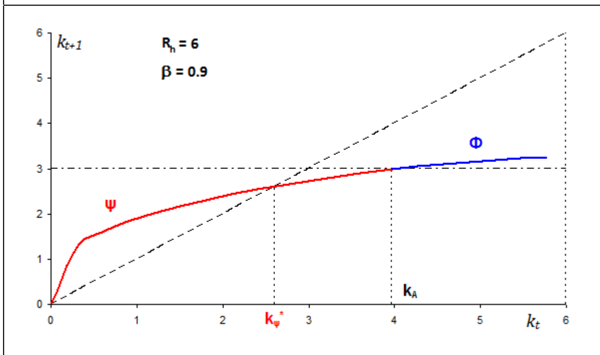
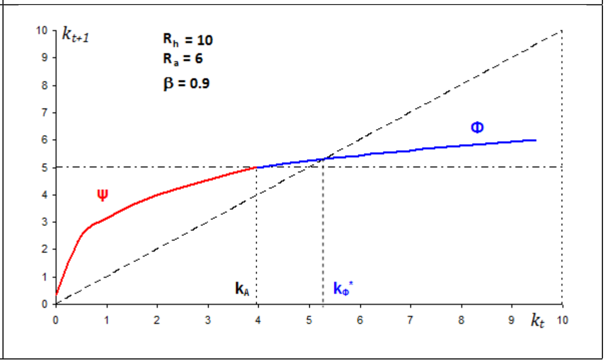


Fig. 1.2



1.3.3. Another choice of parameters: the case $\alpha = \frac{1}{3}$, $\gamma = \frac{2}{5}$

With this choice of parameters, particularly with $0 < \gamma < 1$, the map $G\left(k_{t,} R_h, R_a\right)$ comes from the relation [1.24] and has three branches, the first and the third one increasing, the medium flat.

In order to simplify the expressions, let's set $V_j = \frac{2}{3} A R_j$; $C_j = \beta^{\frac{5}{2}} \left(\frac{AR_j}{3}\right)^{\frac{3}{2}}$;
$H = \frac{R_h - R_a}{2}$.
In this way, we can introduce the following proposition.

Proposition 1.25.
Under the hypotheses $\alpha = \frac{1}{3}$ and $\gamma = \frac{2}{5}$, the law of motion is:

$$\mathbf{G}\left(\mathbf{k_t},\mathbf{R_h},\mathbf{R_a}\right) = \begin{cases} \psi\left(k_t, R_h\right) = \frac{1}{2}\sqrt{C_h^2 + 4V_h C_h k_t^{\frac{1}{3}}} - \frac{1}{2}C_h & if \quad k_t \leq k_A \\ \frac{R_h}{2} & if \quad k_A < k_t < k_B \\ \phi\left(k_t, R_h, R_a\right) = \frac{1}{2}\sqrt{C_a^2 + 4HC_a + 4V_a C_a k_t^{\frac{1}{3}}} - \frac{1}{2}C_a & if \quad k_t \geq k_B \end{cases} \qquad [\mathbf{1.27}]$$

where k_A: $\psi\left(k_A, R_h\right) = \frac{R_h}{2}$ and k_B: $\phi\left(k_B, R_h, R_a\right) = \frac{R_h}{2}$ are:

$$\mathbf{k_A} = \left(\frac{\mathbf{R_h^2 + 2C_h R_h}}{\mathbf{4V_h C_h}}\right)^{\mathbf{3}} \qquad [\mathbf{1.28}]$$

$$\mathbf{k_B} = \left(\frac{\mathbf{R_h^2 + 2C_a R_h - 4HC_a}}{\mathbf{4V_a C_a}}\right)^{\mathbf{3}} = \left(\frac{\mathbf{R_h^2 + 2C_a R_a}}{\mathbf{4V_a C_a}}\right)^{\mathbf{3}} \qquad [\mathbf{1.29}]$$

Remark: as for most of the previous propositions, the proof can be found in the Appendix to Chapter 1.

Proposition 1.26.

The function $G(k_{t,}R_h, R_a)$ [1.27] *has three branches; the first one is increasing and connects* $(0;0)$ *to* $\left(k_A; \frac{R}{2}\right)$*; the second one is flat and stands between* $\left(k_A; \frac{R}{2}\right)$ *and* $\left(k_B; \frac{R}{2}\right)$*; finally the third one is increasing and starts in* $\left(k_B; \frac{R}{2}\right)$*.*

Whenever a feasible steady state exists along a given branch, then it is asymptotically stable, and no other steady states exist on other branches. Moreover, at least a branch must converge. Therefore, $G(k_{t,}R_h, R_a)$ *has always a unique point of steady state.*

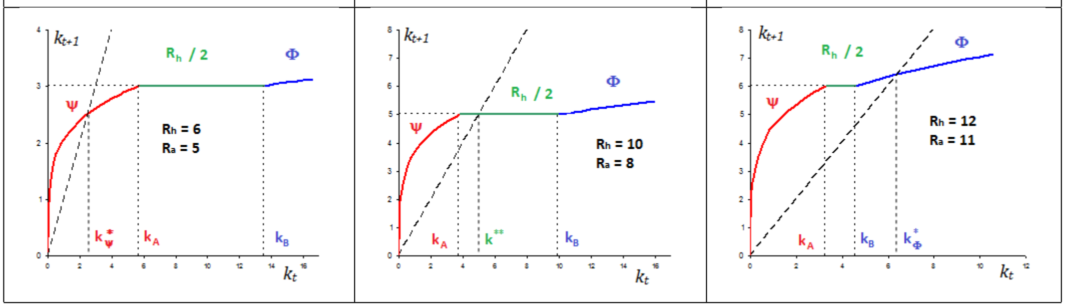
Three are the main situations:

– if $\frac{R_h}{2} \leq k_A$ *then the fixed point of* $G(k_{t,}R_h, R_a)$ *is* $k^*_\psi \leq \frac{R_h}{2}$*. Production of capital goods occurs only in the home Country; interest factor will be* $r_{t+1} = R_h f'(k_{t+1})$*;*

– if $k_A < \frac{R_h}{2} < k_B$ *the fixed point of* $G(k_{t,}R_h, R_a)$ *is* $k^{**} = \frac{R_h}{2}$*. Only agents in the home Country start investment projects, while all agents in the foreign Country are lenders. The interest factor is* $R_a f'\left(\frac{R_h}{2}\right) < r_{t+1} < R_h f'\left(\frac{R_h}{2}\right)$ *and the aggregate savings are* 1*;*

– if $k_B \leq \frac{R_h}{2}$ *the fixed point of* $G(k_{t,}R_h, R_a)$ *is* $k^*_\phi \geq \frac{R_h}{2}$*. Production of capital goods is carried out in each country and the interest factor is* $r_{t+1} = R_a f'(k_{t+1})$*, while aggregate savings become bigger than* 1*.*

Fig. 1.3 *Fig. 1.4* *Fig. 1.5*



In order to prove the assertions above, as usual, let's consider each branch of the curve $G(k_{t,}R_h, R_a)$.

The "branch" $\psi(k_t, R_h)$**.** It is easy to verify that the expression $\psi(k_t, R_h)$,

for $k_t \geq 0$, is an increasing curve, with declining slope and therefore downwardly concave.

In order to prove that, let's determine its first and second derivatives.

$$\frac{\partial\psi(k_t,R_h)}{\partial k_t} = \frac{V_h C_h}{3\left(C_h^2+4V_hC_hk_t^{\frac{1}{3}}\right)^{\frac{1}{2}} k_t^{\frac{2}{3}}}$$

$\frac{\partial^2\psi(k_t,R_h)}{\partial k_t^2} = -\frac{2V_h C_h^2\left(5V_h k_t^{\frac{1}{3}}+C_h\right)}{9\left(C_h^2+4V_h C_h k_t^{\frac{1}{3}}\right)^{\frac{3}{2}} k_t^{\frac{5}{3}}}$

Thus $\frac{\partial\psi(k_t,R_h)}{\partial k_t} > 0$, $\lim_{k_t\to 0^+} \frac{\partial\psi(k_t,R_h)}{\partial k_t} = +\infty$, $\lim_{k_t\to +\infty} \frac{\partial\psi(k_t,R_h)}{\partial k_t} = 0$; while $\frac{\partial^2\psi(k_t,R_h)}{\partial k_t^2} < 0$.

Considering that $\psi(0, R_h) = 0$, it follows that $\psi(k_t, R_h)$ is tangent, on the origin, to the ordinate axis; then the curve rises continuously, but every time more slowly and tends to become flat. Summarizing, the equation $\psi(k_t, R_h) = k_t$ has a unique, not trivially, solution, over the interval $(0; +\infty)$. If the solution falls into the interval $(0; k_A]$, then it is a valid steady state for $G(k_{t,} R_h, R_f)$; as usual, we'll indicate it as k_ψ^*.

The curve $\psi(k_t, R_h)$ passes through the point $A\left(k_A, \frac{R_h}{2}\right)$. Then $k_\psi^* > k_A$ if A lies over the bisector and $k_\psi^* < k_A$ if A stays under the bisector. In conclusion the steady state point k_ψ^* exists if and only if $k_A \geq \frac{R_h}{2}$. Finally, because of the geometry of the curve, if k_ψ^* exists, it is necessary stable.

The value k_A as function of R_h. Because of the previous outcomes, it is worth to study the relation between k_A and R_h, some more deeply.

From [1.28], being $V_h = \frac{2}{3}AR_h$ and $C_h = \beta^{\frac{5}{2}}\left(\frac{AR_h}{3}\right)^{\frac{3}{2}}$ it is:

$k_A = \left(\frac{1+2\beta^{\frac{5}{2}}\left(\frac{A}{3}\right)^{\frac{3}{2}}R_h^{\frac{1}{2}}}{8\beta^{\frac{5}{2}}\left(\frac{A}{3}\right)^{\frac{5}{2}}R_h^{\frac{1}{2}}}\right)^3.$

For simplicity let's set $p = 2\beta^{\frac{5}{2}}\left(\frac{A}{3}\right)^{\frac{3}{2}}$ and $q = 8\beta^{\frac{5}{2}}\left(\frac{A}{3}\right)^{\frac{5}{2}}$, then

$$\mathbf{k_A}\left(\mathbf{R_h}\right) = \left(\frac{\mathbf{1+pR_h^{\frac{1}{2}}}}{\mathbf{qR_h^{\frac{1}{2}}}}\right)^{\mathbf{3}} \qquad [\mathbf{1.30}]$$

It is easy to verify that $k_A(R_h)$ is a continuous, decreasing function, over the interval interval $R_h > 0$ and has two asymptotes, one vertical that is the ordinate axis, and the other horizontal, the line $k = \left(\frac{3}{4A}\right)^3$.

Indeed its first derivative is $\frac{dk_A}{R_h} = \frac{-3}{2qR_h^{\frac{3}{2}}}\left(\frac{1+pR_h^{\frac{1}{2}}}{qR_h^{\frac{1}{2}}}\right)^2 < 0 \quad \forall R_h > 0$ and

$\lim_{R_h\to 0^+} k_A(R_h) = +\infty$, while $\lim_{R_h\to +\infty} k_A(R_h) = \left(\frac{p}{q}\right)^3 = \left(\frac{3}{4A}\right)^3$

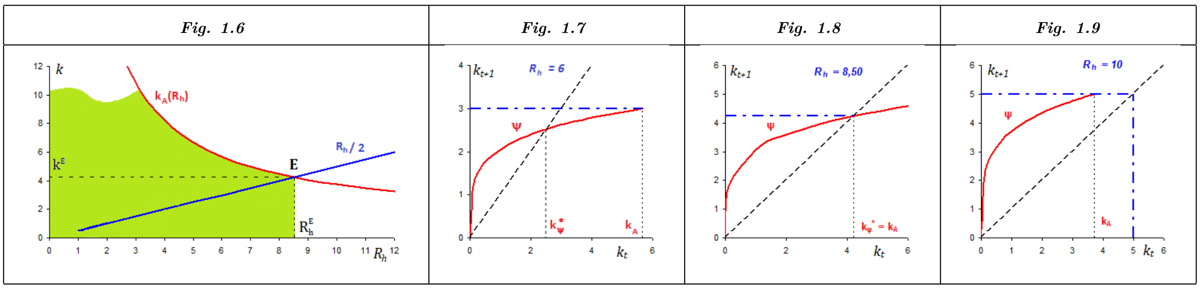
Finally, let us consider the intersection $E\left(R_h^E; k^E\right)$ between $k_A(R_h)$ and the line $k = \frac{R_h}{2}$.

The coordinate R_h^E it's hard to find through algebraical way, but can be determined by numerical methods.

If we accept an approximation of 2 decimal points, under the hypothesis $A = 1$ and $\beta = 0.9$, we obtain $R_h^E = 8,50$.

For our purposes, the main fact is that, when $R_h \leq R_h^E$, then $k_A(R_h) \geq \frac{R_h}{2}$ and the steady point $k_\psi^* = \psi\left(k_\psi^*; R_h\right)$ exists.

On the contrary, when $R_h > R_h^E$ (R_h is too big), the branch $\psi(k_t; R_h)$ ends before reaching its steady point and k_ψ^* is virtual.



The "branch" $\phi(k_t, R_h, R_a)$**.**

We will see that $\phi(k_t, R_h, R_a)$ is, mathematically speaking, quite similar to $\psi(k_t, R_h)$, in the semi-plane $k_t \geq 0$.

$\phi(k_t, R_h, R_a)$ is a continuous function, tangent to the ordinate axis on $(0; \phi_0 > 0)$, increasing, but with declining slope, as k_t becomes bigger, thus downwardly concave. Indeed:

$$\frac{\partial \phi(k_t,R_h,R_a)}{\partial k_t} = \frac{V_a C_a}{3\left(C_a^2+4HC_a+4V_aC_ak_t^{\frac{1}{3}}\right)^{\frac{1}{2}} k_t^{\frac{2}{3}}}$$

$$\frac{\partial^2 \phi(k_t,R_h,R_a)}{\partial k_t^2} = -\frac{2V_aC_a^2\left(5V_ak_t^{\frac{1}{3}}+C_a+4H\right)}{9\left(C_a^2+4HC_a+4V_aC_ak_t^{\frac{1}{3}}\right)^{\frac{3}{2}} k_t^{\frac{5}{3}}}$$

Then

$\phi(0, R_h, R_a) > 0;$

$$\frac{\partial \phi(k_t,R_h,R_a)}{\partial k_t} > 0, \ \lim_{k_t \to 0^+} \frac{\partial \phi(k_t,R_h,R_a)}{\partial k_t} = +\infty, \ \lim_{k_t \to +\infty} \frac{\partial \phi(k_t,R_h,R_a)}{\partial k_t} = 0;$$

$$\frac{\partial^2 \phi(k_t,R_h,R_a)}{\partial k_t^2} < 0.$$

In conclusion, equation $\phi(k_t, R_h, R_a) = k_t$ has a unique solution, over the interval $(0; +\infty)$. If such a solution belongs to $[k_B; +\infty)$, then it is a point of steady state k_ϕ^*, for $G(k_t, R_h, R_a)$.

Considering that $\phi(k_B, R_h, R_a) = \frac{R_h}{2}$, even in this case, the relation between k_B and $\frac{R_h}{2}$ determines whether k^*_ϕ is a valid steady point, or is virtual. Let's assume $B\left(k_B, \frac{R_h}{2}\right)$ the point at the beginning of the curve.

If B is over or, at limit, on the bisector, then $k^*_\phi \in [k_B; +\infty)$. On the other hand, if B is under the bisector, then $k^*_\phi \in (0, k_B)$. In conclusion, k^*_ϕ exists, if and only if, it is $k_B \le \frac{R_h}{2}$.

Remembering that, for a given $R_h > R_a$, it is $k_B > k_A$, it follows that, if $k_A \ge \frac{R_h}{2}$, then $k_B > \frac{R_h}{2}$ and k^*_ϕ is virtual.[10]

On the contrary, if $k_A < \frac{R_h}{2}$, then k_B may be or may not be less than $\frac{R_h}{2}$; *it depends on* R_a. Only if $k_B \le \frac{R_h}{2}$, then k^*_ϕ is real.

Hence we will study the relation between k_B and R_a.

From [1.29] it is $k_B(R_a) = \left(\frac{R_h^2}{4V_aC_a} + \frac{3}{4A}\right)^3 = \left(\frac{R_h^2}{8\left(\frac{A}{3}\right)^{\frac{5}{2}}\beta^{\frac{5}{2}}R_a^{\frac{5}{2}}} + \frac{3}{4A}\right)^3$.

Therefore, setting $m = \frac{R_h^2}{8\left(\frac{A}{3}\right)^{\frac{5}{2}}\beta^{\frac{5}{2}}}$ and $n = \frac{3}{4A}$:

$$\mathbf{k_B}\left(\mathbf{R_a}\right) = \left(\frac{\mathbf{m}}{\mathbf{R_a^{\frac{5}{2}}}} + \mathbf{n}\right)^{\mathbf{3}} \qquad [\mathbf{1.31}]$$

$k_B(R_a)$ is a continuous, decreasing function of R_a , in the semi-plane $R_a > 0$.

The curve has a vertical asymptote corresponding to the ordinate axis and an horizontal asymptote that is $k = \left(\frac{3}{4A}\right)^3$.

Indeed:

$$\frac{dk_B(R_a)}{dR_a} = -\frac{\frac{15}{2}m}{R_a^{\frac{7}{2}}}\left(\frac{m}{R_a^{\frac{5}{2}}} + n\right)^2 < 0$$

$$\lim_{R_a \to 0^+} k_B(R_a) = +\infty; \lim_{R_a \to +\infty} k_B(R_a) = \left(\frac{3}{4A}\right)^3$$

Remark. From [1.28] and [1.29] the condition $k_A \le k_B$ implies:

$R_h^2(V_aC_a - V_hC_h) \le 2C_hC_a(V_hR_a - V_aR_h) = 0 \Rightarrow$

$V_aC_a \le V_hC_h \Leftrightarrow R_a \le R_h$.

Remembering the hypothesis $R_a < R_h$, we conclude that $k_A \le k_B$, while their limits are equal:

$\underset{R_h \to +\infty}{lim} k_A(R_h) = \underset{R_a \to +\infty}{lim} k_B(R_a)$.

[10] In such case, condition $k_A \ge \frac{R_h}{2}$ implies that k^*_ψ is real.

Now, suppose it is $k_A < \frac{R_h}{2}$, for a given R_h. Then the point k_ψ^* is virtual.

If $k_B(R_a) \leq \frac{R_h}{2}$, then there exists the steady point k_ϕ^* generated by $\phi(k_t, R_h, R_a)$. This happens when $k_B^{-1}\left(\frac{R_h}{2}\right) \leq R_a < R_h$.

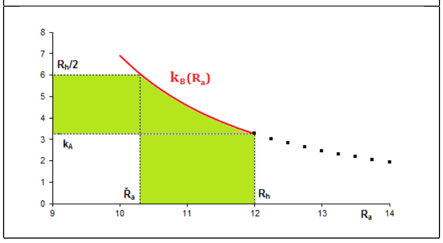
For $R_a < k_B^{-1}\left(\frac{R_h}{2}\right)$ instead, it's $k_B > \frac{R_h}{2}$ and the steady point of the map is $\frac{R_h}{2}$ generated by the horizontal line $y = \frac{R_h}{2}$.

Finally, we have to determine $\tilde{R}_a = k_B^{-1}\left(\frac{R_h}{2}\right)$.

From $\frac{R_h}{2} = \left(\frac{m}{\tilde{R}_a{}^{\frac{5}{2}}} + n\right)^3$ we obtain:

$$\mathbf{\tilde{R}_a} = \left(\frac{\mathbf{R_h^2}}{\mathbf{8}\left(\frac{\mathbf{A}}{\mathbf{3}}\right)^{\frac{\mathbf{5}}{\mathbf{2}}}\beta^{\frac{\mathbf{5}}{\mathbf{2}}}\left(\left(\frac{\mathbf{R_h}}{\mathbf{2}}\right)^{\frac{\mathbf{1}}{\mathbf{3}}} - \frac{\mathbf{3}}{\mathbf{4A}}\right)}\right)^{\frac{\mathbf{2}}{\mathbf{5}}} \qquad [\mathbf{1.32}]$$

Fig. 1.10



In fig. 1.10, is $A = 1$, $\beta = 0.9$, $R_h = 12$. Approximating at second decimal, $\tilde{R}_a$ is $10,32$ and k_A almost $3,26$.

Into the colored zone, for $\tilde{R}_a < R_a < R_h$, is $k_A < k_B < \frac{R_h}{2}$. The function $\phi(k_t, R_h, R_a)$ intersects the bisector in a steady point k_ϕ^* for $G(k_{t,} R_h, R_a)$.

Differently, when $R_a < \tilde{R}_a$, the curve $\phi(k_t, R_h, R_a)$ doesn't intersect the bisector in a valid point. In that case, the steady point for $G(k_{t,} R_h, R_a)$ is $\frac{R_h}{2}$.

1.4. Conclusions

The last figures 1.3, 1.4 and 1.5 portrait essentially the main phenomenon described in this first part of the thesis.

Depending on quantity of capital accumulation, the law of motion of new capital goes on towards its steady state point and two situations may materialize: the one described in figures 1 and 2, in which the production of capital goods occurs only in the most productive country, the other described in figure 3, in which the production of capital goods takes place in the two countries. In each case, the opening of markets (of factors and loans) obviously assures that there will be the same interest rate in each country, but not always is the return of investment equal to the cost of savings, sometimes it is greater. Finally, despite different capacities on producing capital goods, the law of motion doesn't know any discontinuity point.

Now there are sufficient elements to give an answer to the main question, which we began this work with.

What happens when economic barriers fall down and a country opens itself to foreign exchanges?

In order to ease the explanation, let's firstly suppose that the financial markets are initially closed; then the two countries open their markets.

We can make a comparison between the two situations just described.

Supposing that there is different efficiency in producing capital goods, $R_h > R_a$, initially (when the frontiers are closed) there are two interest factors, $r_h = R_h f^{'}(k_{t+1})$ greater than $r_a = R_a f^{'}(k_{t+1})$, that reflect the heterogeneity of the countries. The production of capital goods takes place in each country and savings are exactly paid their marginal return; non arbitrage condition makes agents indifferent whether becoming lenders or borrowers.

Let's suppose that during the year t, when the two countries decide to open their financial frontiers, it is $W(k_t) < \frac{1}{2}$. In this way, whichever is the interest rate, there are no doubts that aggregate savings $s_t^h + s_t^a$ will be less than one.

During year t various changes will occur: production of capital goods will concentrate only in the home Country; interest factor will rise to $R_h f^{'}(k_{t+1})$ also in the foreign Country; for $h - agents$ will be indifferent whether to become depositors or to start an investment project, while $a - agents$ will strictly prefer to lend their savings.

Depending on R_h, the law of motion of aggregate economy can reach its steady point $k_{\psi}^{*} \leq \frac{R_h}{2}$, in its first "branch"$\psi(k_t, R_h)$. In that case, the new situation will be "frozen" on its initial features.

If R_h is too big (if parameter $A = 1$, we know it does, when $R_h > 8,50$), then the steady point of $\psi(k_t, R_h)$ is virtual and system reaches its stability, depending on the second parameter R_a.

If R_a is not far from R_h, then the world economy converges at its steady point k^*_ϕ, on the third "branch" $\phi\,(k_t, R_h, R_a)$.

This means that capital goods are produced in each country, the cost of savings is $R_a f^{'}\,(k_{t+1})$, but the return of investments is bigger, $R_h f^{'}\,(k_{t+1})$. It follows that, agents in the home Country strictly prefer to start an investment project, differently from agents in the foreign Country.

If differences between the two countries are bigger, i.e. if R_a is a lot lower than R_h, the system converges to its "flat branch" $\frac{R_h}{2}$.

Production of capital goods is concentrated only in the home Country, but savings are paid less than $R_h f^{'}\,(k_{t+1})$ (even if more than $R_a f^{'}\,(k_{t+1})$).

For agents in h-Country it is more convenient to become entrepreneurs, while agents in a-Country strictly prefer to become lenders.

Concluding: for the two countries, consequences of opening their markets may depend on their specific characteristics or on their relative characteristics.

If efficiency in producing capital goods is generally low, then there is no other option, except obtaining capital factors from the more productive country. If efficiency is generally high, producing will still remain on the two countries, even after the opening of the frontiers.

Part II.

A two-Country OLG model, with credit market imperfections

In the two following chapters, we will adopt a model proposed by Matsuyama (Matsuyama [10]) and adapted to the two-country case by Kikuchi and Stachurski (Kikuchi & Stachurski [8]). In order to do that, we will introduce some new hypotheses on the model presented in the first part.

The main change is a new axiom concerning credit market imperfection. We will suppose that lenders are suspicious about borrowers and believe that they will not repay their debts, if the cost of the obligation is greater than the cost of the default, which is taken to be a fraction $\lambda \in (0,1)$ of the project output. Therefore, agents take the interest rate as given but, at the equilibrium rate, borrowers cannot obtain any amount of credit they like. Consequently, this hypothesis introduces a "borrowing constraint", close to the usual profitability constraint.

We continue to consider an overlapping generations model, with a constant, normalized population, consisting in old and young agents. Now we suppose that the young don't consume their wage, but supply it totally, in order to assure their final consumption. In this way, their optimization choice consists only in whether to become lenders, or entrepreneurs (borrowers). Again, in a more general way, we suppose that the consistency of each generation will be a fraction $0 < L < 1$.

The main results.

In Chapter two, after introduced the model, we will analyze the case of two perfectly symmetric open economies, with integrated financial markets.

We will develop a global analysis, in the hypothesis of Cobb-Douglas production function and, adopting the particular choice of setting the share of income paid to capital as $\alpha = \frac{1}{2}$, we will obtain a complete description of the economic environment. Particularly we will prove the existence of multiple steady states and the presence of a Milnor attractor. These outcomes allow to predict consequences of credit market integration, for two countries identical in their economic features and, by extension for two quite similar ones.

In Chapter three we will explore consequences due to heterogeneity of the two countries in their economic features, when they open their financial markets. Various cases will be investigated: "quasi-symmetry", supposing differences only in population size; "complete heterogeneity" in population, technology and credit market imperfection; "quasi-heterogeneity", with equal populations, but differences in technology and credit market imperfection.

The first set of hypotheses will allow to obtain formal results which, in some cases, exhibit analogies with the perfect symmetrical situation but, as in Kikuchi & Stachurski ([8]), via numerical simulation, it will also be possible to prove the existence of periodic phenomena.

Adopting the hypotheses of complete heterogeneity, we will argue that, periodic phenomena are not due only by differences in population size (as in Kikuchi &

Stachurski), but also in technologies and credit market imperfections. We will analyze deeply some periodic dynamics, involving cyclical phenomena and a crater bifurcation. Finally showing the presence of a crater bifurcation also in the case of "quasi-heterogeneity" (therefore with equal populations and different parameters of technology and credit market imperfection) we will consider definitely proved our thesis about differences in technology and credit market imperfection as cause of periodic and quasi-periodic phenomena.

2. Two open economies, with credit market imperfections.

2.1. A new axiomatic picture. The model.

We consider an overlapping generations model, with two generations of old and young, each one consisting of people having mass $0 < L < 1$ and living for two periods.

As considered before, there is only one type of goods suitable for every economic dealing and two productive processes: the first, of capital goods, covering two periods; the second, of consumer goods, starting and ending in the same period.

Capital goods producing technology is again linear and requires a minimum investment, normalized to one:

$$G(z) = \begin{cases} 0 & if\ z \in [0,1) \\ R\epsilon\mathbb{R}^{+} & if\ z \in [1,+\infty) \end{cases}$$

The production of consumer goods is obtained by an intensive, per capita function $f(k_t)$, with the same axiomatic features of Chapter 1, which takes in input the two factors of labor and capital, and consume they entirely, at the end.

Young agents are endowed with one unit of labor and they inelastically supply it to the competitive labor market. They receive an income w_t and they invest it entirely, in order to assure availability of goods during the older part of their life (the second one), assumed to be the only season in which people consume.

The choice of the young agent. In this reformulation of the axiomatic picture, people consume only in the second part of their life. So, after received his income, the young agent has to decide whether to become a depositor, or an entrepreneur. In the first case he lends his savings in the first period and will receive, in the later one, $c_{2t+1} = w_t r_{t+1}$. He will entirely consume it and exit.

On the contrary, if the agent becomes an entrepreneur, he has to run a discrete, indivisible project, that requires a minimum supply of one unit of starting investment, in period t and returns R units of capital factors in $t+1$. The latter may be supplied to the productive process of consumer goods and finally allows to obtain a return of $Rf^{'}(k_{t+1})$ units of goods available for the last consumption.

We suppose that:

$i_{2.1})$ $R \in \left(0, R^{(+)}\right)$, where $R^{(+)} : W\left(R^{(+)}\right) = 1$.

In this way, because of the properties of the production function $f(.)$, supposing $k_t \in (0, R)$, it will be $W(k_t) < W(R)$. This means that agents need to borrow, in order to start their investment projects.

$i_{2.2})$ Factors are not tradeable and agents cannot start an investment project abroad.

$i_{2.3})$ Each agent cannot start more than one investment project.

Comparing the two outcomes for lenders and borrowers, we can conclude that agents are willing to become entrepreneurs, if and only if $Rf^{'}(k_{t+1}) - (1 - w_t)\, r_{t+1} \geq w_t r_{t+1}$; hence:

$$\mathbf{Rf^{'}(k_{t+1}) \geq r_{t+1}} \qquad \textit{profitability constraint} \qquad [v_1]$$

The credit market imperfection. Now we introduce a particular hypothesis, about credit market imperfection. It consists in supposing that *borrowers will not repay their debts, if the cost of the obligation is bigger than the cost of the default.*

The cost of the default is taken to be $\lambda \in (0,1)$ fraction of the project output: $\lambda Rf^{'}(k_{t+1})$.

In this way, agents take the interest rate as given (and in that sense the credit market is freely competitive) but borrowers cannot obtain any amount of credit they like. In fact an agent who needs an amount $(1 - W(k_t))$ of credit, is able to obtain it only if $(1 - W(k_t))\, r_{t+1} \leq \lambda Rf^{'}(k_{t+1})$.

So the young entrepreneur can have the loan only if:

$$\mathbf{Rf^{'}(k_{t+1}) \geq \frac{(1-W(k_t))r_{t+1}}{\lambda}} \qquad \textit{borrowing constraint} \qquad [v_2]$$

In conclusion the young agents will start their investment projects, when they are *willing to borrow* (v_1) and *able to borrow* (v_2) .

Every time one of the two constraints is binding.

The limit value $k(\lambda)$. Let $k(\lambda)$ be the capital value s.t. $1 - W(k(\lambda)) = \lambda$.

If $k_t < k(\lambda)$, then the borrowing constraint v_2 is binding, because it is stronger than v_1.

If $k_t \geq k(\lambda)$, the profitability constraint v_1 is binding.

Let's suppose $\lambda = 1$. Consequently $W(k(1)) = 0$ and agents are able to borrow whatever they want. Then it must be $k(1) = 0$.

Then, suppose it is $\lambda = 0$. $W(k(0)) = 1$. Agents will never obtain credit. From $i_{2.1}$ we know it is $k(0) = R^{(+)}$.

Differentiating relative to λ each side of $1 - W(k(\lambda)) = \lambda$, we obtain:

$\frac{\partial k(\lambda)}{\partial \lambda} = -\frac{1}{W'(k(\lambda))} < 0$ because of the hypothesis $f''(k_t) < 0$. Thus $k(\lambda)$ is a decreasing function of λ.

Then, supposing $0 < \lambda < 1$ it follows that $0 < k(\lambda) < R^{(+)}$.

2.2. Autarky.

Let's assume initially that the financial markets are closed, so that there is no international borrowing and lending and each country operates in autarky.

The law of motion. Being $W(k_t)$ the total per capita amount of savings existing at time t, since savings must equalize investment, the rule by which new capital can be obtained will be:

$$\mathbf{k_{t+1} = RW(k_t)} \qquad [\mathbf{2}.\mathbf{1}]$$

This equation completely describes the dynamics of capital formation.

The interest factor. In each time, one of the constraint v_1 or v_2 is binding, depending whether it is $k_t \geq k(\lambda)$ or not and we may assume them with the symbol of " = ".

In each period, the interest rate will adjust, so that domestic investment will be equal to domestic savings:

$$\mathbf{r_{t+1}} = \begin{cases} \frac{\lambda R}{1-W(k_t)} f'(RW(k_t)) & if\ k_t < k(\lambda) \\ Rf'(RW(k_t)) & if\ k_t \geq k(\lambda) \end{cases} \qquad [\mathbf{2}.\mathbf{2}]$$

Remarks. The λ degree of imperfection of the financial market affects the interest rate, but not the total amount of capital produced.

Proposition 2.1.

$(0; R)$ *is a forward invariant set for* k_{t+1}.

Indeed, supposing $k_t \in (0, R)$, from [2.1] and $i_{2.1}$ it follows:

$k_{t+1} = RW(k_t) < RW(R) < R$. So $k_{t+1} \in (0, R)$.

Proposition 2.2.

In the long run, economy converges to a unique, steady state

$$\mathbf{k^{\star} = RW(k^{\star}) = K^{\star}(R)} \qquad [\mathbf{2}.\mathbf{3}]$$

(See proposition 3.1 of Kikuchi & Stachurski [8]).

2.3. Two open economies

Later on, we will assume the following hypotheses:

− The world economy is only made by two countries, indicated by "h" for "*home*" and "a" for "*abroad*".

− Populations are, respectively, $0 < L_h < 1$ and $0 < L_a < 1$; they are supposed to be complementary, so $L_h + L_a = 1$.

− Markets factors are closed, and agents cannot start an investment project abroad.

− Financial markets are integrated.

Now, since savings can freely flow throughout the world, the law of formation of new capital, in each country, is affected by the degree of imperfection of the financial markets.

In the Country j (where $j = h\,,\, a$), according to which constraint is binding, the new capital will be derived from v_1 or v_2.

Setting $\phi = \left(f'\right)^{-1}$, it will be:

$$\mathbf{k_{jt+1}} = \psi_\mathbf{j}\left(\mathbf{k_{jt}}, \mathbf{r_{t+1}}\right) = \begin{cases} \phi\left(\frac{r_{t+1}}{R_j}\left(\frac{1-W(k_{jt})}{\lambda_j}\right)\right) & if\ k_{jt} < k\left(\lambda_j\right) \\ \phi\left(\frac{r_{t+1}}{R_j}\right) & if\ k_{jt} \geq k\left(\lambda_j\right) \end{cases} \qquad [\mathbf{2.4}]$$

Remark. Remembering that $f'(.)$ is a decreasing function, the relation [2.4] can also be expressed as:

$$\mathbf{k_{jt+1}} = \psi_\mathbf{j}\left(\mathbf{k_{jt}}, \mathbf{r_{t+1}}\right) = \mathbf{min}\left\{\phi\left(\frac{\mathbf{r_{t+1}}}{\mathbf{R_j}}\left(\frac{\mathbf{1-W\left(k_{jt}\right)}}{\lambda_\mathbf{j}}\right)\right), \phi\left(\frac{\mathbf{r_{t+1}}}{\mathbf{R_j}}\right)\right\} \qquad [\mathbf{2.4b}]$$

The interest factor must endogenously equate savings and world investment. This brings the relation of closure:

$$\mathbf{L_h}\frac{\mathbf{k_{ht+1}}}{\mathbf{R_h}} + \mathbf{L_a}\frac{\mathbf{k_{at+1}}}{\mathbf{R_a}} = \mathbf{L_h W}\left(\mathbf{k_{ht}}\right) + \mathbf{L_a W}\left(\mathbf{k_{at}}\right) \qquad [\mathbf{2.5}]$$

Now it is possible to substitute $\psi_j\left(k_{jt}, r_{t+1}\right)$ into the [2.5], in order to derive the interest factor as a function of the capital stocks:

$$\mathbf{r_{t+1}} = \mathbf{\Phi}\left(\mathbf{k_{ht}}, \mathbf{k_{at}}\right) \qquad [\mathbf{2.6}]$$

Finally, substituting the relation [2.6] into the [2.4], we obtain the law describing the evolution of the world economy, under integrated financial markets. It is made by a two-dimensional dynamical system:

$$\mathbf{M}\left(\mathbf{k_{ht}},\mathbf{k_{at}}\right)\ :\ \begin{cases} k_{ht+1}=\psi_h\left(k_{ht},\Phi\left(k_{ht},k_{at}\right)\right)=m_h\left(k_{ht},k_{at}\right)\\ k_{at+1}=\psi_a\left(k_{at},\Phi\left(k_{ht},k_{at}\right)\right)=m_a\left(k_{ht},k_{at}\right)\end{cases} \qquad [\mathbf{2.7}]$$

2.3.1. Production functions from Cobb-Douglas family

Now we will assume that the productive function comes from the Cobb-Douglas family.

For $j=h\,,\,a$ it is:

$i_{2.4})\quad \mathbf{f}\left(\mathbf{k_{jt}}\right)=\mathbf{A_j k_{jt}^{\alpha}}$

Moreover, in order to reduce the number of the parameters, we introduce the following substitutions:

$i_{2.5})\quad$ for $j=h\,,\,a$: $\mathbf{y_{jt}}=\frac{\mathbf{k_{jt}}}{\mathbf{R_j}}$ and $\mathbf{\Gamma_j}=\mathbf{A_j R_j^{\alpha}}$ [1]

Remark. Applying the substitutions above, we obtain:

$f\left(y_{jt}\right)=\Gamma_j y_{jt}^{\alpha}$; $W_j\left(y_{jt}\right)=\left(1-\alpha\right)\Gamma_j y_{jt}^{\alpha}$.

Again, $\overline{Y_j}=\frac{K(\lambda_j)}{R_j}=\left(\frac{1-\lambda_j}{(1-\alpha)A_j R_j^{\alpha}}\right)^{\frac{1}{\alpha}}=\left(\frac{1-\lambda_j}{(1-\alpha)\Gamma_j}\right)^{\frac{1}{\alpha}}$ is the border value of constrained-not constrained credit.

Particularly, from [2.3], the steady state of autarky in the j-Country is:

$$\mathbf{Y}^{\star}\left(\mathbf{R_j}\right)=\frac{\mathbf{K}^{\star}\left(\mathbf{R_j}\right)}{\mathbf{R_j}}=\left[\left(\mathbf{1}-\alpha\right)\mathbf{\Gamma_j}\right]^{\frac{\mathbf{1}}{\mathbf{1}-\alpha}} \qquad [\mathbf{2.8}]$$

The law of motion. A piecewise continuous map.

Applying $i_{2.4}$ and $i_{2.5}$ from [2.4] to [2.7] we have:

$$\mathbf{M}\left(\mathbf{y_{ht}},\mathbf{y_{at}}\right)\ :\ \begin{cases} y_{ht+1}=\frac{F_h(y_{ht})}{\tau(y_{ht},y_{at})}\\ y_{at+1}=\frac{F_a(y_{at})}{\tau(y_{ht},y_{at})}\end{cases} \qquad [\mathbf{2.9}]$$

[1] The variable y_{jt} represents the amount of investment i_{jt-1} needed in period $t-1$ to obtain the capital k_{jt} in period t. From now on, the law of motion of the economy will describe the relations in terms of investment, not yet of capital.

Γ_j is a composed parameter, that joins the coefficient of technology A_j, with the one of efficiency R_j. Hence it is possible to fix one parameter, say R_j, and vary solely the other one. For example we can set $R_h=R_a=1$ and assume $\Gamma_j=A_j$ in quantitative terms. In this way, the two countries may differ only relative to the efficiency in producing the consumer goods and any other heterogeneity about the production of capital goods can be omitted.

Where:

$$F_j\left(y_{jt}\right)=\begin{cases}\left(\frac{\alpha\Gamma_j\lambda_j}{1-W_j(y_{jt})}\right)^{\frac{1}{1-\alpha}} & if\ y_{jt}<\overline{Y_j}\\ \left(\alpha\Gamma_j\right)^{\frac{1}{1-\alpha}} & if\ y_{jt}\geq\overline{Y_j}\end{cases}$$

and $\tau\left(y_{ht},y_{at}\right)=\frac{L_hF_h(y_{ht})+L_aF_a(y_{at})}{L_hW_h(y_{ht})+L_aW_a(y_{at})}$

being $L_h+L_a=1$.

Four regions. The map [2.9] can be decomposed in four pieces, each one defined in a different region of the phase space $\Omega=(y_{ht}\geq 0)\cap(y_{at}\geq 0)$.

− Region h_0a_0. Each country is borrowing constrained; $0\leq y_{ht}<\overline{Y_h}$ and $0\leq y_{at}<\overline{Y_a}$

− Region h_0a_1. The h-Country is borrowing constrained, the a-Country is not; $0\leq y_{ht}<\overline{Y_h}$ and $\overline{Y_a}\leq y_{at}$

− Region h_1a_0. The h-Country is not borrowing constrained, the a-Country is; $\overline{Y_h}\leq y_{ht}$ and $0\leq y_{at}<\overline{Y_a}$

− Region h_1a_1. No country is borrowing constrained; $\overline{Y_h}\leq y_{ht}$ and $\overline{Y_a}\leq y_{at}$

$$\mathbf{M_{00}}\left(\mathbf{y_{ht}},\mathbf{y_{at}}\right)\ :\ \begin{cases}y_{ht+1}=\left[\frac{(1-\alpha)\left(L_h\Gamma_hy_{ht}^{\alpha}+L_a\Gamma_ay_{at}^{\alpha}\right)}{L_h\left(\frac{\Gamma_h\lambda_h}{1-(1-\alpha)\Gamma_hy_{ht}^{\alpha}}\right)^{\frac{1}{1-\alpha}}+L_a\left(\frac{\Gamma_a\lambda_a}{1-(1-\alpha)\Gamma_ay_{at}^{\alpha}}\right)^{\frac{1}{1-\alpha}}}\right]\left(\frac{\Gamma_h\lambda_h}{1-(1-\alpha)\Gamma_hy_{ht}^{\alpha}}\right)^{\frac{1}{1-\alpha}}\\ y_{at+1}=\left[\frac{(1-\alpha)\left(L_h\Gamma_hy_{ht}^{\alpha}+L_a\Gamma_ay_{at}^{\alpha}\right)}{L_h\left(\frac{\Gamma_h\lambda_h}{1-(1-\alpha)\Gamma_hy_{ht}^{\alpha}}\right)^{\frac{1}{1-\alpha}}+L_a\left(\frac{\Gamma_a\lambda_a}{1-(1-\alpha)\Gamma_ay_{at}^{\alpha}}\right)^{\frac{1}{1-\alpha}}}\right]\left(\frac{\Gamma_a\lambda_a}{1-(1-\alpha)\Gamma_ay_{at}^{\alpha}}\right)^{\frac{1}{1-\alpha}}\end{cases}$$

[**2.9 a**]

$$\mathbf{M_{01}}\left(\mathbf{y_{ht}},\mathbf{y_{at}}\right)\ :\ \begin{cases}y_{ht+1}=\left[\frac{(1-\alpha)\left(L_h\Gamma_hy_{ht}^{\alpha}+L_a\Gamma_ay_{at}^{\alpha}\right)}{L_h\left(\frac{\Gamma_h\lambda_h}{1-(1-\alpha)\Gamma_hy_{ht}^{\alpha}}\right)^{\frac{1}{1-\alpha}}+L_a\Gamma_a^{\frac{1}{1-\alpha}}}\right]\left(\frac{\Gamma_h\lambda_h}{1-(1-\alpha)\Gamma_hy_{ht}^{\alpha}}\right)^{\frac{1}{1-\alpha}}\\ y_{at+1}=\left[\frac{(1-\alpha)\left(L_h\Gamma_hy_{ht}^{\alpha}+L_a\Gamma_ay_{at}^{\alpha}\right)}{L_h\left(\frac{\Gamma_h\lambda_h}{1-(1-\alpha)\Gamma_hy_{ht}^{\alpha}}\right)^{\frac{1}{1-\alpha}}+L_a\Gamma_a^{\frac{1}{1-\alpha}}}\right]\Gamma_a^{\frac{1}{1-\alpha}}\end{cases}$$

[**2.9 b**]

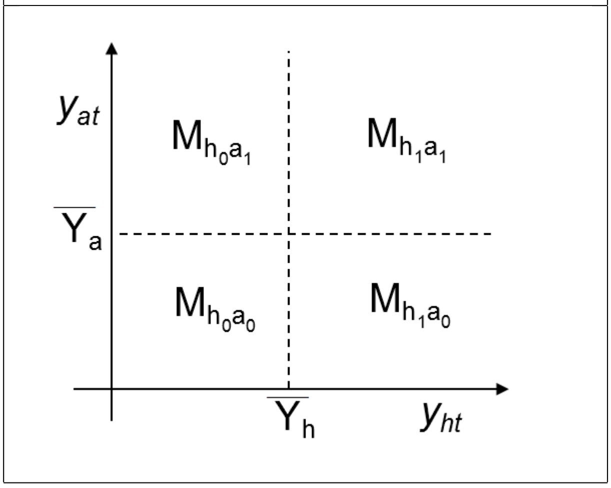
$$\mathbf{M_{10}}\left(\mathbf{y_{ht}},\mathbf{y_{at}}\right)\ :\ \begin{cases} y_{ht+1}=\left[\frac{(1-\alpha)\left(L_h\Gamma_h y_{ht}^{\alpha}+L_a\Gamma_a y_{at}^{\alpha}\right)}{L_h\Gamma_h^{\frac{1}{1-\alpha}}+L_a\left(\frac{\Gamma_a\lambda_a}{1-(1-\alpha)\Gamma_a y_{at}^{\alpha}}\right)^{\frac{1}{1-\alpha}}}\right]\Gamma_h^{\frac{1}{1-\alpha}} \\ y_{at+1}=\left[\frac{(1-\alpha)\left(L_h\Gamma_h y_{ht}^{\alpha}+L_a\Gamma_a y_{at}^{\alpha}\right)}{L_h\Gamma_h^{\frac{1}{1-\alpha}}+L_a\left(\frac{\Gamma_a\lambda_a}{1-(1-\alpha)\Gamma_a y_{at}^{\alpha}}\right)^{\frac{1}{1-\alpha}}}\right]\left(\frac{\Gamma_a\lambda_a}{1-(1-\alpha)\Gamma_a y_{at}^{\alpha}}\right)^{\frac{1}{1-\alpha}} \end{cases}$$

[**2.9 c**]

$$\mathbf{M_{11}}\left(\mathbf{y_{ht}},\mathbf{y_{at}}\right)\ :\ \begin{cases} y_{ht+1}=\left[\frac{(1-\alpha)\left(L_h\Gamma_h y_{ht}^{\alpha}+L_a\Gamma_a y_{at}^{\alpha}\right)}{L_h\Gamma_h^{\frac{1}{1-\alpha}}+L_a\Gamma_a^{\frac{1}{1-\alpha}}}\right]\Gamma_h^{\frac{1}{1-\alpha}} \\ y_{at+1}=\left[\frac{(1-\alpha)\left(L_h\Gamma_h y_{ht}^{\alpha}+L_a\Gamma_a y_{at}^{\alpha}\right)}{L_h\Gamma_h^{\frac{1}{1-\alpha}}+L_a\Gamma_a^{\frac{1}{1-\alpha}}}\right]\Gamma_a^{\frac{1}{1-\alpha}} \end{cases}$$

[**2.9 d**]

Fig. 2.1



Proposition 2.3.

The map [2.9] *is continuous.*

In order to prove the proposition, consider that $W_j\left(\overline{Y_j}\right)=1-\lambda_j$ so the function $F_j\left(y_{jt}\right)$ is continuous. This proves the continuity of the map $M\left(y_{ht},y_{at}\right)$.

2.4. The symmetric case with a generic α

Later on, we will consider the case of two identical countries.

First we will consider a generic value for the capital income share α of the Cobb-Douglas production function.

In the second part, we will derive some outcomes under the particular hypothesis $\alpha = \frac{1}{2}$.

2.4.1. The symmetric case. Basic features.

The map. Let's suppose the two countries are identical:

$L_h = L_a = \frac{1}{2}$; $\Gamma_h = \Gamma_a = \Gamma$; $\lambda_h = \lambda_a = \lambda$

The map [2.9] becomes $\mathbf{M}\left(\mathbf{y_{ht}}, \mathbf{y_{at}}\right) : \begin{cases} y_{ht+1} = \frac{F(y_{ht})}{\tau(y_{ht}, y_{at})} \\ y_{at+1} = \frac{F(y_{at})}{\tau(y_{ht}, y_{at})} \end{cases}$ **[2.9 s]**

Where:

$F\left(y_{jt}\right) = \begin{cases} \left(\frac{\alpha\Gamma\lambda}{1-W(y_{jt})}\right)^{\frac{1}{1-\alpha}} & if\ y_{jt} < \overline{Y} \\ (\alpha\Gamma)^{\frac{1}{1-\alpha}} & if\ y_{jt} \geq \overline{Y} \end{cases}$ and $\tau\left(y_{ht}, y_{at}\right) = \frac{F(y_{ht}) + F(y_{at})}{W(y_{ht}) + W(y_{at})}$,

being $W\left(y_{jt}\right) = (1-\alpha)\,\Gamma y_{jt}^{\alpha}$ and $\overline{Y} = \left(\frac{1-\lambda}{(1-\alpha)\Gamma}\right)^{\frac{1}{\alpha}}$

Invariant sets and attractors. We can prove that, in the case of symmetry, these features emerge:

- the set of points $\begin{cases} y_{ht} = y_{at} \\ y_{jt} \geq 0 \end{cases}$ is invariant for the map;

- attractors are symmetric regarding the bisector $y_{ht} = y_{at}$.

Proposition 2.4.

The half-line $(y_{ht} = y_{at}) \cap (y_{jt} \geq 0)$ *is an invariant set of points.*

Applying the condition $y_{ht} = y_{at} = y_t$ to the map [2.9 s], we have:

$\tau\left(y_t, y_t\right) = \frac{F(y_t)}{W(y_t)}$ hence, $y_{ht+1} = y_{at+1} = W\left(y_t\right)$.

This proves that the image of a point on the bisector, lies on the bisector.

Moreover, a point (y_{t+1}, y_{t+1}) on the bisector has a pre-image (y_t, y_t) on the same bisector, which is simply obtained by taking $y_t = W^{-1}\left(y_{t+1}\right)$ and we are done.

Proposition 2.5.

If a point $Q_1(\mu,\nu)$ is a steady state for the map, then the symmetric point $Q_2(\nu,\mu)$ is a steady state too.

Moreover, their stable sets are symmetric relative to the bisector.

Preliminary it is to be noted that $\tau(y_{ht}, y_{at}) = \tau(y_{at}, y_{ht})$.

Moreover, being $Q_1(\mu,\nu)$ *a steady state*, from [2.9 s] it must be $\mu = \frac{F(\mu)}{\tau(\mu,\nu)}$ and $\nu = \frac{F(\nu)}{\tau(\mu,\nu)}$.

Considering the image of the point $Q_2(\nu,\mu)$, we obtain:

$y_{ht+1}(Q_2) = \frac{F(\nu)}{\tau(\nu,\mu)} = \frac{F(\nu)}{\tau(\mu,\nu)} = \nu$ and $y_{at+1}(Q_2) = \frac{F(\mu)}{\tau(\nu,\mu)} = \frac{F(\mu)}{\tau(\mu,\nu)} = \mu$.

That proves the first part of the thesis.

Now, let's suppose that $Q_1(\mu,\nu)$ and $Q_2(\nu,\mu)$ are two, symmetric stable points.

Let $A_1(\mu_0,\nu_0)$ be a point belonging to the stable set of Q_1.

Then it will be $\lim_{n\to\infty}(\mu_n,\nu_n) = Q_1$, where

$(\mu_n,\nu_n) = M(\mu_{n-1},\nu_{n-1}) = \left(\frac{F(\mu_{n-1})}{\tau(\mu_{n-1},\nu_{n-1})}, \frac{F(\nu_{n-1})}{\tau(\mu_{n-1},\nu_{n-1})}\right)$, for $n \geq 1$.

Again invoking the "parity" of $\tau(.,.)$, it is immediate to notice that a chain starting from the symmetric point $A_2(\nu_0,\mu_0)$, ends in Q_2.

Indeed, $\nu_n = \frac{F(\nu_{n-1})}{\tau(\nu_{n-1},\mu_{n-1})} = \frac{F(\nu_{n-1})}{\tau(\mu_{n-1},\nu_{n-1})}$ and $\mu_n = \frac{F(\mu_{n-1})}{\tau(\nu_{n-1},\mu_{n-1})} = \frac{F(\mu_{n-1})}{\tau(\mu_{n-1},\nu_{n-1})}$.

So $(\nu_n,\mu_n) = M(\nu_{n-1},\mu_{n-1})$. Then $\lim_{n\to\infty}(\nu_n,\mu_n) = Q_2$ *q.e.d.*

2.4.2. The steady state of autarky $\mathbf{P}^\star$ in the symmetric case: a Milnor attractor.

Let's consider the point $\mathbf{P}^\star\left([(\mathbf{1}-\alpha)\,\boldsymbol{\Gamma}]^{\frac{1}{1-\alpha}}, [(\mathbf{1}-\alpha)\,\boldsymbol{\Gamma}]^{\frac{1}{1-\alpha}}\right)$ lying on the bisector $y_{ht} = y_{at}$ and belonging to the region h_1a_1, or h_0a_0, depending whether is

$\Gamma \geq \frac{(1-\lambda)^{1-\alpha}}{1-\alpha}$ or not.

We will see that $P^\star$ is a steady state of the system [2.9 s]. It is to be noted that when the two countries lie in $P^\star$, they are in the same equilibrium as in autarky. For this reason we call $P^\star$ *the steady state of autarky.*[2]

According to the values of parameters, we will see some general outcomes in the next proposition. For brevity we set $p = [(1-\alpha)\,\Gamma]^{\frac{1}{1-\alpha}}$.

Proposition 2.6.

Depending whether is $\Gamma \geq \frac{(1-\lambda)^{1-\alpha}}{1-\alpha}$ or not, the steady state of autarky $P^\star$ lies on the region h_1a_1, or h_0a_0.

Into the region h_1a_1 $P^\star$ is a stable node, while, into h_0a_0, it may be a stable node or a saddle point. In each case, its stable set includes, as minimum configuration of points, the set $\left]\,O, \overline{Y}\,\right[\cup h_1a_1$, that is the region h_1a_1 plus the segment of bisector

$\left]\,O, \overline{Y}\,\right[= \left\{(y_{ht}, y_{at}) \ : \ y_{ht} = y_{at} \text{ and } 0 < y_{jt} < \overline{Y}\right\}$. For this reason we assert that $P^\star$ is a Milnor attractor.

Depending on the parameters of the system, we can have the following situations:

If** $\alpha > \lambda$ **then:

$\Gamma = \frac{1}{(1-\alpha)^\alpha}$ and $\Gamma = \frac{(1-\lambda)^{1-\alpha}}{1-\alpha}$ are bifurcation values.

– For $\Gamma < \frac{1}{(1-\alpha)^\alpha}$ the point $P^\star$ stays into the region h_0f_0 and is a stable node.

– For $\frac{1}{(1-\alpha)^\alpha} < \Gamma < \frac{(1-\lambda)^{1-\alpha}}{1-\alpha}$ the point $P^\star$ remains into h_0f_0, but becomes a saddle point.

– For $\Gamma \geq \frac{(1-\lambda)^{1-\alpha}}{1-\alpha}$ the point $P^\star$ enters the region h_1f_1 and becomes a stable node.

If** $\alpha \leq \lambda$**:

– For $\Gamma < \frac{(1-\lambda)^{1-\alpha}}{1-\alpha}$ the point $P^\star$ stays into the region h_0f_0 and is a stable node.

– For $\Gamma \geq \frac{(1-\lambda)^{1-\alpha}}{1-\alpha}$ the point $P^\star$ enters the region h_1f_1 and remains a stable node.

[2]Of course this is a misuse of language, because the two economies are not in autarky, but their equilibrium is the same as in autarky.

The pictures in Figures 2.2 and 2.3 summarize the results.

For this and the following propositions, when the proof doesn't follow immediately, it can be found in the Appendix to Chapter 2.

Fig. 2.2

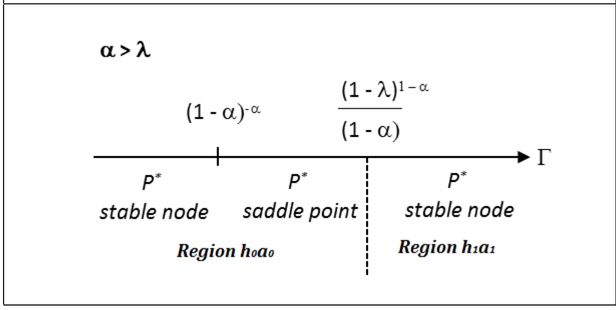
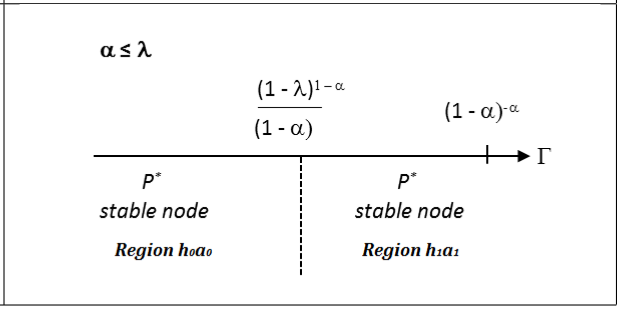


Fig. 2.3



Remark. We have seen that, even when $P^\star$ is a saddle point (of the region h_0a_0) its stable set includes, as minimum configuration, the set of points $\left]O, \overline{Y}\right[\cup h_1a_1$ and therefore has a positive Lebesgue measure. This means that $P^\star$ can be considered as a Milnor attractor. This notion has been introduced by Milnor [13] to underline, in general terms, the possible existence of an invariant set that can attract many points, even if it is not an attractor in the usual sense.

An economic intuition, behind this result, is that the financial globalization can destabilize the autarky steady state, however, there still exists a set of initial conditions into the regions h_0a_1, h_1a_1, h_1a_0, whose trajectories will continue to converge towards $P^\star$. In other words, imperfection in the credit market can imply symmetry breaking to occur, but this result largely depends on the initial conditions of the world economy during the time of the financial globalization.

2.5. The symmetric case with $\alpha = \frac{1}{2}$

We consider now a particular choice of the Cobb-Douglas production function and we look for the fixed points in each of the four regions: we set to $\alpha = \frac{1}{2}$ the share of income paid to capital.

The map [2.9 s] can be re-written considering its components in this particular case:

$$F\left(y_{jt}\right) = \begin{cases} \frac{\frac{1}{4}\Gamma^2\lambda^2}{\left(1-\frac{1}{2}\Gamma\sqrt{y_{jt}}\right)^2} & if\ y_{jt} < \overline{Y} \\ \frac{1}{4}\Gamma^2 & if\ y_{jt} \geq \overline{Y} \end{cases}$$

being $\overline{Y} = 4\left(\frac{1-\lambda}{\Gamma}\right)^2$

while $\tau\left(y_{ht}, y_{at}\right) = \frac{2}{\Gamma}\left(\frac{F(y_{ht})+F(y_{at})}{\sqrt{y_{ht}}+\sqrt{y_{at}}}\right)$.

Firstly, we will see the number and the nature of the fixed points of the regions h_1a_1 and h_0a_0.

Then we will extend the analysis to the remaining regions.

2.5.1. The fixed points of the regions h_1a_1 and h_0a_0

The region h_1a_1 can only contain the steady state of autarky $\mathbf{P}^{\star}\left(\frac{\mathbf{\Gamma^2}}{\mathbf{4}}, \frac{\mathbf{\Gamma^2}}{\mathbf{4}}\right)$ (stable node).

The region h_0a_0 may contain $\mathbf{P}^{\star}\left(\frac{\mathbf{\Gamma^2}}{\mathbf{4}}, \frac{\mathbf{\Gamma^2}}{\mathbf{4}}\right)$ and other two symmetric steady states:

$$\mathbf{Q_1}\left(\frac{\frac{1}{2}\mathbf{\Gamma^2}+2\sqrt{\frac{1}{2}\mathbf{\Gamma^2}-1}}{\mathbf{\Gamma^2}}, \frac{\frac{1}{2}\mathbf{\Gamma^2}-2\sqrt{\frac{1}{2}\mathbf{\Gamma^2}-1}}{\mathbf{\Gamma^2}}\right) ; \mathbf{Q_2}\left(\frac{\frac{1}{2}\mathbf{\Gamma^2}-2\sqrt{\frac{1}{2}\mathbf{\Gamma^2}-1}}{\mathbf{\Gamma^2}}, \frac{\frac{1}{2}\mathbf{\Gamma^2}+2\sqrt{\frac{1}{2}\mathbf{\Gamma^2}-1}}{\mathbf{\Gamma^2}}\right).$$

While $\mathbf{P}^{\star}$ in h_0a_0 may be a saddle point or a stable node, the points $\mathbf{Q_1}$ and $\mathbf{Q_2}$ are always stable and may be nodes or foci.

To be more precise we will prove the following Proposition.

Proposition 2.7.

For $\lambda \geq \frac{1}{2}$ only the fixed point $P^{\star}$ exists. It is an attractive node and it belongs to h_1a_1 when $\Gamma \geq 2\sqrt{1-\lambda}$ and to h_0a_0 in the opposite case.

For $\lambda < \frac{1}{2}$ then the parameter Γ determines if there are one or three fixed points; definitely:

– if $\Gamma > 2\sqrt{1-\lambda}$ the point $P^{\star}$ belongs to h_1a_1 and no fixed points exist in h_0a_0;

– if $2\sqrt{2\lambda^2-2\lambda+1} < \Gamma < 2\sqrt{1-\lambda}$ the point $P^{\star}$ is again the unique fixed point into the two regions we are considering, but it is a saddle point and it stays in h_0a_0. Its stable set has positive measure and includes the set $\left]O, \overline{Y}\right[\cup h_1a_1$;

– *if* $2\sqrt{\frac{2}{7}\left(2\sqrt{2}-1\right)} < \Gamma < 2\sqrt{2\lambda^2 - 2\lambda + 1}$ *there are three fixed points in* h_0a_0*:* $P^\star$, Q_1 *and* Q_2*, while there are no stable points in* h_1a_1*;* $P^\star$ *is a saddle point,* Q_1 *and* Q_2 *are stable foci;*

– *if* $\sqrt{2} < \Gamma < 2\sqrt{\frac{2}{7}\left(2\sqrt{2}-1\right)}$ *then* $P^\star$, Q_1 *and* Q_2 *are in* h_0a_0*, while* h_1a_1 *has no fixed points;* $P^\star$ *is a saddle point,* Q_1 *and* Q_2 *are stable nodes;*

– *if* $\Gamma < \sqrt{2}$ *there is the only stable node* $P^\star$ *in* h_0a_0 *and no steady points in* h_1a_1*.*

Fig. 2.4

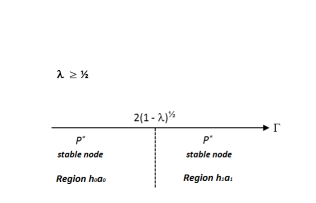
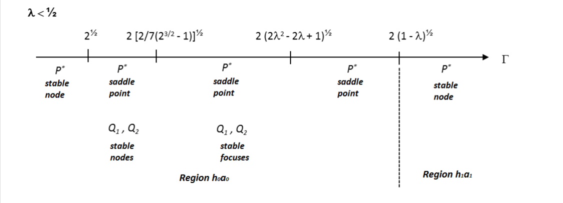


Fig. 2.5



2.5.2. The fixed points of the regions h_0a_1 and h_1a_0

In each of the regions h_0a_1 and h_1a_0 there may symmetrically be two/one/none fixed points.

We will prove some analytical conditions.

The figures 2.6 and 2.8 are drawn in the plane (λ, Γ^2). The figures 2.7 and 2.9 are the graphical representation of the conditions of steady state $M\,(y_{ht}, y_{at}) = (y_{ht}, y_{at})$ for some particular choices of the parameters; their aim is merely an explanatory one.

Proposition 2.8.

One steady state exists in each region h_0a_1, h_1a_0, when:

$$\begin{cases} \Gamma^2 \geq 8\lambda^2 - 8\lambda + 4 \\ \Gamma^2 < 4\,(1-\lambda) \\ 0 < \lambda < \frac{1}{2} \end{cases}$$

Fig. 2.6

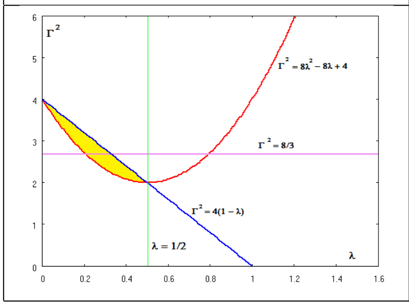
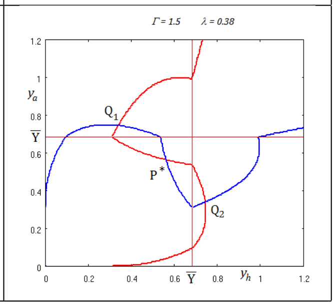


Fig. 2.7



Two steady states exist in each region h_0a_1, h_1a_0, when:

$$\begin{cases} \Gamma^2 \geq 8\lambda^2 - 8\lambda + 4 \\ \Gamma^2 > \frac{8}{3} \\ \Gamma^2 > 4\,(1-\lambda) \\ 0 < \lambda \leq \lambda_T\,(\Gamma) < \frac{1}{2} \end{cases}$$

Fig. 2.8

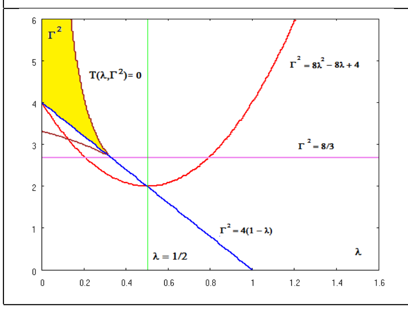
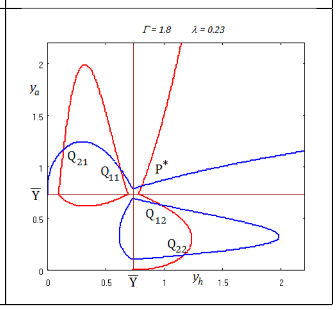


Fig. 2.9



Since in this case, an explicit expression of the fixed points couldn't be obtained, we will try to derive them from a numerical simulation.

2.6. A numerical simulation

Later on we will view a numerical simulation obtained by fixing λ and decreasing Γ. The other parameters will be $\alpha = \frac{1}{2}$ and $L_h = L_a = \frac{1}{2}$. We will find all the outcomes previously described.

The figures that follow are of two types. The colored ones represent the basins of attraction of the stable points; different colors identify different basins. The others complete the description of the dynamics and represent the conditions of steady state $M(y_{ht}, y_{at}) = (y_{ht}, y_{at})$.

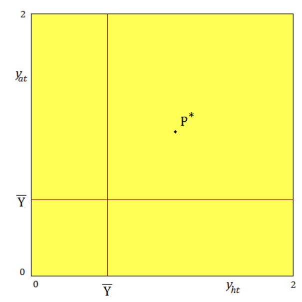
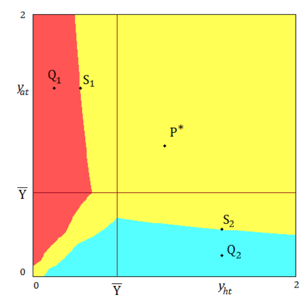
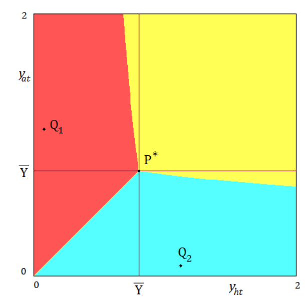
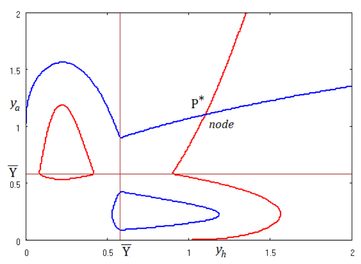
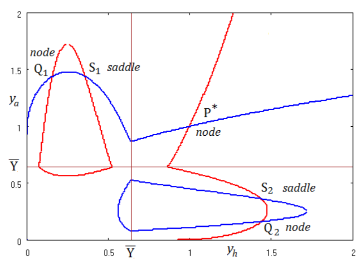
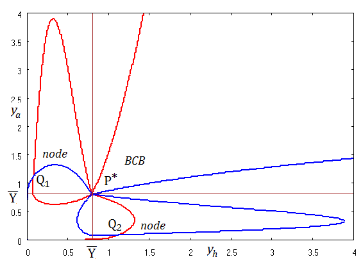
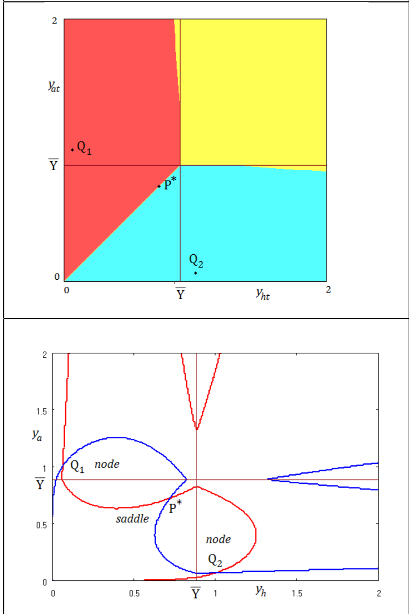
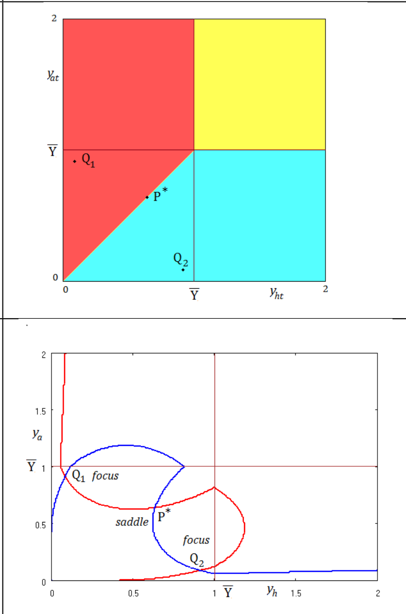
Fig. 2.10	***Fig. 2.11***	***Fig. 2.12***
		
		
$\Gamma = 2.1$; $\lambda = 0.20$	$\Gamma = 2.0$; $\lambda = 0.20$	$\Gamma = 1.788854381999832$; $\lambda = 0.20$

Fig. 2.13



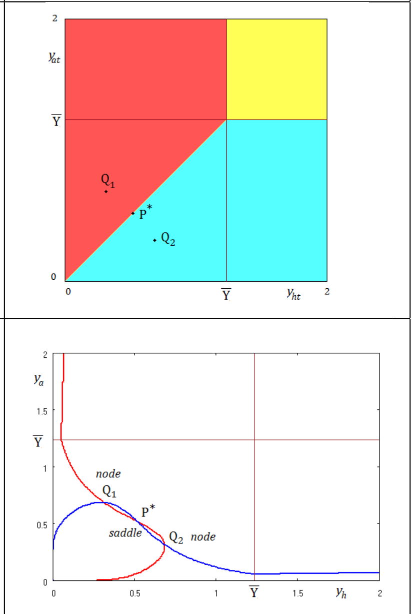
Γ = 1.7 ; λ = 0.20

Fig. 2.14



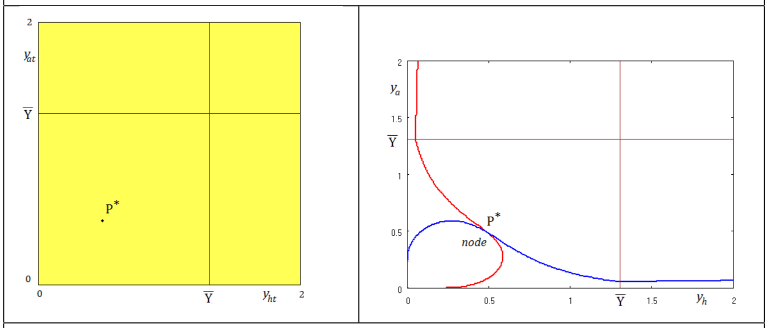
Γ = 1.6 ; λ = 0.20

Fig. 2.15



Γ = 1.44 ; λ = 0.20

Fig. 2.16



Γ = 1.40 ; λ = 0.20

At the beginning (Fig. 2.10) the unique stable point is the asymptotically attractive node $P^{\star}$ of the region h_1a_1 (the stable state of autarky).

Diminishing the parameter Γ, a pair of steady points appear in h_0a_1 and, symmetrically, in h_1a_0 (Fig. 2.11). Precisely, they are stable nodes and saddles. The stable sets of the saddles separate the basins of attraction of the three coexisting stable nodes.

Decreasing Γ, the node $P^{\star}$ tends towards the bifurcation point $\left(\overline{Y},\overline{Y}\right)$ and the same thing do the two saddle points S_1 and S_2. At the border collision bifurcation (BCB) these three steady points merge (Fig. 2.12). Therefore the BCB if of pitchfork type.

Continuing in diminishing Γ, the steady state of autarky $P^{\star}$ enters the region h_0a_0 and becomes a saddle, but its stable set has always positive measure and its frontier is given by the preimages of any rank of the two constraints $y_h = \overline{Y}$ and $y_a = \overline{Y}$ (Fig. 2.13).

In the same time, the two stable points of h_0a_1 and h_1a_0 move towards the lines separating the regions, until they enter h_0a_0 too. Initially they are foci, then nodes, approaching more and more $P^{\star}$ (Fig. 2.14 and 2.15). Finally they merge to $P^{\star}$ that becomes a node and remains the unique steady state of the map, because of the occurrence of the pitchfork bifurcation (Fig. 2.16).

Bifurcations. Summarizing, these are the bifurcation dynamics:

– A border collision bifurcation of subcritical pitchfork type, that occurs when $P^{\star}$ is crossing from h_1a_1 to h_0a_0 and merges with the saddle points S_j.

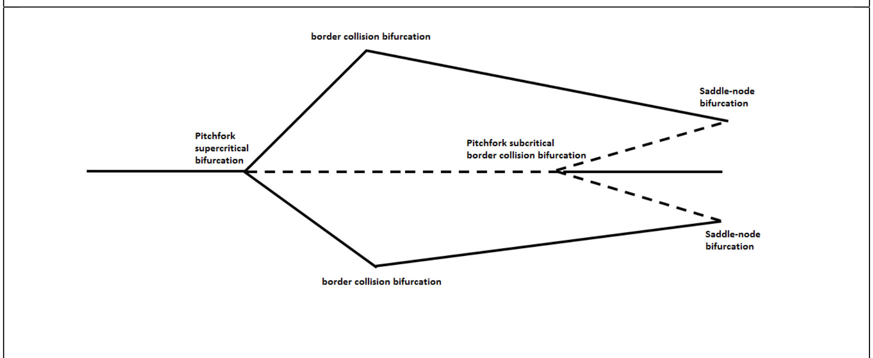
– A supercritical pitchfork bifurcation, when the two nodes Q_j merge with the saddle $P^{\star}$ into the region h_0a_0.

– Two symmetrical saddle-node bifurcations into the regions h_1a_0 and h_0a_1, when two steady points, respectively a node Q_j and a saddle S_j, appear in each region.

– Two symmetrical border collision bifurcations, when the the points Q_j enter the region h_0a_0.

See figure 2.17 that summarizes the results.

Fig. 2.17



2.7. Conclusions

This chapter has been concentrated essentially on the consequences of credit market integration, between two countries, assuming that the other markets are closed. In this way the aim of the work is to picture some possible scenarios due to financial globalization.

After having introduced the model and derived the main formal implications, the last part of the Chapter has been dedicated to a general description of financial integration, in case of perfect identity between the two countries.

In order to make a comparison between the situation before and after the credit market integration, great importance has been given to the steady state of autarky $P^\star$ that marks a situation in which the two countries, after their financial integration, lie in the same condition as before.

Even though referred to a particular parameter choice, the description made for the case $\alpha = \frac{1}{2}$ can be useful to understand the dynamics in general terms.

Observing the figure 2.17, we can see that $P^\star$ may change its nature, from stable to unstable, but the most relevant result is that, even when $P^\star$ is a saddle, its stable set has positive measure (see, for example, Fig. 2.13) and it includes, at least, the entire not borrowing constrained region $h_1 a_1$. Again we can observe that $P^\star$ may coexist with other steady states (up to four and two of them).

All these facts have great consequences about the opportunity for the countries to join their financial markets.

Let's suppose, for instance, that the steady state $P^\star$ is very closed to the border point $\left(\overline{Y}, \overline{Y}\right)$ and, before the financial integration, the two countries lie in a point A of the neighborhood of $P^\star$. If A is situated into the stable set of $P^\star$(for example all the region $y_{jt} > \overline{Y}$) integration doesn't really matter in the long run, because the two countries will eventually go to $P^\star$. On the contrary, if A lies in the basin of attraction of some Q_{2j} (see Fig. 2.13) through time, integration will be convenient for one country, but harmful for the other.

In conclusion, it is to be noticed that, until perfect symmetry between the two countries persists, no endogenous periodic dynamics may be found. These last will be studied in the next Chapter, under a new set of hypotheses.

3. Credit market imperfections. The cases of “quasi-symmetry”, “heterogeneity” and “quasi-heterogeneity”.

In this last chapter, we continue the analysis begun in the second one and we gradually introduce heterogeneity between the two countries. As declared in the previous parts of the work, our aim is to demonstrate that endogenous fluctuations may occur not only because of differences in the population size, as in Kikuchi and Stachurski ([8]), but also due to heterogeneity in other economic features like technology and credit market imperfection. Hence we start with the case of “quasi-symmetry”, supposing the two countries to be identical in all their characteristic parameters, except in the size of their populations. Firstly we study the consequences for the steady state of autarky and we depict a global analysis, via a numerical simulation. Then, like Kikuchi and Stachurski, we prove the existence of periodic dynamics, in this case consisting in a Neimark-Sacker bifurcation of supercritical type. Continuing our exploration, we assume the hypotheses of total heterogeneity in populations, technologies, and credit market imperfections. We deeply analyze some periodic phenomena and depict the evolution of a crater bifurcation. Finally we consider the case of “quasi-heterogeneity”, assuming that countries are different in all their features, except for the size of their populations, these last supposed to be equal. The occurrence of a crater bifurcation even in this case, prove definitely our thesis, that the causes of periodic dynamics can be found in different magnitudes not only of population, but also of technology and credit market imperfection.

The hypotheses considered will be the same as in the previous chapter:

− There are only two countries, “h” for “*home*” and “a” for “*abroad*”.

− Their populations $0 < L_h < 1$ and $0 < L_a < 1$ are supposed to be complementary: $L_h + L_a = 1$.

− Markets factors are closed, and agents cannot start an investment project abroad.

− Financial markets are integrated, but imperfect.

3.1. The case of “quasi-symmetry”.

We will suppose that the two countries are identical in all their specific features, but their populations are different:

$L_h \neq L_a$; $\Gamma_h = \Gamma_a = \Gamma$, $\lambda_h = \lambda_a = \lambda$.

The map. Under these assumptions the system map becomes:

$$\mathbf{M}\left(\mathbf{y_{ht}}, \mathbf{y_{at}}\right) \; : \; \begin{cases} y_{ht+1} = \frac{F(y_{ht})}{\tau(y_{ht}, y_{at})} \\ y_{at+1} = \frac{F(y_{at})}{\tau(y_{ht}, y_{at})} \end{cases}$$

Where:

$$F\left(y_{jt}\right) = \begin{cases} \left(\frac{\alpha\Gamma\lambda}{1-W(y_{jt})}\right)^{\frac{1}{1-\alpha}} & if\ y_{jt} < \overline{Y} \\ \left(\alpha\Gamma\right)^{\frac{1}{1-\alpha}} & if\ y_{jt} \geq \overline{Y} \end{cases}$$

and $\tau\left(y_{ht}, y_{at}\right) = \frac{L_h F(y_{ht}) + L_a F(y_{at})}{L_h W(y_{ht}) + L_a W(y_{at})}$

being $L_h + L_a = 1$, $W\left(y_{jt}\right) = \left(1-\alpha\right)\Gamma y_{jt}^{\alpha}$, $\overline{Y} = \left(\frac{1-\lambda}{(1-\alpha)\Gamma}\right)^{\frac{1}{\alpha}}$

The main results. We will see that the half-line corresponding to the positive segment of the bisector still remain an invariant set and the features relative to the steady state of autarky $P^\star$ are preserved.

But now different attractors are not symmetric.

Proposition 3.1.

The half-line $(y_{ht} = y_{at}) \cap (y_{jt} \geq 0)$ *is an invariant set of points.*

Indeed, applying the condition $y_{ht} = y_{at} = y_t$, we obtain the same outcomes as in case of perfect symmetry:

$\tau\left(y_{ht}, y_{at}\right) = \frac{F(y_t)}{W(y_t)}$ hence $y_{ht+1} = y_{at+1} = W\left(y_t\right)$.

Again $y_t = W^{-1}\left(y_{t+1}\right)$ is a pre-image of y_{t+1}, then each point $\left(y_{t+1}, y_{t+1}\right)$ has almost a pre-image $\left(y_t, y_t\right)$ on the bisector.

The steady state of autarky $\mathbf{P}^\star\left(\left[\left(\mathbf{1}-\alpha\right)\mathbf{\Gamma}\right]^{\frac{\mathbf{1}}{\mathbf{1}-\alpha}}, \left[\left(\mathbf{1}-\alpha\right)\mathbf{\Gamma}\right]^{\frac{\mathbf{1}}{\mathbf{1}-\alpha}}\right)$ behaves in the same way as in the case of perfect symmetry (see the proposition 2.6 of Chapter 2).

Indeed $P^\star$ belongs to the bisector $y_{at} = y_{ht}$ and there the map is not affected by the parameters L_j. About the basin of attraction of $P^\star$, it is to be noted that the points of the region $h_1 a_1$ reach the bisector just at the first iteration (as in symmetry) and, by this way, they tend to the point $P^\star$.

We summarize these features in the following proposition.

Proposition 3.2.

For $\Gamma \geq \frac{(1-\lambda)^{1-\alpha}}{1-\alpha}$ *the steady state of autarky* $P^\star$ *is an asymptotically stable node of the region* h_1a_1.

For $\Gamma < \frac{(1-\lambda)^{1-\alpha}}{1-\alpha}$ *the steady state of autarky* $P^\star$ *lies into the region* h_0a_0*, where it may be an attractive node, or a saddle point.*

Even if it is a saddle, its stable set has positive measure. It certainly includes all the points of the region h_1a_1*, plus the segment of bisector* $\left]O, \overline{Y}\right[$:

$\left]O, \overline{Y}\right[= \left\{(y_{ht}, y_{at}) \ : \ y_{ht} = y_{at} \text{ and } 0 < y_{jt} < \overline{Y}\right\}$.

The following configuration of parameters exists:

If $\alpha > \lambda$ ***then:***

$\Gamma = \frac{1}{(1-\alpha)^\alpha}$ *and* $\Gamma = \frac{(1-\lambda)^{1-\alpha}}{1-\alpha}$ *are bifurcation values.*

– *For* $\Gamma < \frac{1}{(1-\alpha)^\alpha}$ *the point* $P^\star$ *stays into the region* h_0f_0 *and is a stable node.*

– *For* $\frac{1}{(1-\alpha)^\alpha} < \Gamma < \frac{(1-\lambda)^{1-\alpha}}{1-\alpha}$ *the point* $P^\star$ *remains into* h_0f_0*, but becomes a saddle point.*

– *For* $\Gamma \geq \frac{(1-\lambda)^{1-\alpha}}{1-\alpha}$ *the point* $P^\star$ *enters the region* h_1f_1 *and becomes a stable node.*

If $\alpha \leq \lambda$***:***

– *For* $\Gamma < \frac{(1-\lambda)^{1-\alpha}}{1-\alpha}$ *the point* $P^\star$ *stays into the region* h_0f_0 *and is a stable node.*

– *For* $\Gamma \geq \frac{(1-\lambda)^{1-\alpha}}{1-\alpha}$ *the point* $P^\star$ *enters the region* h_1f_1 *and remains a stable node.*

(See the figures that follow for a synthetic summary)

Fig. 3.1

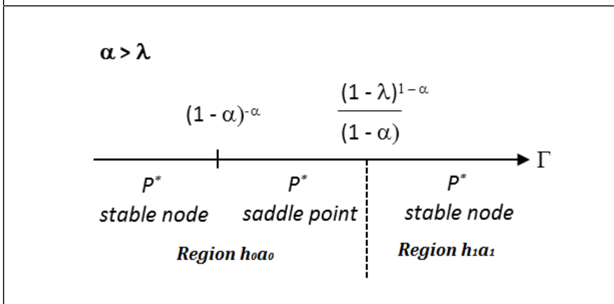
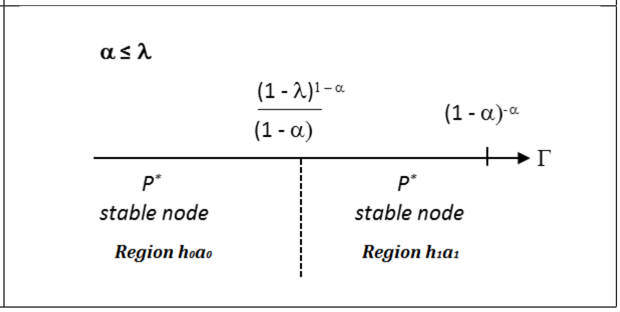


Fig. 3.2



Finally, out of the bisector, attractors are no longer symmetric, as we can easy see in the following numerical simulations.

Later on we will propose some numerical simulations. The figures that follow in all this chapter are essentially of two types (the same types already appeared in the previous chapter). The colored ones represent the basins of attraction of the stable points (different colors identify different basins), while the other ones are the graphical representation of the conditions of steady state $M(y_{ht}, y_{at}) = (y_{ht}, y_{at})$.

3.1.1. A first set of numerical simulations

Let’s suppose $\alpha = \frac{1}{3}$, $L_h = 0.25$ for the h-Country and $L_a = 0.75$ for the a-Country.

To assume $\alpha = \frac{1}{3}$ means that the share of income paid to capital is approximately one third of the total output. This is generally considered a quite realistic hypothesis in economic literature. Besides, a significant difference in the dimensions of the countries is hypothesized.

This section has essentially an introductive aim. Even if under particular hypotheses, we will draw a general scenario, we will study the evolution of the steady state of autarky P^*, and we will introduce a first example of periodic phenomena. These last will be the basic matter of the next section.

Hence, we will see two different dynamic behaviors. In each one of them we will fix the parameter λ at some constant value and vary Γ over a significant numerical interval.

3.1.1.1. The first dynamic scenario: $\lambda = 0.10$, $\Gamma \in (0, 1.7)$.

Let’s fix the parameter $\lambda = 0.10$ and varying Γ over the set of values $(0, 1.7)$.

From the proposition 3.2 we know that the interval $\Gamma > 0$ can be divided into three sub-intervals as it follows:

– for $\Gamma \in \left(\frac{9\sqrt[3]{30}}{20}, \infty\right)$ the steady point $P^\star$ is a stable node of the region h_1a_1;

– for $\Gamma \in \left(\frac{\sqrt[3]{12}}{2}, \frac{9\sqrt[3]{30}}{20}\right)$ the steady point $P^\star$ is a saddle point of the region h_0a_0;

– for $\Gamma \in \left(0, \frac{\sqrt[3]{12}}{2}\right)$ the steady point $P^\star$ is a stable node of the region h_0a_0.

In order to follow the evolution of the map, see the figures from 3.3 to 3.11.

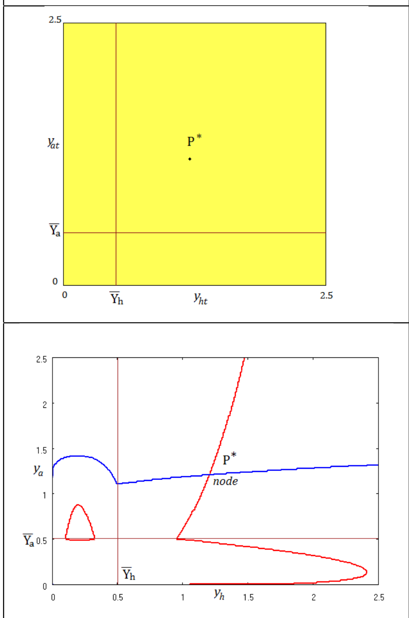
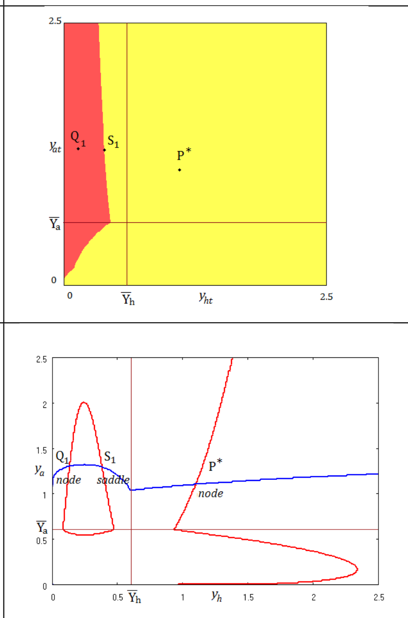
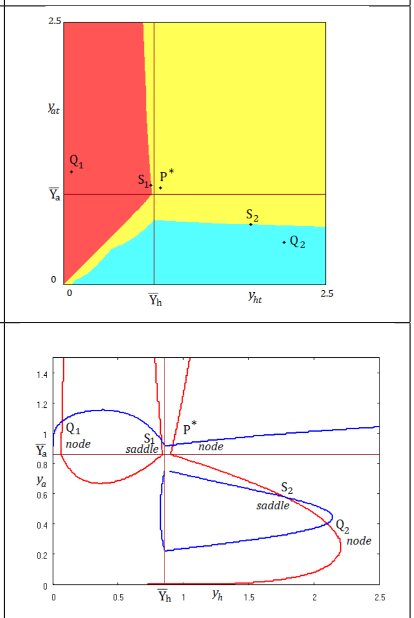
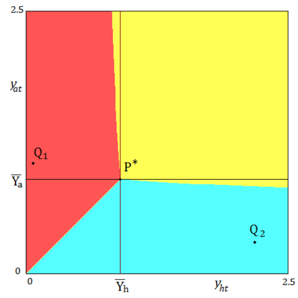
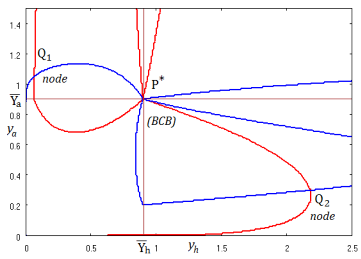
Fig. 3.3	*Fig. 3.4*	*Fig. 3.5*
		
$\Gamma_h = \Gamma_a = 1.70$; $\lambda_h = \lambda_a = 0.10$	$\Gamma_h = \Gamma_a = 1.60$; $\lambda_h = \lambda_a = 0.10$	$\Gamma_h = \Gamma_a = 1.42$; $\lambda_h = \lambda_a = 0.10$

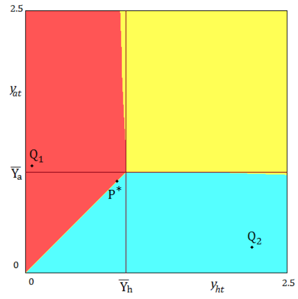
Fig. 3.6

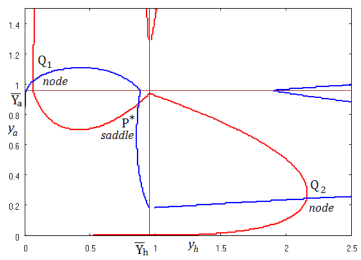




$\Gamma_h = \Gamma_a = 1.398254627679237$; $\lambda_h = \lambda_a = 0.10$

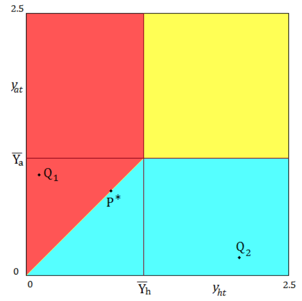
Fig. 3.7

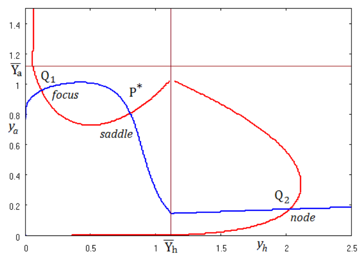




$\Gamma_h = \Gamma_a = 1.37$; $\lambda_h = \lambda_a = 0.10$

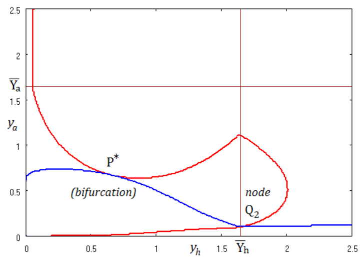
Fig. 3.8





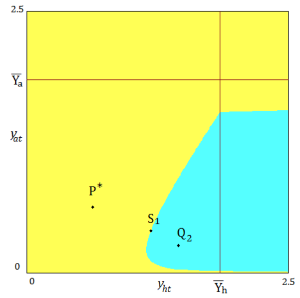
$\Gamma_h = \Gamma_a = 1.30$; $\lambda_h = \lambda_a = 0.10$

Fig. 3.9



$\Gamma_h = \Gamma_a = 1.144714242553332$; $\lambda_h = \lambda_a = 0.10$

Fig. 3.10



$\Gamma_h = \Gamma_a = 1.10$; $\lambda_h = \lambda_a = 0.10$

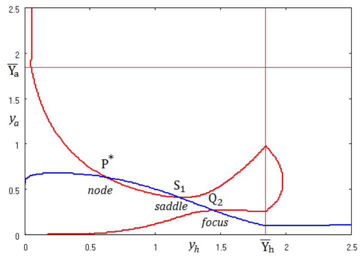
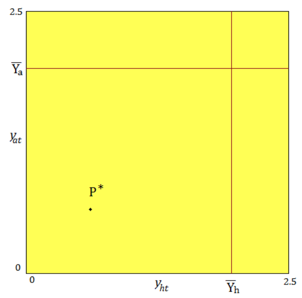
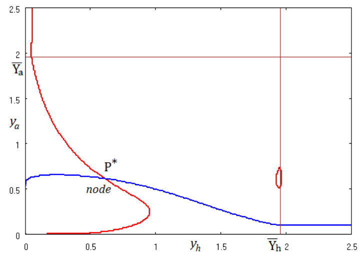


Fig. 3.11



$\Gamma_h = \Gamma_a = 1.08$; $\lambda_h = \lambda_a = 0.10$



At the beginning (Fig. 3.3) there is the only stable state of autarky $P^\star$.

It lies into the region $h_1 a_1$ and is a node, asymptotically attractive for all the space $(y_{ht} > 0) \cap (y_{at} > 0)$.

Then a *saddle-node bifurcation* occurs and two further steady states appear into h_0a_1. They are a stable node and a saddle point (Q_1 and S_1 of Fig. 3.4).

$P^\star$ remains a node of h_1a_1, but it starts to slide towards the border point $\left(\overline{Y}, \overline{Y}\right)$.

Hence the basins of attraction of Q_1 and $P^\star$ are separated by the stable set of the saddle S_1.

Later on, another similar *saddle-node bifurcation* takes place in h_1a_0, with Q_2 node and S_2 saddle.

Now the stable sets of S_1 and S_2 separate the basins of attraction of the three stable points $P^\star$, Q_1 and Q_2 (see Fig. 3.5).

Until Γ is diminishing, the two saddles move towards the point $P^\star$ and finally they merge at the *border collision bifurcation* of Fig. 3.6 always of pitchfork type.

Crossing the border $\left(\overline{Y}, \overline{Y}\right)$ $P^\star$ becomes a saddle point, but its stable set still has positive measure: all the points of the region h_1a_1, plus two narrow strips “debording” into h_0a_1 and h_1a_0, plus the points of the positive half-line of bisector $y_{at} = y_{ht}$ are attracted by $P^\star$. The narrow strips have, as boundaries, the union of the preimages of any rank of the lines that separate the regions where the map changes definition.

In the meantime, the two nodes Q_j move towards the border lines.

When they cross the border lines, they become foci; then, they will return to be nodes again.

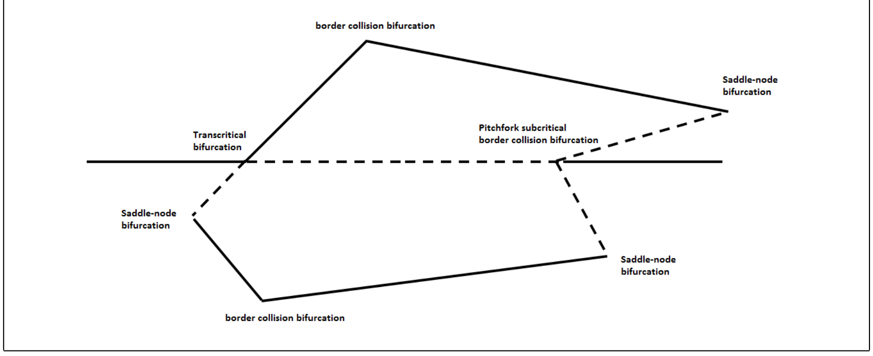
The first to enter the region h_0a_0 is the one of Nord-West, Q_1, that starts to approach $P^\star$ (Fig. 3.8) (during that time Q_1 will change its nature, from focus to node).

In Fig. 3.9 a *transcritical bifurcation* takes place, with a change of stability between $P^\star$ and Q_1: the first becomes a node and continues its running towards the origin, while the second becomes a saddle and starts to approach the other stable point Q_2, until they merge and disappear, in a *saddle-node bifurcation* (after Q_2 having become a node).

Finally only the stable node $P^\star$ remains, as in Fig. 3.11.

The following picture summarizes the results, in a qualitative view:

Fig. 3.12


border collision bifurcation
Saddle-node
bifurcation
Transcritical
bifurcation
Pitchfork subcritical
border collision bifurcation
Saddle-node
bifurcation
Saddle-node
bifurcation
border collision bifurcation

3.1.1.2. The second dynamic scenario: a supercritical Neimark-Sacker bifurcation.

In the following, we continue to move the parameter Γ, being the others fixed, but now we set the imperfect financial market parameter λ at a lower value; exactly, $\lambda = 0.05$.

In such a case, in order to investigate the nature of the steady state of autarky $P^{\star}$, these are the divisions of the interval $\Gamma > 0$:

− for $\Gamma \in \left(\frac{3\sqrt[3]{7220}}{40}, \infty\right)$ the steady point $P^{\star}$ is a stable node of the region $h_1 a_1$;

− for $\Gamma \in \left(\frac{\sqrt[3]{12}}{2}, \frac{3\sqrt[3]{7220}}{40}\right)$ the steady point $P^{\star}$ is a saddle point of the region $h_0 a_0$;

− for $\Gamma \in \left(0, \frac{\sqrt[3]{12}}{2}\right)$ the steady point $P^{\star}$ is a stable node of the region $h_0 a_0$.

Figures from 3.13 to 3.17 explain the dynamical phenomenon.

At the beginning, we set $\Gamma = 1.16$. The system has three fixed points, exactly the node Q_1, the saddle $P^{\star}$ and the focus Q_2 (Fig. 3.13).

We are interested in the trajectories starting close to Q_2 that, for $\Gamma = 1.16$, is an attractive focus.

Increasing Γ, the eigenvalues of Q_2 become in modulus more and more close to 1. Consequently, the attractive strength of the point is decreasing, until the bifurcation value, approximately $\Gamma = 1.163426938722213$, is reached.

At the bifurcation value, the modulus of the eigenvalues becomes equal to 1 and a *Neimark-Sacker bifurcation* (NS) occurs.

Indeed, after a slight increasing of Γ, the focus Q_2 becomes repulsive and an attractive closed curve appears around it (Fig. 3.15, 3.16).

The term $\rho_2(\lambda)$ in figures 3.14, 3.15, 3.16 indicates the modulus of the complex eigenvalues of Q_2. Figure 3.14 photographs a dynamics quite near the bifurcation point: the focus Q_2 is still attractive and there are no closed curves around it. In figures 3.15 and 3.16, indeed, Q_2 has become unstable and a closed attractive curve appears around it. The Neimark-Sacker bifurcation has been of *supercritical type*.

A 6-cycle. Increasing Γ until 1.19, a cycle of period 6 appears around the repulsive focus Q_2 (Fig. 3.17 *a* and *b*; periodic points move in an clockwise motion).

From an economic point of view, such a phenomenon is quite interesting.

Suppose that, before the financial integration, the two countries were near their steady state of autarky equilibrium. If they were over the bisector, after the integration, they would go towards the steady point Q_1, but if they were under the bisector, they would be captured by the Sud-East cycle and, during the time, they would know endogenous fluctuations around the border line $y_{ht} = \overline{Y}$.

Particularly, the h-Country would continuously cross from a situation of borrowing constrained to the one of no borrowing constrained, while the a-Country always lies on low levels of y_{at}.

Fig. 3.13 a

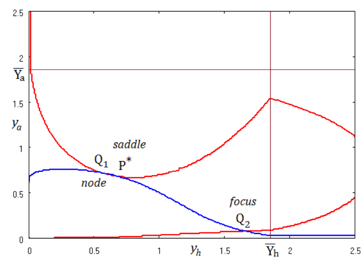
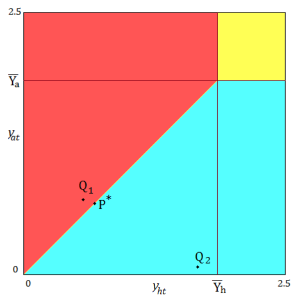
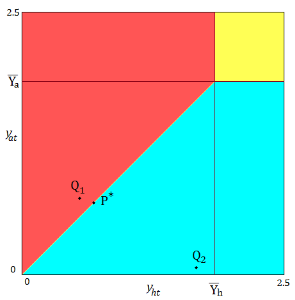


Fig. 3.13 b



$\Gamma_h = \Gamma_a = 1.16$; $\lambda_h = \lambda_a = 0.05$

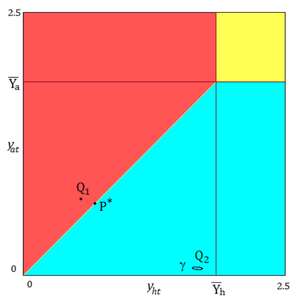
Fig. 3.14



$\Gamma_h = \Gamma_a = 1.163425$; $\lambda_h = \lambda_a = 0.05$

$\rho_2(\lambda) = 0.99999366124131$

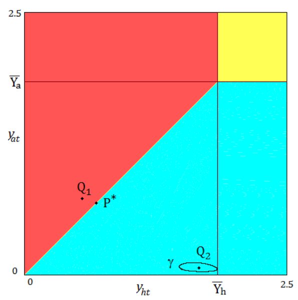
Fig. 3.15



$\Gamma_h = \Gamma_a = 1.16343$; $\lambda_h = \lambda_a = 0.05$

$\rho_2(\lambda) = 1.000010009026156$

Fig. 3.16



$\Gamma_h = \Gamma_a = 1.16347$; $\lambda_h = \lambda_a = 0.05$

$\rho_2(\lambda) = 1.000140792532607$

Fig. 3.17 a

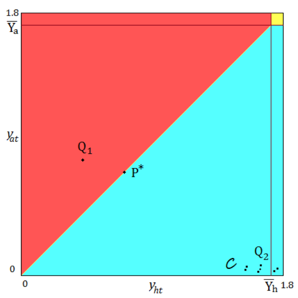
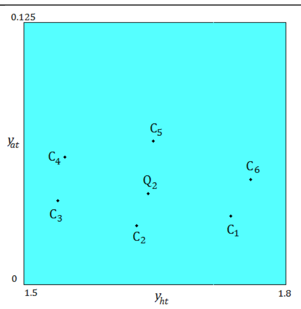


Fig. 3.17 b



$\Gamma_h = \Gamma_a = 1.19$; $\lambda_h = \lambda_a = 0.05$; $\rho_2(\lambda) = 1.088227518080795$

3.2. Heterogeneity

In this section we will consider the case of complete heterogeneity between the two countries.

Therefore we will assume $L_h \neq L_a$, $\Gamma_h \neq \Gamma_a$, $\lambda_h \neq \lambda_a$.

In this case, many features due to some symmetrical behavior will disappear, but the steady state of autarky $P^\star$ keeps some properties, as we will see in the following proposition.

Proposition 3.3.

The steady state of autarky $P^\star$ may lie into the regions h_0a_0, or h_1a_1; in the latter case, $P^\star$ is a stable node.

The conditions under which $P^\star$ lies in h_0a_0 are:

$$\begin{cases} \frac{\lambda_h}{1-[(1-\alpha)\Gamma_h]^{\frac{1}{1-\alpha}}} = \frac{\lambda_a}{1-[(1-\alpha)\Gamma_a]^{\frac{1}{1-\alpha}}} \\ \Gamma_h < \frac{(1-\lambda_h)^{1-\alpha}}{1-\alpha} \\ \Gamma_a < \frac{(1-\lambda_a)^{1-\alpha}}{1-\alpha} \end{cases} \qquad [\mathbf{3}.\mathbf{1}]$$

The conditions under which $P^\star$ lies in h_1a_1 are:

$$\begin{cases} \Gamma_h \geq \frac{(1-\lambda_h)^{1-\alpha}}{1-\alpha} \\ \Gamma_a \geq \frac{(1-\lambda_a)^{1-\alpha}}{1-\alpha} \end{cases} \qquad [\mathbf{3}.\mathbf{2}]$$

(For a complete proof, see the Appendix to Chapter 3).

3.2.1. Some numerical simulations

In the following steps we will observe some dynamical phenomena obtained via numerical simulations.

In each case it is $L_h = 0.35$ and, consequently, $L_a = 0.65$. As usual, the share of income paid to capital is $\alpha = \frac{1}{3}$.

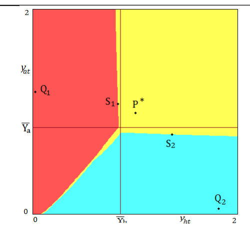
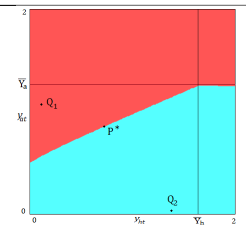
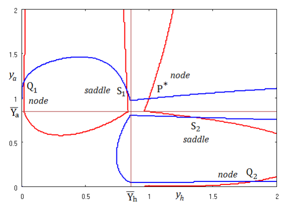
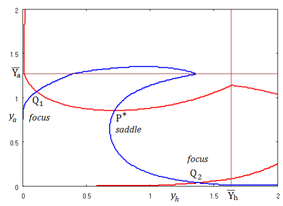
3.2.1.1. The first dynamic scenario.

We begin with the dynamics described in figures 3.18 *a* and *b*.

In the first picture, conditions [3.2] are satisfied and P^* is an asymptotically stable node of the region h_1a_1. Then there are other four stable points; respectively a node and a saddle into the regions h_0a_1 and into h_1a_0. The stable sets of the saddles separate the basins of attraction of the three nodes.

In the second picture, conditions [3.1] are worth and P^* is a saddle point. Its stable set has null measure (not yet as in case of symmetry) and it separates the basins of attraction of two stable steady points, that are attractive foci.

The bisector $y_{at} = y_{ht}$ is no longer a stable set.

Fig. 3.18 a	*Fig. 3.18 b*
	
	
$\Gamma_h = 1.50$; $\Gamma_a = 1.49$; $\lambda_h = 0.05$; $\lambda_a = 0.06$	$\Gamma_h = 1.2085$; $\Gamma_a = 1.35$; $\lambda_h = 0.05$ $\lambda_a = 0.026402233503035$

In the last situation, let's suppose that, before their financial integration the two countries lay in the neighborhood of the autarky point $P^\star$. Then, when the financial integration is complete, they may be attracted significantly away from their initial position (respectively by Q_1 or Q_2) and their conditions may differ a lot from one another. We can see that integration will be convenient for one country, but not at all for the other one.

3.2.1.2. The second dynamic scenario. A "crater" bifurcation.

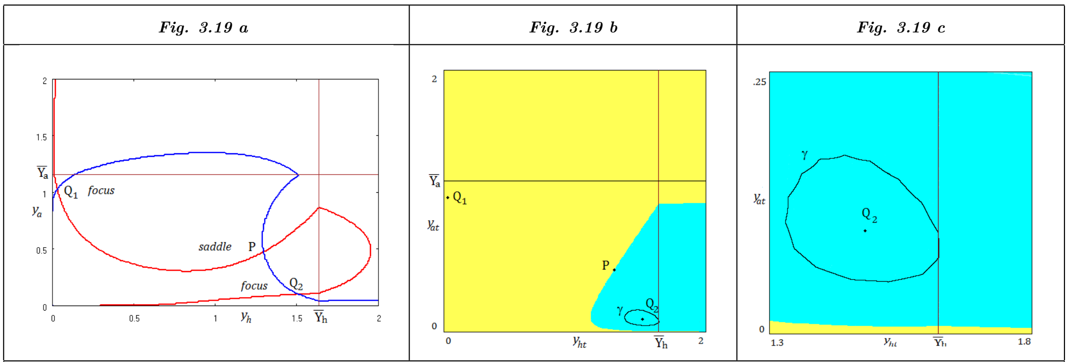
Now we will modify the parameters in the following manner: $\Gamma_h = 1.2085$, $\lambda_h = 0.05$, $\Gamma_a = 1.35$, $\lambda_a = 0.0564$.

The point $P^\star$ is no longer a fixed point for the system.

Moreover, a Neimark-Sacker bifurcation now occurs, as illustrated in the following figures.

As it is known, a NS bifurcation may be of supercritical, or of subcritical type. In the first case a closed, attractive curve starts to form at bifurcation, then it grows up, while the central focus becomes unstable. On the contrary, when the NS is of subcritical type, a repulsive closed curve appears around the focus when it is still stable, it decreases and it merges with the fixed point at bifurcation, leaving it in an unstable equilibrium state. In the latter case, enlarging the analysis from a local to a global point of view, a phenomenon known in literature as "crater bifurcation" may occur: two curves, one attracting, the other repelling, coexist with a stable focus inside. The attracting curve surrounds the repelling one; the latter disappears at bifurcation point, leaving the attracting curve as the unique attractor.[1]

In Fig. 3.19 *b* we see two attractors; they are respectively, the stable focus Q_1 and the closed curve γ, that surrounds the unstable focus Q_2. Then there is the steady state P that is a saddle, whose stable set separates the basins of attraction of Q_1 and γ. The complex eigenvalues of Q_2 have modulus $\rho_2 = 1.000052389992438$, then we can suppose that bifurcation has just occurred. We argue that the NS bifurcation we are considering is of subcritical type. In such case, the closed curve γ doesn't come from the bifurcation point, but preexists at it and remains the unique attractor, after the focus Q_2 has become unstable and the repelling curve has disappeared.



$\Gamma_h = 1.2085$; $\Gamma_a = 1.35$; $\lambda_h = 0.05$; $\lambda_a = 0.0564$

Later on we will try to analyze these periodic phenomena.

In order to do that, we will do a two dimensional bifurcation diagram for $\Gamma_a \in (1.3, 1.5)$ and $\lambda_a \in (0.05, 0.065)$.

See Fig. 3.20. Observing such a figure, we can see that, as generally occurs when a NS bifurcation takes place, different periodicity regions exist: the Arnold's tongues.

[1]For an example of crater bifurcation see Agliari [2].

Their shape is typical of piece-wise maps: each region involves a stable cycle of a certain period and its boundary is given by saddle-node and/or border collision bifurcations, that are responsible of the disappearance of the cycle itself. Generally, if the NS bifurcation is of supercritical type, regions of different periodicity come from the boundary of the stable set of the focus. On the contrary, if the NS bifurcation is of subcritical type, the periodicity regions may open below the bifurcation curve.

The one-dimensional bifurcation diagram of Fig. 3.21 ($\Gamma_a = 1.38047112,$) proves that the NS bifurcation we are analyzing is of subcritical type: immediately after the destabilization of the fixed point, the trajectory is attracted by a large closed invariant curve. Then we can argue that such a curve does not appear at the bifurcation, but it exists even when the fixed point is still stable. From the same figure we can also observe that different cyclical phenomena appear around Q_2, as well as strange attractors and that such attractors cross the border line $y_{ht} = \overline{Y}$.

To confirm our intuition we present in Fig. 3.22 an enlargement of the bifurcation diagram. Such enlargement shows that a stable cycle of period 7 coexists with Q_2 still stable.

Fig. 3.20

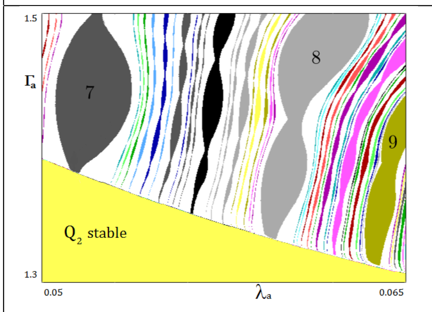


Fig. 3.21

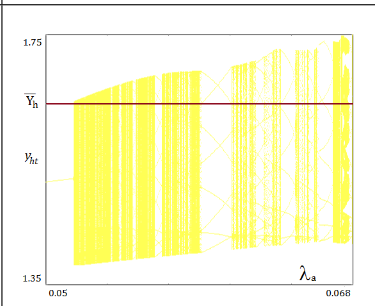
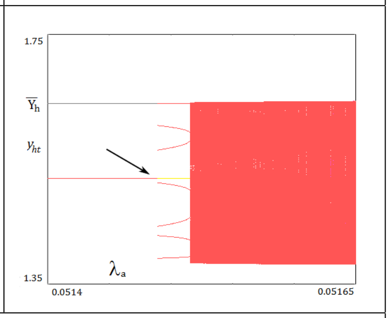


Fig. 3.22



$\Gamma_h = 1.2085$; $\lambda_h = 0.05$

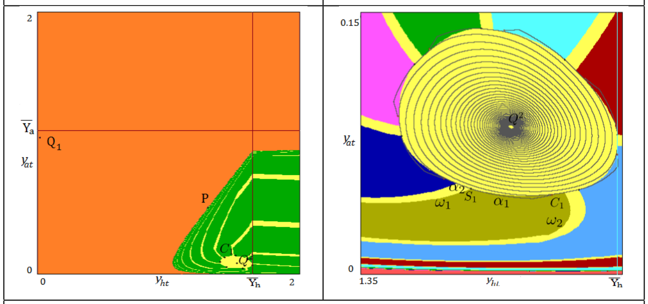
The cycle of period 7. Now we will study more deeply the nature of the 7-cycle appeared in the last figure.

These are the parameter values:

$\Gamma_h = 1.2085$, $\lambda_h = 0.05$, $\Gamma_a = 1.38047112$, $\lambda_a = 0.0514928$.

See Fig. 3.23 a, b.

Fig. 3.23 a | *Fig. 3.23 b*



$\Gamma_h = 1.2085$; $\Gamma_a = 1.38047112$; $\lambda_h = 0.05$; $\lambda_a = 0.0514928$

The node-cycle C coexists with the fixed, attracting points Q_1 and Q_2.

This cycle has appeared in the phase space through a saddle-node bifurcation, together with a saddle cycle S of the same period. The stable set of S separates the basins of attraction of C and Q_2, while the boundary of the basin of attraction of Q_1 has still given by the stable set of the saddle steady state P.

The yellow strips include points that go to the attractive focus Q_2, while the green zone is made by points that tend towards the cycle C.

Look at the Fig. 3.23b and consider the points C_1 and S_1 that belong, respectively, to C and S.

The stable set of S_1 is made by the two branches ω_1 and ω_2. As we previously said, these branches separate the stable set of C_1 from the one of Q_2.

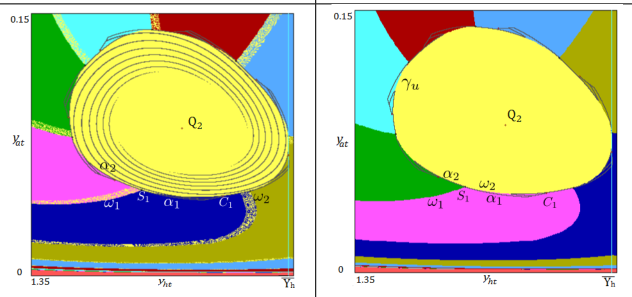
The unstable set of S_1 are the gray lines α_1 and α_2. The branch α_1 connects S_1 with C_1, while α_2 is spiral-shaped and moves towards Q_2.

An homoclinic bifurcation. Let's see now the sequence of Fig. 3.24 obtained by increasing λ_a.

Looking at Fig. 3.24b, we can observe that the branch α_2 of the unstable set of the periodic point S_1 of saddle cycle has modified its behavior. Now it reaches a periodic point of the stable cycle. Then, considering the whole unstable set of the saddle cycle S, we obtain that it connects the periodic points of the stable cycle C, giving birth to an invariant closed curve. The branch ω_2 of the stable set of S_1 as well has modified its behavior, since now it exits from a repelling closed curves surrounding the stable focus Q_2. The change in the dynamic behavior of the stable and unstable sets of the saddle cycle S suggests that a homoclinic bifurcation has occurred, having as consequence the appearance of two invariant closed curves: one attracting, given by the saddle-node connection; one repelling, bounding the basin of attraction of Q_2. The latter, as λ_a increases, is involved in the subcritical NS bifurcation of the focus. Now Q_2 still remains an attractive focus, but it is "trapped" in a very circumscribed area. Points next to the repelling curve, but outside of it, will be attracted by the stable cycle C and will never reach Q_2.

Fig. 3.24a offers an illustration of the homoclinic tangle associated to the occurrence of the bifurcation just described. However, it is important to observe that such a global bifurcation is associated with a piecewise map (the periodic points of the cycles belong to different regions of definition of the map) then we cannot conclude that a chaotic attractor exists during the development of the homoclinic tangle.

Fig. 3.24 a	*Fig. 3.24 b*



λ_a = 0.05149299918	λ_a = 0.051493

Remark. Let's consider the last phenomena described in Fig. 3.23 and 3.24.

Imaging that, before their financial markets integration, the two countries lie in a point P_0 quite close to the saddle P.

If the starting point P_0 lies in the space *over* the stable set of P, then the destiny of the countries, after their financial integration, is quite predictable: they will move towards Q_1.

More interesting is the situation in which P_0 lies *under* the stable set of P.

In case of Fig. 3.23, the destiny of the two countries is hardly predictable.

Indeed they can start along the "yellow" strips and, in such case, after their financial markets have become integrated, they will go towards the attractive point Q_2.

But they can also start along the "green" region and then being attracted by the 7-cycle C.

In this last case, through time, they will know endogenous fluctuations around Q_2, but they will never reach it.

On the contrary, if the parameter values are those of Fig. 3.34 b, the two countries, starting from P_0 located under the stable set of P, will be attracted by the cycle and involved in endogenous fluctuation, without reaching Q_2.

3.3. The case of “quasi-heterogeneity”.

Finally, we consider a last case in which parameters describing technology, efficiency and credit market imperfections are different, but population size is equal.

Therefore we set $L_h = L_a = \frac{1}{2}$.

This is an intermediate situation between the case of “quasi-symmetry” and the case of heterogeneity.

We prove that, even in this case, there are periodic equilibria and endogenous fluctuations.

3.3.1. A “crater” bifurcation.

We set the parameter values as it follows.

$L_h = L_a = \frac{1}{2}$; $\Gamma_h = 1.31$; $\lambda_h = 0.05$; $\lambda_a = 0.0783$.

Modifying the parameter Γ_a, we can obtain the sequence described in Fig. 3.25, 3.26, 3.27.

Fig. 3.25 a

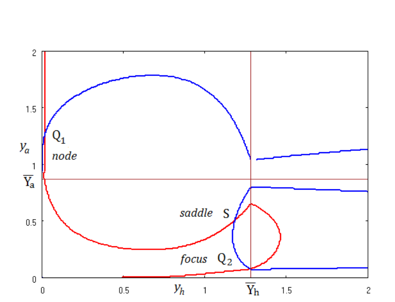
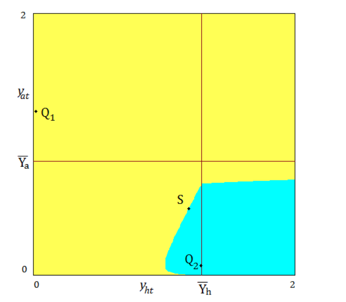
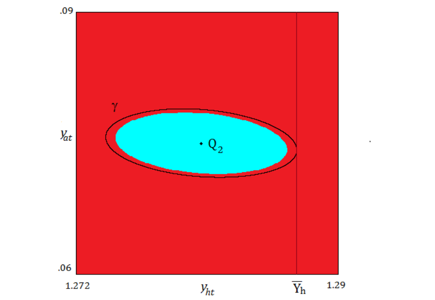


Fig. 3.25 b



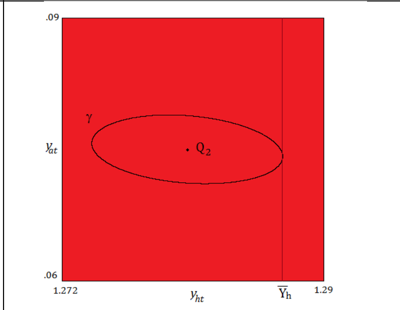
$\Gamma_a = 1.4485$ $\rho_2(\lambda) = 0.99995798501051$

Fig. 3.26



$\Gamma_a = 1.4485248202$; $\rho_2(\lambda) = 0.99998104308917$

Fig. 3.27



$\Gamma_a = 1.44856$; $\rho_2(\lambda) = 1.000013725884191$

In the first picture there are three fixed points. Two of them are attractive, while the third is a saddle and its stable set separates the basins of attraction of the other two.

The modulus of the complex eigenvalues of the focus Q_2 is $\rho_2(\lambda) = 0.99995798501051$; it means that the point is quite near its bifurcation step.

We increase the parameter Γ_a from the initial value 1.4485 to 1.4485248202, as in figure 3.26. Now it is $\rho_2(\lambda) = 0.99998104308917$, therefore the focus Q_2 is still attractive, but nearer its bifurcation.

Now a closed invariant curve (γ) appears around Q_2. The curve γ is stable and inside it another repelling curve separates the basin of attraction of Q_2 from the one of γ itself.

Increasing again Γ_a we obtain the picture of figure 3.27, in which Q_2 has become unstable ($\rho_2(\lambda) = 1.000013725884191$), the repelling curve has disappeared and the only attractor is the curve γ.

A subcritical Neimark-Sacker bifurcation has occurred and, in the meantime, a “crater” bifurcation also occurred.

This is a dynamics quite similar to the one described above, in the case of complete heterogeneity.

To sum up, the case of “quasi heterogeneity” confirms the phenomenon pointed out by the case of complete heterogeneity: periodic and quasi-periodic equilibria may exist not only due to a different size of populations, but also under the effects of differences in technology and productive efficiency.

3.3.2. Conclusion

In the previous simulations, we have proved the existence of persistent, endogenous fluctuations, after the integration of financial markets between the two countries has occurred.

In case of “quasi-symmetry”, differentiation between population dimensions is sufficient to produce phenomena of endogenous fluctuations.

Differently, heterogeneity in efficiency and technological levels (synthesized by Γ_j) together with financial imperfection (λ_j) may also cause periodic and quasi-periodic phenomena.

This fact is definitively proved by the case of “quasi-heterogeneity”, in which the populations are supposed to have the same consistency, while the other parameters are different.

It is to be noted that, in general terms, the absence of endogenous fluctuations makes a model quite predictable about its long-run evolution. On the contrary when periodic dynamics occur, the future conditions of the countries involved in the process of financial integration are hardly predictable and this, of course, has some consequences about convenience, for a single country, to participate to the reform process.

This work belongs to the branch of study developed by Matsuyama and other researchers, on macroeconomic effects produced by credit market imperfections, in integrated financial economies. It extends the results of Kikuchi and Stachurski, enlarging the set of causes that may originate endogenous periodic dynamics into the economic environment of the integrated countries: not only different populations, but also heterogeneity in technology and credit market imperfection.

4. Appendix to Chapter I

4.1. Proofs of propositions of Chapter 1

4.1.1. Proposition 1.1. Proof.

Relation [1.3] **and** [1.4]**.**

Being $\gamma = 1$, young agents have to solve the following problem:

$$\begin{cases} max \quad U\left(c_{1t}, c_{2t+1}\right) = ln\left(c_{1t}\right) + \beta ln\left(c_{2t+1}\right) \\ s.t. \quad c_{1t} + \frac{c_{2t+1}}{r_{t+1}} = w_t \end{cases}$$

Setting $L = ln\left(c_{1t}\right) + \beta ln\left(c_{2t+1}\right) + \lambda\left[w_t - \left(c_{1t} + \frac{c_{2t+1}}{r_{t+1}}\right)\right]$ we obtain the optimal consumption choices of the agents, giving:

$$\begin{cases} \frac{\partial L}{\partial c_{1t}} = 0 \\ \frac{\partial L}{\partial c_{2t+1}} = 0 \\ \frac{\partial L}{\partial \lambda} = 0 \end{cases} \begin{cases} c_{1t}^{-1} = \lambda \\ \beta c_{2t+1}^{-1} = \frac{\lambda}{r_{t+1}} \\ c_{1t} + \frac{c_{2t+1}}{r_{t+1}} = w_t \end{cases} \begin{cases} c_{1t} = \frac{1}{\lambda} \\ c_{2t+1} = \frac{\beta r_{t+1}}{\lambda} \\ \frac{1}{\lambda} + \frac{\beta}{\lambda} = w_t \end{cases} \Rightarrow \begin{cases} \mathbf{\hat{c}_{1t}} = \frac{\mathbf{w_t}}{\mathbf{1+\beta}} \\ \mathbf{\hat{c}_{2t+1}} = \frac{\beta}{\mathbf{1+\beta}}\mathbf{w_t r_{t+1}} \\ \lambda = \frac{1+\beta}{w_t} \end{cases}$$

Consequently the optimal savings become $\hat{s}_t = w_t - \hat{c}_{1t} = w_t - \frac{w_t}{1+\beta} = \left(\frac{\beta}{1+\beta}\right) w_t$ and the marginal propensity of savings is $\mathbf{S}\left(\beta\right) = \frac{\beta}{\mathbf{1+\beta}}$.

In particular, it must be noted that neither $\hat{c}_{1t}$, nor $\hat{s}_t$ depend on the interest factor r_{t+1}, while it is $\hat{c}_{1t} = \hat{c}_{1t}\left(\overset{+}{w_t}\right)$ and $\hat{s}_t \equiv \hat{s}_t\left(\overset{+}{w_t}\right)$.[1]

From $\beta > 0$, it follows $0 < S\left(\beta\right) < 1$; $S\left(.\right)$ is an injective function, with domain R^+ and range $(0; 1)$.

Relation [1.5] **and** [1.6]**.**

Now we consider $0 < \gamma < 1$.

The optimization process becomes:

[1]From now on, we will write $f\left(\overset{+/-}{x}\right)$ to mean that $f\left(.\right)$ is positively/negatively correlated to x.

$$\begin{cases} max \quad U\left(c_{1t}, c_{2t+1}\right) = \frac{c_{1t}^{1-\gamma}-1}{1-\gamma} + \beta\frac{c_{2t+1}^{1-\gamma}-1}{1-\gamma} \\ s.t. \quad c_{1t} + \frac{c_{2t+1}}{r_{t+1}} = w_t \end{cases}$$

The lagrangian expression becomes $L = \frac{c_{1t}^{1-\gamma}-1}{1-\gamma} + \beta\frac{c_{2t+1}^{1-\gamma}-1}{1-\gamma} + \lambda\left[w_t - \left(c_{1t} + \frac{c_{2t+1}}{r_{t+1}}\right)\right]$ and the first order conditions are:

$$\begin{cases} \frac{\partial L}{\partial c_{1t}} = 0 \\ \frac{\partial L}{\partial c_{2t+1}} = 0 \\ \frac{\partial L}{\partial \lambda} = 0 \end{cases} \quad \begin{cases} c_{1t}^{-\gamma} = \lambda \\ \beta c_{2t+1}^{-\gamma} = \frac{\lambda}{r_{t+1}} \\ c_{1t} + \frac{c_{2t+1}}{r_{t+1}} = w_t \end{cases} \quad \begin{cases} c_{1t}^{-\gamma} = \lambda \\ \beta c_{2t+1}^{-\gamma} = \frac{c_{1t}^{-\gamma}}{r_{t+1}} \\ c_{1t} + \frac{c_{2t+1}}{r_{t+1}} = w_t \end{cases}$$

$$\begin{cases} c_{1t}^{-\gamma} = \lambda \\ \frac{c_{2t+1}}{c_{1t}} = \left(\beta r_{t+1}\right)^{\frac{1}{\gamma}} \\ c_{1t} + \frac{c_{2t+1}}{r_{t+1}} = w_t \end{cases} \quad \begin{cases} c_{2t+1} = \left(\beta r_{t+1}\right)^{\frac{1}{\gamma}} c_{1t} \\ c_{1t} = \frac{w_t}{1+\frac{\left(\beta r_{t+1}\right)^{\frac{1}{\gamma}}}{r_{t+1}}} \end{cases} \Rightarrow \begin{cases} \mathbf{\hat{c}_{1t}} = \frac{\mathbf{w_t}}{\mathbf{1}+\frac{\left(\beta \mathbf{r_{t+1}}\right)^{\frac{1}{\gamma}}}{\mathbf{r_{t+1}}}} \\ \mathbf{\hat{c}_{2t+1}} = \frac{\mathbf{w_t}\left(\beta \mathbf{r_{t+1}}\right)^{\frac{1}{\gamma}}}{\mathbf{1}+\frac{\left(\beta \mathbf{r_{t+1}}\right)^{\frac{1}{\gamma}}}{\mathbf{r_{t+1}}}} \end{cases}$$

$$\hat{s}_t = w_t - \hat{c}_{1t} = w_t - \frac{w_t}{1+\frac{\left(\beta r_{t+1}\right)^{\frac{1}{\gamma}}}{r_{t+1}}} = w_t\left(\frac{\left(\beta r_{t+1}\right)^{\frac{1}{\gamma}}}{r_{t+1}+\left(\beta r_{t+1}\right)^{\frac{1}{\gamma}}}\right)$$

From that, we obtain the marginal propensity of savings $\mathbf{S}\left(\mathbf{r_{t+1}}\right) = \frac{\left(\beta \mathbf{r_{t+1}}\right)^{\frac{1}{\gamma}}}{\mathbf{r_{t+1}}+\left(\beta \mathbf{r_{t+1}}\right)^{\frac{1}{\gamma}}}$.

Particularly, let's note:

i) The first period optimal consumption is positively correlated to the wage w_t and negatively correlated to the second period lending/borrowing factor r_{t+1}:

$\hat{c}_{1t} = \hat{c}_{1t}\left(\overset{+}{w}_t, \overset{-}{r}_{t+1}\right)$.

Indeed $\frac{\partial \hat{c}_{1t}}{\partial w_t} = \frac{r_{t+1}}{r_{t+1}+\left(\beta r_{t+1}\right)^{\frac{1}{\gamma}}} > 0$, while $\frac{\partial \hat{c}_{1t}}{\partial r_{t+1}} = \frac{-w_t\left(\frac{1}{\gamma}-1\right)\left(\beta r_{t+1}\right)^{\frac{1}{\gamma}}}{\left(r_{t+1}+\left(\beta r_{t+1}\right)^{\frac{1}{\gamma}}\right)^2} < 0$.

ii) The optimal savings are positively correlated with the interest factor r_{t+1} as can be derived observing that $\hat{s}_t = w_t - \hat{c}_{1t} = w_t - \hat{c}_{1t}\left(\overset{+}{w}_t, \overset{-}{r}_{t+1}\right)$.

Then $\hat{s}_t = \hat{s}_t\left(\overset{+}{r}_{t+1}\right)$.

For successive uses consider the following assertion:

4.1.2. Proposition 1.A1

For $0 < \gamma < 1$, *the marginal propensity of savings* $S\left(r_{t+1}\right) = \frac{\left(\beta r_{t+1}\right)^{\frac{1}{\gamma}}}{r_{t+1}+\left(\beta r_{t+1}\right)^{\frac{1}{\gamma}}}$ *is an increasing function of* $r_{t+1} > 0$, *with range* $(0, 1)$.

Proof.

The thesis comes quite immediately observing that, for $x > 0$ and $0 < \gamma < 1$, it is:

$\lim_{x \to 0^+} \frac{(\beta x)^{\frac{1}{\gamma}}}{x+(\beta x)^{\frac{1}{\gamma}}} = 0$; $\lim_{x \to +\infty} \frac{(\beta x)^{\frac{1}{\gamma}}}{x+(\beta x)^{\frac{1}{\gamma}}} = 1$; $\frac{\partial}{\partial x}\left[\frac{(\beta x)^{\frac{1}{\gamma}}}{x+(\beta x)^{\frac{1}{\gamma}}}\right] = \frac{\left(\frac{1}{\gamma}-1\right)(\beta x)^{\frac{1}{\gamma}}}{\left(x+(\beta x)^{\frac{1}{\gamma}}\right)^2} > 0$

4.1.3. Proposition 1.2. Proof.

We know that $W(k_t)$ in an increasing function of k_t (see note on assumption i_7).

Let's set $\varphi(k) = \frac{k}{S\left(Rf'(k)\right)}$.

$\varphi(.)$ is a monotonic, increasing function of $k \in (0, +\infty)$. Indeed:

$\frac{\partial \varphi(k)}{\partial k} = \frac{S\left(Rf'(k)\right) - kS'\left(Rf'(k)\right)Rf''(k)}{\left[S\left(Rf'(k)\right)\right]^2} > 0$

being $Rf'(k) > 0$, $S\left(Rf'(k)\right) > 0$, $S'\left(Rf'(k)\right) \geq 0$, $f''(k) < 0$ for $k \in (0, +\infty)$ (see proposition 1.A1 and assumptions i_2, i_3)

Therefore there will be a unique k_{t+1} s.t. $\varphi(k_{t+1}) = RW(k_t)$.

Being $W(k_t)$ increasing in k_t, if k_t or R increase, also $\varphi(k_{t+1})$ will rise and, being $\varphi(.)$ increasing, it follows that k_{t+1} is increasing too.

4.1.4. Proposition 1.3. Proof.

The second part of the proposition, can be extrapolated from the corollary 1 of Wendner [16].

Hence we will proof that $(0, R)$ in an invariant interval for k_{t+1}.

Step 1.

First we will prove that $\psi(0, R) = 0 \quad \forall R > 0$, where $\psi(0, R) = \lim_{k_t \to 0^+} \psi(k_t, R)$.

In order to prove that, let's consider the increasing function $\varphi(k) = \frac{k}{S\left(Rf'(k)\right)}$

(see demonstration of proposition 1.2).

From assumption i_4 and remembering that $0 < S(x) < 1$ is an increasing function, it comes that: $\varphi(0) = \lim_{k_t \to 0^+} \frac{k_t}{S\left(Rf'(k_t)\right)} = 0$

Moreover, from [1.7]:

$\varphi(\psi(0, R)) = \varphi\left(\lim_{k_t \to 0^+} \psi(k_t, R)\right) = \lim_{k_t \to 0^+} \varphi(\psi(k_t, R)) =$

$= \lim\limits_{k_t \to 0^+} RW(k_t) = R \lim\limits_{k_t \to 0^+} W(k_t)$

Remembering assumptions i_1, i_6 and i_7:

$\lim\limits_{k_t \to 0^+} W(k_t) = \lim\limits_{k_t \to 0^+} \left[f(k_t) - k_t f'(k_t) \right] = \lim\limits_{k_t \to 0^+} f(k_t) \left[1 - \frac{k_t f'(k_t)}{f(k_t)} \right] = 0 \; .$

Then it is proved that $\varphi(\psi(0, R)) = 0$ and, being $\varphi(0) = 0$, because of monotonicity of $\varphi(.)$, it must be $\psi(0, R) = 0$.

Step 2.

Now we'll try to identify a superior extreme value for R, say $R^{(+)}$, such that,

if $R < R^{(+)}$, then $k_{t+1} < R$.

In order to do this, let's consider the function $\Omega(x) = W(x) S\left(x f'(x)\right)$, that is continuous in R^+, because of the product of two continuous functions.

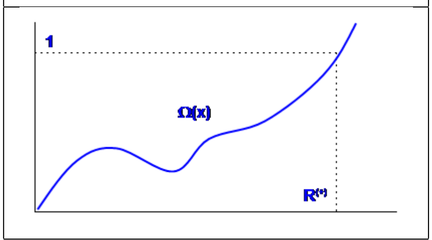
It can be immediately proved that $\Omega(0) = 0$. Effectively $W(0) = 0$ and $0 < S(x) < 1$.

Let $R^{(+)}$ be the smallest number for which equation $\Omega(x) = 1$ is satisfied (if this equation doesn't have solutions, then $R^{(+)} = +\infty$).

Because of continuity of $\Omega(.)$ and being proved that $\Omega(0) = 0$, $\forall R \in \left(0; R^{(+)}\right)$ it follows:

$$W(R) S\left(R f'(R)\right) < 1 \qquad [\mathbf{1}.\mathbf{A1}]$$

Fig. 1.A1



Now, let's assume $k_t \in (0; R)$ and $R \in \left(0; R^{(+)}\right)$.

Observing that $\varphi(\psi(R, R)) = RW(R)$ and remembering that $\varphi(.)$ is a monotonic increasing function of $k_{t+1} = \psi\left(\overset{+}{k_t}, \overset{+}{R}\right)$, it is:

$\varphi(k_{t+1}) = \varphi\left(\psi\left(\overset{+}{k_t}, \overset{+}{R}\right)\right) < \varphi(\psi(R, R)) = RW(R).$

From relation $W(R)\,S\left[Rf'(R)\right] < 1$, it follows that $RW(R) < \frac{R}{S\left[Rf'(R)\right]}$,

so $\varphi(k_{t+1}) = RW(k_t) < RW(R) < \frac{R}{S\left[Rf'(R)\right]} = \varphi(R)$.

Finally, $\varphi(k_{t+1}) < \varphi(R)$. Then it follows that $k_{t+1} < R$. *q.e.d.*

4.1.5. Proposition 1.4. Proof.

Let k_t^h and k_t^a be the concentrations of capital respectively in h and a.

Because of the opening of the markets, the return of capital must be the same: $f'\left(k_t^h\right) = f'(k_t^a)$. This involves $k_t^h = k_t^a = k_t$

From [1.1] it follows that $r_t^h = R_h f'(k_t)$ and $r_t^a = R_a f'(k_t)$, then, from the hypothesis $R_h > R_a$, it must be $r_t^h > r_t^a$.

The interest rates of the two countries are not the same, but the closure of their financial markets preserves the differential.

Finally, because of different interest rates, never savings are the same, except in the case $\gamma = 1$. Indeed $s_t^h = W(k_t)\,S\left(R_h f'(k_{t+1})\right) \geq s_t^a = W(k_t)\,S\left(R_a f'(k_{t+1})\right)$.[2]

4.1.6. Proposition 1.5. Proof.

From relation [1.2] and proposition 1.4, it follows that, in each country, capital will be produced in different quantities: $R_h s_t^h > R_a s_t^a$; but free trade will level them:

$k_{t+1}^h = k_{t+1}^a = k_{t+1}$.

Therefore, substituting the optimal values into the resource constraint, we obtain:

$2k_{t+1} = R_h s_t^h + R_f s_t^a = W(k_t)\left[R_h S\left(R_h f'(k_{t+1})\right) + R_a S\left(R_a f'(k_{t+1})\right)\right]$ *q.e.d.*

4.1.7. Proposition 1.6. Proof.

Remembering that $S(.)$ is a non decreasing function and $R_a < R_h$, it follows:

$R_h S\left(R_h f'(k_{t+1})\right) + R_a S\left(R_a f'(k_{t+1})\right) < 2R_h S\left(R_h f'(k_{t+1})\right)$.

From [1.9] we have $k_{t+1} < W(k_t)\,R_h S\left(R_h f'(k_{t+1})\right)$.

Therefore $\varphi(k_{t+1}) = \frac{k_{t+1}}{R_h S\left(R_h f'(k_{t+1})\right)} < W(k_t)$.

Now suppose it is $k_t \in (0; R_h)$ and $R_h \in \left(0; R^{(+)}\right)$.

[2] This relation occurs with sign ">" when $0 < \gamma < 1$ and with "=" when $\gamma = 1$; in the latter case it is $\mathrm{S}(r_{t+1}) = \frac{\beta}{1+\beta}$.

From [1.A1] we know that $W(R_h)\, S\left(R_h f'(R_h)\right) < 1$.

Remembering that $W(.)$ is an increasing function (see note to assumption i_7) we have:

$\varphi(k_{t+1}) = \frac{k_{t+1}}{R_h S\left[R_h f'(k_{t+1})\right]} < W(k_t) < W(R_h) < \frac{R_h}{R_h S\left[R_h f'(R_h)\right]} = \varphi(R_h)$.

Being $\varphi(.)$ is an increasing function too (see proposition 1.1) it follows that

$k_{t+1} < R_h$.

Asymptotic stability comes from proposition 1.3.

4.1.8. Proposition 1.9. Proof.

From $s_t \leq \frac{1}{2}$, it follows $k_{t+1} = R_h s_t \leq \frac{R_h}{2}$.

We know that the left hand side of [1.10] is an increasing function in k_{t+1} (see proof of proposition 1.2). The right hand side is an increasing function of k_t (see note to i_7).

Then $k_A = W^{-1}\left(\frac{1}{2S\left(R_h f'\left(\frac{R_h}{2}\right)\right)}\right)$ is the maximum value that k_t can reach.

Finally note that $\psi(k_A, R_h) = \frac{R_h}{2}$.

4.1.9. Proposition 1.10. Proof.

From proposition 1.8, we can write $r_{t+1} = R_h f'(\psi(k_t, R_h))$.

It follows that $r_{t+1}(k_t)$ is a decreasing, continuous function of k_t, into the interval $(0, k_A]$, because $\psi(k_t)$ is a continuous, increasing function of k_t and $f'(\psi)$ is a continuous, decreasing function of ψ. Consequently, the minimum value $r_{t+1}(k_t)$ can reach on $(0, k_A]$ is $r_{t+1}(k_A) = R_h f'\left(\frac{R_h}{2}\right)$.

4.1.10. Proposition 1.11. Proof.

In home Country: $U\left(c_1^h ; c_2^h\right) = \frac{\left(W(k_t)-s_t^h\right)^{1-\gamma}-1}{1-\gamma} + \beta \frac{\left[R_h f'(k_{t+1}) - \left(1-s_t^h\right) R_a f'(k_{t+1})\right]^{1-\gamma}-1}{1-\gamma}$

F.O.C. : $\frac{dU}{ds_t^h} = 0 \Rightarrow$

$$-\left(W(k_t) - s_t^h\right)^{-\gamma} + \beta\left[R_h f'(k_{t+1}) - \left(1 - s_t^h\right) R_a f'(k_{t+1})\right]^{-\gamma} R_a f'(k_{t+1}) = 0$$

$$\frac{\beta R_a f'(k_{t+1})}{\left[R_h f'(k_{t+1})-\left(1-s_t^h\right)R_a f'(k_{t+1})\right]^{\gamma}} = \frac{1}{\left(W(k_t)-s_t^h\right)^{\gamma}}$$

$$\frac{\left(\beta R_a f'(k_{t+1})\right)^{\frac{1}{\gamma}}}{R_h f'(k_{t+1})-\left(1-s_t^h\right)R_a f'(k_{t+1})} = \frac{1}{W(k_t)-s_t^h}$$

$$\left(W\left(k_t\right)-s_t^h\right)\left(\beta R_a f'\left(k_{t+1}\right)\right)^{\frac{1}{\gamma}} = R_h f'\left(k_{t+1}\right)-\left(1-s_t^h\right)R_a f'\left(k_{t+1}\right)$$

$$s_t^h\left\{R_a f'\left(k_{t+1}\right)+\beta^{\frac{1}{\gamma}}\left[R_a f'\left(k_{t+1}\right)\right]^{\frac{1}{\gamma}}\right\} = W\left(k_t\right)\left[\beta R_a f'\left(k_{t+1}\right)\right]^{\frac{1}{\gamma}}-f'\left(k_{t+1}\right)\left(R_h-R_a\right)$$

$$s_t^h R_a f'\left(k_{t+1}\right)\left\{1+\beta^{\frac{1}{\gamma}}\left[R_a f'\left(k_{t+1}\right)\right]^{\frac{1}{\gamma}-1}\right\} =$$

$$= W\left(k_t\right)\left[\beta R_a f'\left(k_{t+1}\right)\right]^{\frac{1}{\gamma}}-f'\left(k_{t+1}\right)\left(R_h-R_a\right)$$

$$s_t^h\left\{1+\beta^{\frac{1}{\gamma}}\left[R_a f'\left(k_{t+1}\right)\right]^{\frac{1}{\gamma}-1}\right\} = W\left(k_t\right)\beta^{\frac{1}{\gamma}}\left[R_a f'\left(k_{t+1}\right)\right]^{\frac{1}{\gamma}-1}-\left(\frac{R_h-R_a}{R_a}\right) \Rightarrow$$

$$\mathbf{\hat{s}_t^h} = \frac{\mathbf{W(k_t)}\beta^{\frac{1}{\gamma}}\left[\mathbf{R_a f'(k_{t+1})}\right]^{\frac{1}{\gamma}-1}-\left(\frac{\mathbf{R_h-R_a}}{\mathbf{R_a}}\right)}{\mathbf{1}+\beta^{\frac{1}{\gamma}}\left[\mathbf{R_a f'(k_{t+1})}\right]^{\frac{1}{\gamma}-1}}$$

In the foreign Country: $U\left(c_1^a\,;\,c_2^a\right) = \frac{\left(W(k_t)-s_t^a\right)^{1-\gamma}-1}{1-\gamma}+\beta\frac{\left[s_t^a R_a f'(k_{t+1})\right]^{1-\gamma}-1}{1-\gamma}$

F.O.C. : $\frac{dU}{ds_t^a} = 0 \Rightarrow$

$$-\left(W\left(k_t\right)-s_t^a\right)^{-\gamma}+\beta\left(s_t^a R_a f'\left(k_{t+1}\right)\right)^{-\gamma}R_a f'\left(k_{t+1}\right) = 0$$

$$\frac{\beta R_a f'(k_{t+1})}{\left(s_t^a R_a f'(k_{t+1})\right)^{\gamma}} = \frac{1}{\left(W(k_t)-s_t^a\right)^{\gamma}}$$

$$\frac{\left(\beta R_a f'(k_{t+1})\right)^{\frac{1}{\gamma}}}{s_t^a R_a f'(k_{t+1})} = \frac{1}{W(k_t)-s_t^a}$$

$$\beta^{\frac{1}{\gamma}}\left(R_a f'\left(k_{t+1}\right)\right)^{\frac{1}{\gamma}}\left(W\left(k_t\right)-s_t^a\right) = s_t^a R_a f'\left(k_{t+1}\right)$$

$$\mathbf{\hat{s}_t^a} = \frac{\beta^{\frac{1}{\gamma}}\mathbf{W(k_t)}\left(\mathbf{R_a f'(k_{t+1})}\right)^{\frac{1}{\gamma}-1}}{\mathbf{1}+\beta^{\frac{1}{\gamma}}\left(\mathbf{R_a f'(k_{t+1})}\right)^{\frac{1}{\gamma}-1}}$$

Now we can express the optimal choices of savings, in term of marginal propensity. Agents of the a-Country will compare with the interest factor $r_{t+1} = R_a f'(k_{t+1})$. Therefore their optimal propensity of savings will be:

$$S\left(R_a f'(k_{t+1})\right) = \frac{\hat{s}_t^a}{W(k_t)} = \frac{\beta^{\frac{1}{\gamma}}\left(R_a f'(k_{t+1})\right)^{\frac{1}{\gamma}-1}}{1+\beta^{\frac{1}{\gamma}}\left(R_a f'(k_{t+1})\right)^{\frac{1}{\gamma}-1}}$$

For successive developments, notice that $1 - S\left(R_a f'(k_{t+1})\right) = \frac{1}{1+\beta^{\frac{1}{\gamma}}\left(R_a f'(k_{t+1})\right)^{\frac{1}{\gamma}-1}}$

Substituting the last expressions into $\hat{s}_t^h$, we obtain:

$$\hat{s}_t^h = \frac{W(k_t)\beta^{\frac{1}{\gamma}}\left[R_a f'(k_{t+1})\right]^{\frac{1}{\gamma}-1}}{1+\beta^{\frac{1}{\gamma}}\left[R_a f'(k_{t+1})\right]^{\frac{1}{\gamma}-1}} - \frac{\frac{R_h-R_a}{R_a}}{1+\beta^{\frac{1}{\gamma}}\left[R_a f'(k_{t+1})\right]^{\frac{1}{\gamma}-1}} =$$

$$= W(k_t)\, S\left(R_a f'(k_{t+1})\right) - \left[1 - S\left(R_a f'(k_{t+1})\right)\right]\left(\frac{R_h-R_a}{R_a}\right)$$

Finally, we have to substitute the optimal values of savings into the resource constraint and we are done:

$$2k_{t+1} = (1)\, R_h + \left(\hat{s}_t^h + \hat{s}_t^a - 1\right) R_a \Rightarrow$$

Now substituting in the resource constraint:

(for brevity let's set $S \equiv S\left(R_a f'(k_{t+1})\right)$ and $W_t = W(k_t)$)

$$2k_{t+1} = R_h + \left(W_t S - (1-S)\left(\frac{R_h-R_a}{R_a}\right) + W_t S - 1\right) R_a$$

$$2k_{t+1} = 2R_a W_t S + S\left(R_h - R_a\right)$$

$$k_{t+1} - \left(\frac{R_h-R_a}{2}\right) S = R_a W_t S \Rightarrow$$

$$\frac{\mathbf{k_{t+1}} - \left(\frac{\mathbf{R_h-R_a}}{\mathbf{2}}\right)\mathbf{S}\left(\mathbf{R_a f'(k_{t+1})}\right)}{\mathbf{S}\left(\mathbf{R_a f'(k_{t+1})}\right)} = \mathbf{R_a W(k_t)}$$

4.1.11. Proposition 1.12. Proof.

The left hand side of [1.13] is a continuous, increasing function of k_{t+1}. Indeed:

$$\frac{\partial}{\partial k_{t+1}}\left(\frac{k_{t+1} - \left(\frac{R_h-R_a}{2}\right) S\left(R_a f'(k_{t+1})\right)}{S\left(R_a f'(k_{t+1})\right)}\right) =$$

$$\frac{\left(1-\left(\frac{R_h-R_a}{2}\right)S'\left(R_a f'(k_{t+1})\right)R_a f''(k_{t+1})\right)S\left(R_a f'(k_{t+1})\right) - \left(k_{t+1}-\left(\frac{R_h-R_a}{2}\right)S\left(R_a f'(k_{t+1})\right)\right)R_a S'\left(R_a f'(k_{t+1})\right)f''(k_{t+1})}{\left[S\left(R_a f'(k_{t+1})\right)\right]^2} =$$

$= \frac{S\left(R_a f'(k_{t+1})\right) - k_{t+1} R_a S'\left(R_a f'(k_{t+1})\right) f''(k_{t+1})}{\left[S\left(R_a f'(k_{t+1})\right)\right]^2} > 0$ (because of assumption i_3 and the positivity of $S'\left(R_a f'(k_{t+1})\right)$) .

The right hand side is a continuous, increasing function of k_t (see note of assumption i_7).

4.1.12. Proposition 1.13. Proof.

For brevity, let's set $S_a = S\left(R_a f'\left(\frac{R_h}{2}\right)\right)$ and $S_h = S\left(R_h f'\left(\frac{R_h}{2}\right)\right)$.
Now, let's consider that k_B is the first period capital stock for which
$\frac{\frac{R_h}{2} - \left(\frac{R_h - R_a}{2}\right) S\left(R_a f'\left(\frac{R_h}{2}\right)\right)}{S\left(R_a f'\left(\frac{R_h}{2}\right)\right)} = R_a W(k_B)$.
From [1.13], it means that $\phi(k_B, R_h, R_a) = \frac{R_h}{2}$.
Then $W(k_B) = \frac{\frac{R_h}{2} - \left(\frac{R_h - R_a}{2}\right) S_a}{R_a S_a}$.
Remembering the previous proposition 1.8, it is $\psi(k_A, R_h) = \frac{R_h}{2}$.
Thus $W(k_A) = \frac{1}{2S_h}$.
We are ready to prove that $W(k_B) > W(k_A)$.
Indeed $W(k_B) > W(k_A) \Leftrightarrow \frac{\frac{R_h}{2} - \left(\frac{R_h - R_a}{2}\right) S_a}{R_a S_a} > \frac{1}{2S_h} \Leftrightarrow \frac{R_h S_h}{R_a S_a} > \frac{1 - S_h}{1 - S_a}$.
The latter relation is true, because, for hypotheses, $R_h > R_a$ and $S_h > S_a$ and we are done.

4.1.13. Proposition 1.14. Proof.

From the main hypothesis $r_{t+1} = R_a f'(k_{t+1})$ and from proposition 1.12, it follows that $r_{t+1} = R_a f'(\phi(k_t, R_h, R_a))$, where $\phi(k_t)$ is a continuous, increasing function of k_t.
$f'(\phi)$ is a continuous, decreasing function of ϕ (assumption i_3).
It follows that $r_{t+1}(k_t)$ is a continuous, decreasing function of k_t. The maximum value it can reach over the interval $[k_B; R)$ is $r_{t+1}(k_B) = R_a f'\left(\frac{R_h}{2}\right)$.

4.1.14. Proposition 1.16. Proof.

First, let's note that
$c_{1t}^h = W(k_t) - s_t^h$; $c_{2t+1}^h = (1) R_h f'\left(\frac{R_h}{2}\right) - \left(1 - s_t^h\right) r_{t+1}$
$c_{1t}^a = W(k_t) - s_t^a$; $c_{2t+1}^a = s_t^a r_{t+1}$

In home Country: $U\left(c_{1t}^{h}\,,\,c_{2t+1}^{h}\right)=\frac{\left(W(k_t)-s_t^h\right)^{1-\gamma}-1}{1-\gamma}+\beta\frac{\left(R_h f'\left(\frac{R_h}{2}\right)-\left(1-s_t^h\right)r_{t+1}\right)^{1-\gamma}-1}{1-\gamma}$

F.O.C.: $\frac{dU}{ds_t^h}=0\Rightarrow$

$$-\left(W\left(k_t\right)-s_t^h\right)^{-\gamma}+\beta r_{t+1}\left(R_h f'\left(\frac{R_h}{2}\right)-\left(1-s_t^h\right)r_{t+1}\right)^{-\gamma}=0$$

$$\frac{\beta r_{t+1}}{\left(R_h f'\left(\frac{R_h}{2}\right)-\left(1-s_t^h\right)r_{t+1}\right)^{\gamma}}=\frac{1}{\left(W(k_t)-s_t^h\right)^{\gamma}}$$

$$\beta^{\frac{1}{\gamma}}r_{t+1}^{\frac{1}{\gamma}}\left(W\left(k_t\right)-s_t^h\right)=R_h f'\left(\frac{R_h}{2}\right)-\left(1-s_t^h\right)r_{t+1}$$

$$s_t^h\left(\beta^{\frac{1}{\gamma}}r_{t+1}^{\frac{1}{\gamma}}+r_{t+1}\right)=W\left(k_t\right)\beta^{\frac{1}{\gamma}}r_{t+1}^{\frac{1}{\gamma}}+r_{t+1}-R_h f'\left(\frac{R_h}{2}\right)\Rightarrow$$

$$\mathbf{\hat{s}_t^h}=\frac{\mathbf{W(k_t)}\beta^{\frac{1}{\gamma}}\mathbf{r_{t+1}^{\frac{1}{\gamma}}+r_{t+1}-R_h f'\left(\frac{R_h}{2}\right)}}{\beta^{\frac{1}{\gamma}}\mathbf{r_{t+1}^{\frac{1}{\gamma}}+r_{t+1}}}$$

In foreign Country: $U\left(c_{1t}^{a}\,,\,c_{2t+1}^{a}\right)=\frac{\left(W(k_t)-s_t^a\right)^{1-\gamma}-1}{1-\gamma}+\beta\frac{\left(s_t^a r_{t+1}\right)^{1-\gamma}-1}{1-\gamma}$

F.O.C.: $\frac{dU}{ds_t^a}=0\Rightarrow$

$$-\left(W\left(k_t\right)-s_t^a\right)^{-\gamma}+\beta r_{t+1}\left(s_t^a r_{t+1}\right)^{-\gamma}=0$$

$$\frac{\beta r_{t+1}}{\left(s_t^a r_{t+1}\right)^{\gamma}}=\frac{1}{\left(W(k_t)-s_t^a\right)^{\gamma}}\;;\;\beta^{\frac{1}{\gamma}}r_{t+1}^{\frac{1}{\gamma}}\left(W\left(k_t\right)-s_t^a\right)=s_t^a r_{t+1}\;;$$

$$s_t^a\left(\beta^{\frac{1}{\gamma}}r_{t+1}^{\frac{1}{\gamma}}+r_{t+1}\right)=W\left(k_t\right)\beta^{\frac{1}{\gamma}}r_{t+1}^{\frac{1}{\gamma}}\Rightarrow$$

$$\mathbf{\hat{s}_t^a}=\frac{\mathbf{W(k_t)}\beta^{\frac{1}{\gamma}}\mathbf{r_{t+1}^{\frac{1}{\gamma}}}}{\beta^{\frac{1}{\gamma}}\mathbf{r_{t+1}^{\frac{1}{\gamma}}+r_{t+1}}}$$

The interest factor. Applying the condition $\hat{s}_t^h+\hat{s}_t^a=1$, we obtain:

$$\frac{W(k_t)\beta^{\frac{1}{\gamma}}r_{t+1}^{\frac{1}{\gamma}}+r_{t+1}-R_h f'\left(\frac{R_h}{2}\right)}{\beta^{\frac{1}{\gamma}}r_{t+1}^{\frac{1}{\gamma}}+r_{t+1}}+\frac{W(k_t)\beta^{\frac{1}{\gamma}}r_{t+1}^{\frac{1}{\gamma}}}{\beta^{\frac{1}{\gamma}}r_{t+1}^{\frac{1}{\gamma}}+r_{t+1}}=1\;;$$

$$2W\left(k_t\right)\beta^{\frac{1}{\gamma}}r_{t+1}^{\frac{1}{\gamma}}-\beta^{\frac{1}{\gamma}}r_{t+1}^{\frac{1}{\gamma}}=R_h f'\left(\frac{R_h}{2}\right)\Rightarrow$$

$$\mathbf{r_{t+1}\left(k_t\right)}=\left[\frac{\mathbf{R_h f'\left(\frac{R_h}{2}\right)}}{\beta^{\frac{1}{\gamma}}\mathbf{(2W(k_t)-1)}}\right]^{\gamma}\qquad q.e.d.$$

4.1.15. Proposition 1.17. Proof.

The interest factor r_{t+1} over the interval (k_A, k_B).

First we consider the expression $r_{t+1}(k_t) = \left[\frac{R_h f'\left(\frac{R_h}{2}\right)}{\beta^{\frac{1}{\gamma}}(2W(k_t)-1)}\right]^{\gamma}$ over the interval (k_A, k_B).

Because $s_t^h + s_t^a = 1$ and considering that savings are proper fraction of incomes, it must be $W(k_t) > \frac{1}{2}$.

Thus $r_{t+1}(k_t)$ is continuous.

Its first derivative is:

$\frac{dr_{t+1}(k_t)}{dk_t} = -\frac{2\gamma W'(k_t)\left(R_h f'\left(\frac{R_h}{2}\right)\right)^{\gamma}}{\beta(2W(k_t)-1)^{\gamma+1}} < 0.$

Then $r_{t+1}(k_t)$ is decreasing.

Now consider what it follows:

$W(k_A) = \frac{1}{2S\left(R_h f'\left(\frac{R_h}{2}\right)\right)}$ and $S(r_{t+1}) = \frac{\beta^{\frac{1}{\gamma}} r_{t+1}^{\frac{1}{\gamma}-1}}{1+\beta^{\frac{1}{\gamma}} r_{t+1}^{\frac{1}{\gamma}-1}}$ from which $\frac{S(r_{t+1})}{1-S(r_{t+1})} = \beta^{\frac{1}{\gamma}} r_{t+1}^{\frac{1}{\gamma}-1}$

$W(k_B) = \frac{\frac{R_h}{2} - \left(\frac{R_h - R_a}{2}\right) S\left(R_a f'\left(\frac{R_h}{2}\right)\right)}{R_a S\left(R_a f'\left(\frac{R_h}{2}\right)\right)}$

By these relations we can extend the expression above to the limit point k_A and k_B:

$l_A = \lim_{k_t \to k_A^+} r_{t+1}(k_t) = \left[\frac{R_h f'\left(\frac{R_h}{2}\right) S\left(R_h f'\left(\frac{R_h}{2}\right)\right)}{\beta^{\frac{1}{\gamma}}\left(1 - S\left(R_h f'\left(\frac{R_h}{2}\right)\right)\right)}\right]^{\gamma} = \left[R_h f'\left(\frac{R_h}{2}\right)\right]^{\gamma} {l_A}^{1-\gamma} \Rightarrow$

$\mathbf{l_A = R_h f'\left(\frac{R_h}{2}\right)}.$

$l_B = \lim_{k_t \to k_B^-} r_{t+1}(k_t) = \left[\frac{R_a f'\left(\frac{R_h}{2}\right) S\left(R_a f'\left(\frac{R_h}{2}\right)\right)}{\beta^{\frac{1}{\gamma}}\left(1 - S\left(R_a f'\left(\frac{R_h}{2}\right)\right)\right)}\right]^{\gamma} = \left[R_a f'\left(\frac{R_h}{2}\right)\right]^{\gamma} (l_B)^{1-\gamma} \Rightarrow$

$\mathbf{l_B = R_a f'\left(\frac{R_h}{2}\right)}.$

The interest factor r_{t+1} over the interval $(0, k_A]$.

From proposition 1.10 we know that, when $k_t \le k_A$, $r_{t+1}(k_t) = R_h f'(\psi(k_t, R_h))$ is a continuous, decreasing function of k_t. Its minimum value is $r_{t+1}(k_A) = R_h f'\left(\frac{R_h}{2}\right)$.

The interest factor r_{t+1} over the interval $[k_B, +\infty)$.

From proposition 1.12, when $k_t \ge k_B$, the expression of the interest factor is

$r_{t+1}(k_t) = R_a f'(\phi(k_t, R_h, R_a))$.

For $k_t = k_B$ it is $r_{t+1}(k_B) = R_a f'\left(\frac{R_h}{2}\right)$. As above, $r_{t+1}(k_t)$ is continuous and decreasing.

Summarizing: $\mathbf{r_{t+1}}\left(\mathbf{k_t}\right) = \begin{cases} R_h f^{'}\left(\psi\left(k_t, R_h\right)\right) & if \quad k_t \leq k_A \\ \left[\frac{R_h f^{'}\left(\frac{R_h}{2}\right)}{\beta^{\frac{1}{\gamma}}\left(2W(k_t)-1\right)}\right]^{\gamma} & if \quad k_A < k_t < k_B \\ R_a f^{'}\left(\phi\left(k_t, R_h, R_a\right)\right) & if \quad k_t \geq k_B \end{cases}$

4.1.16. Proposition 1.19. Proof.

From $s_t = \left(\frac{\beta}{1+\beta}\right) W\left(k_t\right)$ and being $W\left(k_t\right)$ an increasing function (see note to assumption i_7) it follows that, for $k_t \leq k_A$, it is $s_t \leq \frac{1}{2}$ and production of capital goods will be concentrated only in home Country.

Interest factor will necessary be $r_{t+1} = R_h f^{'}\left(k_{t+1}\right)$. For agents in home Country will be indifferent between borrowing or lending, while for those in foreign Country, lending will strictly more convenient than borrowing.

From the resource constraint $2k_{t+1} = \left(2s_t\right) R_h$ and expression of s_t, comes immediately the relation [1.20].

4.1.17. Proposition 1.22. Proof.

From the assumption $f\left(k_t\right) = Ak_t^{\alpha}$, with $0 < \alpha < 1$, it follows:

$f^{'}\left(k_t\right) = \alpha A k_t^{\alpha-1}$;

$f^{''}\left(k_t\right) = -\alpha\left(1-\alpha\right) A k_t^{\alpha-2}$;

$\frac{k_t f^{'}\left(k_t\right)}{f\left(k_t\right)} = \frac{\alpha A k_t^{\alpha}}{A k_t^{\alpha}} = \alpha$;

$W\left(k_t\right) = \left(1-\alpha\right) A k_t^{\alpha}$;

$W^{'}\left(k_t\right) = \alpha\left(1-\alpha\right) A k_t^{\alpha-1}$;

$\sigma\left(k\right) = \frac{\alpha(1-\alpha)A^2 k_t^{2\alpha-1}}{\alpha(1-\alpha)A^2 k_t^{2\alpha-1}} = 1$

Therefore, all the assumptions from i_0 to i_9 are verified.

4.1.18. Proposition 1.23. Proof.

Let's see some useful transformations:

From $f\left(k_t\right) = Ak_t^{\alpha}$ it follows:

$R_j f^{'}\left(k_{t+1}\right) = \alpha A R_j k_{t+1}^{\alpha-1}$.

From [1.4] and [1.6] it is:

$$S\left(R_j f^{'}\left(k_{t+1}\right)\right) = \frac{\beta^{\frac{1}{\gamma}}\left(R_j f^{'}\left(k_{t+1}\right)\right)^{\frac{1}{\gamma}-1}}{1+\beta^{\frac{1}{\gamma}}\left(R_j f^{'}\left(k_{t+1}\right)\right)^{\frac{1}{\gamma}-1}} = \frac{\left(\alpha A\right)^{\frac{1-\gamma}{\gamma}} \beta^{\frac{1}{\gamma}} R_j^{\frac{1-\gamma}{\gamma}} k_{t+1}^{-\frac{(1-\alpha)(1-\gamma)}{\gamma}}}{1+\left(\alpha A\right)^{\frac{1-\gamma}{\gamma}} \beta^{\frac{1}{\gamma}} R_j^{\frac{1-\gamma}{\gamma}} k_{t+1}^{-\frac{(1-\alpha)(1-\gamma)}{\gamma}}}$$

The relation [1.10] becomes:

$$\frac{1+(\alpha A)^{\frac{1-\gamma}{\gamma}}\beta^{\frac{1}{\gamma}}R_h^{\frac{1-\gamma}{\gamma}}k_{t+1}^{-\frac{(1-\alpha)(1-\gamma)}{\gamma}}}{(\alpha A)^{\frac{1-\gamma}{\gamma}}\beta^{\frac{1}{\gamma}}R_h^{\frac{1-\gamma}{\gamma}}k_{t+1}^{-\frac{(1-\alpha)(1-\gamma)}{\gamma}}}k_{t+1} = (1-\alpha)\,R_h A k_t^{\alpha} \Rightarrow$$

$$\mathbf{k_{t+1}^{\frac{\gamma+(1-\alpha)(1-\gamma)}{\gamma}}} + (\alpha\mathbf{A})^{\frac{1-\gamma}{\gamma}}\beta^{\frac{1}{\gamma}}\mathbf{R_h^{\frac{1-\gamma}{\gamma}}k_{t+1}} = (\mathbf{1}-\alpha)\,(\alpha)^{\frac{1-\gamma}{\gamma}}\mathbf{R_h^{\frac{1}{\gamma}}A^{\frac{1}{\gamma}}}\beta^{\frac{1}{\gamma}}\mathbf{k_t^{\alpha}}$$

The relation [1.13] becomes:

$$\frac{k_{t+1}-\left(\frac{R_h-R_a}{2}\right)\frac{(\alpha A)^{\frac{1-\gamma}{\gamma}}\beta^{\frac{1}{\gamma}}R_a^{\frac{1-\gamma}{\gamma}}k_{t+1}^{-\frac{(1-\alpha)(1-\gamma)}{\gamma}}}{1+(\alpha A)^{\frac{1-\gamma}{\gamma}}\beta^{\frac{1}{\gamma}}R_a^{\frac{1-\gamma}{\gamma}}k_{t+1}^{-\frac{(1-\alpha)(1-\gamma)}{\gamma}}}}{\frac{(\alpha A)^{\frac{1-\gamma}{\gamma}}\beta^{\frac{1}{\gamma}}R_a^{\frac{1-\gamma}{\gamma}}k_{t+1}^{-\frac{(1-\alpha)(1-\gamma)}{\gamma}}}{1+(\alpha A)^{\frac{1-\gamma}{\gamma}}\beta^{\frac{1}{\gamma}}R_a^{\frac{1-\gamma}{\gamma}}k_{t+1}^{-\frac{(1-\alpha)(1-\gamma)}{\gamma}}}} = R_a\,(1-\alpha)\,A k_t^{\alpha}$$

$$\mathbf{k_{t+1}^{\frac{\gamma+(1-\alpha)(1-\gamma)}{\gamma}}} + (\alpha\mathbf{A})^{\frac{1-\gamma}{\gamma}}\beta^{\frac{1}{\gamma}}\mathbf{R_a^{\frac{1-\gamma}{\gamma}}k_{t+1}} = (\alpha\mathbf{A})^{\frac{1-\gamma}{\gamma}}\beta^{\frac{1}{\gamma}}\mathbf{R_a^{\frac{1-\gamma}{\gamma}}}\left(\mathbf{\frac{R_h-R_f}{2}} + (\mathbf{1}-\alpha)\,\mathbf{AR_a k_t^{\alpha}}\right)$$

Similarly, relations [1.20] and [1.21] become:

$$\mathbf{k_{t+1}} = \left(\frac{\beta}{\mathbf{1}+\beta}\right)(\mathbf{1}-\alpha)\,\mathbf{AR_h k_t^{\alpha}}$$

$$\mathbf{k_{t+1}} = \mathbf{\frac{R_h-R_f}{2}} + \left(\frac{\beta}{\mathbf{1}+\beta}\right)(\mathbf{1}-\alpha)\,\mathbf{AR_a k_t^{\alpha}}$$

4.1.19. Proposition 1.24. Proof.

The first branch of $G\,(k_{t,}R_h, R_a)$. In order to prove the above assertion, let's consider the "branch" $\psi\,(k_t, R_h)$, as function of k_t , with parameter R_h.

Solving the system $\begin{cases} k_{t+1} = \psi\,(k_t, R_h) \\ k_{t+1} = k_t \end{cases}$ we obtain the unique solution $\mathbf{k}_\psi^* = \left[\mathbf{\frac{2}{3}AR_h}\left(\frac{\beta}{\mathbf{1}+\beta}\right)\right]^{\frac{3}{2}}$.

Deriving the function and substituting the expression of k_ψ^*, we have:

$$\frac{\partial\psi(k_t,R_h)}{\partial k_t} = \frac{2}{9}AR_h\left(\frac{\beta}{1+\beta}\right)k_t^{-\frac{2}{3}} \Rightarrow \frac{\partial\psi\left(k_\psi^*,R_h\right)}{\partial k_t} = \frac{1}{3}$$

This function $\psi\,(k_t, R_h)$ is defined for $k_t \leq k_A$ and the condition $k_\psi^* \leq k_A$ involves $\frac{R_h}{2} \leq k_A$.

Summarizing, if $\frac{R_h}{2} \leq k_A$, then the first branch of $G\,(k_{t,}R_h, R_a)$ converges to a stable steady state k_ψ^*.

In such case the production of capital goods takes place only in home Country.

The second branch of $G(k_{t,}R_h, R_a)$. Now we are considering the branch $\phi(k_t, R_h, R_a)$, as function of k_t , with parameters R_h and R_a.

Computing its first derivative, we obtain:

$\frac{\partial\phi(k_t,R_h,R_a)}{\partial k_t} = \frac{2}{9}AR_a\left(\frac{\beta}{1+\beta}\right)k_t^{-\frac{2}{3}} > 0$; $\lim_{k_t \to 0^+} \frac{\partial\phi(k_t,R_h,R_a)}{\partial k_t} = +\infty$ and

$\lim_{k_t \to +\infty} \frac{\partial\phi(k_t,R_h,R_a)}{\partial k_t} = 0.$

Moreover its second derivative is $\frac{\partial^2\phi(k_t,R_h,R_a)}{\partial k_t^2} = -\frac{4}{27}AR_a\left(\frac{\beta}{1+\beta}\right)k_t^{-\frac{5}{3}} < 0$.

Therefore this function is an increasing one, but with declining slope.

It starts from the point $A\left(k_A; \frac{R_h}{2}\right)$, where the value of its derivative is

$\frac{\partial\phi(k_A,R_h,R_a)}{\partial k_t} = \frac{R_a}{6}\left(\frac{4A\beta}{3(1+\beta)}\right)^3 = \frac{R_a}{6k_A}$.

Now suppose it is $\frac{R_h}{2} \leq k_A$. With this condition the first branch of $G(k_{t,}R_h, R_a)$ has a unique steady state $k_\psi^* \leq k_A$.

Then the point A is under the bisector or, at limit, on the bisector and the slope of the function $\phi(k_t, R_h, R_a)$ in A is less than 1, as it can be easy proved. [3]

Because of declining slope of $\phi(k_t, R_h, R_a)$ the function will never intercept the bisector into the interval $k_t > k_A$. So, the map $G(k_{t,}R_h, R_a)$ has the unique steady state k_ψ^*.

On the other hand, suppose it is $k_A < \frac{R_h}{2}$. In such case the first branch of the map ends before to intercept the bisector.

The second branch $\phi(k_t, R_h, R_a)$ starts always in A, but now that point is over the bisector.

Because of declining slope of the curve, it has to intercept the bisector in a point on the right relative to A. In this way, the function $\phi(k_t, R_h, R_a)$ has a unique steady state $k_\phi^* > k_A$.

Again, this point is stable, because it must necessary be $\frac{\partial\phi\left(k_\phi^*,R_h,R_a\right)}{\partial k_t} < 1$; if not the curve $\phi(k_t, R_h, R_a)$ couldn't have intercepted the bisector.

That proves the thesis.

4.1.20. Proposition 1.25. Proof.

With the choice of parameters $\alpha = \frac{1}{3}$ and $\gamma = \frac{2}{5}$, the relation [1.24] becomes:

[3] Suppose it is $\frac{R_a}{6k_A} \geq 1$ and $k_A \geq \frac{R_h}{2}$. Then $R_a \geq 6k_A \geq 3R_h$. This is absurd, because, for hypothesis, $R_a < R_h$

$$\begin{cases} k_{t+1}^2 + \left(\frac{AR_h}{3}\right)^{\frac{3}{2}} \beta^{\frac{5}{2}} k_{t+1} = 2\left(\frac{AR_h}{3}\right)^{\frac{5}{2}} \beta^{\frac{5}{2}} k_t^{\frac{1}{3}} & if \quad k_t \le k_A \\ \frac{R_h}{2} & if \quad k_A < k_t < k_B \\ k_{t+1}^2 + \left(\frac{AR_a}{3}\right)^{\frac{3}{2}} \beta^{\frac{5}{2}} k_{t+1} = \left(\frac{R_h - R_a}{2}\right)\left(\frac{AR_a}{3}\right)^{\frac{3}{2}} \beta^{\frac{5}{2}} + 2\left(\frac{AR_a}{3}\right)^{\frac{5}{2}} \beta^{\frac{5}{2}} k_t^{\frac{1}{3}} & if \quad k_t \ge k_B \end{cases}$$

and substituting $V_j = \frac{2}{3}AR_j$, $C_j = \beta^{\frac{5}{2}}\left(\frac{AR_j}{3}\right)^{\frac{3}{2}}$, $H = \frac{R_h - R_a}{2}$:

$$\begin{cases} k_{t+1}^2 + C_h k_{t+1} - V_h C_h k_t^{\frac{1}{3}} = 0 & if \quad k_t \le k_A \\ \frac{R_h}{2} & if \quad k_A < k_t < k_B \\ k_{t+1}^2 + C_a k_{t+1} - HC_a - V_a C_a k_t^{\frac{1}{3}} = 0 & if \quad k_t \ge k_B \end{cases}$$

Finally, solving and omitting the negative values, we obtain the [1.27]

Now we can obtain the separating values for k_t, imposing the conditions:

k_A: $\psi(k_A, R_h) = \frac{R_h}{2}$ and k_B: $\phi(k_B, R_h, R_a) = \frac{R_h}{2}$

$$\frac{1}{2}\sqrt{C_h^2 + 4V_h C_h k_t^{\frac{1}{3}}} - \frac{1}{2}C_h = \frac{R_h}{2} \Rightarrow k_A = \left(\frac{R_h^2 + 2C_h R_h}{4V_h C_h}\right)^3$$

$$\frac{1}{2}\sqrt{C_a^2 + 4HC_a + 4V_a C_a k_t^{\frac{1}{3}}} - \frac{1}{2}C_a = \frac{R_h}{2} \Rightarrow k_B = \left(\frac{R_h^2 + 2C_a R_h - 4HC_a}{4V_a C_a}\right)^3 = \left(\frac{R_h^2 + 2C_a R_a}{4V_a C_a}\right)^3$$

Remark. It was proved in general terms that, if $R_a < R_h$, then $k_A < k_B$ (see proposition 1.13).

We can easily verify that result here.

From [1.28] and [1.29], $k_A < k_B \Rightarrow \frac{R_h^2 + 2C_h R_h}{4V_h C_h} < \frac{R_h^2 + 2C_a R_a}{4V_a C_a}$, from which:

$R_h^2 V_a C_a + 2V_a C_a C_h R_h < R_h^2 V_h C_h + 2C_a R_a V_h C_h$

$2C_a C_h (V_a R_h - V_h R_a) < R_h^2 (V_h C_h - V_a C_a)$.

But $V_a R_h - V_h R_a = \frac{2}{3}AR_a R_h - \frac{2}{3}AR_h R_a = 0$, while $V_h C_h - V_a C_a > 0$ whenever $R_h > R_a$ and we are done.

5. Appendix to Chapter II

5.1. Proofs of propositions of Chapter 2

5.1.1. Proposition 2.6. Proof.

Being $P^{\star}\left(\left[\left(1-\alpha\right)\Gamma\right]^{\frac{1}{1-\alpha}},\left[\left(1-\alpha\right)\Gamma\right]^{\frac{1}{1-\alpha}}\right)$ and $\overline{Y}=\left(\frac{1-\lambda}{(1-\alpha)\Gamma}\right)^{\frac{1}{\alpha}}$ it follows that

$P^{\star}\in h_1a_1$, when it is $\left[\left(1-\alpha\right)\Gamma\right]^{\frac{1}{1-\alpha}}\geq\left(\frac{1-\lambda}{(1-\alpha)\Gamma}\right)^{\frac{1}{\alpha}}$; that involves $\Gamma\geq\frac{(1-\lambda)^{1-\alpha}}{1-\alpha}$.

On the opposite case, $P^{\star}\in h_0a_0$.

The steady point $P^{\star}$into the region h_1a_1.

Into region h_1a_1 the map becomes $\begin{cases} y_{ht+1}=\frac{(1-\alpha)\Gamma}{2}\left(y_{ht}^{\alpha}+y_{at}^{\alpha}\right)\\ y_{at+1}=\frac{(1-\alpha)\Gamma}{2}\left(y_{ht}^{\alpha}+y_{at}^{\alpha}\right)\end{cases}$ and $P^{\star}$ is its unique steady state solution.

The jacobian matrix of the system calculated in $P^{\star}$ is:

$J_{h_1a_1}\left(P^{\star}\right)=\frac{\alpha}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$; $detJ_{h_1a_1}\left(P^{\star}\right)=0$; $trJ_{h_1a_1}\left(P^{\star}\right)=\alpha$

Therefore its eigenvalues are $\mu_1=0$ and $\mu_2=\alpha$ and its eigenvectors are $[1,-1]^T$ and $[1,1]^T$.

This proves that $P^{\star}$ is a node, asymptotically attractive.

Whichever is the starting point $(y_{h0},y_{a0})\in h_1a_1$, just at the first iteration, the system reaches the bisector and, by this way, the steady point $P^{\star}$.

On the other hand, because in the region h_0a_0 and along the bisector $y_{ht}=y_{at}=y_t$ the map is $\begin{cases} y_{ht+1}=\left(1-\alpha\right)\Gamma y_t^{\alpha}\\ y_{at+1}=\left(1-\alpha\right)\Gamma y_t^{\alpha}\end{cases}$, it can trivially be noted that $y_{jt+1}\gtrless y_t$

if $y_t\lesseqgtr\left[\left(1-\alpha\right)\Gamma\right]^{\frac{1}{1-\alpha}}$. It means that a point starting along the bisector, into the region h_0a_0, moves towards the steady point $P^{\star}$ of h_1a_1. *This proves that the basin of attraction of $P^{\star}$ must to include the set* $\left]O,\overline{Y}\right[\cup h_1a_1$.

The steady point $P^\star$ into the region h_0a_0.

In the region h_0a_0 the map is:

$$M_{h_0a_0}\left(y_{ht},y_{at}\right):\begin{cases} y_{ht+1}=m\left(y_{ht},y_{at}\right)=\frac{1}{\tau(y_{ht},y_{at})}\left(\frac{\alpha\Gamma\lambda}{1-(1-\alpha)\Gamma y_{ht}^{\alpha}}\right)^{\frac{1}{1-\alpha}} \\ y_{at+1}=n\left(y_{ht},y_{at}\right)=\frac{1}{\tau(y_{ht},y_{at})}\left(\frac{\alpha\Gamma\lambda}{1-(1-\alpha)\Gamma y_{at}^{\alpha}}\right)^{\frac{1}{1-\alpha}} \end{cases}$$

Now we are going to study the nature of the point $P^\star$. In order to do that, we will calculate the jacobian matrix of the system:

$$J_{h_0a_0}=\begin{bmatrix} \frac{\partial m}{\partial y_{ht}} & \frac{\partial m}{\partial y_{at}} \\ \frac{\partial n}{\partial y_{ht}} & \frac{\partial n}{\partial y_{at}} \end{bmatrix}$$

$$\frac{\partial m(y_{ht},y_{at})}{\partial y_{ht}}=\frac{1}{\tau(y_{ht},y_{at})}\left(\frac{\alpha\Gamma\lambda}{1-(1-\alpha)\Gamma y_{ht}^{\alpha}}\right)^{\frac{1}{1-\alpha}}\left[\frac{\alpha\Gamma y_{ht}^{\alpha-1}}{1-(1-\alpha)\Gamma y_{ht}^{\alpha}}-\frac{1}{\tau(y_{ht},y_{at})}\frac{\partial\tau(y_{ht},y_{at})}{\partial y_{ht}}\right]$$

$$\frac{\partial n(y_{ht},y_{at})}{\partial y_{at}}=\frac{1}{\tau(y_{ht},y_{at})}\left(\frac{\alpha\Gamma\lambda}{1-(1-\alpha)\Gamma y_{at}^{\alpha}}\right)^{\frac{1}{1-\alpha}}\left[\frac{\alpha\Gamma y_{at}^{\alpha-1}}{1-(1-\alpha)\Gamma y_{at}^{\alpha}}-\frac{1}{\tau(y_{ht},y_{at})}\frac{\partial\tau(y_{ht},y_{at})}{\partial y_{at}}\right]$$

$$\frac{\partial m(y_{ht},y_{at})}{\partial y_{at}}=-\frac{1}{\tau^2(y_{ht},y_{at})}\left(\frac{\alpha\Gamma\lambda}{1-(1-\alpha)\Gamma y_{ht}^{\alpha}}\right)^{\frac{1}{1-\alpha}}\frac{\partial\tau(y_{ht},y_{at})}{\partial y_{at}}$$

$$\frac{\partial n(y_{ht},y_{at})}{\partial y_{ht}}=-\frac{1}{\tau^2(y_{ht},y_{at})}\left(\frac{\alpha\Gamma\lambda}{1-(1-\alpha)\Gamma y_{at}^{\alpha}}\right)^{\frac{1}{1-\alpha}}\frac{\partial\tau(y_{ht},y_{at})}{\partial y_{ht}}$$

$$\frac{\partial F(y_{jt})}{\partial y_{jt}}=\frac{\alpha\Gamma y_{jt}^{\alpha-1}}{\left(1-(1-\alpha)\Gamma y_{jt}^{\alpha}\right)}\left(\frac{\alpha\Gamma\lambda}{1-(1-\alpha)\Gamma y_{jt}^{\alpha}}\right)^{\frac{1}{1-\alpha}}$$

$$\tau\left(y_{ht},y_{at}\right)=\frac{\left(\frac{\alpha\Gamma\lambda}{1-(1-\alpha)\Gamma y_{ht}^{\alpha}}\right)^{\frac{1}{1-\alpha}}+\left(\frac{\alpha\Gamma\lambda}{1-(1-\alpha)\Gamma y_{at}^{\alpha}}\right)^{\frac{1}{1-\alpha}}}{(1-\alpha)\Gamma\left(y_{ht}^{\alpha}+y_{at}^{\alpha}\right)}$$

$$\frac{\partial\tau(y_{ht},y_{at})}{\partial y_{ht}}=\frac{\frac{\alpha\Gamma y_{ht}^{\alpha-1}}{\left(1-(1-\alpha)\Gamma y_{ht}^{\alpha}\right)}\left(\frac{\alpha\Gamma\lambda}{1-(1-\alpha)\Gamma y_{ht}^{\alpha}}\right)^{\frac{1}{1-\alpha}}\left(y_{ht}^{\alpha}+y_{at}^{\alpha}\right)-\left[\left(\frac{\alpha\Gamma\lambda}{1-(1-\alpha)\Gamma y_{ht}^{\alpha}}\right)^{\frac{1}{1-\alpha}}+\left(\frac{\alpha\Gamma\lambda}{1-(1-\alpha)\Gamma y_{at}^{\alpha}}\right)^{\frac{1}{1-\alpha}}\right]\alpha y_{ht}^{\alpha-1}}{(1-\alpha)\Gamma\left(y_{ht}^{\alpha}+y_{at}^{\alpha}\right)^2}$$

Substituting the symmetric coordinates of $P^\star\left(p,p\right)$, we obtain:

$$\frac{\partial m(P^\star)}{\partial y_{ht}}=\frac{1}{\tau(P^\star)}\left(\frac{\alpha\Gamma\lambda}{1-[(1-\alpha)\Gamma]^{\frac{1}{1-\alpha}}}\right)^{\frac{1}{1-\alpha}}\left[\frac{\alpha\Gamma p^{\alpha-1}}{1-[(1-\alpha)\Gamma]^{\frac{1}{1-\alpha}}}-\frac{1}{\tau(P^\star)}\frac{\partial\tau(P^\star)}{\partial y_{ht}}\right]$$

$$\frac{\partial n(P^\star)}{\partial y_{at}}=\frac{1}{\tau(P^\star)}\left(\frac{\alpha\Gamma\lambda}{1-[(1-\alpha)\Gamma]^{\frac{1}{1-\alpha}}}\right)^{\frac{1}{1-\alpha}}\left[\frac{\alpha\Gamma p^{\alpha-1}}{1-[(1-\alpha)\Gamma]^{\frac{1}{1-\alpha}}}-\frac{1}{\tau(P^\star)}\frac{\partial\tau(P^\star)}{\partial y_{at}}\right]$$

$$\frac{\partial m(P^\star)}{\partial y_{at}}=-\frac{1}{\tau^2(P^\star)}\left(\frac{\alpha\Gamma\lambda}{1-[(1-\alpha)\Gamma]^{\frac{1}{1-\alpha}}}\right)^{\frac{1}{1-\alpha}}\frac{\partial\tau(P^\star)}{\partial y_{at}}$$

$$\frac{\partial n(P^\star)}{\partial y_{ht}}=-\frac{1}{\tau^2(P^\star)}\left(\frac{\alpha\Gamma\lambda}{1-[(1-\alpha)\Gamma]^{\frac{1}{1-\alpha}}}\right)^{\frac{1}{1-\alpha}}\frac{\partial\tau(P^\star)}{\partial y_{ht}}$$

$$\frac{\partial\tau(P^\star)}{\partial y_{ht}}=\frac{\partial\tau(P^\star)}{\partial y_{at}}=\frac{\alpha\left[\frac{\Gamma p^{\alpha}}{1-(1-\alpha)\Gamma p^{\alpha}}-1\right]\left(\frac{\alpha\Gamma\lambda}{1-(1-\alpha)\Gamma p^{\alpha}}\right)^{\frac{1}{1-\alpha}}}{2(1-\alpha)\Gamma p^{\alpha+1}}=\frac{\alpha\left[\frac{\Gamma[(1-\alpha)\Gamma]^{\frac{\alpha}{1-\alpha}}}{1-[(1-\alpha)\Gamma]^{\frac{1}{1-\alpha}}}-1\right]\left(\frac{\alpha\Gamma\lambda}{1-[(1-\alpha)\Gamma]^{\frac{1}{1-\alpha}}}\right)^{\frac{1}{1-\alpha}}}{2[(1-\alpha)\Gamma]^{\frac{2}{1-\alpha}}}$$

$\tau\left(P^{\star}\right)=\frac{1}{[(1-\alpha)\Gamma]^{\frac{1}{1-\alpha}}}\left(\frac{\alpha\Gamma\lambda}{1-[(1-\alpha)\Gamma]^{\frac{1}{1-\alpha}}}\right)^{\frac{1}{1-\alpha}}$

Hence:

$\eta=\frac{\partial m(P^{\star})}{\partial y_{ht}}=\frac{\partial n(P^{\star})}{\partial y_{at}}=\frac{\alpha}{2}\left(\frac{1+\Gamma[(1-\alpha)\Gamma]^{\frac{\alpha}{1-\alpha}}-[(1-\alpha)\Gamma]^{\frac{1}{1-\alpha}}}{1-[(1-\alpha)\Gamma]^{\frac{1}{1-\alpha}}}\right)$

and $\beta=\frac{\partial m(P^{\star})}{\partial y_{at}}=\frac{\partial n(P^{\star})}{\partial y_{ht}}=\frac{\alpha}{2}\left(\frac{1-\Gamma[(1-\alpha)\Gamma]^{\frac{\alpha}{1-\alpha}}-[(1-\alpha)\Gamma]^{\frac{1}{1-\alpha}}}{1-[(1-\alpha)\Gamma]^{\frac{1}{1-\alpha}}}\right)$

The jacobian matrix becomes:

$J_{h_0a_0}\left(P^{\star}\right)=\begin{bmatrix}\eta & \beta\\ \beta & \eta\end{bmatrix}$

Then: $Tr\left[J_{h_0f_0}\left(P^{\star}\right)\right]=2\eta$; $det\left[J_{h_0f_0}\left(P^{\star}\right)\right]=\eta^2-\beta^2$

The eigenvalues can be obtained from $\mu^2-2\eta\mu+\eta^2-\beta^2=0$.

Therefore: $\mu_1=\eta+\beta=\alpha$ and $\mu_2=\eta-\beta=\frac{\alpha\Gamma[(1-\alpha)\Gamma]^{\frac{\alpha}{1-\alpha}}}{1-[(1-\alpha)\Gamma]^{\frac{1}{1-\alpha}}}$

The eigenvectors derive from $\left(J_{h_0f_0}\left(P^{\star}\right)-\mu_kI\right)u_k=0\Rightarrow$

$\begin{bmatrix}a-\mu_k & b\\ b & a-\mu_k\end{bmatrix}\begin{bmatrix}u_k^1\\ u_k^2\end{bmatrix}=0$ for $k=1$, 2.

For $k=1$ it is $\mu_1=a+b$ then $\begin{bmatrix}-b & b\\ b & -b\end{bmatrix}\begin{bmatrix}u_1^1\\ u_1^2\end{bmatrix}=0\Rightarrow u_1=\begin{bmatrix}1\\ 1\end{bmatrix}$

For $k=2$ it is $\mu_2=a-b$ then $\begin{bmatrix}b & b\\ b & b\end{bmatrix}\begin{bmatrix}u_2^1\\ u_2^2\end{bmatrix}=0\Rightarrow u_2=\begin{bmatrix}1\\ -1\end{bmatrix}$

Now, the first eigenvalue is certainly positive and less than 1. So we have to consider the second one.

Let's observe, at the beginning, that the condition $P^{\star}\in h_0a_0$ involves $[(1-\alpha)\Gamma]^{\frac{1}{1-\alpha}}<(1-\lambda)<1$, then μ_2 is positive.

The condition $\mu_2<1$ implies $\Gamma<\frac{1}{(1-\alpha)^{\alpha}}$.

Comparing this last condition, with $\Gamma<\frac{(1-\lambda)^{1-\alpha}}{1-\alpha}$ by which $P^{\star}$ belongs to h_0a_0, it follows that if $\alpha>\lambda$, then $\frac{1}{(1-\alpha)^{\alpha}}<\frac{(1-\lambda)^{1-\alpha}}{1-\alpha}$ and $P^{\star}\in h_0a_0$ may be a stable node, or a saddle point, while if $\alpha\leq\lambda$, it is $\frac{1}{(1-\alpha)^{\alpha}}\geq\frac{(1-\lambda)^{1-\alpha}}{1-\alpha}$ and $P^{\star}\in h_0a_0$ is always a stable node.

Finally, even if $P^{\star}$ is a saddle point, *its stable set has positive measure* and includes $\left]O,\overline{Y}\right[\cup h_1a_1$. In fact its stable path coincides with the bisector and we know that a point starting in h_1a_1 immediately reaches the bisector and, by this way, the point $P^{\star}$.

That proves the thesis.

5.1.2. Proposition 2.7. Proof. The regions h_1a_1 and h_0a_0.

The region h_1a_1.

About the region h_1a_1 it is sufficient to invoke the general outcomes we have found above.

This region, at the most, can only contain the fixed point $P^\star\left(\frac{\Gamma^2}{4}, \frac{\Gamma^2}{4}\right)$ that is a stable node.

The only condition required is $\Gamma \geq 2\sqrt{1-\lambda}$, that assures $P^\star \in h_1a_1$.

The region h_0a_0.

Now we will consider the region h_0a_0, about which there are some more articulate results.

Under the hypothesis $\alpha = \frac{1}{2}$ the map system becomes:

$$\mathbf{M_{h_0a_0}}\left(\alpha = \frac{1}{2}\right) : \begin{cases} y_{ht+1} = \left[\dfrac{\frac{1}{2}\Gamma\left(\sqrt{y_{ht}}+\sqrt{y_{at}}\right)}{\frac{1}{\left(1-\frac{1}{2}\Gamma\sqrt{y_{ht}}\right)^2}+\frac{1}{\left(1-\frac{1}{2}\Gamma\sqrt{y_{at}}\right)^2}}\right]\dfrac{1}{\left(1-\frac{1}{2}\Gamma\sqrt{y_{ht}}\right)^2} & y_{ht} < 4\left(\frac{1-\lambda}{\Gamma}\right)^2 \\ y_{at+1} = \left[\dfrac{\frac{1}{2}\Gamma\left(\sqrt{y_{ht}}+\sqrt{y_{at}}\right)}{\frac{1}{\left(1-\frac{1}{2}\Gamma\sqrt{y_{ht}}\right)^2}+\frac{1}{\left(1-\frac{1}{2}\Gamma\sqrt{y_{at}}\right)^2}}\right]\dfrac{1}{\left(1-\frac{1}{2}\Gamma\sqrt{y_{at}}\right)^2} & y_{at} < 4\left(\frac{1-\lambda}{\Gamma}\right)^2 \end{cases}$$

In order to find the fixed points of the map, we will assume $y_{jt+1} = y_{jt} = y_j$

Null solution will be ignored.

$$\begin{cases} y_h = \left[\dfrac{\frac{1}{2}\Gamma\left(\sqrt{y_h}+\sqrt{y_a}\right)}{\frac{1}{\left(1-\frac{1}{2}\Gamma\sqrt{y_h}\right)^2}+\frac{1}{\left(1-\frac{1}{2}\Gamma\sqrt{y_a}\right)^2}}\right]\dfrac{1}{\left(1-\frac{1}{2}\Gamma\sqrt{y_h}\right)^2} & y_h < 4\left(\frac{1-\lambda}{\Gamma}\right)^2 \\ \sqrt{y_h}\left(1-\frac{1}{2}\Gamma\sqrt{y_h}\right) = \sqrt{y_a}\left(1-\frac{1}{2}\Gamma\sqrt{y_a}\right) & y_a < 4\left(\frac{1-\lambda}{\Gamma}\right)^2 \end{cases}$$

Let's substitute $x_j = \sqrt{y_j}$ for $j = h\ ,\ a$. Hence $0 < x_j < \frac{2}{\Gamma}(1-\lambda)$

$$\begin{cases} x_h^2 = \left[\dfrac{\frac{1}{2}\Gamma(x_h+x_a)}{\frac{1}{\left(1-\frac{1}{2}\Gamma x_h\right)^2}+\frac{1}{\left(1-\frac{1}{2}\Gamma x_a\right)^2}}\right]\dfrac{1}{\left(1-\frac{1}{2}\Gamma x_h\right)^2} & x_h < \frac{2}{\Gamma}(1-\lambda) \\ \left[\frac{1}{2}\Gamma\left(x_h+x_a\right)-1\right]\left(x_h-x_a\right) = 0 & x_a < \frac{2}{\Gamma}(1-\lambda) \end{cases}$$

By this way we find, at first, the already known solution *of autarky* $\mathbf{P}^\star\left(\frac{\mathbf{\Gamma^2}}{\mathbf{4}}, \frac{\mathbf{\Gamma^2}}{\mathbf{4}}\right)$.

Then there are two other fixed points:

$$\begin{cases} x_h^2 = \left[\dfrac{\frac{1}{2}\Gamma(x_h+x_a)}{\frac{1}{\left(1-\frac{1}{2}\Gamma x_h\right)^2}+\frac{1}{\left(1-\frac{1}{2}\Gamma x_a\right)^2}}\right]\dfrac{1}{\left(1-\frac{1}{2}\Gamma x_h\right)^2} & x_h < \frac{2}{\Gamma}(1-\lambda) \\ x_h + x_a = \frac{2}{\Gamma} & x_a < \frac{2}{\Gamma}(1-\lambda) \end{cases}$$

$$\begin{cases} \frac{1}{2}\Gamma^2 x_h^2 - \Gamma x_h + 1 - \frac{1}{4}\Gamma^2 = 0 & x_h < \frac{2}{\Gamma}(1-\lambda) \\ x_h + x_a = \frac{2}{\Gamma} & x_a < \frac{2}{\Gamma}(1-\lambda) \end{cases}$$

$$\begin{cases} x^\star_{h1,2} = \frac{1 \pm \sqrt{\frac{1}{2}\Gamma^2 - 1}}{\Gamma} \\ x^\star_{a1,2} = \frac{1 \mp \sqrt{\frac{1}{2}\Gamma^2 - 1}}{\Gamma} \end{cases} \Rightarrow \begin{cases} y^\star_h = \frac{\frac{1}{2}\Gamma^2 \pm 2\sqrt{\frac{1}{2}\Gamma^2 - 1}}{\Gamma^2} \\ y^\star_a = \frac{\frac{1}{2}\Gamma^2 \mp 2\sqrt{\frac{1}{2}\Gamma^2 - 1}}{\Gamma^2} \end{cases}$$

Eventually we obtain two symmetric, fixed points

$$\mathbf{Q_1}\left(\frac{\frac{1}{2}\boldsymbol{\Gamma}^2 + 2\sqrt{\frac{1}{2}\boldsymbol{\Gamma}^2 - 1}}{\boldsymbol{\Gamma}^2}, \frac{\frac{1}{2}\boldsymbol{\Gamma}^2 - 2\sqrt{\frac{1}{2}\boldsymbol{\Gamma}^2 - 1}}{\boldsymbol{\Gamma}^2}\right);\ \mathbf{Q_2}\left(\frac{\frac{1}{2}\boldsymbol{\Gamma}^2 - 2\sqrt{\frac{1}{2}\boldsymbol{\Gamma}^2 - 1}}{\boldsymbol{\Gamma}^2}, \frac{\frac{1}{2}\boldsymbol{\Gamma}^2 + 2\sqrt{\frac{1}{2}\boldsymbol{\Gamma}^2 - 1}}{\boldsymbol{\Gamma}^2}\right).$$

5.1.2.1. The fixed points Q_1 and Q_2. Nature and stability.

The existence of these points preliminary requires $\frac{1}{2}\Gamma^2 - 1 > 0$. That is $\Gamma > \sqrt{2}$ (for $\Gamma = \sqrt{2}$, it is $Q_1 \equiv Q_2 \equiv P^\star$).

In order to assure their positivity, it must be $1 - \sqrt{\frac{1}{2}\Gamma^2 - 1} > 0$. It involves $\Gamma < 2$.

Moreover boundary conditions take $\frac{1 + \sqrt{\frac{1}{2}\Gamma^2 - 1}}{\Gamma} < \frac{2}{\Gamma}(1-\lambda)$, that is

$\sqrt{\frac{1}{2}\Gamma^2 - 1} < 1 - 2\lambda$.

This last condition is not satisfied for $\lambda \geq \frac{1}{2}$, while for $\lambda < \frac{1}{2}$ it becomes

$\Gamma < 2\sqrt{2\lambda^2 - 2\lambda + 1}$.

Therefore:

If $\lambda \geq \frac{1}{2}$ there are not any points Q_1 and Q_2 into the region h_0a_0.

If $\lambda < \frac{1}{2}$ there are three conditions needed for the points Q_1 and Q_2 to exist and stay into h_0a_0:

− to be real and distinct: $\Gamma > \sqrt{2}$

− to be positive: $\Gamma < 2$

− to stay into h_0a_0: $\Gamma < 2\sqrt{2\lambda^2 - 2\lambda + 1}$.

Being for hypothesis $0 < \lambda < 1$ it is easy to prove that the relation

$\sqrt{2\lambda^2 - 2\lambda + 1} < 1$ is always verified; consequently, the second condition may be omitted.

Summarizing, necessary and sufficient conditions for the existence of the two distinct, fixed points Q_1 and Q_2 into h_0a_0 are:

$$\begin{cases} \lambda < \frac{1}{2} \\ \sqrt{2} < \Gamma < 2\sqrt{2\lambda^2 - 2\lambda + 1} \end{cases}$$

Nature. This part requires a straight manipulation of data; we will do it only for Q_1, being the one of Q_2 deducible for symmetry, as we can prove later.

Resuming previous outcomes for a generic α and adapting them to the present case $\alpha = \frac{1}{2}$, we have some results described in the following.

The jacobian matrix is (see demonstration of proposition 2.6):

$$_{\alpha=\frac{1}{2}}J_{h_0a_0} = \begin{bmatrix} \frac{\partial m(Q_1)}{\partial y_{ht}} & \frac{\partial m(Q_1)}{\partial y_{at}} \\ \frac{\partial n(Q_1)}{\partial y_{ht}} & \frac{\partial n(Q_1)}{\partial y_{at}} \end{bmatrix}$$

$$\frac{\partial m(Q_1)}{\partial y_{ht}} = \frac{1}{\tau(Q_1)} \left(\frac{\frac{1}{2}\Gamma\lambda}{1-\frac{1}{2}\Gamma\sqrt{y_{h_{Q_1}}}} \right)^2 \left[\frac{\frac{1}{2}\Gamma}{\sqrt{y_{h_{Q_1}}}\left(1-\frac{1}{2}\Gamma\sqrt{y_{h_{Q_1}}}\right)} - \frac{1}{\tau(Q_1)}\frac{\partial\tau(Q_1)}{\partial y_{ht}} \right]$$

$$\frac{\partial n(Q_1)}{\partial y_{at}} = \frac{1}{\tau(Q_1)} \left(\frac{\frac{1}{2}\Gamma\lambda}{1-\frac{1}{2}\Gamma\sqrt{y_{a_{Q_1}}}} \right)^2 \left[\frac{\frac{1}{2}\Gamma}{\sqrt{y_{a_{Q_1}}}\left(1-\frac{1}{2}\Gamma\sqrt{y_{a_{Q_1}}}\right)} - \frac{1}{\tau(Q_1)}\frac{\partial\tau(Q_1)}{\partial y_{at}} \right]$$

$$\frac{\partial m(Q_1)}{\partial y_{at}} = -\frac{1}{\tau^2(Q_1)} \left(\frac{\frac{1}{2}\Gamma\lambda}{1-\frac{1}{2}\Gamma\sqrt{y_{h_{Q_1}}}} \right)^2 \frac{\partial\tau(Q_1)}{\partial y_{at}}$$

$$\frac{\partial n(Q_1)}{\partial y_{ht}} = -\frac{1}{\tau^2(Q_1)} \left(\frac{\frac{1}{2}\Gamma\lambda}{1-\frac{1}{2}\Gamma\sqrt{y_{a_{Q_1}}}} \right)^2 \frac{\partial\tau(Q_1)}{\partial y_{ht}}$$

$$Q_1\left(\frac{\frac{1}{2}\Gamma^2+2\sqrt{\frac{1}{2}\Gamma^2-1}}{\Gamma^2}, \frac{\frac{1}{2}\Gamma^2-2\sqrt{\frac{1}{2}\Gamma^2-1}}{\Gamma^2} \right)$$

In order to simplify the relations above, we set $s = 1+\sqrt{\frac{1}{2}\Gamma^2-1}$, $t = 1-\sqrt{\frac{1}{2}\Gamma^2-1}$ and derive some useful relations:

$s+t=2$; $s^2+t^2=\Gamma^2$; $s^3+t^3 = 2\left(\Gamma^2 - \sqrt{2-\frac{1}{2}\Gamma^2}\right)$; $s^4+t^4 = \Gamma^4+\Gamma^2-4$;

$st = 2-\frac{1}{2}\Gamma^2$; $\frac{1}{t}+\frac{1}{s} = \frac{2}{2-\frac{1}{2}\Gamma^2}$; $\frac{1}{t^2}+\frac{1}{s^2} = \frac{\Gamma^2}{\left(2-\frac{1}{2}\Gamma^2\right)^2}$; $\frac{s}{t}+\frac{t}{s} = \frac{\Gamma^2}{2-\frac{1}{2}\Gamma^2}$; $\frac{t}{s^2}+\frac{s}{t^2} = \frac{3\Gamma^2-4}{\left(2-\frac{1}{2}\Gamma^2\right)^2}$

$\sqrt{\frac{1}{2}\Gamma^2+2\sqrt{\frac{1}{2}\Gamma^2-1}} = s$; $\sqrt{\frac{1}{2}\Gamma^2-2\sqrt{\frac{1}{2}\Gamma^2-1}} = t$

$$\left(\frac{\frac{1}{2}\Gamma\lambda}{1-\frac{1}{2}\Gamma\sqrt{y_{h_{Q_1}}}} \right)^2 = \left(\frac{\frac{1}{2}\Gamma\lambda}{1-\frac{1}{2}\sqrt{\frac{1}{2}\Gamma^2+2\sqrt{\frac{1}{2}\Gamma^2-1}}} \right)^2 = \frac{\Gamma^2\lambda^2}{t^2}$$

$$\left(\frac{\frac{1}{2}\Gamma\lambda}{1-\frac{1}{2}\Gamma\sqrt{y_{a_{Q_1}}}} \right)^2 = \left(\frac{\frac{1}{2}\Gamma\lambda}{1-\frac{1}{2}\sqrt{\frac{1}{2}\Gamma^2-2\sqrt{\frac{1}{2}\Gamma^2-1}}} \right)^2 = \frac{\Gamma^2\lambda^2}{s^2}$$

$$\sqrt{y_{h_{Q_1}}} = \frac{1}{\Gamma}\sqrt{\frac{1}{2}\Gamma^2+2\sqrt{\frac{1}{2}\Gamma^2-1}} = \frac{s}{\Gamma}$$

$$\sqrt{y_{a_{Q_1}}} = \frac{1}{\Gamma}\sqrt{\frac{1}{2}\Gamma^2-2\sqrt{\frac{1}{2}\Gamma^2-1}} = \frac{t}{\Gamma}$$

$$\sqrt{y_{h_{Q_1}}} + \sqrt{y_{a_{Q_1}}} = \frac{1}{\Gamma}\left(\sqrt{\frac{1}{2}\Gamma^2+2\sqrt{\frac{1}{2}\Gamma^2-1}} + \sqrt{\frac{1}{2}\Gamma^2-2\sqrt{\frac{1}{2}\Gamma^2-1}} \right) = \frac{2}{\Gamma}$$

$$1-\tfrac{1}{2}\Gamma\sqrt{y_{h_{Q_1}}}=1-\tfrac{1}{2}\sqrt{\tfrac{1}{2}\Gamma^2+2\sqrt{\tfrac{1}{2}\Gamma^2-1}}=\tfrac{t}{2}$$

$$1-\tfrac{1}{2}\Gamma\sqrt{y_{a_{Q_1}}}=1-\tfrac{1}{2}\sqrt{\tfrac{1}{2}\Gamma^2-2\sqrt{\tfrac{1}{2}\Gamma^2-1}}=\tfrac{s}{2}$$

$$\tau\left(Q_1\right)=\frac{\left(\frac{2\Gamma\lambda}{2-\sqrt{2\Gamma^2-4}}\right)^2+\left(\frac{2\Gamma\lambda}{2+\sqrt{2\Gamma^2-4}}\right)^2}{\frac{\sqrt{2}}{4}\left(\sqrt{2}+\sqrt{\Gamma^2-2}\right)+\frac{\sqrt{2}}{4}\left(\sqrt{2}-\sqrt{\Gamma^2-2}\right)}=\frac{\Gamma^4\lambda^2}{s^2t^2}$$

$$\frac{1}{\tau(Q_1)}\left(\frac{\frac{1}{2}\Gamma\lambda}{1-\frac{1}{2}\Gamma\sqrt{y_{h_{Q_1}}}}\right)^2=\frac{s^2}{\Gamma^2}$$

$$\frac{1}{\tau(Q_1)}\left(\frac{\frac{1}{2}\Gamma\lambda}{1-\frac{1}{2}\Gamma\sqrt{y_{a_{Q_1}}}}\right)^2=\frac{t^2}{\Gamma^2}$$

$$\frac{\frac{1}{2}\Gamma}{\sqrt{y_{h_{Q_1}}}\left(1-\frac{1}{2}\Gamma\sqrt{y_{h_{Q_1}}}\right)}=\frac{\frac{1}{2}\Gamma}{\frac{1}{\Gamma}\sqrt{\frac{1}{2}\Gamma^2+2\sqrt{\frac{1}{2}\Gamma^2-1}}\left(1-\frac{1}{2}\sqrt{\frac{1}{2}\Gamma^2+2\sqrt{\frac{1}{2}\Gamma^2-1}}\right)}=\frac{\Gamma^2}{st}$$

$$\frac{\partial\tau(Q_1)}{\partial y_{ht}}=\frac{\frac{\Gamma\left(\sqrt{y_{h_{Q_1}}}+\sqrt{y_{a_{Q_1}}}\right)}{1-\frac{1}{2}\Gamma\sqrt{y_{h_{Q_1}}}}\left(\frac{\frac{1}{2}\Gamma\lambda}{1-\frac{1}{2}\Gamma\sqrt{y_{h_{Q_1}}}}\right)^2-\left(\frac{\frac{1}{2}\Gamma\lambda}{1-\frac{1}{2}\Gamma\sqrt{y_{h_{Q_1}}}}\right)^2-\left(\frac{\frac{1}{2}\Gamma\lambda}{1-\frac{1}{2}\Gamma\sqrt{y_{a_{Q_1}}}}\right)^2}{\Gamma\sqrt{y_{h_{Q_1}}}\left(\sqrt{y_{h_{Q_1}}}+\sqrt{y_{a_{Q_1}}}\right)^2}=\frac{\Gamma^4\lambda^2}{4s}\left(\frac{4}{t^3}-\frac{1}{t^2}-\frac{1}{s^2}\right)$$

$$\frac{1}{\tau(Q_1)}\frac{\partial\tau(Q_1)}{\partial y_{ht}}=\frac{st^2}{4}\left(\frac{4}{t^3}-\frac{1}{t^2}-\frac{1}{s^2}\right)=\delta$$

$$\frac{\partial\tau(Q_1)}{\partial y_{at}}=\frac{\frac{\Gamma\left(\sqrt{y_{h_{Q_1}}}+\sqrt{y_{a_{Q_1}}}\right)}{1-\frac{1}{2}\Gamma\sqrt{y_{a_{Q_1}}}}\left(\frac{\frac{1}{2}\Gamma\lambda}{1-\frac{1}{2}\Gamma\sqrt{y_{a_{Q_1}}}}\right)^2-\left(\frac{\frac{1}{2}\Gamma\lambda}{1-\frac{1}{2}\Gamma\sqrt{y_{h_{Q_1}}}}\right)^2-\left(\frac{\frac{1}{2}\Gamma\lambda}{1-\frac{1}{2}\Gamma\sqrt{y_{a_{Q_1}}}}\right)^2}{\Gamma\sqrt{y_{a_{Q_1}}}\left(\sqrt{y_{h_{Q_1}}}+\sqrt{y_{a_{Q_1}}}\right)^2}=\frac{\Gamma^4\lambda^2}{4t}\left(\frac{4}{s^3}-\frac{1}{t^2}-\frac{1}{s^2}\right)$$

$$\frac{1}{\tau(Q_1)}\frac{\partial\tau(Q_1)}{\partial y_{at}}=\frac{s^2t}{4}\left(\frac{4}{s^3}-\frac{1}{t^2}-\frac{1}{s^2}\right)=\varepsilon$$

$$\delta+\varepsilon=\frac{st}{4}\left[4\left(\frac{1}{t^2}+\frac{1}{s^2}\right)-\left(\frac{1}{t}+\frac{1}{s}\right)-\left(\frac{t}{s^2}+\frac{s}{t^2}\right)\right]=\frac{\Gamma^2}{4-\Gamma^2}$$

$$\frac{\partial m(Q_1)}{\partial y_{ht}}=\frac{s^2}{\Gamma^2}\left(\frac{\Gamma^2}{st}-\frac{1}{\tau(Q_1)}\frac{\partial\tau(Q_1)}{\partial y_{ht}}\right)=\frac{s^2}{\Gamma^2}\left(\frac{\Gamma^2}{st}-\delta\right)=\frac{s}{4}\left(\frac{s^2+t^2+4t}{s^2+t^2}\right)$$

$$\frac{\partial n(Q_1)}{\partial y_{at}}=\frac{t^2}{\Gamma^2}\left(\frac{\Gamma^2}{st}-\frac{1}{\tau(Q_1)}\frac{\partial\tau(Q_1)}{\partial y_{at}}\right)=\frac{t^2}{\Gamma^2}\left(\frac{\Gamma^2}{st}-\varepsilon\right)=\frac{t}{4}\left(\frac{s^2+t^2+4s}{s^2+t^2}\right)$$

$$\frac{\partial m(Q_1)}{\partial y_{at}}=-\frac{s^2}{\Gamma^2}\frac{1}{\tau(Q_1)}\frac{\partial\tau(Q_1)}{\partial y_{at}}=-\frac{s^2\varepsilon}{\Gamma^2}$$

$$\frac{\partial n(Q_1)}{\partial y_{ht}}=-\frac{t^2}{\Gamma^2}\frac{1}{\tau(Q_1)}\frac{\partial\tau(Q_1)}{\partial y_{ht}}=-\frac{t^2\delta}{\Gamma^2}$$

$det\left({}_{\alpha=\frac{1}{2}}J_{h_0a_0}\right) = \frac{st}{\Gamma^2}\left[\frac{\Gamma^2}{st} - (\delta+\varepsilon)\right] = \frac{1}{2}$

$tr\left({}_{\alpha=\frac{1}{2}}J_{h_0a_0}\right) = \frac{s}{4}\left(\frac{s^2+t^2+4t}{s^2+t^2}\right) + \frac{t}{4}\left(\frac{s^2+t^2+4s}{s^2+t^2}\right) = \frac{8-\Gamma^2}{2\Gamma^2}$

The eigenvalues of the jacobian matrix will come from: $\mu^2 - \left(\frac{8-\Gamma^2}{2\Gamma^2}\right)\mu + \frac{1}{2} = 0$

$\mu_{1,2} = \frac{8-\Gamma^2 \pm \sqrt{64-7\Gamma^4-16\Gamma^2}}{4\Gamma^2}$

Therefore the eigenvalues are real under the condition $64 - 7\Gamma^4 - 16\Gamma^2 \geq 0$.

Adapting it to our contest, we have $0 < \Gamma \leq 2\sqrt{\frac{2}{7}\left(2\sqrt{2}-1\right)}$.

Remembering that a necessary condition for the existence of Q_1 is $\Gamma > \sqrt{2}$ (and supposing the existence of Q_1) we obtain:

– if $\sqrt{2} < \Gamma < 2\sqrt{\frac{2}{7}\left(2\sqrt{2}-1\right)}$, then Q_1 is a node;

– if it is $2\sqrt{\frac{2}{7}\left(2\sqrt{2}-1\right)} < \Gamma$, then Q_1 is a focus.

Stability. First, supposing that Q_1 is a node, the conditions $-1 < \mu_j < 1$ involve:

$$\begin{cases} -1 < \frac{8-\Gamma^2+\sqrt{64-7\Gamma^4-16\Gamma^2}}{4\Gamma^2} < 1 \\ -1 < \frac{8-\Gamma^2-\sqrt{64-7\Gamma^4-16\Gamma^2}}{4\Gamma^2} < 1 \\ \sqrt{2} < \Gamma < 2\sqrt{\frac{2}{7}\left(2\sqrt{2}-1\right)} \end{cases}$$

$$\begin{cases} \sqrt{64-7\Gamma^4-16\Gamma^2} < 5\Gamma^2 - 8 \\ -3\Gamma^2 - 8 < -\sqrt{64-7\Gamma^4-16\Gamma^2} < 5\Gamma^2 - 8 \\ \sqrt{2} < \Gamma < 2\sqrt{\frac{2}{7}\left(2\sqrt{2}-1\right)} \end{cases}$$

$$\begin{cases} \sqrt{64-7\Gamma^4-16\Gamma^2} < 5\Gamma^2 - 8 \\ \sqrt{64-7\Gamma^4-16\Gamma^2} < 3\Gamma^2 + 8 \\ \sqrt{2} < \Gamma < 2\sqrt{\frac{2}{7}\left(2\sqrt{2}-1\right)} \end{cases}$$

$$\begin{cases} \sqrt{64-7\Gamma^4-16\Gamma^2} < 5\Gamma^2 - 8 \\ \sqrt{2} < \Gamma < 2\sqrt{\frac{2}{7}\left(2\sqrt{2}-1\right)} \end{cases}$$

$$\begin{cases} \sqrt{2} < \Gamma \\ \sqrt{2} < \Gamma < 2\sqrt{\frac{2}{7}\left(2\sqrt{2}-1\right)} \end{cases}$$

Therefore, when Q_1 is a node it is certainly stable.

$$\begin{cases} \left(\frac{8-\Gamma^2}{4\Gamma^2}\right)^2 + \left(\frac{\sqrt{7\Gamma^4+16\Gamma^2-64}}{4\Gamma^2}\right)^2 < 1 \\ \Gamma > 2\sqrt{\frac{2}{7}\left(2\sqrt{2}-1\right)} \end{cases}$$

$$\begin{cases} always\ verified \\ \Gamma > 2\sqrt{\frac{2}{7}\left(2\sqrt{2}-1\right)} \end{cases}$$

In conclusion, even when it is a focus, Q_1 is stable.

The stable point Q_2. At last, we will prove that all the outcomes above can worth even for the point Q_2.

In fact $y_{h_{Q_2}} = y_{a_{Q_1}}$ and $y_{a_{Q_2}} = y_{h_{Q_1}}$ and, from a laborious but trivially manipulation, it can be proved that $\tau\left(Q_2\right) = \tau\left(Q_1\right)$, $\frac{\partial\tau(Q_2)}{\partial y_{ht}} = \frac{\partial\tau(Q_1)}{\partial y_{at}}$, $\frac{\partial\tau(Q_2)}{\partial y_{at}} = \frac{\partial\tau(Q_1)}{\partial y_{ht}}$.

Hence: $\frac{\partial m(Q_2)}{\partial y_{ht}} = \frac{\partial n(Q_1)}{\partial y_{at}}$; $\frac{\partial n(Q_2)}{\partial y_{at}} = \frac{\partial m(Q_1)}{\partial y_{ht}}$; $\frac{\partial m(Q_2)}{\partial y_{at}} = \frac{\partial n(Q_1)}{\partial y_{ht}}$; $\frac{\partial n(Q_2)}{\partial y_{ht}} = \frac{\partial m(Q_1)}{\partial y_{at}}$

The jacobian matrix $J_{h_0a_0}\left(Q_2\right)$ can be obtained from $J_{h_0a_0}\left(Q_1\right)$ changing the first raw/column with the second one; therefore it has the same eigenvalues of the previous matrix.

Existence, nature and stability of Q_2 hold under the same conditions as Q_1.

5.1.3. Proposition 2.8. Proof. The regions h_0a_1 and h_1a_0.

Because of the symmetry of the map [2.9b] we will consider only the region h_0a_1 and extend the results to the opposite region h_1a_0.

Into the region h_0a_1 the map becomes:

$$\mathbf{M_{h_0a_0}}\left(\alpha = \frac{1}{2}\right): \begin{cases} y_{ht+1} = \left[\frac{\frac{\Gamma}{2}\left(\sqrt{y_{ht}}+\sqrt{y_{at}}\right)}{\frac{\lambda^2}{\left(1-\frac{1}{2}\Gamma\sqrt{y_{ht}}\right)^2}+1}\right]\frac{\lambda^2}{\left(1-\frac{1}{2}\Gamma\sqrt{y_{ht}}\right)^2} \\ y_{at+1} = \left[\frac{\frac{\Gamma}{2}\left(\sqrt{y_{ht}}+\sqrt{y_{at}}\right)}{\frac{\lambda^2}{\left(1-\frac{1}{2}\Gamma\sqrt{y_{ht}}\right)^2}+1}\right] \end{cases}$$

with $y_{ht} < 4\left(\frac{1-\lambda}{\Gamma}\right)^2$ and $y_{at} \geq 4\left(\frac{1-\lambda}{\Gamma}\right)^2$

As usual, we assume $y_{jt+1} = y_{jt} = y_j$ and $x_j = \sqrt{y_j}$. Then:

$$\begin{cases} x_h^2 = \left[\frac{\frac{\Gamma}{2}(x_h+x_a)}{\frac{\lambda^2}{\left(1-\frac{\Gamma}{2}x_h\right)^2}+1}\right]\frac{\lambda^2}{\left(1-\frac{\Gamma}{2}x_h\right)^2} \\ x_a^2 = \frac{\frac{\Gamma}{2}(x_h+x_a)}{\frac{\lambda^2}{\left(1-\frac{\Gamma}{2}x_h\right)^2}+1} \end{cases}$$

with $x_h < \frac{2}{\Gamma}(1-\lambda)$ and $x_a \geq \frac{2}{\Gamma}(1-\lambda)$

$$\begin{cases} \left[x_h\left(1-\frac{\Gamma}{2}x_h\right)+\lambda x_a\right]\left[x_h\left(1-\frac{\Gamma}{2}x_h\right)-\lambda x_a\right]=0 \\ x_a^2 = \frac{\frac{\Gamma}{2}(x_h+x_a)}{\frac{\lambda^2}{\left(1-\frac{\Gamma}{2}x_h\right)^2}+1} \end{cases}$$

From the condition $x_h < \frac{2}{\Gamma}(1-\lambda)$, the relation $x_h\left(1-\frac{\Gamma}{2}x_h\right)+\lambda x_a = 0$ does not intercept any point, into the region we are considering and can be omitted. So it is:

$$\begin{cases} x_a = -\frac{\Gamma}{2\lambda}x_h^2 + \frac{1}{\lambda}x_h & Parabola \\ x_h^2 + x_a^2 = \frac{\Gamma}{2}(x_h+x_a) & Circumference \end{cases}$$

Parabola. The first curve is a parabola of the plane $(x_h\,,\,x_a)$. It has the vertex $E\left(\frac{1}{\Gamma},\frac{1}{2\lambda\Gamma}\right)$. It intercepts the $x_a = 0$ axis in $O(0,0)$ and in $X_P\left(\frac{2}{\Gamma},0\right)$; the bisector $x_a = x_h$ in $O(0,0)$ and in $B_P\left(\frac{2}{\Gamma}(1-\lambda)\,,\frac{2}{\Gamma}(1-\lambda)\right)$. The latter is a border point.

A second border point can be obtained intercepting the parabola with the line $x_a = \frac{2}{\Gamma}(1-\lambda)$ and is $D_P\left(\frac{2\lambda}{\Gamma},\frac{2}{\Gamma}(1-\lambda)\right)$.

Circumference. The second curve is a circumference of the plane $(x_h\,,\,x_a)$. Its center is $C\left(\frac{\Gamma}{4},\frac{\Gamma}{4}\right)$ and its ray $\rho = \frac{\sqrt{2}}{4}\Gamma$.

It intercepts the bisector $x_a = x_h$ in $O(0,0)$ and the axes, besides the origin, in $X_C\left(\frac{\Gamma}{2},0\right)$ and $Y_C\left(0,\frac{\Gamma}{2}\right)$.

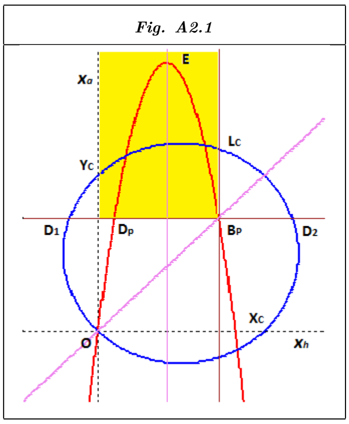
It intercepts the line $x_a = \frac{2}{\Gamma}(1-\lambda)$ in the two points:

$D_C^1\left(\frac{\Gamma}{4}-\frac{1}{4\Gamma}\sqrt{\left[\Gamma^2+8\left(\sqrt{2}+1\right)(1-\lambda)\right]\left[\Gamma^2-8\left(\sqrt{2}-1\right)(1-\lambda)\right]},\frac{2}{\Gamma}(1-\lambda)\right)$ and

$D_C^2\left(\frac{\Gamma}{4}+\frac{1}{4\Gamma}\sqrt{\left[\Gamma^2+8\left(\sqrt{2}+1\right)(1-\lambda)\right]\left[\Gamma^2-8\left(\sqrt{2}-1\right)(1-\lambda)\right]},\frac{2}{\Gamma}(1-\lambda)\right)$.

Finally it intercepts the line $x_h = \frac{2}{\Gamma}(1-\lambda)$ in the border point

$L_C\left(\frac{2}{\Gamma}(1-\lambda)\,,\frac{\Gamma}{4}+\frac{1}{4\Gamma}\sqrt{\left[\Gamma^2+8\left(\sqrt{2}+1\right)(1-\lambda)\right]\left[\Gamma^2-8\left(\sqrt{2}-1\right)(1-\lambda)\right]}\right)$



5.1.3.1. The fixed points.

In order to determine the fixed points of the region h_0a_1 and, by symmetry, of h_1a_0, we have to consider the mutual positions of the two curves above. Of course we are only interested in their segments compatible with the constraints.

In the following, we will identify a set of necessary, basic conditions.

Parabola. About this curve, it is to be noted that its point D_P may be at the left, or at the right of B_P. Comparing the first coordinates of these points, we can see that it depends whether is $\lambda < \frac{1}{2}$, or $\lambda > \frac{1}{2}$, respectively.

When D_P is at the right of B_P, or, as limit position, D_P is on B_P, all the parabola lies out of the boundaries and there are no fixed points into the region h_0a_1. Consequently, we will suppose $0 < \lambda < \frac{1}{2}$. [1]

Circumference. We requires that the circumference intercepts the border lines $x_{a,h} = \frac{2}{\Gamma}(1-\lambda)$, that is $\Gamma^2 \geq 8\left(\sqrt{2}-1\right)(1-\lambda)$.

Again we requires that $D_C^1 \preceq D_P \preceq D_C^2$ (where "$a \preceq b$" means "*a is not at the right of b*").

We will develop them under the hypothesis $0 < \lambda < \frac{1}{2}$:

$$D_C^1 \preceq D_P \Rightarrow \Gamma^2 - 8\lambda \leq \sqrt{\left[\Gamma^2 + 8\left(\sqrt{2}+1\right)(1-\lambda)\right]\left[\Gamma^2 - 8\left(\sqrt{2}-1\right)(1-\lambda)\right]}$$

$$D_P \preceq D_C^2 \Rightarrow \Gamma^2 - 8\lambda \geq -\sqrt{\left[\Gamma^2 + 8\left(\sqrt{2}+1\right)(1-\lambda)\right]\left[\Gamma^2 - 8\left(\sqrt{2}-1\right)(1-\lambda)\right]}$$

It involves $\Gamma^2 \geq 8\lambda^2 - 8\lambda + 4$.

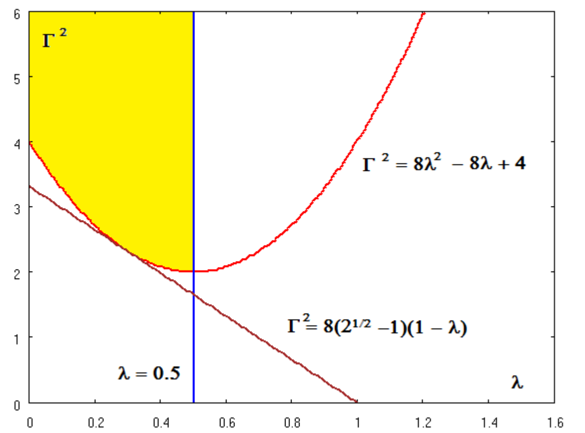
Hence:

[1] It is also to be noted that, for $\lambda < \frac{1}{2}$ the vertex E of the parabola is above the bisector, while for $\lambda > \frac{1}{2}$ E is below the bisector

$$\begin{cases} \Gamma^2 \geq 8\lambda^2 - 8\lambda + 4 \\ \Gamma^2 \geq 8\left(\sqrt{2}-1\right)(1-\lambda) \\ 0 < \lambda < \frac{1}{2} \end{cases}$$

Into the plane (λ, Γ^2), the first relation is a parabola, while the second is a line tangent to the parabola (see the next figure).

Fig. A2.2



Therefore the conditions may be simplified:

$$\begin{cases} \Gamma^2 \geq 8\lambda^2 - 8\lambda + 4 \\ 0 < \lambda < \frac{1}{2} \end{cases} \qquad [\zeta_1]$$

We will assume them from now on.

The useful intervals.

Now, let's suppose to fix Γ and move λ into the interval $\left(0, \frac{1}{2}\right)$. In such a way, of the two curves, only the parabola will move, the circumference not being affected by the parameter λ.

The parabola will always pass through the points O and X_P independent of λ; the abscissa of the vertex will remain stable on $\frac{1}{\Gamma}$, while the ordinate will change, extending or contracting the curve itself.

Exactly, until the parameter λ is diminishing, the vertex E goes up along the vertical line $x_h = \frac{1}{\Gamma}$; on the contrary, when λ is increasing, E goes down.

Moving λ, also the lines $x_{a,h} = \frac{2}{\Gamma}(1-\lambda)$ are affected, contracting towards the origin, or expanding in the opposite direction, respectively when λ is increasing or diminishing. Finally, the point B_P goes up and down along the bisector, in the same way.

In the following, we will consider four steps.

We will see three types of figures. The first ones are drawn in the plane $(x_h\,,\, x_a)$ and are made by the two curves (respectively the parabola and the circumference) generating the fixed points. The second type represents figures drawn in the plane $(\lambda\,,\, \Gamma^2)$; they are relative to the existence conditions. Finally, the third ones come from the steady state conditions $M\,(y_{ht}, y_{at}) = (y_{ht}, y_{at})$ for some particular choices of the parameters and, as usual, they only have an explanatory purpose; they cover the plane $(y_h\,,\, y_a)$.

First step: $\Gamma = 2\sqrt{1-\lambda}$**.**

At the beginning, we may consider the particular situation in which the parabola and the circumference intercept each other on the bisector; this happens when

$\Gamma = 2\sqrt{1-\lambda}$ (or $\lambda = 1 - \frac{\Gamma^2}{4}$) that is compatible with the necessary conditions $[\zeta_1]$ above.

Let's indicate with $Q_1 = B_P = B_c$ the common point between the two curves.

We ask if they also pass through an other point on "Nord-West", inside the boundaries. It depends from the slope of the curves in $Q_1 = B_P = B_c$.

The slope of the tangent at the parabola in B_P is $m_P\,(B_P) = -\frac{1-2\lambda}{\lambda}$.[2]

The slope of the tangent at the circumference in B_C is $m_C\,(B_C) = -1$[3]

Then the two curves have the same slope for $\lambda = \frac{1}{3}$ and $m_P\,(B_P) \gtrless m_C\,(B_C)$ for $\lambda \gtrless \frac{1}{3}$ (the circumference is steeper/less steep than the parabola).

Therefore, for $\lambda \geq \frac{1}{3}$, the two curves, a part from Q_1 and the origin, have not any further interceptions.

Indeed, for $\lambda < \frac{1}{3}$, a new interception Q_2 appears on Nord-West.

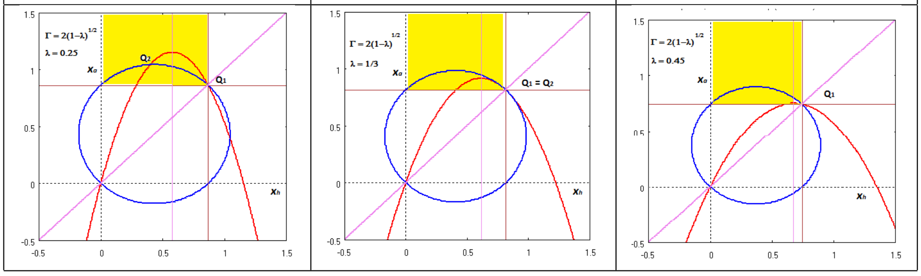
Some further hypotheses, will allow us to know if Q_2 lies into the boundaries, or not.

Meantime, it is to be noted that, for $\Gamma = 2\sqrt{1-\lambda}$, $\lambda \gtreqless \frac{1}{3} \Leftrightarrow \Gamma \lesseqgtr \frac{2}{3}\sqrt{6}$

[2]In fact, deriving with respect to x_h the expression of the parabola, we have $\frac{dx_a}{dx_h} = -\frac{\Gamma}{2\lambda}x_h + \frac{1}{\lambda}$. The result comes substituting the coordinates of B_P.

[3]Deriving by x_h into the equation of the circumference, it is $\frac{dx_a}{dx_h} = \frac{\Gamma - 4x_h}{4x_a - \Gamma}$ and result comes substituting $x_a = x_h$.

Fig. A2.3 *Fig. A2.4* *Fig. A2.5*



Second step: $\Gamma \leq \frac{2}{3}\sqrt{6}$**.**

With this hypothesis, when the two curves pass through the common point $Q_1 = B_P = B_c$ (that is when $\lambda = 1 - \frac{\Gamma^2}{4}$) the circumference is steeper than (or, at least, as steep as) the parabola.

Then, for $\lambda \geq 1 - \frac{\Gamma^2}{4}$ there are not any interceptions inside the boundaries.

On the other hand, for $\lambda < 1 - \frac{\Gamma^2}{4}$ the point Q_2 appears on Nord-West.

Combining the hypothesis $\Gamma \leq \frac{2}{3}\sqrt{6}$ (i.e. $\Gamma^2 \leq \frac{8}{3}$) and the relation $\lambda < 1 - \frac{\Gamma^2}{4}$, with the $[\zeta_1]$, we obtain the conditions by which Q_2 lies into the boundaries:

$$\begin{cases} \Gamma^2 \geq 8\lambda^2 - 8\lambda + 4 \\ \Gamma^2 \leq \frac{8}{3} \\ \Gamma^2 < 4(1-\lambda) \\ 0 < \lambda < \frac{1}{2} \end{cases} \Rightarrow \textit{One steady state exists in each region } h_0a_1,\ h_1a_0$$

The resulting set is nonempty; one steady state appears in h_0a_1 and, symmetrically, in h_1a_0.

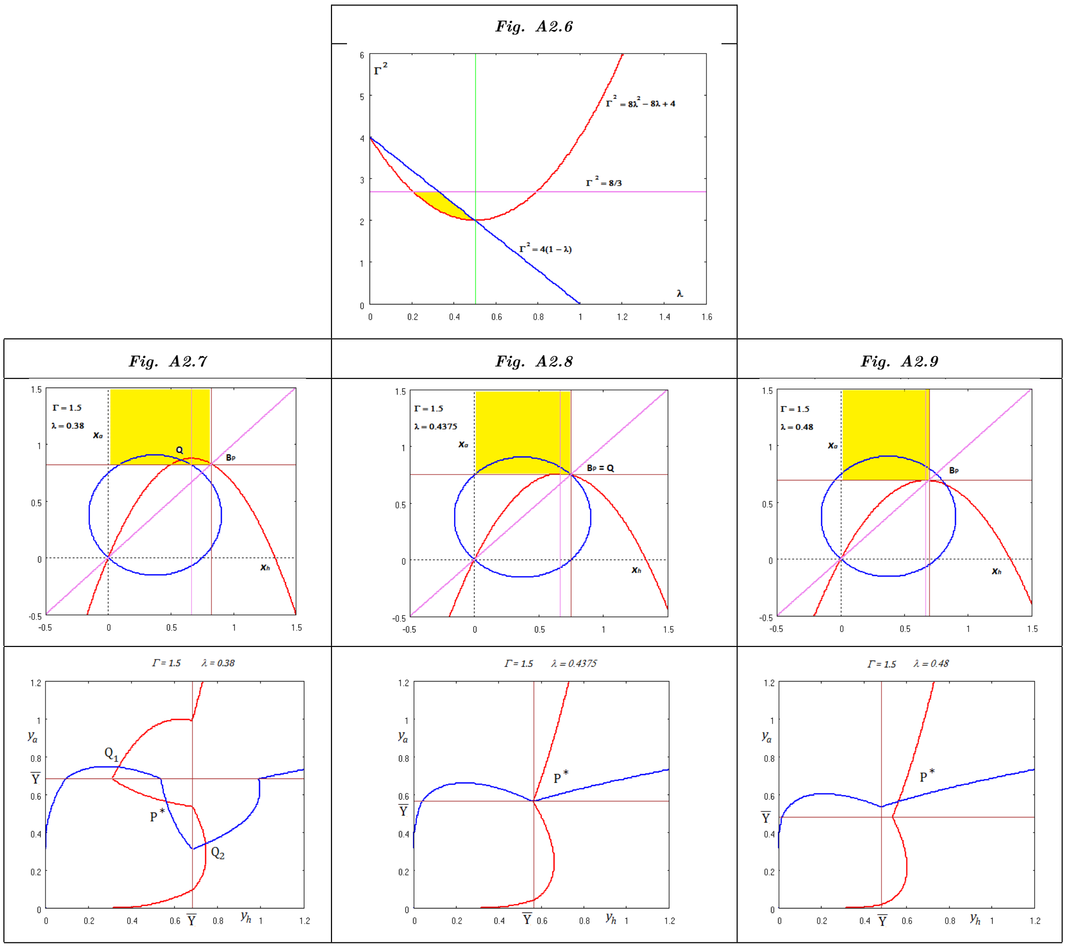


Fig. A2.6

Fig. A2.7 Fig. A2.8 Fig. A2.9

Third step: $\Gamma > \frac{2}{3}\sqrt{6}$ **and** $\lambda \leq 1 - \frac{\Gamma^2}{4}$**.**

As usual, we begin with $\lambda = 1 - \frac{\Gamma^2}{4}$. The parabola intercepts the circumference in $Q_1 = B_P = B_c$ and there, is steeper than the circumference.

It follows that the two curves have another common point Q_2 situated on Nord-West then, into the boundaries.

Diminishing λ, the parabola goes up, the point Q_2 moves towards the left, and it eventually remains the only valid interception between the two curves.

The conditions by which Q_2 stays into the boundaries are:

$$\begin{cases} \Gamma^2 \geq 8\lambda^2 - 8\lambda + 4 \\ \Gamma^2 > \frac{8}{3} \\ \Gamma^2 < 4(1-\lambda) \\ 0 < \lambda < \frac{1}{2} \end{cases} \Rightarrow \textit{One steady state exists in each region } h_0a_1,\ h_1a_0$$

Even in this case the resulting set is nonempty and it involves two symmetric steady states, one in each region.

Fig. A2.10

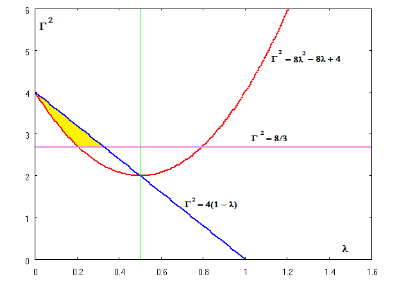
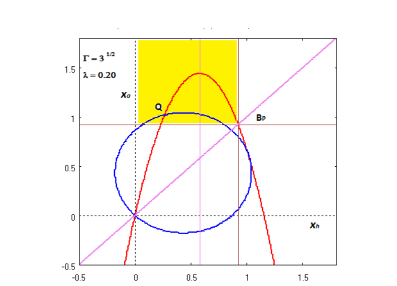
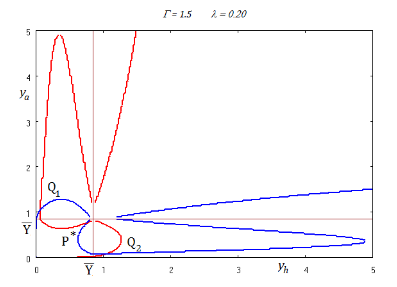


Fig. A2.11





Fourth step: $\Gamma > \frac{2}{3}\sqrt{6}$ and $\lambda > 1 - \frac{\Gamma^2}{4}$.

Let's suppose to increase λ up to $1 - \frac{\Gamma^2}{4}$.

The parabola goes down and the same does the line $x_a = \frac{2}{\Gamma}(1-\lambda)$. Two valid interceptions Q_1 and Q_2 appear into the boundaries.

The dynamic will stop when the two points reach each other and the parabola is tangent to the circumference.

The determination of the point of tangent between the parabola and the circumference.

To continue our development, we have to obtain the relation of tangent between the parabola and the circumference.

In order to do that, we have to consider the third degree equation

$\Gamma^2 x^3 - 4\Gamma x^2 + (\lambda\Gamma^2 + 4\lambda^2 + 4)\, x - 2\lambda\Gamma\,(1+\lambda) = 0.$

It can be reduced substituting $x = w + \frac{4}{3\Gamma}$ and it becomes:

$w^3 + \left(\frac{\lambda\Gamma^2 + 4\lambda^2 - \frac{4}{3}}{\Gamma^2}\right) w + \frac{2}{\Gamma^2}\left[\frac{8}{3\Gamma}\left(\frac{1}{9} + \lambda^2\right) - \lambda\Gamma\left(\frac{1}{3} + \lambda\right)\right] = 0\ .$

Setting $u = \frac{\lambda\Gamma^2 + 4\lambda^2 - \frac{4}{3}}{\Gamma^2}$ and $v = \frac{2}{\Gamma^2}\left[\frac{8}{3\Gamma}\left(\frac{1}{9} + \lambda^2\right) - \lambda\Gamma\left(\frac{1}{3} + \lambda\right)\right]$ the condition under which this equation has two real solutions, is: $\frac{v^2}{4} + \frac{u^3}{27} = 0$.

Applying that, setting $\Upsilon = \Gamma^2$ and simplifying, we obtain the relation that involves the condition of tangent between the two curves:

$T(\lambda, \Gamma^2) =$

$= \lambda (\Gamma^2)^3 + 39 (\Gamma^2)^2 (\lambda^2 + 18\lambda - 1) + 16\Gamma^2 (3\lambda^3 - 9\lambda^2 - 5\lambda - 1) + 64 (\lambda^4 + 2\lambda^2 + 1) = 0$

About this curve, we are interested only on its part into the region $0 < \lambda < \frac{1}{2}$. There it passes through the point $\left(\frac{1}{3}, \frac{8}{3}\right)$ and has two branches at the left, the upper of which is the one compatible with the boundaries, while the inferior is out.

Let's set $\lambda_T(\Gamma)$: $T(\lambda, \Gamma^2) = 0$. For each Γ, when $\lambda > \lambda_T(\Gamma)$, the two interceptions between the parabola and the circumference disappear.

Finally, these are the conditions by which the system has two interceptions:

$$\begin{cases} \Gamma^2 \geq 8\lambda^2 - 8\lambda + 4 \\ \Gamma^2 > \frac{8}{3} \\ \Gamma^2 > 4(1-\lambda) \\ 0 < \lambda \leq \lambda_T(\Gamma) < \frac{1}{2} \end{cases} \Rightarrow \textit{Two steady states exist in each region } h_0 a_1,\ h_1 a_0$$

Fig. A2.12

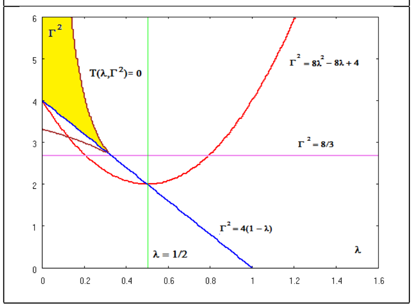


Fig. A2.13

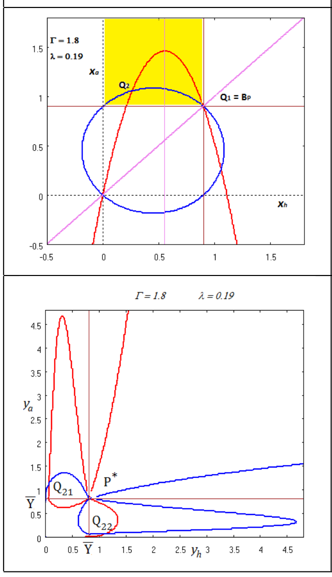


Fig. A2.14

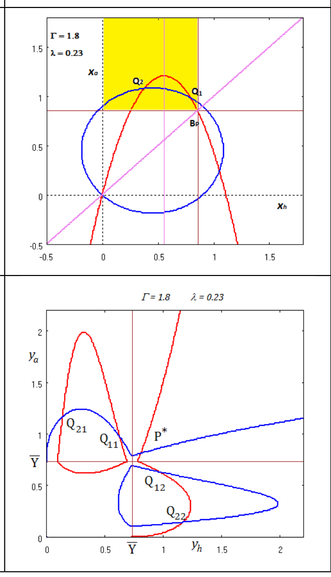


Fig. A2.15

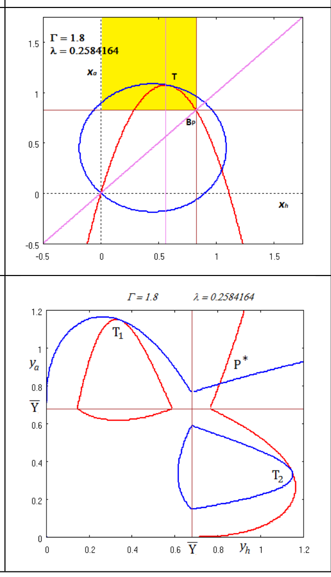
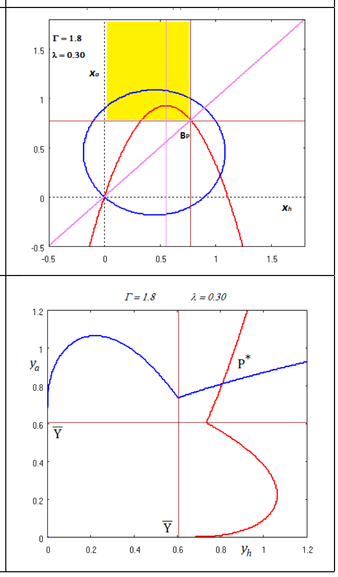


Fig. A2.16



6. Appendix to Chapter III

6.1. Proofs of propositions of Chapter 3

6.1.1. Proposition 3.3. Proof.

The steady state of autarky is $P^{\star}=\left([(1-\alpha)\,\Gamma_h]^{\frac{1}{1-\alpha}},[(1-\alpha)\,\Gamma_a]^{\frac{1}{1-\alpha}}\right)$

We will consider it into each of the four regions of the plane $\Omega=(y_{ht}>0)\cap(y_{at}>0)$.

6.1.1.1. Into the region h_0a_0

Considering the map into this region and substituting the coordinates of the point $P^{\star}$, we obtain:

$$M_{h_0a_0}\left(P^{\star}\right)\ :\ \begin{cases}[(1-\alpha)\,\Gamma_h]^{\frac{1}{1-\alpha}}=\frac{1}{\tau(P^{\star})}\left(\frac{\alpha\Gamma_h\lambda_h}{1-[(1-\alpha)\Gamma_h]^{\frac{1}{1-\alpha}}}\right)^{\frac{1}{1-\alpha}}\\ [(1-\alpha)\,\Gamma_a]^{\frac{1}{1-\alpha}}=\frac{1}{\tau(P^{\star})}\left(\frac{\alpha\Gamma_a\lambda_a}{1-[(1-\alpha)\Gamma_a]^{\frac{1}{1-\alpha}}}\right)^{\frac{1}{1-\alpha}}\end{cases}$$

with the conditions $\Gamma_j<\frac{(1-\lambda_j)^{1-\alpha}}{1-\alpha}$ for $j=h\ ,\ a$,

being $\tau\left(P^{\star}\right)=\dfrac{L_h\left(\frac{\alpha\Gamma_h\lambda_h}{1-[(1-\alpha)\Gamma_h]^{\frac{1}{1-\alpha}}}\right)^{\frac{1}{1-\alpha}}+L_a\left(\frac{\alpha\Gamma_a\lambda_a}{1-[(1-\alpha)\Gamma_a]^{\frac{1}{1-\alpha}}}\right)^{\frac{1}{1-\alpha}}}{L_h[(1-\alpha)\Gamma_h]^{\frac{1}{1-\alpha}}+L_a[(1-\alpha)\Gamma_a]^{\frac{1}{1-\alpha}}}$.

Dividing, member by member, the first relation with the second, it results:

$\frac{\lambda_h}{1-[(1-\alpha)\Gamma_h]^{\frac{1}{1-\alpha}}}=\frac{\lambda_a}{1-[(1-\alpha)\Gamma_a]^{\frac{1}{1-\alpha}}}$

Substituting this last into $\tau\left(P^{\star}\right)$ it is:

$$\tau\left(P^{\star}\right)=\frac{\left(L_h\Gamma_h^{\frac{1}{1-\alpha}}+L_a\Gamma_a^{\frac{1}{1-\alpha}}\right)\left(\frac{\alpha\lambda_a}{1-[(1-\alpha)\Gamma_a]^{\frac{1}{1-\alpha}}}\right)^{\frac{1}{1-\alpha}}}{(1-\alpha)^{\frac{1}{1-\alpha}}\left(L_h\Gamma_h^{\frac{1}{1-\alpha}}+L_a\Gamma_a^{\frac{1}{1-\alpha}}\right)}=\frac{1}{(1-\alpha)^{\frac{1}{1-\alpha}}}\left(\frac{\alpha\lambda_a}{1-[(1-\alpha)\Gamma_a]^{\frac{1}{1-\alpha}}}\right)^{\frac{1}{1-\alpha}}$$

Finally, substituting into any one relation of the system, it's easy to see that it becomes an identity.

This proves that the necessary and sufficient conditions for $P^{\star}$ to stay in h_0a_0 are:

$$\begin{cases} \frac{\lambda_h}{1-[(1-\alpha)\Gamma_h]^{\frac{1}{1-\alpha}}} = \frac{\lambda_a}{1-[(1-\alpha)\Gamma_a]^{\frac{1}{1-\alpha}}} \\ \Gamma_h < \frac{(1-\lambda_h)^{1-\alpha}}{1-\alpha} \\ \Gamma_a < \frac{(1-\lambda_a)^{1-\alpha}}{1-\alpha} \end{cases}$$

6.1.1.2. Into the region h_0a_1 (and h_1a_0)

These are the border conditions required:

– h_0a_1: $\Gamma_h < \frac{(1-\lambda_h)^{1-\alpha}}{1-\alpha}$ and $\Gamma_a \geq \frac{(1-\lambda_a)^{1-\alpha}}{1-\alpha}$

– h_1a_0: $\Gamma_h \geq \frac{(1-\lambda_h)^{1-\alpha}}{1-\alpha}$ and $\Gamma_a < \frac{(1-\lambda_a)^{1-\alpha}}{1-\alpha}$

We will consider only the region h_0a_1, being the results perfectly adaptable to the other one.

The map is

$$M_{h_0a_1}(P^\star) \ : \ \begin{cases} [(1-\alpha)\Gamma_h]^{\frac{1}{1-\alpha}} = \frac{1}{\tau(P^\star)}\left(\frac{\alpha\Gamma_h\lambda_h}{1-[(1-\alpha)\Gamma_h]^{\frac{1}{1-\alpha}}}\right)^{\frac{1}{1-\alpha}} \\ [(1-\alpha)\Gamma_a]^{\frac{1}{1-\alpha}} = \frac{1}{\tau(P^\star)}(\alpha\Gamma_a)^{\frac{1}{1-\alpha}} \end{cases}$$

Dividing side by side the two equations, it comes $\Gamma_h = \frac{(1-\lambda_h)^{1-\alpha}}{1-\alpha}$ that is incompatible with the condition of the system.

So $P^\star$ cannot lie into the region h_0a_1 and, with analogous prove, neither into h_1a_0.

6.1.1.3. Into the region h_1a_1

In this region, the map is: $M_{h_1a_1}(P^\star) \ : \ \begin{cases} [(1-\alpha)\Gamma_h]^{\frac{1}{1-\alpha}} = \frac{(\alpha\Gamma_h)^{\frac{1}{1-\alpha}}}{\tau(P^\star)} \\ [(1-\alpha)\Gamma_a]^{\frac{1}{1-\alpha}} = \frac{(\alpha\Gamma_a)^{\frac{1}{1-\alpha}}}{\tau(P^\star)} \end{cases}$

and the conditions required at borders are $y_{jt} \geq \overline{Y_j}$, for $j = h$, a.

Observing that $\tau(P^\star) = \left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{1-\alpha}}$, it is immediate to verify that $P^\star$ may be a solution.

Therefore, whenever the border conditions are satisfied, the steady state solution $P^\star$ exists.

Consequently, the necessary and sufficient conditions of the problem are:

$$\begin{cases} \Gamma_h \geq \frac{(1-\lambda_h)^{1-\alpha}}{1-\alpha} \\ \Gamma_a \geq \frac{(1-\lambda_a)^{1-\alpha}}{1-\alpha} \end{cases}$$

6.1.1.4. The stability of $P^{\star}$ into the region h_1a_1

Now we will prove that $P^{\star}$ is asymptotically stable into the region h_1a_1.

In fact, the map in that region is:

$$\begin{cases} y_{ht+1} = \frac{(1-\alpha)\Gamma_h^{\frac{1}{1-\alpha}}}{L_h\Gamma_h^{\frac{1}{1-\alpha}}+L_a\Gamma_a^{\frac{1}{1-\alpha}}}\left(L_h\Gamma_h y_{ht}^{\alpha} + L_a\Gamma_a y_{at}^{\alpha}\right) \\ y_{at+1} = \frac{(1-\alpha)\Gamma_a^{\frac{1}{1-\alpha}}}{L_h\Gamma_h^{\frac{1}{1-\alpha}}+L_a\Gamma_a^{\frac{1}{1-\alpha}}}\left(L_h\Gamma_h y_{ht}^{\alpha} + L_a\Gamma_a y_{at}^{\alpha}\right) \end{cases}$$

Consequently, the jacobian matrix of the system calculated in $P^{\star}$ is:

$$J_{h_1a_1}\left(P^{\star}\right) = \frac{\alpha}{L_h\Gamma_h^{\frac{1}{1-\alpha}}+L_a\Gamma_a^{\frac{1}{1-\alpha}}}\begin{bmatrix} L_h\Gamma_h^{\frac{1}{1-\alpha}} & L_a\Gamma_h^{\frac{1}{1-\alpha}} \\ L_h\Gamma_a^{\frac{1}{1-\alpha}} & L_a\Gamma_a^{\frac{1}{1-\alpha}} \end{bmatrix}$$

$detJ_{h_1f_1}\left(P^{\star}\right) = 0$; $trJ_{h_1f_1}\left(P^{\star}\right) = \alpha$

From $\mu^2 - \alpha\mu = 0$ we obtain the eigenvalues $\mu_1 = 0$ and $\mu_2 = \alpha$.

This proves that $P^{\star}$ is a node, asymptotically attractive.

We can also determine the eigenvectors.

For brevity let's set $\delta = \frac{\alpha}{L_h\Gamma_h^{\frac{1}{1-\alpha}}+L_a\Gamma_a^{\frac{1}{1-\alpha}}}$

For $\mu_1 = 0$ the eigenvector can be obtained from $\delta L_h\Gamma_h^{\frac{1}{1-\alpha}}u_{11} + \delta L_a\Gamma_h^{\frac{1}{1-\alpha}}u_{12} = 0$.

Hence $u_1 = \begin{bmatrix} L_a \\ -L_h \end{bmatrix}$.

For $\mu_1 = \alpha$ we have: $\delta L_h\Gamma_h^{\frac{1}{1-\alpha}}u_{21} + \delta L_a\Gamma_h^{\frac{1}{1-\alpha}}u_{22} = \alpha u_{21}$. Then $u_2 = \begin{bmatrix} \Gamma_h^{\frac{1}{1-\alpha}} \\ \Gamma_a^{\frac{1}{1-\alpha}} \end{bmatrix}$.

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